

Noether's Ledger: Interpreting Charge Conservation in non-Abelian Gauge Theories

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Abstract

The application of Noether's theorem to the exact $SU(3)$ color symmetry of quantum chromodynamics results in the conservation of the color charge current. This current takes values in $SU(3)$'s Lie algebra, and it is therefore eight-dimensional. But how can this eight-dimensional space be the right mathematical object for the conservation of the three color charges *red*, *blue*, and *green* and their three corresponding anti-colors? We might have expected a six-dimensional space, or perhaps a nine-dimensional one, but eight is surprising. This paper answers this question through explicit construction of the $SU(3)$ adjoint representation from the two fundamental representations of $SU(3)$. This construction generates principled reasons for interpreting elements of the $SU(3)$ Lie algebra as bearing combinations of color and anti-color. In light of this construction, this paper contrasts mathematical and conceptual features of color charge conservation with electric charge conservation, thereby highlighting some of the challenges and subtleties of interpreting non-Abelian gauge theories.

Keywords: Noether's theorem, color charge, charge conservation, non-Abelian gauge theory, Lie algebra valued quantities.

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1 Introduction

Emmy Noether's two celebrated 1918 theorems give a correspondence between the continuous symmetries of a physical system and the conserved quantities of that system [Noether, 1918]. It is well-known that these theorems provide, when applied in the appropriate contexts, proofs of the conservation of energy, linear and angular momentum, as well as charge. The $U(1)$ gauge symmetry in classical electromagnetism is shown, by way of these theorems, to imply conservation of electric charge. The generalization of this application to the conservation

of new kinds of charge in non-Abelian gauge theories, such as color charge for chromodynamics, is mathematically clear. However, the physical interpretation of the resulting conserved quantity is not straightforward. This aim of this paper is to clarify the physical interpretation of Noether’s charges for non-Abelian gauge theories. We find that the non-Abelian case exhibits the foundational role of the relationship between charge and anti-charge, whereas this relationship is obscured in the Abelian case. Conceptually, conservation requires that charge and its opposite, anti-charge, share a home within a single mathematical space, such that contributions of charge and anti-charge can be sensibly added together. For electric charge the shared mathematical home is \mathbb{R} , and the accounting work behind conservation is simple arithmetic. But non-Abelian theories need the more sophisticated mathematical home of the Lie algebra of the relevant symmetry group. Noether’s non-Abelian ‘ledger’, as it were, is Lie algebra valued. In what follows, we will discuss the sense in which the Lie algebra is the appropriate mathematical space for bringing together charge and anti-charge using tools from Lie group representation theory.

To make the discussion concrete, we will focus on a (classical version of) chromodynamics, the non-Abelian gauge theory of the strong interaction between quarks, anti-quarks, and gluons. In this theory, the charge property is color charge. Philosophers and physicists alike introduce color charge as analogous to electric charge. Moreover, there are three colors, usually called ‘red’, ‘green’, and ‘blue.’¹ Quarks are the sort of particles that can have these three different colors. Moreover, quarks are the leptons in the theory, which means that they are the chromodynamic analogues of electrons (and their heavier cousins, the muons and tauons): as electrons are in electrodynamics, quarks are in chromodynamics. This analogy suggests that conservation of charge in chromodynamics would simply be the conservation of *red* charge, of *blue* charge, and of *green* charge. And frequently in the context of Feynman diagrams, color charge conservation is explicitly discussed as the accounting of such color states at each vertex in the diagram (as shown below in figure 1).

However, Noether’s theorem does not lead to conservation of the three colors. In general, the conserved quantity given by these theorems takes values in the Lie algebra of the symmetry group. In chromodynamics, the relevant Lie algebra is eight-dimensional. So the conserved quantity cannot be valued in the three dimensional space of the three basic color states. But if not these three basic colors, what is the appropriate physical interpretation of this eight-dimensional Noether charge?

