Another philosophical look at twistor theory

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Abstract

Despite its being one of Roger Penrose's greatest contributions to spacetime physics, there is a dearth of philosophical literature on twistor theory. The one exception to this is (Bain, 2006)—but although excellent, there remains much to be said on the foundations and philosophy of twistor theory. In this article, we (a) present for philosophers an introduction to twistor theory, (b) consider how the spacetime–twistor correspondence interacts with the philosophical literature on theoretical equivalence, and (c) explore the bearing which twistor theory might have on philosophical issues such as the status of dynamics, the geometrisation of physics, spacetime ontology, the emergence of spacetime, and symmetry-to-reality inferences. We close with an elaboration of a variety of further opportunities for philosophical investigation into twistor theory.

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1 Introduction

Contemporary philosophy of spacetime has many facets. As one example: there are famous and well-explored debates regarding the ontological status of spacetime: is this a fundamental entity in its own right (*per* the substantivalist), or, rather, is spacetime somehow reducible to material entities and relations between said entities (*per* the relationalist)?[1] And as another: there are debates regarding the geometrisation of physics: what does it mean to geometrise a particular physical theory or effect, and what are the merits of doing so?[2]

Sometimes, these debates are broadly orthogonal to one another. For example: substantivalists and relationalists can debate over the ontological status of the spacetime manifold in general relativity in light of the hole argument (re-introduced into the philosophy literature by Earman and Norton (1987)[3]), but the verdict here is largely independent of questions of geometrisation.[4] And to take another example: one can talk of the geometrisation of Newtonian gravity in Newton-Cartan theory (see e.g. (Fried-

[1] Of course, there is a variety of ways of making more precise the difference between substantivalism and relationalism—see e.g. (Baker, 2021; Dasgupta, 2011; Earman, 1977; North, 2021; Pooley, 2013a)—but for now, this characterisation is sufficient.

[2] For literature on this issue, see (Dürr, 2020; Kalinowski, 1988; Lehmkuhl, 2009).

[3] For some philosophical pre-history of the hole argument, see (Weatherall, 2020).

[4] Or at least, isn't connected in any particularly straightforward way. Perhaps one can argue that it is the geometrisation of gravity in general relativity—in the sense that this is a manifestation of spacetime curvature (although see (Lehmkuhl, 2014) for some contrary thoughts)—which leads to its general covariance, and thereby to the hole argument raising its head. However, every link in this chain of connections here is difficult and controversial: see e.g. (Norton et al., 2023) for discussion.

man, 1983; Knox, 2013)) without calling into question the reality (or otherwise) of the spacetime manifold.

This being said, there are also natural points of convergence between these topics. One such point—the point which, indeed, will constitute our exclusive focus in this article—is twistor theory, developed initially by Roger Penrose (1967). The guiding idea behind twistor theory is that conformally invariant field theory dynamics on a spacetime manifold can be mapped to geometrical statements on an alternative space known as 'twistor space', and *vice versa*. (For recent reviews of twistor theory, see (Adamo, 2018; Atiyah et al., 2017).) Via this map, physical theories not only appear to be purged of their commitment to the basic structure of the spacetime manifold, but also appear to be geometrised in a quite radical sense. In physics and mathematics, twistors have found multifarious applications—for example, to string theory, holography, to the evaluation of perturbation series (see e.g. (Atiyah et al., 2017)), and to the general project of unification in physics (see e.g. (Woit, 2021)). However, within the philosophy of physics, the significance of twistor theory remains lamentably under-explored.

To our knowledge, the only published discussion of twistor theory in the philosophy literature is due to Bain (2006).[5] Although we will engage with Bain's excellent discussion quite substantially in what follows, in our view there remains much regarding twistor theory which warrants greater foundational scrutiny; moreover, there remains space in the market for a more accessible introduction to twistors for philosophers.[6]

With all of this in mind, then, the structure of this article is as follows. In §2, we provide a clear route into twistor theory for philosophers of physics (it should be stressed that this constitutes but one such route—we discuss this more below). In §3, we assess the spacetime–twistor correspondence from the point of view of the modern philosophical literature on theoretical equivalence, in particular categorical equivalence. In §4, we further explore the philosophical and foundational significance of twistor theory with respect to questions of (i) dynamics—is it true that twistor theory is purged of dynamics, as is sometimes claimed?; (ii) geometrisation—in what sense does twistor theory offer a novel approach to the geometrisation of physical theories?; (iii) ontology—what comprise the metaphysical commitments of a theory set on twistor space?; (iv) the emergence of spacetime—does twistor theory offer any novel outlook on this issue?; and (v) symmetry principles—does twistor theory present interesting and novel challenges

[5] There is also (March, 2023), which is approximately contemporaneous with our article; however, since that piece focuses on non-relativistic twistor theory and its foundational applications, we will not discuss it further here.

[6] Since this already involves a significant amount of work, we limit ourselves to what Bain (2006) refers to as 'stone age' twistor theory (i.e., the theory in the period 1967–80), while noting that contemporary twistor theory has even broader applications, on which see e.g. (Atiyah et al., 2017).

to Earman's famous symmetry principles, as presented in (Earman, 1989)? All of this achieved, we close this article in §5.

2 Mathematics of twistors

In this section, we present for philosophers (i) the mathematics of twistor space and some particular paths to the construction of that space (§2.1), and (ii) the Penrose transformations, which map fields in spacetime (and their dynamics) to geometrical statements in twistor space (§2.2). With these preliminaries in hand, we will be in a position to assess in detail in §§3–4 the philosophical significance of twistor theory.

In order to make our exposition as digestible as possible, we attempt here to synthesise aspects of the various expositions of twistor theory found in the standard texts (Penrose and Rindler, 1988a,b; Ward and Wells, 1990), the more informal treatments (Huggett and Tod, 1985; Adamo, 2018), and the recent major review article (Atiyah et al., 2017). Throughout the article, we use notation consistent with (Adamo, 2018).

Before we begin, one further aspect of the twistor theory literature should be flagged for the aspiring philosopher of twistors: much of the literature works in coordinates, and generally we will follow suit. Most of the time, the constructions can be lifted to a coordinate-free description, but this is rarely addressed explicitly. We will see an example of this shortly.

2.1 Twistor space

There are at least three equivalent ways of defining twistors:[7]

1. twistors as α-planes,
2. twistors as spinors for the conformal group,
3. twistors as solutions to the twistor equation.

We turn our attention first to approach (3). And to write down the twistor equation, we should first discuss spinors. These also matter in their own right: as pointed out in (Penrose and Rindler, 1988b, p. 43, emphasis added),

> two-component spinor calculus is a very specific calculus for studying the structure of space-time manifolds. Indeed, the four-dimensionality and

[7] Equivalent in 4-dimensional flat space, at least: see (Atiyah et al., 2017, p. 11), although even there this is not substantiated.

> $(+---)$ signature of space-time, together with the desirable global properties of orientability, time-orientability, and existence of spin structure, may all, in a sense, be regarded as derived from two-component spinors, rather than just given. However [...] there is still only a limited sense in which these properties can be so regarded, because the manifold of space-time points itself has to be given beforehand, even though the nature of this manifold is somewhat restricted by its having to admit the appropriate kind of spinor structure. If we were to attempt to take totally seriously the philosophy that all the space-time concepts are to be derived from more primitive spinorial ones, then we would have to find some way in which the space-time points themselves can be regarded as derived objects. *Spinor algebra by itself is not rich enough to achieve this, but a certain extension of spinor algebra, namely twistor algebra, can indeed be taken as more primitive than space-time itself.*

In light of the above, we will shortly provide a short exposition of spinor algebra and calculus, following the presentation by Fatibene and Francaviglia (2003).

In addition to this, it is worth noting that we here consider from the outset (unlike some other treatments) *complexified* Minkowski space $\mathbb{CM}$ as our base space, in place of the more familiar real Minkowski space $\mathbb{M}$. This is partly for purposes of brevity and partly because "[t]wistors [...] are essentially *complex* objects. To get a proper understanding of twistor geometry, it is therefore necessary to consider complex geometry and, in particular, the [complexification of Minkowski space]" (Penrose and Rindler, 1988b, p. 306, emphasis in original). Complexified Minkowski space is simply the manifold $\mathbb{CM} = (\mathbb{C}^4, \eta)$ with the metric η obtained by holomorphic extension of the real Minkowski metric.[8,9] Note in particular that this is not a case of Wick rotation since all four coordinates are allowed to range over all complex numbers. Note also that this space will play a double role: first in constructing twistors, and later in the correspondence between twistorial objects and fields defined on spacetime. To recover the field as defined on *real* spacetime, it suffices to restrict complexified spacetime to the subset

[8] This means simply that the coordinates in the expression for the metric $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$ are allowed to take complex values. Note in particular that this does not make this metric Hermitian. See (Penrose and Rindler, 1988b, p. 64).

[9] At this point we could still treat $\mathbb{CM}$ as a vector space (or an affine space) rather than as a manifold but soon we will require techniques from differential geometry, so in anticipation we define it as a manifold right off the bat. The reader should be aware that the mathematical literature often vacillates between the vector space and manifold descriptions. Luckily, there is an obvious manifold structure on $\mathbb{CM}$: a standard topology is defined using the Euclidean inner product, and the maximal atlas extends from the identity global coordinate chart.

of real spacetime points, so we will typically not do this explicitly in what follows.[10]

Our first goal is to define spinor bundles on our base manifold, such that we can then define spinor fields as their smooth sections. This is in complete analogy to how we construct a tangent bundle in order to then define vector fields. The difference is that while 2π-rotations of coordinate frames act as the identity on vector components, (famously) they flip the sign of spinor components and we instead need a 4π-rotation to return to the identity. This suggests that rather than an $\mathrm{SO}(1,3)$-principal bundle we need instead a $\mathrm{Spin}(1,3)$-principal bundle, where $\mathrm{Spin}(m,n)$ is the universal double cover of $\mathrm{SO}(m,n)$. How this is implemented formally is reviewed in (Fatibene and Francaviglia, 2003, §9); here we wish merely to draw attention to the underlying geometric structures. We have:

Definition 2.1 (Fatibene and Francaviglia (2003), §9.2)**.** A *spin structure* $\left(\bar{\Sigma},\Lambda\right)$ on a d-dimensional Lorentzian manifold $(\mathcal{M},g)$ is

(i) (spin bundle) a $\mathrm{Spin}(1,d-1)$-principal bundle $\bar{\Sigma}$, and

(ii) (frame bundle) a principal bundle morphism $\Lambda : \bar{\Sigma} \to F$, where F is the orthonormal frame bundle, i.e. an $\mathrm{SO}(1,d-1)$-bundle on $(\mathcal{M},g)$.

In the case in which $(\mathcal{M},g)$ is $\mathbb{CM}$, there is a unique spin structure. Here the frame bundle is an $\mathrm{SO}(4,\mathbb{C})$-bundle on $\mathbb{CM}$ with universal double cover $\mathrm{Spin}(4,\mathbb{C})$, recalling that there is no such thing as signature for complex metrics. We can define the trivial frame bundle $F=\mathbb{C}^4\times\mathrm{SO}(4,\mathbb{C})$ and the trivial spin bundle $\bar{\Sigma}=\mathbb{C}^4\times\mathrm{Spin}(4,\mathbb{C})$. This will be seen to define a spin structure on $\mathbb{CM}$ with Λ given by the double cover $\mathrm{Spin}(4,\mathbb{C})\to\mathrm{SO}(4,\mathbb{C})$.

Spinor fields are then defined as smooth sections of the vector bundle $E_\lambda=\bar{\Sigma}\times_\lambda V$ where λ is a representation of $\mathrm{SO}(4,\mathbb{C})$. We are interested only in the component of the Lie group connected to the identity so we may work on the Lie algebra level. We have that $\mathfrak{so}(4,\mathbb{C})\cong\mathfrak{sl}(2,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})$, as can be checked easily using Dynkin diagrams. Representations of $\mathfrak{sl}(2,\mathbb{C})$ are classified by $j\in\mathbb{Z}/2$. Spinor fields in the $(\frac{1}{2},0)$ representation will be denoted σ^α, where α is an abstract index, i.e. the index denotes only the rank of the object and in particular never assumes any numerical values. At any point of the manifold, a spinor field has two components in a given coordinate frame so we write informally $\sigma^\alpha=(\sigma^0,\sigma^1)$ with $\sigma^0,\sigma^1\in\mathbb{C}$. We also have 'conjugate spinors'

[10] This is the point at which complexified spacetime differs from *complex* spacetime, according to the usage in (Penrose and Rindler, 1988b, pp. 127–8). Complex spacetimes have no privileged subset designated as the 'real' spacetime. Twistor theory is sometimes also constructed on complex spacetimes.

in the representation $\left(0, \frac{1}{2}\right)$ which are denoted $\sigma^{\dot{\alpha}}$. Finally we have the dual spinors σ_α, which are sections of the dual bundle $E_{\left(\frac{1}{2},0\right)}{}^*$, and the conjugate duals.