The answer is that Noetherian color charge is the same sort of color charge carried by gluons: it is a combination of both color and anti-color. The goal of this paper is to develop this interpretation explicitly and in a mathematically principled manner. In so doing, several points of connection between the pure mathematics of Lie group representations and the applications of these representations to particle physics are laid out systematically. There are a number of excellent resources for the applications of group theory to particle physics, such as [Cahn, 2014], [Georgi, 1999], [Baez and Huerta, 2010], [Lichtenberg, 1978], and

more recently [Woit, 2017]. However, the specific interpretive challenges of Lie algebra valued quantities, such as in the case of color charge, are not addressed in these resources. This paper serves to fill in this gap in the literature concerning the physical interpretation of Lie algebra valued quantities in particle physics. Furthermore, much of the philosophical literature has focused on the physical status of gauge symmetry transformations. This paper, instead, focuses on the physical interpretation of the conserved quantity associated with a non-Abelian gauge symmetry, and in this way it pushes the philosophical literature on gauge theories in a new direction.

The remainder of this paper is structured as follows. Section 2 reviews the standard account of Noetherian conservation of electric charge in scalar electrodynamics in terms of the vanishing of the divergence of the charge-current density. It then presents the generalization of this account to non-Abelian scalar field theories using the fiber bundle formalism. Section 3 develops the preliminaries of group representation theory of $SU(3)$ that are necessary for understanding the sense in which the Noether charge for chromodynamics is a combination of both color and anti-color. Section 4 gives the technical punch line, demonstrating the construction of the adjoint representation out of the two fundamental representations. Walking through the mathematical details of this construction has the philosophy payoff of generating principled reasons for assigning combinations of color and anti-color to states in the adjoint representation. Further philosophical discussion of the appropriateness of the adjoint representation for the conceptual work of conservation is discussed in section 5. Concluding remarks are given in section 6.

For a recent mathematical treatment of Noether's theorems and generalization thereof, see [Sardanashvily, 2016]. See also [Kosmann-Schwarzbach, 2011] for an authoritative historical account of Noether's work. For more on the history and philosophy of Noether's work and the relationship between the two theorems, see [Brading, 2002], [Brading and Brown, 2003], and [Brading and Brown, 2000]. For additional recent philosophical discussions of the connections between Noether's theorems, symmetries, and conserved quantities see [Butterfield, 2006] and the entries in [Read and Teh, 2022], especially [Gomes, 2022] for more on the interpretive challenges of and subtleties of Noether's theory in the non-Abelian context.

2 Noether's Theorem in Scalar Field Theories

In this section, we first briefly review the standard account of Noether's theorem as applied in classical scalar electrodynamics, as presented in such places as [Ryder, 1996] and [Brading and Brown, 2000]. We then consider the straightforward generalization of this account to classical scalar chromodynamics. Throughout, we restrict attention to scalar matter for simplicity of presentation.

In scalar electrodynamics, the conservation of electric charge is derived by applying Noether's theorem to the following Lagrangian. (The subscript E on

the Lagrangian reminds us that we are considering electrodynamics.)

$$L_E = D_\mu \phi (D^\mu \phi)^* - m \phi \phi^* \quad (1)$$

where ϕ is a scalar field of mass m carrying non-zero electric charge. D_μ is the covariant derivative,

$$D_\mu = \partial_\mu + iqA_\mu, \quad (2)$$

where A_μ is the vector potential. The real number q has a double role: it is both the coupling constant and the fundamental unit of electric charge.

The charge-current density associated with this Lagrangian is

$$j_\mu = iq(\phi^* D_\mu \phi - \phi (D_\mu \phi)^*). \quad (3)$$

As is well-known, Noether's first theorem shows that the $U(1)$ symmetry of this Lagrangian implies that

$$\partial_\mu j^\mu = 0. \quad (4)$$

This is the so-called continuity equation, and it is the key result that leads to the conservation of electric charge. One defines the total charge Q as,

$$Q := \int j^0 d^3x. \quad (5)$$

From the continuity of j^μ given in eq. (4), it follows that the time derivative of Q is zero. This is how Noether's theorem implies the conservation of total electric charge.

In generalizing this result to the non-Abelian case, we will employ the fiber bundle formalism of gauge theories. This formalism has many appealing features. For our purposes, it is especially helpful since, in this formalism, distinct group representations are very clearly hardwired into the equations of a non-Abelian gauge theory. The gauge field is a connection on a principal fiber bundle, and all such connections are Lie algebra valued. Thus, the gauge field transforms according to the adjoint representation. Moreover, matter fields are sections of vector bundles associated to the principal bundle. The typical fiber of an associated bundle is a carrier space for a given irreducible representation of the principal bundle's Lie group. Different types of matter generally transform according to different irreducible representations. In what follows, our goal will be to explain how these distinct representations relate to each other, and to use those relations to better understand charge conservation. For more on the fiber bundle formalism, see [Kobayashi and Nomizu, 1969], [Nakahara, 2003], [Bleecker, 2013], [Weatherall, 2016], [Hamilton, 2017], and references therein.

Here we adopt the abstract index notion developed in [Wald, 1984] along with the further notational conventions of [Weatherall, 2016]. Let (M, g_{ab}) be a relativistic spacetime. Vectors and tensors tangent to M have lower-case Latin indices a, b, c ; vectors and tensors tangent to the total space P have lower-case Greek indices; and upper-case Fraktur indices are used for vectors with a Lie algebra structure.

Let us first recast the conservation of electric charge in the fiber bundle formalism before turning to the non-Abelian case. Fix a relativistic spacetime (M, g_{ab}) and an $U(1)$ principal bundle $U(1) \rightarrow P \xrightarrow{\varrho} M$ over M . We then define the total current J_α on the total space of this principal bundle. Given a section $\sigma : M \rightarrow P$, we can pull back J_α along σ , resulting in a local representation J_a on spacetime. The current J_a is conserved in the sense that it is divergence-free.

Because $U(1)$ is Abelian, this resulting current J_a on spacetime is independent of the choice of section, and thus the charge-current density in electromagnetism is gauge-independent. While a current for a non-Abelian theory defined on the total space of the bundle can still be pulled back along a choice of local section to the base space, the resulting local representation of the current depends upon the choice of section. Thus, for non-Abelian theories, the charge-current density is a gauge-*dependent* quantity.

This gauge dependence poses the usual interpretive challenges concerning the physical significance of a quantity that changes with arbitrary choices of gauge. But prior to pulling back the current along a choice of section, one is struck by a different interpretive challenge. In the case of non-Abelian gauge theories, as we shall see presently, Noether's conserved current is manifestly Lie algebra valued. (In the Abelian case of electrodynamics, the current is also Lie algebra valued, but this is easily overlooked since the Lie algebra of $U(1)$ is simply \mathbb{R} .) We are accustomed to interpreting real valued quantities such as electric charge and mass without needing to give any special attention to the structure of Lie algebras. But in the non-Abelian case, our interpretation of the physical quantity of charge must take into account Lie algebra structure.

Let us use the specific case of (a classical version of) chromodynamics to see how Lie algebra structure arises for the charge current density. In chromodynamics, the relevant symmetry group is $SU(3)$, whose eight-dimensional Lie algebra we denote as $\mathfrak{su}(3)$. In addition to the index conventions set above, we further specify for our purposes here that indices i, j , etc. will be used to label vectors in the carrier space V of a representation of $SU(3)$ used to construct an associated bundle. Using these conventions, fix a relativistic spacetime (M, g_{ab}) and an $SU(3)$ principal bundle $P \xrightarrow{\varrho} M$ over M . In addition, fix a principal connection $\omega^\mathfrak{a}_\alpha$ on P and an inner product $k_{\mathfrak{a}\mathfrak{b}}$ on $\mathfrak{su}(3)$.

Since the charge-current density will take values in the Lie algebra, we give it the appropriate index structure as $J^\mathfrak{a}_\alpha$. The definition of $J^\mathfrak{a}_\alpha$ relies upon an inner product $k_{\mathfrak{a}\mathfrak{b}}$ on $\mathfrak{su}(3)$, as well as an inner product h_{ij} on the carrier space V of the representation of G used to describe the relevant matter field. Fix a basis $\{e^\mathfrak{a}\}$ of the Lie algebra $\mathfrak{su}(3)$. Then, following [Bleecker, 2013] 5.1.2., the current $J^\mathfrak{a}_\alpha$ is given by

$$J^\mathfrak{a}_\alpha = k^\mathfrak{a}\mathfrak{b} e_{\mathfrak{b}} h_{ij} \tilde{\psi}^j \overset{\omega}{D}_\alpha \psi^i, \quad (6)$$

where $\tilde{\psi}^j = \rho_*(e_{\mathfrak{a}}) \triangleright \psi^j$. That is, $\tilde{\psi}^j$ is the result of transforming ψ^j under the representation ρ_* of $\mathfrak{su}(3)$ on V induced by the representation ρ of G on V (see [Hamilton, 2017] 2.1.12.)