As with vectors, we can take tensor products to obtain objects of higher rank. We adopt the common convention that the order among the dotted and undotted indices does not matter, thus e.g. $\sigma_\alpha{}^{\dot{\alpha}} = \sigma^{\dot{\alpha}}{}_\alpha$. There is a way to identify vectors with two-index spinors and we write informally $v^a = v^{\alpha\dot{\alpha}}$, where v^a is a vector. To motivate this, recall that vectors transform in the $(\frac{1}{2}, \frac{1}{2})$ representation of $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$. Following (Adamo, 2018, § 1.2) and (Ward and Wells, 1990, § 4.2), associate to every vector $v^a = (v^0, v^1, v^2, v^3) \in \mathbb{CM}$—seen here as a vector space—a matrix

$$v^{\alpha\dot{\alpha}} := \frac{1}{\sqrt{2}} \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix} . \tag{2.1.1}$$

Incidentally, in flat spacetime, every point is associated with a vector which is then manifestly represented as a spinor–conjugate spinor pair.

With objects of higher rank, the prescription is simply to replace every tensorial index with a pair of spinorial indices, exactly one of which should be dotted. One makes much use of symmetrised and antisymmetried indices,[11] so it is important to note that skewing over more than two spinor indices always gives zero, essentially because spin space is two-dimensional.[12] We also make a choice of a non-zero skew two-index spinor $\varepsilon_{\alpha\beta}$ which allows for the identification of spinors and dual spinors in the following way: $\sigma_\alpha = \sigma^\beta \varepsilon_{\beta\alpha}$ and $\sigma^\alpha = \varepsilon^{\alpha\beta}\sigma_\beta$. Note that, unlike the metric tensor, $\varepsilon_{\alpha\beta}$ is skew so index placement matters. The usual convention is summarised by the mnemonic 'adjacent indices, descending to the right'. The Levi-Civita connection ∇_a on $\mathbb{M}$ can be extended to a connection on spin bundles $\nabla_{\alpha\dot{\alpha}}$ and we write informally $\nabla_{\alpha\dot{\alpha}} = \nabla_a$.[13] In a particular coordinate frame, this amounts to $\nabla_{\alpha\dot{\alpha}} = \frac{\partial}{\partial x^a}$ where $x^{\alpha\dot{\alpha}} = x^a$ is the position vector relative to some origin. Finally, this allows us to write down the 'twistor equation',

$$\nabla_{\dot{\alpha}}{}^{(\alpha}\omega^{\beta)} = 0, \tag{2.1.2}$$

where brackets denote symmetrisation and of course $\nabla_{\dot{\alpha}}{}^{\alpha} = \varepsilon^{\alpha\beta}\nabla_{\beta\dot{\alpha}}$. Solutions to (2.1.2) can be found by proceeding *à la* (Penrose and Rindler, 1988b, pp. 45–6). First,

[11] It is perhaps instructive to illustrate what permuting abstract indices amounts to. Tensors can be defined uniquely by their action on vectors and covectors. Consider a tensor T_{ab} which maps from the set of pairs of vectors to an algebraic field. For any two vectors v^a, u^a, we can define T_{ba} via $T_{ba}(v^a, u^a) = T_{ab}(u^a, v^a)$, i.e. by flipping which argument is passed to which 'slot'. Such contractions are more commonly written $T_{ba}v^b u^a = T_{ab}u^a v^b$.

[12] To see this, expand the antisymmetrisation in any spin frame and realise that out of any three indices, two will necessarily be numerically equal, as in (Penrose and Rindler, 1988a, p. 136).

[13] See (Huggett and Tod, 1985, p. 28), although strictly they claim this only for real manifolds.

consider the object $\nabla_{\dot{\alpha}}{}^{\alpha}\nabla_{\dot{\beta}}{}^{\beta}\omega^{\gamma}$, where ω^{γ} solves the twistor equation. The object is skew in the indices $\beta\gamma$:

$$\nabla_{\dot{\alpha}}{}^{\alpha}\nabla_{\dot{\beta}}{}^{\beta}\omega^{\gamma} = \nabla_{\dot{\alpha}}{}^{\alpha}\nabla_{\dot{\beta}}{}^{(\beta}\omega^{\gamma)} - \nabla_{\dot{\alpha}}{}^{\alpha}\nabla_{\dot{\beta}}{}^{\gamma}\omega^{\beta} = -\nabla_{\dot{\alpha}}{}^{\alpha}\nabla_{\dot{\beta}}{}^{\gamma}\omega^{\beta}\,. \tag{2.1.3}$$

In flat space, derivatives commute so the object is also skew in $\alpha\gamma$. Hence it is skew in all three indices $\alpha\beta\gamma$, meaning that it vanishes since (recall) spinorial objects can only be non-trivially skew in up to two indices. Therefore, $\nabla_{\dot{\beta}}{}^{\beta}\omega^{\gamma}$ is constant. Since it is skew in the indices $\beta\gamma$, it has to be proportional to the unique skew two-index spinor $\varepsilon^{\beta\gamma}$. Hence

$$\nabla_{\beta\dot{\alpha}}\omega^{\gamma} = -i\varepsilon_{\beta}{}^{\gamma}\mu_{\dot{\alpha}}\,, \tag{2.1.4}$$

where $\mu_{\dot{\alpha}}$ is a constant spinor and $-i$ is inserted for later convenience. Integrating (2.1.4) then yields the general solution to the twistor equation

$$\omega^{\alpha} = \lambda^{\alpha} - ix^{\alpha\dot{\alpha}}\mu_{\dot{\alpha}}\,, \tag{2.1.5}$$

with λ^{α} a constant of integration and $x^{\alpha\dot{\alpha}}$ the position vector relative to some origin. The solutions of the twistor equation are determined fully by the spinors λ^{α} and $\mu_{\dot{\alpha}}$, i.e. by four complex numbers. Now define twistor space $\mathbb{T}$ to be the vector space of the solutions to the twistor equation, making it a 4-dimensional complex vector space coordinatised by the two spinors λ^{α} and $\mu_{\dot{\alpha}}$ as in (Huggett and Tod, 1985, p. 54). A twistor $Z^{A} \in \mathbb{T}$ can be written as[14]

$$Z^{A} = (\omega^{\alpha}, \mu_{\dot{\alpha}})\,, \tag{2.1.6}$$

and is also coordinatised by four complex numbers so we write informally $Z^{A} = (Z^{1}, Z^{2}, Z^{3}, Z^{4})$ with $Z^{1}, Z^{2}, Z^{3}, Z^{4} \in \mathbb{C}$.

We can establish a correspondence between (complex) spacetime points and twistors by considering the points at which the field ω^{α} vanishes. This yields the so-called 'incidence relation' (also sometimes called the 'Klein correspondence'[15])

$$\lambda^{\alpha} = ix^{\alpha\dot{\alpha}}\mu_{\dot{\alpha}}. \tag{2.1.7}$$

For a given twistor $Z^{A} = (\lambda^{\alpha}, \mu_{\dot{\alpha}})$, what is the locus of spacetime points $x^{\alpha\dot{\alpha}}$ that satisfies the incidence relation? It can be shown—see (Huggett and Tod, 1985, p. 56)—that this locus is a 2-plane with the additional properties that

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[14]Note that Adamo (2018) has $Z^{A} = (\lambda^{\alpha}, \mu_{\dot{\alpha}})$ here. We take this to be a typographic error.

[15]Note, though, that typically the terminology 'Klein correspondence' is reserved for the context in which one is working with compactified complexified Minkowski space $\mathbb{CM}^{C}$, more on which below.

1. every tangent is null,
2. any two tangents are orthogonal,
3. the tangent bivector is self-dual.[16]

Such an object is referred to as an 'α-plane'. Now it is clear how definition (1) links to definition (3). Notice also that (2.1.7) is invariant under rescalings of the twistor by a non-zero complex number. This motivates the definition of projective twistor space $\mathbb{PT}$, coordinatised by the homogeneous coordinates $Z^1 : Z^2 : Z^3 : Z^4$.[17] Commonly, $\mathbb{PT}$ is referred to as 'twistor space' whereas $\mathbb{T}$ is called 'non-projective twistor space'. Since we had $\mathbb{T} \cong \mathbb{C}^4$, then also $\mathbb{PT} \cong \mathbb{CP}^3$.

It is useful to divide $\mathbb{PT}$ into three subspaces $\mathbb{PT}^+$, $\mathbb{PN}$, and $\mathbb{PT}^-$ depending upon whether the twistor 'norm'

$$\|Z^A\| = \bar{Z}^2 Z^0 + \bar{Z}^3 Z^1 + \bar{Z}^0 Z^2 + \bar{Z}^1 Z^3 \tag{2.1.8}$$

is positive, negative or zero. One can show from the incidence relation that a null twistor ($\|Z^A\| = 0$, i.e. $Z^A \in \mathbb{PN}$) corresponds to a real null geodesic (light ray) in $\mathbb{M}$, i.e. that the twistor space of real Minkowski space is $\mathbb{PN}$.[18]

There is one complication that we have not yet dealt with. Looking at the incidence relation (2.1.7), it is clear that if $\mu_{\dot{\alpha}} = 0$, then λ^α also vanishes but then the entire twistor vanishes, meaning that it cannot be represented using homogeneous coordinates. Unless, that is, $x^{\alpha\dot{\alpha}}$ is allowed to be infinity. We can indeed achieve this through compactification, so what we have actually shown is that $\mathbb{CP}^3$ is the twistor space for compactified complexified Minkowski space $\mathbb{CM}^C$:

$$\mathbb{PT} := \mathbb{PT}(\mathbb{CM}^C) = \mathbb{CP}^3 \,. \tag{2.1.9}$$

[16] 'Self-duality' is a notion that appears all over the literature on twistor theory, so let us be clear about what it means. Recall the Hodge star operator, familiar from e.g. the construction of the dual Faraday tensor in classical electrodynamics. On 2-forms, the action of the operator is simply $F_{ab} \mapsto *F_{ab} = \frac{1}{2}\varepsilon_{ab}{}^{cd}F_{cd}$, where ε is a choice of non-zero skew 4-index tensor that represents a choice of orientation. Since $** = -1$, the eigenspace of the operator decomposes into the subspace with eigenvalue $+i$ (self-dual two-forms) and that with eigenvalue $-i$ (anti-self-dual two-forms). The operator generalises to arbitrary k-forms.

[17] A brief refresher on projective spaces. Given a real vector space $\mathbb{V}$, we can construct the projective space $\mathbb{PV}$ by identifying nonzero real multiples of vectors, i.e. $\mathbb{PV} = \{[v] : v' \in [v] \text{ if } v' = \lambda v \text{ for } \lambda \in \mathbb{R} \setminus 0\}$. Projective spaces have a natural manifold structure. But clearly homogeneous coordinates are not a coordinate system, as they are many-to-one: indeed there is no global coordinate system for projective spaces, as is typical of manifolds.

[18] See (Adamo, 2018, § 2.1).