As in the case of electromagnetism, we can take the pull back of $J^\mathfrak{a}_\alpha$ along a

choice of local section σ to get the local current $J^{\mathfrak{a}}_a$. Note, however, that unlike in electromagnetism, the chromodynamic current gauge-dependent. Finally, $J^{\mathfrak{a}}_a$ is divergence-free, and thus color charge is conserved. For more on Noether's theorem applied in non-Abelian gauge theories, especially issues concerning definitions for spacetime-region specific conservation laws, see [Gomes, 2022].

The definition of the current in eq. (6) suggests that, in general, the charge quantities for charged matter fields that contribute to $J^{\mathfrak{a}}_a$ are Lie-algebra-valued. This is *prima facie* surprising. In discussions of the conservation of color charge, most textbook discussions proceed at the level of specific quark and anti-quark color states, $(r, b, g; \bar{r}, \bar{b}, \bar{g})$. Often in conjunction with Feynman diagrams, physicists speak of the conservation of color in terms of, separately, the conservation of each of these kinds of color at each vertex in a diagram (see figure 1). Since $(r, b, g; \bar{r}, \bar{b}, \bar{g})$ are not elements of $\mathfrak{su}(3)$, this gives a very different way of thinking of the conservation of color charge than the sense of conservation given by Noether's theorem.² We might have reasonably expected that the application of Noether's theorem in chromodynamics would give us a clear sense in which *red*, *blue*, and *green* are conserved. But instead Noether's theorem tells us that the conserved color current $J^{\mathfrak{a}}_a$ associated with the $SU(3)$ symmetry is eight-dimensional.

What, then, is the relationship between the eight-dimensional conserved current $J^{\mathfrak{a}}_a$ and the three colors of *red*, *blue*, and *green*? How exactly do the corresponding anti-colors, *anti-red*, *anti-blue*, and *anti-green* play a role in color charge conservation? Answering these questions is the central task of this paper. The answer lies in understanding the relationship between, on the one hand, the role of the Lie algebra in the adjoint representation and, on the other hand, the description of the basic colors $(r, b, g; \bar{r}, \bar{b}, \bar{g})$ in the two fundamental representations of $SU(3)$. What we find is a principled way of associating specific combinations of color and anti-color states to Lie algebra elements. This is developed in section 4. But first, in the next section, we need to clearly set out several moving pieces from the mathematics of group representation theory.

3 Representation Theory for $SU(n)$

Our target is to understand the relationship between the Lie algebra-valued Noether charge $J^{\mathfrak{a}}_a$ on the one hand and the (anti)color properties of quarks and anti-quarks, $(r, b, g; \bar{r}, \bar{b}, \bar{g})$, on the other. The key to understanding this relationship lies in showing how three distinct irreducible representations of $SU(3)$ relate to each other. These three irreducible representations are the two fundamental representations (used for quark and anti-quark color states) and the adjoint representation (used for gluon color states). We will first show in concrete detail how the adjoint representation of $SU(3)$ may be built out of the two fundamental representations, and then discuss the physical interpretation of adjoint charge states afforded by this construction.