If we instead wish to work with the twistor space for $\mathbb{CM}$, we simply have to remove the points in the full twistor space corresponding to spacetime infinity: $\mathbb{PT}(\mathbb{CM}) = \{(\lambda^\alpha, \mu_{\dot\alpha}) \in \mathbb{CP}^3 : \mu_{\dot\alpha} \neq 0\}$. This gives a subset of the full twistor space which is commonly written[19]

$$\mathbb{PT}(\mathbb{CM}) = \mathbb{CP}^3 - \mathbb{CP}^1 \,. \tag{2.1.10}$$

There is another way of looking at this, presented in (Adamo, 2018, § 2.3). Notice that the full twistor space $\mathbb{PT} = \mathbb{CP}^3$ represents the entire class of conformally flat spacetimes, essentially because the twistor equation is conformally invariant. To pick out the desired conformal structure, one introduces an object called the infinity bi-twistor I_{AB}. In this way, one can for example show that the twistor space of Euclidean anti-de Sitter spacetime (a conformally flat spacetime) is $\mathbb{PT}^+$.[20]

In the midst of all this formalism, it is easy to lose sight of what the underlying idea behind twistors is supposed to be. Recall that the original goal was to unify spacetime and spinor degrees of freedom. The natural object to consider, then, is

$$G_2(\mathbb{C}^4) = \{\text{2-complex-dimensional subspaces of } \mathbb{C}^4\} \,, \tag{2.1.11}$$

which is called the 'Grassmannian'. Here, $\mathbb{C}^4$ is to be understood as a vector space, but in fact $G_2(\mathbb{C}^4)$ will have a canonical manifold structure. Since spinors are represented locally by two complex numbers, and (complexified) spacetime points by four, the Grasmannian seemingly provides the spinor degrees of freedom at each point of complexified spacetime. The Grasmannian also has a canonical manifold structure, and in fact we have $G_2(\mathbb{C}^4) = \mathbb{CM}^C$. Compactified complexified spacetime $\mathbb{CM}^C$ provides spinor degrees of freedom for each point of $\mathbb{C}^4$. Then, as Woit (2021, p. 16) puts it, "[a] space-time point is thus a $\mathbb{C}^2$ in $\mathbb{C}^4$ which tautologically provides the spinor degree of freedom at that point. The spinor bundle S is the tautological two-dimensional complex vector bundle over $[G_2(\mathbb{C}^4)]$ whose fiber S_m at a point $m \in [G_2(\mathbb{C}^4)]$ is the $\mathbb{C}^2$ that defines the point." To link to twistors, we proceed along the lines of (Ward and Wells, 1990, §1.2), and first define the 'flag manifold'

$$\begin{aligned} \mathbb{F}_{d_1 \dots d_n} := \{(S_1, \dots, S_n) : S_j \text{ are } d_j\text{-dimensional subspaces of } \mathbb{T} \\ \text{and } S_1 \subset \cdots \subset S_n\} \end{aligned} \tag{2.1.12}$$

We use these flag manifolds to define complex manifolds of twistor geometry. Consider three special twistor manifolds (twistor manifolds because the vector space chosen

[19] E.g. in (Atiyah et al., 2017, p. 7).

[20] The reader will notice that we do not discuss definition (2) from the list. This is because we think that it does not contribute to the questions raised in this article, although it is useful to know it exists for the sake of completeness.

to construct the flag manifold is the twistor space $\mathbb{T}$), $\mathbb{F}_{12}$, $\mathbb{F}_1$ and $\mathbb{F}_2$. We define two maps α and β such that

$$\mathbb{F}_1 \overset{\alpha}{\leftarrow} \mathbb{F}_{12} \overset{\beta}{\rightarrow} \mathbb{F}_2. \tag{2.1.13}$$

These mappings are projection mappings and (2.1.13) is called a 'double fibration'. A correspondence between $\mathbb{F}_1$ and $\mathbb{F}_2$ is an assignment of a subspace $c(p) \subset \mathbb{F}_2$ for each subspace p of $\mathbb{F}_1$ (since $\mathbb{F}_1$ is a collection of subspaces), where $c = \alpha^{-1} \circ \beta$.[21] A double fibration (2.1.13) always yields a correspondence between $\mathbb{F}_1$ and $\mathbb{F}_2$. Now, since $\mathbb{F}_1$ is a collection of 1-complex-dimensional subspaces of $\mathbb{T}$, by definition it is a projective space. So $\mathbb{F}_1 = \mathbb{PT}$. $\mathbb{F}_2$ is a collection of 2-complex-dimensional spaces. So from what we have remarked in this section, $\mathbb{F}_2 = G_2(\mathbb{C}^4) = \mathbb{CM}^C$. We denote $\mathbb{F}_{12}$ simply as $\mathbb{F}$. $\mathbb{F}$ is nothing but the correspondence space between $\mathbb{PT}$ and $\mathbb{CM}^C$. So one could rewrite the double fibration (2.1.13) as

$$\mathbb{PT} \overset{\alpha}{\leftarrow} \mathbb{F} \overset{\beta}{\rightarrow} \mathbb{CM}^C. \tag{2.1.14}$$

The double fibration (2.1.14) allows us to transform from $\mathbb{PT}$ to $\mathbb{CM}^C$ and vice-versa; it is therefore a geometrical statement of the incidence relation (2.1.7).

2.2 Penrose transformations

'Penrose transformation' is an umbrella term denoting the transformations from twistorial (geometric) objects to solutions of various (dynamical) classical field theories defined on a spacetime manifold, and *vice versa*. The original ('the') Penrose transformation describes solutions of the massless free (non-interacting) field equation of arbitrary rank and will be discussed in §2.2.1. This can be extended (under some conditions) to gauge potentials generating these fields in what is known as the 'Sparling transform'. Another case of interest to high energy physics is the 'Ward transform', which pertains Yang-Mills theories, again under some rather limiting conditions leading to what is called the 'Googly problem'. In this article, we will skip over these transformations; however, an accessible account is given by Adamo (2018). We will, however, in §2.2.2 sketch a proof of 'Kerr's theorem', which relates shear-free null congruences in $\mathbb{CM}$ and homogeneous, holomorphic functions on $\mathbb{PT}$. Finally we describe in §2.2.3 the 'non-linear graviton': a construction due to Penrose that describes solutions to the Einstein field equations, again only in some limited cases.[22]

[21] This 'correspondence' is a generalisation of a function in the sense that it maps objects in the domain to subsets of the range.

[22] For further philosophical discussion of these transformations, see (Bain, 2006).

2.2.1 Zero rest-mass Penrose transform (ZRMPT)

To set the stage for the original Penrose transformation—the 'zero rest-mass Penrose transformation' (ZRMPT)—we first provide a short discussion of classical field theory and in particular of massless (zero rest-mass) fields. A simple (and hopefully familiar) example is that of a Maxwell field in classical electrodynamics. The field strength tensor in spinor indices is

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \partial_{\alpha\dot{\alpha}} A_{\beta\dot{\beta}} - \partial_{\beta\dot{\beta}} A_{\alpha\dot{\alpha}}. \tag{2.2.1}$$

It is easy to observe that this is antisymmetric in $\alpha\dot{\alpha}$ and $\beta\dot{\beta}$. This antisymmetry can be attained if $F_{\alpha\dot{\alpha}\beta\dot{\beta}}$ is symmetric in the dotted indices and antisymmetric in the undotted indices or vice versa. This property of the field strength tensor results in the decomposition

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \varepsilon_{\alpha\beta} F'_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}} \tilde{F}_{\alpha\beta}. \tag{2.2.2}$$

Here, $F'_{\dot{\alpha}\dot{\beta}}$ and $\tilde{F}_{\alpha\beta}$ are the 'self dual' (SD) and 'anti-self dual' (ASD) parts of the field strength tensor, respectively.[23] It should be remarked upon that in the context of *classical* fields, this decomposition is not the most natural: both F' and $\tilde{F}$ are complex since we have $F^{\pm} = \frac{1}{2}(F \mp i * F)$ as can be verified easily. Further, if the overall field F is to be real—as it is, classically—then the two components have to be complex conjugates of each other, as is likewise verified easily. With a view to quantum theory, however, these components represent the right- and left-handed photons respectively and are indeed independent of each other since F is complex. The first source-free Maxwell equation in terms of spinor indices is then given as

$$\partial^{\alpha\dot{\alpha}} F_{\alpha\dot{\alpha}\beta\dot{\beta}} = 0. \tag{2.2.3}$$

On substituting the decomposition given by (2.2.2) into the Maxwell equation (2.2.3), we get

$$\partial^{\dot{\alpha}}_{\beta} F'_{\dot{\alpha}\dot{\beta}} + \partial^{\alpha}_{\dot{\beta}} \tilde{F}_{\alpha\beta} = 0. \tag{2.2.4}$$

The Bianchi identity in terms of SD/ASD decomposition of the field strength is given as

$$\partial^{\dot{\alpha}}_{\beta} F'_{\dot{\alpha}\dot{\beta}} - \partial^{\alpha}_{\dot{\beta}} \tilde{F}_{\alpha\beta} = 0. \tag{2.2.5}$$

(2.2.4) and (2.2.5) give rise to the zero rest-mass (ZRM) equations in the case of the electromagnetic field (spin 1, helicity +1,-1):

$$\partial^{\dot{\alpha}}_{\beta} F'_{\dot{\alpha}\dot{\beta}} = 0, \tag{2.2.6}$$

$$\partial^{\alpha}_{\dot{\beta}} \tilde{F}_{\alpha\beta} = 0. \tag{2.2.7}$$

[23] See footnote 16.

So we obtain free-field equations expressed entirely in terms of SD and ASD components of the field strength tensor. $F_{\alpha\beta}$ (undotted indices) is a ZRM field of negative helicity and $F'_{\dot{\alpha}\dot{\beta}}$ (dotted indices) is a ZRM field of positive helicity.

We obtain a similar set of ZRM equations in the case of the linearised vacuum gravitational field resulting from a similar decomposition of the Weyl tensor φ_{abcd}. This decomposition contains components that encode the SD and ASD parts of the Weyl curvature. We obtain the following equations (after linearizing—i.e. after replacing the covariant derivative with a partial derivative):

$$\partial^{\dot{\alpha}}_{\beta}\varphi'_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\phi}} = 0, \tag{2.2.8}$$

$$\partial^{\alpha}_{\dot{\beta}}\varphi_{\alpha\beta\gamma\phi} = 0. \tag{2.2.9}$$

In general, a ZRM field ϕ of helicity h (having $2|h|$ dotted or undotted indices, depending on whether it is a negative or positive helicity field), obeys the differential equations

$$\partial^{\beta\dot{\alpha}_1}\phi_{\dot{\alpha}_1\ldots\dot{\alpha}_{2|h|}} = 0 \qquad (h > 0), \tag{2.2.10}$$

$$\partial^{\alpha_1\dot{\beta}}\phi_{\alpha_1\ldots\alpha_{2|h|}} = 0 \qquad (h < 0). \tag{2.2.11}$$

One of the key properties of ZRM equations is that they are conformally invariant (Adamo, 2018, §3.1).[24] Also, it is worth noting that conformal invariance is encoded in twistor space. So this motivates the construction of ZRM fields in terms of twistorial objects. From the incidence relation (2.1.7), we observe that a fixed point in $\mathbb{CM}$ corresponds to a twistor line (X $\cong \mathbb{CP}^1$) in $\mathbb{PT}$. So in order to express a field which is local on $\mathbb{CM}$ in terms of twistorial objects, it is natural to integrate over $X \cong \mathbb{CP}^1$ and excise the $\mathbb{CP}^1$ degrees of freedom. The following is one natural such integral construction (considering a negative helicity field):

$$\phi_{\alpha_1\ldots\alpha_{2|h|}}(x) = \int_X \mathrm{D}\mu \wedge \mu_{\alpha_1}\ldots\mu_{\alpha_{2|h|}} f_X(Z^A). \tag{2.2.12}$$

There is a lot to explain in order to justify the above construction. Here $\mathrm{D}\mu$ ($= \mu^\alpha \mathrm{d}\mu_\alpha$) is a holomorphic differential form on $\mathbb{CP}^1$: $\mathrm{D}\mu \in \Omega^{(1,0)}(\mathbb{PT}, \mathscr{O}(2))$ (i.e. $(1,0)$ form on $\mathbb{PT}$), homogeneous of projective weight 2.[25] We wedge it with $\mu_{\alpha_1}\ldots\mu_{\alpha_{2|h|}}$ because μ_α is the most natural choice to account for the $2|h|$ indices on the LHS. Ignoring for now the f term in the integrand, the ingredients that we have discussed

[24] Following Wald (1984), a field equation for Ψ is conformally invariant of conformal weight $s \in \mathbb{R}$ provided that ψ is a solution with metric g_{ab} iff $\Omega^s\Psi$ is a solution with metric $\Omega^2 g_{ab}$.

[25] By projective weight 2, we mean $f(rZ^A) = r^2 f(Z^A)$).

so far make the integrand a $(1,0)$ form of homogeneity $2|h|+2$. But, in order for the integral to make sense, the integrand should be a $(1,1)$ form of homogeneity 0 (the homogeneity necessarily has to be zero for the integral to be well-defined, because we are integrating over the projective space). So we add an extra ingredient $f(Z^A)$ (where the subscript X denotes f being restricted to the twistor line over which the integral is performed) in order to make the integral consistent and well-defined. Thus $f \in \Omega^{(0,1)}(\mathbb{PT}, \mathscr{O}(-2|h|-2))$. So, after constructing an integral that is consistent and well-defined, the obvious question we ask is whether (2.2.12) satisfies the ZRM equation (2.2.11). It turns out that this is indeed the case, given that f is a holomorphic function (i.e. independent of the conjugated twistor variables). This holomorphicity condition on $\mathbb{PT}$ can be expressed as $\bar{\partial} f = 0$, where $\bar{\partial}$ is the Dolbeault operator, the generealisation of the exterior derivative to complex manifolds.[26] However, there are some trivial solutions of which one wishes to dispose. Note that $\bar{\partial}^2 = 0$, as for the exterior derivative. This implies that $f = \bar{\partial} g$ also satisfies the holomorphicity condition $\bar{\partial} f = 0$. However, on substituting $f = \bar{\partial} g$ into the integral (2.2.12), the ZRM field vanishes identically. So we should exclude functions f of the form $f = \bar{\partial} g$ from the space of functions f that are holomorphic. So, we conclude that

$$f \in \left\{ h \in \Omega^{(0,1)}(\mathbb{PT}, \mathscr{O}(-2|h|-2)) : \bar{\partial} h = 0, h \neq \bar{\partial} g \right\} \tag{2.2.13}$$

in order for the integral (2.2.12) to represent a nontrivial ZRM field. Further, from the linearity of the integral, we know that any two functions that differ by an $\bar{\partial}$-exact form will evaluate to the same field, so we wish to treat them as equivalent. So we are working with a cohomology group $\ker\bar{\partial}/\mathrm{im}\bar{\partial}$, in this case the first Dolbeault cohomology group, $H^{(0,1)}(\mathbb{PT}, \mathscr{O}(-2|h|-2))$.[27] We can carry out a similar procedure for ZRM fields of positive helicity. What we have shown here is that helicity h ZRM fields on $\mathbb{CM}$ can be specified by cohomology classes on twistor space. One can prove that this holds the other way around as well—see e.g. (Eastwood et al., 1981)—although we will not go into the details here. The final result is an isomorphism relation between helicity h ZRM fields on $\mathbb{CM}$ and Dolbeault cohomology classes on twistor space:

$$\text{Helicity } h \text{ ZRM fields on } \mathbb{CM} \cong H^{(0,1)}(\mathbb{PT}, \mathscr{O}(2h-2)). \tag{2.2.14}$$

In fact, the isomorphism is for any $U \subset \mathbb{M}^C$ open and convex, as verified in (Ward and Wells, 1990, §§ 7.1–2), which also specifies precisely in what sense this is an *iso-*

[26] See (Adamo, 2018, p. 15) for more details.

[27] This is also isomorphic to the first Čech cohomology group $\check{H}^1(\mathfrak{U}, \mathscr{O}(-2|h|-2))$, which is a sheaf cohomology. The isomorphism goes through via Dolbeault's theorem with $\mathfrak{U}$ a good open cover of $\mathbb{PT}$—see (Maddock, 2009).

morphism.[28] We refer to this construction as the 'zero rest-mass Penrose transform' (ZRMPT). If we are interested in solutions on real Minkowski space $\mathbb{M}$, we can find a solution on $\mathbb{CM}$ an restrict it to real coordinates, but there is a caveat: since the d'Alembertian ('wave') operator is hyperbolic, we need to extend our solution space to generalised functions in order to recover the well-known non-smooth solutions to the wave equation. More discussion on this point can be found in (Ward and Wells, 1990, § 7.4). At the end of the day, we obtain the Penrose transformation:

$$\text{Helicity } h \text{ ZRM fields on } \mathbb{M} \;\cong\; H^{(0,1)}_{\mathcal{A}}(\mathbb{PN}, \mathscr{O}(2h-2)), \tag{2.2.15}$$

where $\mathcal{A}$ denotes that we are working with distributions, i.e. generalised functions. This is Eq. 7.4.10 in (Ward and Wells, 1990). Note that, unlike for (2.2.14), Ward and Wells (1990) do not prove (indeed, do not even claim) that (2.2.15) is an isomorphism, in the above sense. For an isomorphism between real Minkowski space and twistor space, one would need to use *second* cohomologies—see (Ward and Wells, 1990, Thm. 7.4.5). This makes it less obvious that to recover fields on real Minkowski space it is sufficient to restrict the constructions on complexified Minkowski space, as claimed by e.g. Adamo (2018) and as discussed above. Still, one can work with second cohomologies and presumably recover the desired results in that way. We discuss some of these issues further in §3.

2.2.2 Kerr's theorem

Kerr's theorem establishes a correspondence between holomorphic functions on twistor space and shear-free null congruences in Minkowski space. A null congruence is a set of non-intersecting null geodesics in an open region U, such that there is a geodesic passing through each point in U. The tangent vectors to these null geodesics define a vector field K^a (up to a scale) and so a spinor field μ^α (also up to a scale). Then the geodesic equation in terms of the spinor field μ^α is given as

$$\mu^\alpha K^a \nabla_a \mu_\alpha \equiv \mu^\alpha \mu^\beta \bar{\mu}^{\dot{\beta}} \nabla_{\beta\dot{\beta}} \mu_\alpha = 0. \tag{2.2.16}$$

Notice, in spinorial terms, that the directional derivative is along the $\mu^\beta \bar{\mu}^{\dot{\beta}}$ direction since the geodesic is in the $\mu^\beta \bar{\mu}^{\dot{\beta}}$ direction. The shear of the null congruence is given

[28] In particular, the result to which we refer is Theorem 7.2.3. Note that this result is expressed in terms of sheaf cohomology, which we are free to convert to Dolbeault cohomology via Dolbeault's theorem discussed in footnote 27. In addition, instead of talking about the set of ZRM fields, one talks about sections of sheaves $\Gamma(\mathcal{Z})$ in order to define a sense of isomorphism.

as[29]

$$\sigma = \mu^\alpha \mu^\beta \bar{\lambda}^{\dot{\beta}} \nabla_{\beta\dot{\beta}} \mu_\alpha, \tag{2.2.17}$$

with the spinor λ^α satisfying $\mu_\alpha \lambda^\alpha = 1$ and $\mu^\beta \bar{\mu}^{\dot{\beta}} \nabla_{\beta\dot{\beta}} \lambda^\alpha = 0$. If the congruence is shear-free, then $\sigma = 0$, in which case we can combine the two equations to obtain an equation for a shear-free congruence,

$$\mu^\alpha \mu^\beta \nabla_{\beta\dot{\beta}} \mu_\alpha = 0. \tag{2.2.18}$$

(In what follows, we use 'adapted' coordinates, replacing covariant derivatives with partial derivatives.) Solving for spaces with conformal curvature has certain difficulties discussed in (Huggett and Tod, 1985, p. 49); in line with that work we now proceed to work with conformally flat spacetimes. In order to solve (2.2.17), we pick a constant and normalized dyad $(\kappa_\alpha, \rho_\beta)$ and coordinatise μ^α as

$$\mu^\alpha = t(\kappa^\alpha + L\rho^\alpha); \qquad L = \mu_0/\mu_1 = -\mu^1/\mu^0. \tag{2.2.19}$$

On substituting in (2.2.18), we obtain

$$\partial_{0\dot{\alpha}} L - L \partial_{1\dot{\alpha}} L = 0. \tag{2.2.20}$$

Recalling (2.1.1), we can write these equations in Minkowski coordinates represented by a pair of $SL(2, \mathbb{C})$ Weyl spinors:

$$x^{\alpha\dot{\alpha}} = \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} = \begin{pmatrix} \tau & \varsigma \\ \bar{\varsigma} & \omega \end{pmatrix}. \tag{2.2.21}$$

Then we obtain the equations:

$$\frac{\partial L}{\partial \tau} - L \frac{\partial L}{\partial \bar{\varsigma}} = 0 \quad (\dot{\alpha} = 0), \tag{2.2.22}$$

$$\frac{\partial L}{\partial \varsigma} - L \frac{\partial L}{\partial \omega} = 0 \quad (\dot{\alpha} = 1). \tag{2.2.23}$$

Using the method of characteristics to solve these equations, we arrive at

$$F(L\tau + \bar{\varsigma}, L\varsigma + \bar{\tau}, L) = 0. \tag{2.2.24}$$

The solution of our equations is determined by (2.2.24). Making use of (2.2.19), we can rewrite (2.2.24) in terms of a homogeneous holomorphic function of four variables, as

$$f\left(-ix^{\alpha\dot{\alpha}} \mu_\alpha, \mu_\alpha\right) = 0. \tag{2.2.25}$$

[29] We don't prove this here—for details, see (Huggett and Tod, 1985).

So we see that a shear-free null congruence is determined by a homogeneous holomorphic function of four variables. This should certainly tempt one to invoke twistors in the analysis. Consider the point on $\mathbb{PN}$, $Z^A = (\pi^\alpha, \mu_{\dot\alpha})$, where $\pi^\alpha = x^{\alpha\dot\alpha}\mu_{\dot\alpha}$ ($x^{\alpha\dot\alpha}$ being real). Consider the zero locus of an analytic function on twistor space, $f(Z^A)$, given by

$$f(Z^A) = 0. \tag{2.2.26}$$

The intersection set of the surface traced out by the zero set and $\mathbb{PN}$ is given by

$$f(ix^{\alpha\dot\alpha}\mu_{\dot\alpha}, \mu_{\dot\alpha}) = 0, \tag{2.2.27}$$

where again $x^{\alpha\dot\alpha}$ is real. So, clearly from the above analysis each point belonging to the intersection set defines a null geodesic in the direction of $\mu_\alpha\bar\mu_{\dot\alpha}$, passing through $x^{\alpha\dot\alpha}$ and the intersection set defines a shear-free null congruence in the Minkowski space. This result is what is known as Kerr's theorem.

2.2.3 The nonlinear graviton

So far, we have discussed two different Penrose transformations in the context of conformally flat spacetime physics. One can extend such transformations to spacetimes with conformal curvature, although typically this is not straightforward. The twistors are well-defined for conformally flat spacetimes (i.e. $\varphi_{\alpha\beta\gamma\delta} = 0$ and $\bar\varphi_{\dot\alpha\dot\beta\dot\gamma\dot\delta} = 0$) because the solutions of the twistor equation are constrained by the condition $\varphi_{\alpha\beta\gamma\delta}\omega^\delta = 0$. We can impose the conditions $\varphi_{\alpha\beta\gamma\delta} = 0$ and $\bar\varphi_{\dot\alpha\dot\beta\dot\gamma\dot\delta} \neq 0$ on the complexified spacetime. Note that such a condition cannot be imposed on real spacetime because the dotted and the undotted Weyl spinors are complex conjugates and thus not independent of each other. In the complex picture, we can treat the dotted and undotted Weyl spinors as independent quantities and thus can impose the above conditions. This requires that the Weyl tensor is anti-self-dual and thus we define a spacetime $\mathcal{M}$ which is referred to as anti-self-dual. For such an $\mathcal{M}$, it is possible to construct a projective twistor space $P\mathfrak{T}$ (note that this is not the same as the projective twistor space $\mathbb{PT}$). We do not sketch a proof of the construction in this section, but we formally state the twistor correspondence known as the 'non-linear graviton Penrose transform' (for a proof, see (Huggett and Tod, 1985)). The statement is this: from a solution $\mathcal{M}$ of vacuum Einstein equations with anti-self-dual Weyl curvature, we can construct a four dimensional complex manifold $\mathfrak{T}$, equipped with the following:

1. a 'homogeneity operator' $\Upsilon = \mu_{\alpha'}\frac{\partial}{\partial\mu_{\alpha'}}$,
2. a projection π on the dotted spin space ,

3. 2-forms $\tau = \varepsilon^{\dot{\alpha}\dot{\beta}} d\pi_{\dot{\alpha}} \wedge d\pi_{\dot{\beta}}$ and $\mu = \varepsilon_{\alpha\beta} X^{\alpha\dot{\alpha}} Y^{\beta\dot{\beta}} \pi_{\dot{\alpha}} \pi_{\dot{\beta}}$ (where X and Y are tangent vectors to fibre of $\mathfrak{T}$ containing a particular α-plane) on each fibre over the dotted spin space ,

4. a four-parameter family of holomorphic curves which are compact and have normal bundle $O(1) \oplus O(1)$ in P$\mathfrak{T}$.