This section reviews well-known definitions and results from Lie group repre-

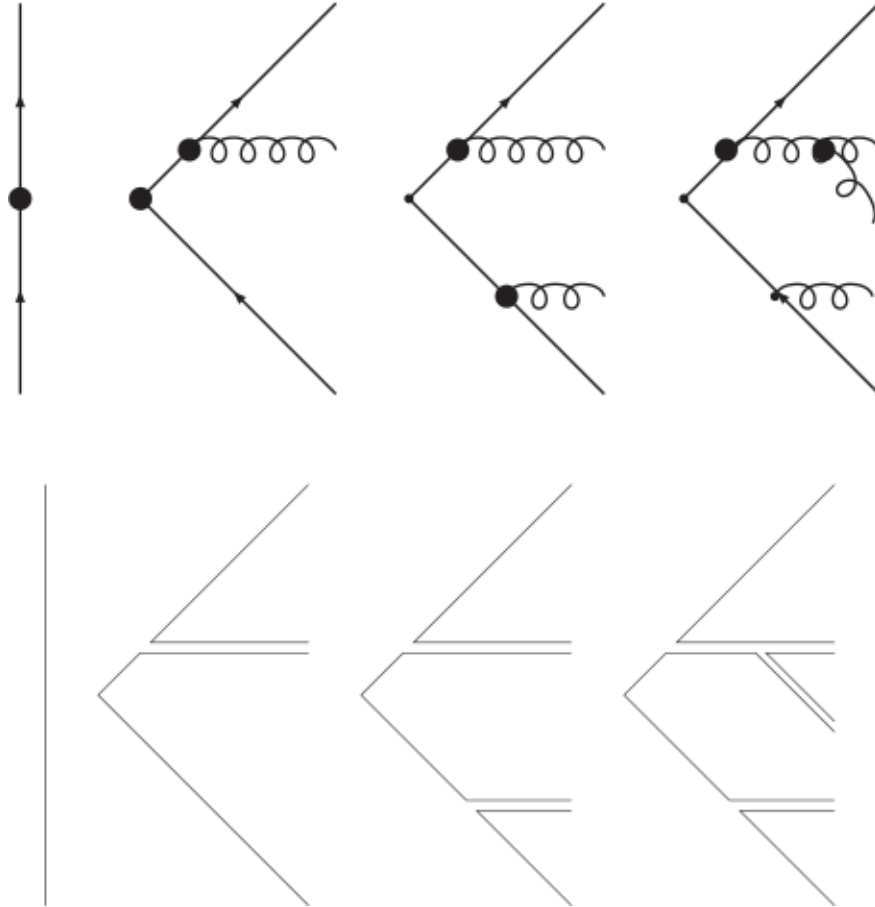


Figure 1: “Some Feynman diagrams for the emission of multiple gluons by a quark pair contrasted with corresponding leading colour diagrams.” The flow lines in the lower diagram capture an intuitive notion of color charge conservation. Quark color states are given by single lines, and gluon states are given by double lines. Time-reverse flow corresponds to anti-color states, yielding an interpretation of the double gluon lines as combinations of color and anti-color. Conservation amounts to the rule that no color line can be broken. For example, a quark in the r state may emit a $r\bar{b}$ gluon, and thereby transform into a quark in the b state. From *The Black Book of Quantum Chromodynamics: A Primer for the LHC Era*. (p. 30), by John Campbell, Joey Huston, and Frank Krauss, 2017, Oxford: Oxford University Press. Licensed under CC BY 4.0, a copy of which is available at <https://creativecommons.org/licenses/by/4.0/>.

sentation theory, which will prove to be key for understanding the interpretive significance of the Lie algebra in the context of Noether’s theorem for non-Abelian gauge theories. There is nothing new as far as group theory is concerned in this section. However, the presentation here aims to be more thorough and mathematically precise than those given in most physics resources. Conversely, it also aims to preserve normalization and choice of bases conventions common to physics sources. The increased attention to mathematical precision serves here to facilitate careful thinking about the intricacies and of the $SU(3)$ color symmetry that will inform our understanding of the Noether current J^{a} . Readers familiar with the details of Lie group representations may proceed to section 3.3.

3.1 Representations of $SU(2)$ and spin

It is instructive to first review the standard application of the classification of representations in the case of spin and $SU(2)$. This case exemplifies the basic framework for distinguishing between the representations themselves and interpretation of the physical properties associated with various aspects of the representations. Recall, for example, that spin- $\frac{1}{2}$ particles, transform according to $SU(2)$ doublets,

$$|\uparrow\rangle = +\frac{1}{2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = -\frac{1}{2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (7)$$

whereas spin-1 particles, such as photons, are described by an $SU(2)$ triplet of states,

$$1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad -1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (8)$$

These sets of states transform according to two different representations of the same group. While the group $SU(2)$ is often defined as the group of 2×2 special unitary matrices, thinking of the group this way is, strictly speaking, to already think of it *in a representation*. Indeed, writing out basis elements of the group as special unitary matrices of the relevant dimension is sometimes called the “defining” representation of the group. We will proceed here instead by locating the defining characterization of a group in its abstract commutation relations. This clarifies the sense in which the 2×2 matrices which act on the states in eq. (7) can implement the same symmetries as the 3×3 matrices which act on the states in eq. (8), since both sets of matrices obey the group’s commutation relations.

Furthermore, it is an artifact of $SU(2)$ that no two inequivalent irreducible representations are of equal dimension; this is what allows for the individuation of representations with terms such as ‘doublet’ and ‘triplet’. This is no longer the case is $SU(3)$. In particular, the quark and anti-quark representations are inequivalent, even though they are both triplets. Thus, it will be helpful from the outset to draw a sharp distinction between the group itself—defined abstractly by the commutation relations—and its myriad concrete representations.