3 Theoretical equivalence

So much for the technical background on twistor theory. In this section, we explore the sense in which a theory set on twistor space is 'equivalent' to a relativistic theory set on a Lorentzian spacetime (in particular $\mathbb{CM}$, for reasons discussed below); to do this, we draw on philosophers' recent work on theoretical equivalence, and on different ways of cashing out this notion, with particular attention given to categorical equivalence (since, being the weakest 'mainstream' notion of theoretical equivalence, a failure of categorical equivalence implies also a failure of other notions of theoretical equivalence—e.g., definitional equivalence or Morita equivalence—see (Weatherall, 2019a,b)).[30,31] For the sake of keeping the discussion tractable while nevertheless seeking to make an interesting philosophical point, most of our considerations in this section focus upon the case of the zero rest-mass Penrose transformations (ZRMPTs), introduced in the previous section. Specifically, in §3.1, we provide some background to the existing philosophical literature on categorical equivalence, and in §3.2 we consider categorical equivalence in the case of ZRMPTs.

[30]For a recent generalisation of the notion of categorical equivalence to theory kinematics, see (March, 2024b).

[31]A referee has correctly pointed out to us that in e.g. the physics literature on dualities, 'equivalence' is often cashed out in terms of sameness of correlation functions, etc. (See e.g. Read (2016) for discussion on this.) We agree; however, there are three points to make here. First: this understanding of equivalence has to do primarily with *empirical* equivalence, since (say) correlation functions are (in the terminology of Van Fraassen (1980)) the 'empirical substructures' of the relevant models; this notion of equivalence is weaker than theoretical or physical equivalence. Second: understanding equivalence in terms of 'sameness of correlation functions' makes sense only in the context of quantum theories, which are not our concern here. Third: since Weatherall (2016a) cashes out theoretical equivalence as categorical equivalence *such that* empirical consequences are retained via the translation functors, the categorical equivalence programme can in fact subsume this particular understanding of empirical substructures.

3.1 Background on categorical equivalence

Weatherall (2016a) has proposed a criterion of equivalence of physical theories, according to which two given theories are equivalent just in case (a) their associated categories of models are *equivalent*, and (b) the functors realising this equivalence preserve empirical content. The category of models associated with a theory $\mathcal{T}$ is a category the objects of which are models of $\mathcal{T}$, and the morphisms of which relate models regarded as having the 'same structure'.[32]

Two categories **A** and **B** are equivalent if and only if there exist functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{A}$ such that $FG \cong 1_{\mathbf{B}}$, and $GF \cong 1_{\mathbf{A}}$. Equivalently, the categorical equivalence of **A** and **B** amounts to the existence of a functor relating them which is:

Full: For all objects $a, b \in \mathbf{A}$, the map $(f : a \to b) \mapsto (F(f) : F(a) \to F(b))$ induced by F is surjective.

Faithful: For all objects $a, b \in \mathbf{A}$, the map $(f : a \to b) \mapsto (F(f) : F(a) \to F(b))$ induced by F is injective.

Essentially surjective: For every object $x \in \mathbf{B}$, there is some object $a \in \mathbf{A}$ and arrows $f : F(a) \to x$ and $f^{-1} : x \to F(a)$ such that $f \circ f^{-1} = 1_x$.

A functor 'forgets structure*' just in case it is not full; 'forgets stuff' just in case it is not faithful, and 'forgets properties' just in case it is not essentially surjective.[33]

3.2 Categorical equivalence and massless free fields

With this lightning-speed review of categorical equivalence in hand, in this subsection we now apply to twistor theory the kinds of considerations on categorical equivalence propounded by Nguyen et al. (2020). In particular, we are interested in whether the twistor-theoretic formulation of field theory is categorically equivalent to the standard spacetime formulation, or whether there is a way twistors in which can realise the latter with less overall structure*.[34]

Recall from the discussion in §2.2.1 of the ZRMPT that there is a one-to-one correspondence between helicity h ZRM fields on $\mathbb{CM}$ and Dolbeault cohomology classes

[32] This will be an interpretative matter—see e.g. (March, 2024a). For relevant background on category theory, see (Mac Lane, 1998).

[33] For more detail on the interpretation of 'structure*', 'stuff', and 'properties', see (Baez et al., 2004). Here, we follow Nguyen et al. (2020) in writing 'structure*' rather than just 'structure', to make clear that the term is being deployed in a technical sense.

[34] Since amount of structure* is a functor-relative notion, there might exist one functor which preserves structure* and another which does not.

on $\mathbb{PT}$. Since the correspondence is an isomorphism for (2.2.14) but not for (2.2.15), we focus on the former in what follows. That is to say: in this subsection we consider only the categorical equivalence of a field theory set on $\mathbb{CM}$ and twistor space; clearly, this sense of equivalence is not as *directly* pertinent to questions of which of two theories is most apt to describe the actual world, although note that theories which involve computations on $\mathbb{CM}$ do still have operational and empirical significance—consider, e.g., theories obtained via Wick rotation.

The fact that one obtains the same field from any choice of class representative leads one to conjecture that this choice is related to the usual gauge freedom wherein the same field is generated by an entire class of gauge potentials. This is supported by (Sparling, 1990, p. 173), which holds that the "freedom in the choice of solutions [of the ZRM equation] corresponds precisely to the gauge freedom [in the gauge potential]"—although for certain specific reasons we will see below that this claim is in fact problematic. Recall from (Ward and Wells, 1990, §7.2)[35] that the potential for a negative helicity field $\phi_{\alpha_1\alpha_2\ldots\alpha_{2|h|}}$ is a spinor field $\psi_{\alpha_1}^{\dot{\alpha}_2\ldots\dot{\alpha}_{2|h|}}$ such that

$$\nabla^{\alpha_1(\dot{\alpha}_1}\psi_{\alpha_1}^{\dot{\alpha}_2\ldots\dot{\alpha}_{2|h|})} = 0, \tag{3.2.1}$$

$$\phi_{\alpha_1\alpha_2\ldots\alpha_{2|h|}} = \nabla_{\dot{\alpha}_{2|h|}(\alpha_{2|h|}}\cdots\nabla_{\dot{\alpha}_2\alpha_2)}\psi_{\alpha_1}^{\dot{\alpha}_2\ldots\dot{\alpha}_{2|h|}}, \tag{3.2.2}$$

where the symmetrisation in (3.2.2) is to be performed over the undotted indices. Notice the gauge symmetry

$$\psi_{\alpha_1}^{\dot{\alpha}_2\ldots\dot{\alpha}_{2|h|}} \to \psi_{\alpha_1}^{\dot{\alpha}_2\ldots\dot{\alpha}_{2|h|}} + \nabla_{\alpha_1}^{(\dot{\alpha}_2}\gamma^{\dot{\alpha}_3\ldots\dot{\alpha}_{2|h|})}, \tag{3.2.3}$$

where $\gamma^{\dot{\alpha}_3\ldots\dot{\alpha}_{2|h|}}$ is any spinor field.

Now we are in a position to define our categories. We consider affine complexified Minkowski space $\mathbb{CM}$ and its twistor space $\mathbb{PT}$. Let **Twist** be the category of all the individual members of the equivalence classes from (2.2.14). Let the morphisms be provided by the $\bar{\partial}$-exact forms relating individual members of a class. Then morphisms exist only between members of a given equivalence class. For a given helicity h, the models are then given by $\left(\mathbb{PT}, \bar{\partial}, f\right)$, where f is any representative $f \in [f] \in H^{0,1}\left(\mathbb{PT}, \mathscr{O}(2h-2)\right)$. For completeness, consider also the category **Class** whose objects $\left(\mathbb{PT}, \bar{\partial}, [f]\right)$ are provided by entire equivalence classes and the only morphisms are the identity arrows.[36] Conversely, the category **Field** has objects $(\mathbb{CM}, \phi)$, where ϕ solves the helicity h ZRM

[35] Although note that this section in particular is riddled with misprints.

[36] These categories are analogous to those used by Nguyen et al. (2020) for classical electromagnetism.

field equation, and the morphisms are given by diffeomorphisms[37] $\chi : \mathbb{CM} \to \mathbb{CM}$ that preserve the field: $\chi^*\phi = \phi$. In addition, construct the category **Gauge** with objects being the gauge potentials ψ generating the helicity h field ϕ via (3.2.2), and the arrows being provided by both the diffeomorphisms that preserve the gauge potential and the gauge transformations themselves.

We note that there is some disagreement in the literature about whether diffeomorphisms should be part of the models. In response to Weatherall (2016b) who considers them as such, Nguyen et al. (2020) omit them, stating that the focus of their article is on gauge structure *simpliciter*. It seems, however, that in order to establish something like theory equivalence, one should carefully consider which models are treated as representing the same physical state by the theory. If one considers Leibniz equivalence to be part of the theory formulation, one rather should consider diffeomorphically-related models as such. Without committing to this view, we elect to include diffeomorphisms at this point, in order to try to obtain the strongest form of equivalence possible.

Let us summarise in the following table:

	Ob	Mor
Twist	$\left(\mathbb{PT}, \bar{\partial}, f\right)$	$f \mapsto f + \bar{\partial} g$
Class	$\left(\mathbb{PT}, \bar{\partial}, [f]\right)$	$[f] \mapsto [f]$
Field	$(\mathbb{CM}, \phi)$	$\phi \mapsto \chi^*\phi$
Gauge	$(\mathbb{CM}, \psi)$	$\psi \mapsto \chi^*\psi, \psi \mapsto \psi + \nabla\gamma$

We now look for the following functors:

$\longrightarrow$	**Twist**	**Class**
Field	F_T	F_C
Gauge	G_T	G_C

Let us first construct F_C. Let the action on objects be provided by the ZRMPT, that is $F_C : (\mathbb{CM}, \phi) \mapsto \left(\mathbb{PT}, \bar{\partial}, [f]\right)$ such that the cohomology class $[f]$ generates the field ϕ such as in §2.2.1. The only arrows in **Class** are identities so we map all the diffeomorphisms of a given field to the same identity acting on the cohomology class onto

[37] Of course, a diffeomorphism of manifolds generates a unique isometry of (pseudo-)Riemannian manifolds so these terms are often used interchangeably. Here we stick to the more evocative 'diffeomorphism'.

which the fields map. There is a question of well-definedness: do all diffeomorphically related fields map to the same cohomology class? Indeed they do, because we choose precisely those diffeomorphisms that preserve the field.

Proposition 3.1. *F_C is a functor.*

Proof. The objects have been provided. Identity and composition are both easy to verify because all the arrows in **Field** are identities in the sense of category theory. □

We now propose:

Proposition 3.2. *F_C forgets stuff.*

Proof. We need to show that F_C is not faithful. Consider a field $\phi \in \mathrm{Ob}(\textbf{Field})$ and the associated $F_{C_{\phi,\phi}} : \hom_{\textbf{Field}}(\phi, \phi) \to \hom_{\textbf{Class}}(F_C\phi, F_C\phi)$, $F_{C_{\phi,\phi}} : \phi \mapsto F_C\phi$. It is clear that $F_{C_{\phi,\phi}}$ is not injective since there is only one identity on $F_C\phi \equiv [f]$ whereas there are in general several diffeomorphisms preserving a given field on spacetime. It follows that F_C is not faithful. □

A short philosophical intermezzo: here it is clear how the choice of arrows makes or breaks equivalence. One is reminded of Weatherall (2016b) and the response offered by Nguyen et al. (2020). Weatherall considers the isometries to be structure-preserving maps between models of classical electromagnetism, whereas Nguyen et al. dispense with them in favour of focussing exclusively on the issue of gauge. We are of the opinion that such a step is more significant than it is perhaps made out to be, as the argument is strongly dependent upon which arrows are available in which category. So: are there any additional structure-preserving maps on twistor space that we might want to consider? We have diffeomorphisms on twistor space, but there is apparently no canonical way to map them to spacetime isometries.[38,39] Another way of seeing this is as a problem of the category-theoretic method of establishing equivalence, in that there is a degree of arbitrariness involved in how arrows are defined. Alternatively, this is a strength of the approach since it allows one to very precisely define which models are treated as equivalent in an 'interpretation' of a theory by carefully selecting the arrows.[40] For

[38] We are grateful to Lionel Mason for conversations on this point.

[39] The only substantial paper on this topic is (Davidov, 2023), which considers diffeomorphisms between twistor spaces for two elements of a conformal class of metrics, and concludes these diffeomorphisms must be the identity map. Since we are interested in breaking such conformal classes by introducing an infinity twistor—see discussion below (2.1.10)—we will not engage with these results further here.

[40] For another illustration of this, see (March, 2024a).

well-understood theories, however, relatively uncontroversial choices of arrows can be made which then allows the approach to derive relatively undisputable results, such as the equivalence of standard GR and Einstein algebras (Rosenstock et al., 2015). Perhaps, then, this limitation is only applicable to novel theories with less established standards of equivalence. We'll return below to issues related to these.

A similar story can be told about G_C. Map all potential fields generating the same gauge field to the cohomology class related to it via the ZRMPT. Then map all arrows between different gauge potentials (diffeomorphisms, gauge transformations) to the identity on the cohomology class related to the gauge potentials via the gauge field they generate.

Proposition 3.3. *G_C is a functor.*

Proof. As above. □

Proposition 3.4. *G_C forgets stuff.*

Proof. Much like above. □

Let us now construct F_T. Let a field be mapped to any single representative of the cohomology class related to it via the ZRMPT.[41] As to how diffeomorphisms should be mapped to $\bar{\partial}$-exact forms, this is in general unclear, and indeed there is no obvious way of encoding spacetime isometries in terms of twistors.[42] Certainly, twistor space has its own structure-preserving maps but they allegedly do not have an obvious relation to spacetime diffeomorphisms. We might then wish to consider the category **Field′** which is just **Field** stripped of diffeomorphisms (by which we mean: take one arbitrary representative of each equivalence class of diffeomorphism-related models—this would be one way of making good on Nguyen et al. (2020) dispensing with isometries, as mentioned above). Of course, identities have to remain in order to satisfy the category axioms. Then we wish to construct a functor F'_T : **Field′** $\to$ **Twist**. We will map fields to cohomology classes as before, and map the identities to identities.

Proposition 3.5. *F'_T is a functor.*

Proof. As above. □

Proposition 3.6. *F'_T forgets structure*.*

[41] Here we potentially require the axiom of choice, because there is no canonical representative of the class.

[42] Again, we are grateful to Lionel Mason for discussion on this point.

Proof. By construction, we never map to any of the $\bar{\partial}$-exact form transformations. So $F'_{T\,a,a'}$: $\text{hom}_{\textbf{Field}}\,(a,a') \to \text{hom}_{\textbf{Class}}\,(F'_T a, F'_T a')$ is not surjective. In fact, $\text{hom}_{\textbf{Field}}\,(a,a')$ is even empty for $a \neq a'$ since our only morphisms are identities. Hence F'_T is not full. □

As for G_T, we will again want to purge **Gauge** of diffeomorphisms to obtain **Gauge**′ and we now look for G'_T : **Gauge**′ → **Twist**. In order to establish equivalence, we need a way of mapping gauge transformations to $\bar{\partial}$-exact forms. One sometimes finds the claim in the literature that there is a canonical such map for $h > 0$—for example, Adamo (2018) calls this map the 'Sparling transform'. This is claimed by Adamo (2018) to be demonstrated in (Sparling, 1990) but we were not able to recover the result from that paper. Instead, the only explicit constructions which we were able to obtain are for the case $h = 1$ in (Adamo, 2018, pp. 27–8) and for the case $h = 2$ in (Mason and Skinner, 2010). In the absence of a general construction, we will here consider only the case $h = 1$ as a 'proof of concept'.

Let us provide a brief account of the construction. Let $f \in [f] \in H^{0,1}(\mathbb{PT}, \mathscr{O}(0))$ be a cohomology class representative. To obtain the gauge potential at some spacetime point x, consider the restriction of f to the line $X \cong \mathbb{CP}^1 \in \mathbb{PT}$ related to this spacetime point via the Klein correspondence. Then $f|_X \in [f|_X] \in H^{0,1}(X, \mathscr{O}(0))$. But $H^{0,1}(X, \mathscr{O}(0)) \cong H^{0,1}(\mathbb{CP}^1, \mathscr{O}(0))$ since diffeomorphic manifolds have isomorphic cohomology, and it turns out that the latter is empty. Therefore $f|_X$ is exact and we can write $f|_X = \bar{\partial}|_X h(x, \lambda, \hat{\lambda})$ for some h of homogeneity degree zero in λ, $\hat{\lambda}$ which are the holomorphic and the antiholomorphic coordinates on the $\mathbb{CP}^1$ subspace. Now, since $f \in H^{0,1}(\mathbb{PT}, \mathscr{O}(0))$, it can only depend on x via $x^{\alpha\dot{\alpha}}\lambda_\alpha$ (this simply follows from the incidence relation (2.1.7)). Now consider

$$\bar{\partial}|_X(\lambda^\alpha \partial_{\alpha\dot{\alpha}} h) = \lambda^\alpha \partial_{\alpha\dot{\alpha}} \bar{\partial}|_X h = \lambda^\alpha \partial_{\alpha\dot{\alpha}} f|_X \propto \lambda^\alpha \lambda_\alpha = 0\,, \tag{3.2.4}$$

recalling that λ^α is the homogeneous coordinate on $\mathbb{CP}^1$ and hence $\lambda^\alpha\lambda_\alpha = 0$. Using the extended Liouville's theorem,[43] we conclude that the function $\lambda^\alpha \partial_{\alpha\dot{\alpha}} h$ must take the form

$$\lambda^\alpha \partial_{\alpha\dot{\alpha}} h = \lambda^\alpha A_{\alpha\dot{\alpha}}(x), \tag{3.2.5}$$

where $A_{\alpha\dot{\alpha}}$ is the Maxwellian gauge potential.

[43] The statement of the extended Liouville theorem is as follows: If g is a holomorphic function and for sufficiently large z, $|g(z)| \leqslant A + B|z|^k$, where A and B are positive constants and $k \in \mathbb{N}$, then g is a polynomial of degree at most k. In our case, the holomorphicity of the function $\lambda^\alpha \partial_{\alpha\dot{\alpha}} h$ is ensured by (3.2.4) and one can prove that the bound exists for $k = 1$. So the function can be a linear polynomial of λ^α. The coefficient is imposed by the index structure to be of the form $A_{\alpha\dot{\alpha}}(x)$ and can only depend on x and not on λ, since the RHS has to be a linear polynomial in λ.

In light of this, there is a map from Dolbeault-exact $(0,1)$-forms on $\mathbb{PT}$ and gauge potentials $A_{\alpha\dot{\alpha}}$ on $\mathbb{CM}$. This suffices to establish the existence of a map $(G'_T)^{-1}$: **Twist** $\to$ **Gauge**$'$. There do, however, remain open two issues. First: it is not completely clear that a different choice of Dolbeault-exact $(0,1)$-form on $\mathbb{PT}$ would yield a different gauge potential on $\mathbb{CM}$—so it remains open whether this map is one-one. And second: it also remains open whether any map G'_T : **Gauge**$'$ $\to$ **Twist** is one-one—that, is, whether gauge potentials can be mapped one-one to elements of the relevant class of forms on $\mathbb{PT}$. As far as we can tell, there is no particularly natural such map which one can identify with such properties. In principle, one could try to construct such a map by fiat: to associate Dolbeault-exact $(0,1)$-forms to gauge transformations, it suffices to specify arbitrary choices for each such mapping, but since both spaces are continua there must exist a bijection between them. This would presumably yield a functor that establishes equivalence between the two categories, but one would be hard-pressed to conclude that this is an interesting case of theoretical equivalence since the choice of functor is so arbitrary. It could be seen as a methodological drawback of categorical approaches to theoretical equivalence that one can define such *fiat* constructions in the cases of continua and suitably simple categories, but one should note that proponents of categorical equivalence are careful to point out that one *should* expect there to be a natural enough functor establishing a purported equivalence, rather than one that is simply based on the equal cardinality of both sides, and then made to respect the internal structure of the theory also by fiat. Preserving internal structure by fiat, however, will only be possible for similar enough structures, e.g. in the case of **Gauge**$'$ and **Twist** we are dealing with groupoids[44] defined on continua so it is not inconceivable that there is an equivalence-generating functor between them, whereas in general one would not obtain this level of similarity between two categories representing two distinct theories.

In any case, so much for the $h = 1$ case. One could apply the same technique to the other cases of twistor correspondence, such as the correspondence for Yang-Mills, and the non-linear graviton for the sourceless Einstein equation. Stepping back somewhat, the situation regarding the equivalence of spacetime and twistor space physics, as we see it, is this. First: such equivalence is only relative to a choice of Penrose transformation—and often, such transformations (e.g. the ZRMPT which we have considered here) are somewhat restricted in scope (in the sense that they do not pertain to the entire solution space of GR—note indeed that the ZRMPT considers fields on $\mathbb{CM}$ and so in fact has more to do with SR than GR!). Second: even focusing on a specific Penrose transformation, whether there indeed is equivalence—in the sense of categorical equivalence—will

[44] Groupoids are categories with all morphisms invertibile.

depend upon particular choices as to how to formulate spacetime physics and twistor space physics: as illustrated above in our specific choices of categories on each case, and the problems for demonstrations of equivalence between **Gauge**′ and **Twist**. In our opinion, therefore, any *ab initio* declarations of the equivalence of spacetime physics and twistor space physics should be tempered with a certain degree of caution.

4 Philosophical issues

Having now presented the relevant background on twistor theory, as well as some regimentation of the spacetime–twistor correspondence using the resources of categorical equivalence, we turn now to an exploration of five topics pertaining to twistor theory and its philosophical significance. In §4.1, we consider the issue of whether there are dynamics on twistor space—a claim which one often finds denied in the existing literature; see e.g. (Bain, 2006). In §4.2, we explore the sense in which the move to twistor space offers a novel form of 'geometrisation' of a physical theory. In §4.3, we try to understand the ontology of twistor space on its own terms. In §4.4, we assess whether twistor theory presents a case of spacetime emergence in physics. Finally, in §4.5, we consider how twistor theory interacts with some 'symmetry principles' which are widely discussed in the philosophy of physics.

4.1 Dynamics on twistor space

According to Bain's (2006) account of twistor theory, Penrose transformations can be said to 'geometrise away' dynamics. Specifically, Bain writes that "the dynamical information represented by the differential equations in the tensor formalism gets encoded in geometric structures in the twistor formalism. Advocates of the twistor formalism emphasize this result—they observe that, in the twistor formalism, there are no dynamical equations; there is just geometry" (Bain, 2006, p. 40). Be that as it may, this is simply not something that appears to be widely claimed by twistor advocates: the claim, for example, is not found in (Penrose and Rindler, 1988a,b; Ward and Wells, 1990; Penrose, 2005). Still, Bain's claim appears *prima facie* plausible and is certainly worthy of attention.

The key notion to unpack here is, of course, that of 'dynamics'. The way in which Bain uses the word seems to be tied closely to the use of a derivative operator in a physical theory. For example, he talks about the "dynamical role" of the manifold having to do with "the support structure on which derivative operators are defined" (Bain, 2006, p. 39). Later in his article, he adopts a somewhat different stance saying that the dynamical role of the manifold is "a local back-drop on which differential equations can be defined that govern the dynamical behavior of fields" (Bain, 2006, p. 47). That is to say, rather than dynamics having to do with a derivative operator *per se*, dynamical equations are *some* class of differential equations, i.e. equations deploying a derivative operator. But then the question of what we take dynamics to be is not really addressed beyond saying that they are a kind of differential equation. In fact, there are several ways of characterising 'dynamics', as has been discussed by e.g. Linnemann and Read (2021). These are:[45]

1. Dynamics as some representation of a system evolving diachronically.
2. Dynamics cashed out in terms of quantities which vary between possibilities according to the theory, *à la* Curiel (2016).[46]
3. Dynamics as having to do with hyperbolic (rather than e.g. elliptic) differential equations.

(The relations between all three of these notions are not entirely straightforward—see (Linnemann and Read, 2021).) The question to be addressed here is this: does a theory set on twistor space have 'dynamics' in any of the above three senses?

Sense (1) can be dealt with fairly swiftly. Time does not feature explicitly in twistor theory so with that goes any hope of grounding its dynamics in temporal evolution. Indeed, the twistor programme regards twistors as pre-spatiotemporal (see §4.4), so this is hardly surprising. Now, of course, there are questions lurking in the background here regarding what it means to identify (perhaps by way of functionalist considerations) time in a given physical theory—various options here are canvassed by Callender (2017). However, we take it that in none of these senses of time is present in a theory set in twistor space as defined in the previous section. For example (to take Callender's preferred functional definition of time) there are no equations in such theories which admit of well-posed Cauchy problems—at least to our knowledge.

Indeed, since it is only hyperbolic partial differential equations which admit of well-posed Cauchy problems, this suffices to address also sense (3) above, and so just leaves sense (2)—more can be said about this. Curiel (2016) differentiates kinematical equations from dynamical equations by their "particular form" being invariant across pos-

[45]Cf. (Read and Cheng, 2022).

[46]There are various ways in which this might be expounded—see (Linnemann and Read, 2021). For an alternative approach to identifying dynamical possibilities according to a theory to that offered by Curiel (2016), see (March, 2024b).

sibilities according to the theory. An example makes this clear: consider the four equations of Maxwellian electrodynamics:

$$\begin{aligned}\vec{\nabla} \times \vec{E} &= -\partial_t \vec{B}, \\ \vec{\nabla} \cdot \vec{B} &= 0, \\ \vec{\nabla} \cdot \vec{E} &= \rho, \\ \vec{\nabla} \times \vec{B} &= \vec{J} + \partial_t \vec{E}.\end{aligned} \tag{4.1.1}$$

The first two equations here have the same form no matter the application, making them kinematical, whereas the third and the fourth require one to substitute for the charge density and the current density respectively in order to fix their form, making them dynamical. One worry with calling twistor theory 'non-dynamical' according to this criterion, though, is that we have so far only translated vacuum electrodynamics in the twistor formalism. *In vacuo*, we have $\rho, \vec{J} = 0$, thereby making all the four equations kinematical in the strict sense. How could we then possibly expect to obtain dynamics if we already started from a system that is not dynamical?[47]

On the other side of the same coin is the following fairly natural point. A given theory set in spacetime (for the time being assuming conformal invariance, so that theory is presumed to be amenable to a Penrose transformation) will in general have many solutions (conformally inequivalent to one another), and it needn't be the case that all such solutions—*qua* models of spacetime geometry—map to the same twistor space geometries (indeed this won't be the case, if the spacetime models are conformally inequivalent). That however means—*pace* Bain—that theories set in twistor space *do* have dynamics according to criterion (2)—because not all dynamical possibilities of twistor theory will be equivalent: some quantities will vary between possibilities according to the theory. Of course, we are not claiming that criterion (2) should be endorsed—ultimately, our point here is just that the 'no dynamics' claim becomes delicate, once one attempts to cash out more precisely what's meant by 'dynamics'.

4.2 Geometrisation

Related closely to the question of whether there are dynamics on twistor space is that of whether, in any precise sense, twistor theory 'geometrises' a given set of dynamics on spacetime. Again, for Bain (2006) it seems that twistor theory offers prospects for a novel form of geometrisation of a physical theory—as he writes, "the local dynamics in

[47] This worry will not apply on the understanding of the kinematics/dynamics distinction offered by March (2024b).

the spacetime formulation gets encoded in a global "static" geometric structure in the twistor description" (p. 47).

In order to get clearer on the sense (if any) in which twistor theory 'geometrises' a physical theory formulated in spacetime, we of course first need to get clearer on what it means to 'geometrise' a physical theory. On this, surprisingly little has been written in the existing philosophical literature—exceptions are (Dürr, 2020; Kalinowski, 1988; Lehmkuhl, 2009); we'll begin with the tripartite classification of degrees of geometrisation debeloped by Lehmkuhl (2009), and summarised succinctly by Dürr (2020) as follows:

- **Strength-1** geometrisation dresses up field theories in geometric clothing. The fiber bundle formulation of electromagnetism is a case in point: while in such a *representation* everything looks geometric, we have prima facie little reason to regard the theory as describing anything inherently related to spacetime geometry.
- In **strength-2** geometrisation, physical degrees of freedom can be accounted for in terms of geometric properties (e.g. topology, curvature, or torsion) of *augmented* spacetime structure. An example is Weyl's (1918, 1919) unified field theory. In it, the electromagnetic field is reconceptualised ("strength-2 geometrised") as a manifestation of what Weyl called "length curvature" of a non-Riemannian spacetime (i.e. a spacetime in whose geometry parallel transport of vectors alters their length).
- **Strength-3** geometrisation, paradigmatically instantiated by GR's geometric interpretation, is essentially *eliminative*: a geometric theory of strength-3 reduces physical degrees of freedom to manifestations of (a universal) inertial structure—a preferred path structure of "natural", uncaused/default motion that, for instance, force-free test particles trace out. (Dürr, 2020, p. 6)

In our own words, we'd put the classification like this: strength-1 geometrisation regards writing physical theories in (differential-)geometric formalism; strength-2 geometrisation regards reconceptualising physical effects/forces/etc. in terms of (novel) geometrical effects; strength-3 geometrisation regards reducing physical effects/forces/etc. to existing geometric quantities (e.g., curvature of a Levi-Civita connection in the case of gravity in GR). The question to be addressed now is this: does the geometrisation in the case of twistor theory amount to geometrisation in any of the above three senses, or is it in fact a *novel* form of geometrisation (of course assuming that we have a case of geometrisation here at all!)?

The kind of geometrisation offered by twistor theory is clearly more than merely strength-1—although it is true that twistor theory is articulated using differential/algebraic geometrical methods, there is good reason to think that there is more at stake than just this. With respect to strength-2 geometrisation, however, it is not obvious to us that it is correct to view the geometric setting of twistor space as an *augmented* version of a more traditional spacetime setting (e.g., Minkowski space, possibly complexified), for the geometric arena of twistor space $\mathbb{PT}$ is clearly simply different from e.g. $\mathbb{CM}$. With respect to strength-3 geometrisation, on the other hand, it seems to us that this notion is at least in some sense satisfied by the twistorial equivalents of spacetime theories, for—as already explained above—facts about spacetime dynamics are encoded in geometrical facts about cohomology classes on twistor space.[48] Note, though, that the kind of geometrisation at play here is rather more extreme and thoroughgoing than what Lehmkuhl (2009) and Dürr (2020) seem to have in mind when it comes to strength-3 geometrisation—for in the case of the transition from a spacetime model to a twistor space model, it is not *merely* (as in Dürr's GR-inspired example) that forces such as gravity are absorbed into a new derivative operator; rather, *all* dynamics are absorbed into facts about twistor geometry. This demonstrates that there is some ambiguity in the scope of strength-3 geometrisation; twistor geometrisation (and here we concur with the spirit of what Bain (2006) writes) appears to lie at the more extreme end of what it would mean to strength-3 geometrise a theory.

4.3 Ontology of twistor space

What would an interpretation of twistor theory 'on its own terms'—what de Haro (2019) might call an 'internal interpretation'—look like?[49] In this subsection, we address this question with particular attention directed towards the status of spacetime in (the physical interpretation of the models of) twistor theory. Clearly, there are broader

[48] That said, it bears stressing that the claim that the map to twistor space constitutes a case of strength-3 geometrisation can't be *exactly* correct, since there is no straightforward sense in which these geometrical facts on twistor space count as 'inertial structure'.

[49] Roughly, for de Haro, an internal interpretation of a given theory does not invoke specific structure unavailable in that theory *per se*; not so for an external interpretation. Presumably, it is internal interpretations which Weatherall (2018) has in mind when he asserts that general relativity and hole-diffeomorphic models thereof 'do not generate a philosophical problem'; for critical engagement with this claim, see (Pooley and Read, 2021). Of course, mention of the hole argument here invites questions as to how this would pan out in twistor theory; we'll return to these questions below.

questions at play in the background here: interpreted *per se*, does twistor theory invite a substantivalist or a relationalist ontology?[50]

Regarding the status of spacetime in twistor theory, Bain (2006) writes that "the twistor constructions indicate that the differentiable manifold is not essential" (p. 46)—by this, we take Bain to mean that twistor theory interpreted *per se* does not invite a commitment to a manifold of points. We confess that we find it a little difficult to make sense of this claim, for a couple of reasons. First: twistor space $\mathbb{PT}$ is still built upon the mathematical structure of differentiable manifolds; hence, interpreted unto itself it does not appear that the theory liberates us from a commitment to (the physical correlates of) such structures. Second, and relatedly: we have seen above that—at least in certain restricted contexts—one might argue that there is an equivalence between a proper subset of the models of relativistic spacetime theories and models formulated in twistor space.[51] But in that case, there is a sense in which twistor theory is committed to (spacetime) differentiable manifold structure after all. Perhaps—to be charitable—all Bain has in mind here is that with twistor space $\mathbb{PT}$ in hand and armed with various Penrose transformations, one need not treat the *spacetime* formulation of a physical theory as being ontologically fundamental.

Compare here the case of the equivalence between models of GR and Einstein algebras.[52] In that case, one side of the 'duality' (i.e., the Einstein algebras side) does not involve a differentiable manifold (at least in any direct sense); the interpretation of those models therefore—Rosenstock et al. (2015) claim—invites a relationalist ontology; by contrast, of course, models of GR do involve a differentiable manifold and so invite a substantivalist ontology.[53] With respect to the first of the above two points, then, the GR–twistor relation is clearly disanalogous to the GR–algebras case, for the 'alternative' formulation still helps itself to a differentiable manifold. And with regard to the

[50]For background on the substantivalism/relationalism debate more generally, see (Pooley, 2013b).

[51]Note here that we have moved to considering the correspondence between twistor theory and the *general* theory of relativity—a correspondence already acknowledged to be delicate and piecemeal in §2. This poses a further obstacle to any straightforward ontological fundamentality claim made on the basis of twistor theory in the context of general relativity, as we will discuss further below.

[52]This equivalence was first presented by Geroch (1972), before being taken up in the context of the philosophical debate regarding substantivalism and relationalism by Earman (1977). More recently, Rosenstock et al. (2015) proved a categorical equivalence between models of GR and Einstein algebras; Wu and Weatherall (2023) demonstrate that this equivalence breaks down when one liberalises the models of GR by dropping the Hausdorff condition. Bain (2006) also assesses the case of Einstein algebras alongside the case of twistor theory.

[53]Here, following Earman and Norton (1987), we have in mind *manifold* substantivalism, which is the position that it is the differentiable manifold in a model of a physical theory which represents physical spacetime.

second of the above two points: one can profess a certain ambivalence about the status of manifold points in GR given the existence of an equivalence formulation of the theory in terms of Einstein algebras (cf. (Rosenstock et al., 2015)); clearly, one cannot do this in the twistor case, if the twistor space formalism invites itself to the same kinds of mathematical objects anyway!

So, *vis-à-vis* the status of spacetime in twistor theory, we are somewhat sceptical of claims made by Bain (2006). Nevertheless, in his defence, it is worth registering that those working in the mathematics of twistor theory do sometimes make similar claims. For example, Atiyah et al. (2017) write that

> [i]n the twistor approach, space-time is secondary with events being derived objects that correspond to compact holomorphic curves in a complex threefold—the twistor space. (Atiyah et al., 2017, p. 1)

On this, we say the following. First, one has to distinguish the fact that point *events*—e.g., intersections of worldlines—in the spacetime formulation are mapped to non-local (i.e., extended) objects in twistor space; true enough, however, this does not detract from the fact that—as explained above—twistor theory remains committed to (the physical correlate of) a differentiable manifold of points. Second, one must recall lessons from Teh (2013) made in the case of dualities: the existence of a mathematical mapping between two theories does not in itself invite any metaphysical asymmetry between the two—thus, the above claim that the spacetime formulation is subordinate to the twistorial formulation seems *prima facie* to be specious. On this latter issue, Bain writes the following:

> The tensor formalism suggests a commitment to local fields and spacetime points, whereas the twistor formalism suggests a commitment to twistors, which themselves admit diverse interpretations. The traditional realist might respond by claiming that the Penrose Transformation just shows that solutions to certain field equations behave in spacetime as if they were geometric/algebraic structures that quantify over twistors. In other words, we should not read the twistor formalism literally—it merely amounts to a way of encoding the behavior of the real objects, which are fields in spacetime, and which are represented more directly in the tensor formalism. In other words, we should only be semantic realists with respect to the tensor formalism. This strategy smacks a bit of *ad hocness*. All things being equal [...], what, we may ask, privileges the tensor formalism over the twistor formalism? (Bain, 2006, p. 50)

What Bain is countenancing here is a ('traditional realist') position according to which the spacetime formulation is preferred over the twistor formulation. In fact, we *agree* with Bain that, in the absence of further details/argument, such a position would seem to be just as *ad hoc* (and contrary to the morals of Teh (2013)) as the twistors-first view adumbrated by Atiyah et al. (2017) in the above-quoted passage. That said, we do think that there are ways in which one might be able to break this interpretative symmetry/impasse in one manner or another—for example, one could appeal to (i) functionalism *à la* Knox (2017), (ii) considerations of surplus 'gauge' degrees of freedom (see e.g. Weatherall (2016b)), (iii) considerations to do with the existence of dynamics in one formulation versus the other (which could be regarded as the *sine qua non* of physical theorising)—cf. §4.1, (iv) descriptive/explanatory power.

Let us go into a little more detail here. On (i), recall that, for Knox (2011, 2013), if one theory can be mapped to another theory which better picks out a "structure of local inertial frames", then that latter theory offers the superior roster of ontological commitments with which to associate even the former theory. In the case of the GR–twistor correspondence, the lack of dynamical equations on the twistor side, as discussed above, militates in favour of an interpretation *per* Knox on which even in the twistor space context, it is really the spacetime interpretation which is fundamental. On (ii), if one has an interpretation of the twistor space setting according to which objects in the cohomology classes are distinct (see §3), then one might be able to argue that the twistor space setting has more gauge freedom than the spacetime setting, thereby favouring the latter over the former. On (iii), if one maintains that there are indeed no dynamics in the twistor space context, then this might again weigh in favour of the spacetime interpretation. And on (iv), of course ultimately explanatory considerations will depend upon the definition of scientific explanation in play (see (Woodward and Ross, 2021) for a survey of the options here), but one thing which can be said immediately is that the piecemeal nature of the map from the GR solution space to models of twistor theory via various Penrose transformations might also be taken to detract from thinking that the latter is somehow more fundamental, ontologically speaking, than the former.

4.4 Emergence of spacetime

In the literature on twistor theory, it is often claimed that spacetime is supposed to be *emergent* from some underlying twistor space ontology.[54] As one representative example of this among many, Penrose writes that in twistor theory,

[54] Related to this, there is a by-now quite large philosophical literature on the emergence of spacetime in quantum gravity—see e.g. (Huggett and Wüthrich, 2013) for an introduction.

> spacetime points are deposed from their primary role in physical theory. Spacetime is taken to be a (secondary) construction from the more primitive twistor notions. (Penrose, 2005, p. 962)

(We have, indeed, already seen something of this in the previous subsection.) Give the preponderance of statements of this kind, it is worth pausing on whether it really is plausible to understand general relativistic spacetime as 'emerging' from some more fundamental twistorial depiction of reality—or whether, instead, 'emergence' might be an inappropriate classification of what is going on here. Moreover, one can go on to ask: if 'emergence' is indeed inappropriate here, what would in fact be a suitable categorization?

Before proceeding further on these issues, it will help to fix terminology; we will follow the lead of Castellani and de Haro (2020). First, let us say that a *duality* is "a bijective map between the states and quantities of two theoretical descriptions, such that the dynamics and the values of the quantities are preserved" (Castellani and de Haro, 2020, p. 199).[55] Second, let us say that one has a case of *emergence* in physics when there are entities which "arise out of more fundamental entities and yet are 'novel' or 'irreducible' with respect to them" (O'Connor and Wong, 2005).[56] One might also wish to distinguish *strong* from *weak* cases of emergence. The former "is the lack of derivability in principle—the theory simply lacks the resources to derive whatever is emergent from it" (Castellani and de Haro, 2020, p. 201); the latter is "the lack of derivability in practice—some derivations may be available, but they are difficult to carry out within the theory's methods or resources, so that the situation is, in practice, as if one was dealing with strong emergence" (Castellani and de Haro, 2020, p. 201). As Castellani and de Haro (2020) correctly go on to point out, if a duality is exact, then one can at best have a case of weak emergence; if, however, a duality is approximate, then one can have both weak and strong cases of emergence.[57]

Turning now to twistor theory, we have seen that, at least in certain circumstances (more restricted, admittedly, than one might have initially thought/hoped—recall again §3), there is an exact, one-to-one correspondence between the twistor space description and the spacetime description; moreover, the correspondence preserves empirical content. For the relevant particular subsector of GR, then, we seem to have an exact duality between said subsector and twistor space. For this particular subsector, in turn,

[55] For further philosophical literature on dualities, see e.g. (de Haro and Butterfield, 2017) and references therein.

[56] Following (Castellani and de Haro, 2020), we focus in this section exclusively upon what has come to be known as 'epistemic emergence', which regards novelty 'in the description' rather than 'in the world'. For further philosophical background on emergence in general, see e.g. (Batterman, 2009).

[57] For further background on exact versus approximate dualities, see (de Haro et al., 2016).

only weak emergence seems to be possible—*not* strong emergence, as the above passage from Penrose would seem to suggest. Each Penrose transform (and its inverse) provides a derivation mechanism and the existence of Penrose transforms clearly eliminates the case of strong emergence in twistor theory. Indeed, in the case of twistor theory, it is not even obvious to us that there is a serious case of *weak* emergence here, because the theory has all the required resources to derive the spacetime notions and all the descriptions have an associated (inverse) Penrose transform that provides a well-defined mechanism to derive the corresponding spacetime notions.

It's also not clear to us that any discussion of 'fundamentality' in the context of twistor theory is particularly relevant, because when ones talks about fundamentality typically one has two theories which one classifies as a 'top' theory and a 'bottom' theory, and one considers the 'top' theory to be less fundamental if it can be derived (at least partly) from the more fundamental 'bottom' theory but not the other way around (Castellani and de Haro, 2020). But in twistor theory, each Penrose transform is a two-way implication (i.e. one can also derive the twistorial objects from physical spacetime notions). And hence it wouldn't be appropriate to compare the relative fundamentality of the twistor space and spacetime—as mentioned in the previous subsection, this point has indeed already been made by Teh (2013) in the context of holography.

From this, what we see is that if twistor proponents are to maintain that it indeed *is* the twistor description which is fundamental, they will need to give further justification for this claim than what has been proffered thus far. Here is one possible such argument which they might give. Since points in twistor space are supposed to correspond to light cones (i.e., the trajectories of possible light rays emanating from a point), and since the latter are sometimes argued to have immediate operational significance (see e.g. (Ehlers et al., 2012)), one could perhaps argue that, on operationalist grounds, the twistorial description is to be preferred. Any such reasoning, however, faces some immediate questions: (i) is the operationalism upon which it is predicated actually plausible?; (ii) what of the GR solutions which do not have obvious twistor correlates but which seem to be important for the physical modelling of the actual world? Etc.

Now, if neither description here is to be regarded as being more fundamental than the other, then there arise for the scientific realist straightforward issues of underdetermination: what is the structure of the world really like, in those circumstances? Here, the usual roster of interpretative options arise: one could either try to find some further reasons which privilege one of the two descriptions, *per* the above; one could try to identify the mathematical 'common core' of the descriptions (cf. (Le Bihan and Read, 2018)); or one could embrace some kind of geometric conventionalism (on which see e.g. (Dürr and Read, 2023)). We will leave for another day further exploration of the merits of each of these options.[58]

4.5 Symmetry principles

As we have discussed, the twistor equation is conformally invariant, so if we want to represent metric structure on twistor space, we have to introduce further structure in the form of the infinity twistor. This allows us to construct the twistor space of Minkowski space $\mathbb{PT}(\mathbb{M}) = \mathbb{PN}$. On the other hand, when we study field theory we have seen that twistor theory only provides convincing examples of correspondence for conformally invariant field theories such as the massless free field (ZRMPT) and self-dual Yang-Mills (via the Ward transform), apart from some partial results such as those presented by Eastwood (1981).

Focussing on the former, we have seen that massless fields on $\mathbb{CM}$ correspond one-to-one to cohomology classes on $\mathbb{PT}(\mathbb{CM})$. The more general statement is that this holds for arbitrary subsets U of $\mathbb{CM}^C$ and the twistor spaces thereof, pending some conditions on U as shown by Eastwood et al. (1981). Imagine then that we want to study the massless free field on the compactified complexified Minkowski space $\mathbb{CM}^C$. Then we have $\mathbb{PT} = \mathbb{CP}^3$. But in fact the first cohomology on this space vanishes so we can only recover the trivial field. If we instead break the conformal invariance and consider $\mathbb{PT}(\mathbb{CM}) = \mathbb{CP}^3 - \mathbb{CP}^1$, or even impose $\mathbb{PT}(\mathbb{M}) = \mathbb{PN}$, we get much more cohomology and recover a rich variety of fields on $\mathbb{M}$.

But now we are in a peculiar situation: we started studying the conformally invariant zero rest mass equation but ended up breaking the conformal invariance of spacetime such that we can obtain any interesting solutions. Our spacetime symmetry group is smaller than our dynamical symmetry group in apparent violation of Earman's principle SP1 (Earman, 1989, ch. 3). What does this tell us about twistor theory? Clearly the problem is that there are no convincing twistor results for non-conformally-invariant theories. At the end of the day, we want twistor theory to reproduce our standard description of the world which consists of non conformally invariant field theory on a manifold with metric structure. But even if we try to move in the realm of the conformally invariant, it seems that twistor theory forces us to break conformal invariance on the spacetime side.

[58] Of course, twistor theory might also be connected to the emergence of spacetime in a less direct way—for example, via ambitwistor string compactification (see (Atiyah et al., 2017) for further discussion). Our thanks to an anonymous referee for inviting us to point this out.

5 Outlook

In this article, we have presented what we hope is a reasonably accessible introduction to twistor theory for philosophers (§2). We have then shown that one can bring to bear recent philosophical work on theoretical equivalence in order to shed some light on the spacetime–twistor correspondence (§3). Further, we have explored how twistor theory bears on a range of contemporary issues in the foundations of spacetime theories (§4). We'll close now by briefly outlining some further areas for possible research into the philosophy of twistor theory:

1. Just as philosophers have asked whether the hole argument (on which see (Norton et al., 2023)) can be generated when one moves from the manifold-based formalism of general relativity to an algebras-based formalism (see e.g. (Rynasiewicz, 1992)), one might likewise ask of the status of the hole argument when one moves to twistor theory. Of course, the answer to this question will hinge upon how diffeomorphisms in the spacetime formalism translate to maps in the twistor formalism—an issue which we have already seen in §3 to be rather delicate. Moreover, the issue will hinge upon whether one can really regard the twistor formalism are more 'fundamental' than the spacetime formalism—a matter on which we have already expressed our scepticism in §4.

2. It would be valuable to probe the extent to which twistor theory can offer any kind of unification in physics. We'll illustrate with three examples:

   (a) Woit (2021) has proposed that twistor theory provides a novel framework for gravi-weak unification, and moreover a unification of all the four fundamental forces. However, there is no physical correlation between the forces being unified, in the sense that e.g. there is no non-trivial coupling of said forces in some mutual dynamics. According to Maudlin (1996), this kind of physical correlation (which he calls "nomic correlation") is one of the key criteria required in order to regard a theory as being unificatory. So, following Maudlin's lead, we would suggest that there is no unification in a 'true' physical sense—though, of course, this ultimately warrants further investigation.

   (b) The significance of twistor theory in string theory is becoming increasingly well-appreciated (in particular see e.g. work on 'ambitwistor strings')—see (Atiyah et al., 2017) for a recent survey.

   (c) In order to establish the ZRMPT in §2.2.1, we adopted the Dolbeault cohomology description, with cohomology classes $f(Z^A)$ containing $(0, 1)$-

forms. However, one can also adopt the Čech cohomology description to establish the ZRMPT (Huggett and Tod, 1985).[59] The cohomology classes of the Čech cohomology group contain holomorphic functions. Recall from §2.2.2 that these holomorphic functions also appear in Kerr's theorem, where the null set of some holomophic function defines a shear-free null congruence. So, if we pick a cohomology class that defines a ZRM field on $\mathbb{CM}$, its null set will also define a shear free null congruence on $\mathbb{CM}$. Hence there is a common mathematical object that describes the two physical notions. Interestingly, there is previous work which establishes physical correlations between these two notions. For example, (Robinson, 1961) deals with connections between the Maxwell field and shear-free null congruences. So one can conjecture that twistor theory might be used to realise such connections between various notions in the spacetime formalism.

3. There is a dearth of philosophical literature on the geometrical foundations of spinor fields—one admirable exception being (Pitts, 2012). Given their close relation to spinors, twistors could well be brought to bear on future foundational explorations in this direction.

4. In §3 of this article, we focussed on the equivalence of spacetime formulations and twistor formulations via the ZRMPT. This leaves open for future exploration such equivalences when one considers e.g. the non-linear graviton, or Yang-Mills fields on spacetime, etc.

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[59] Cf. footnote 27.

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