So far, this is enough to illustrate the two main ingredients of any (matrix) representation of a group. The first ingredient is a vector space V (to be interpreted as a space of physical states). The second ingredient is an appropriate mapping $\rho : G \rightarrow \text{GL}(V)$ of the abstract group elements into matrices that act on that vector space. The sense of “appropriate” mapping is that of a group homomorphism: $\rho(g_1) \circ \rho(g_2) = \rho(g_1 \cdot g_2)$. In this sense, the matrices represent the group by concretely enacting the same commutation relations as those that define the group abstractly. The space V on which the represented group elements act is called the *carrier space*.

The groups associated with charge properties are in the class of groups called Lie groups. Lie groups have, in addition to their group structure, manifold structure. In particular, this means that we can construct a tangent space attached to any group element g in G . When we take the tangent space at the identity element, e , the resulting space $T_e G$ forms the Lie algebra associated to the Lie group G .³

For any simply connected Lie group, such as the $SU(n)$ groups, there is a one-to-one correspondence between the group’s representations and the representations of its Lie algebra \mathfrak{g} (see [Hall, 2015] theorem 5.6). It is often convenient to first work with the representations of the Lie algebra, and then move to the corresponding group representation. A representation of a Lie algebra is defined in parallel with a representation of a group. The representation is a mapping of algebra elements into a set of endomorphisms on a carrier space V that preserves the algebraic structure. Thus, a representation of a Lie algebra \mathfrak{g} on a vector space V is a Lie algebra homomorphism $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$. Given a basis for the carrier space V , we can write each algebra element under the representation as a matrix. For $\mathfrak{su}(n)$ these matrices representing Lie algebra elements are always traceless and skew-Hermitian.

These Lie algebra elements are also called “generators” of the group. This is because elements of the group are generated by exponentiation of the Lie algebra elements. Indeed, one way of defining the Lie algebra of a Lie group is that it consists of all matrices X such that e^{tX} is an element of the group, for all real numbers t (see, for example, [Hall, 2015] §3.3).⁴ Using the Lie algebra is convenient because, as the generators of the group, it gives an infinitesimal description of the group. We will therefore focus attention on representations for Lie algebras, keeping in mind that these have unique partner representations of the group (for the $SU(n)$ groups of present interest).

The Lie algebra $\mathfrak{su}(2)$, just like the Lie group $SU(2)$, has both a defining representation as the set 2×2 traceless hermitian matrices, and a more abstract characterization. In the defining representation, $\mathfrak{su}(2)$ is spanned by the Pauli spin matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (9)$$

This is an arbitrary choice of basis for $\mathfrak{su}(2)$. However, for any choice of basis, only one such basis matrix can be diagonalized at a time. In any basis, the

diagonalized matrix (here given by σ_3) has a special role to play in distinguishing representations of the group.

More abstractly, the key properties of $\mathfrak{su}(2)$ as a Lie algebra are captured by the commutation relations

$$[\sigma_i, \sigma_j] = 2i\sum_k \epsilon_{ij}^k \sigma_k \quad (10)$$

where ϵ_{ijk} is the totally anti-symmetric Levi-Civita tensor. The factors of $2i\epsilon_{ijk}$ are called the *structure constants* of $\mathfrak{su}(2)$. Any set of matrices that obeys these commutation relations, together with a carrier space, can serve as a representation for $\mathfrak{su}(2)$. For instance, in the triplet representation, the Pauli matrices may be written as

$$\rho(\sigma_1) = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(\sigma_2) = \frac{\sqrt{2}i}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (11)$$

and one may easily verify that these matrices obey the commutations relations in eq. (10).

In this context, we develop the standard interpretation of $SU(2)$ group structure for spin properties as follows. We first exploit the fact that $SU(2)$ is the double-cover of $SO(3)$. $SU(2)$ and $SO(3)$ are therefore locally isomorphic. This entails that their Lie algebras are isomorphic: $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. This provides a way to use the group $SU(2)$ to describe the $SO(3)$ rotational symmetry properties which may be aligned in either the x , y , or z directions. For spin properties aligned in each of these three directions, we use eigenvectors of the Pauli matrices,

$$J_x = \frac{1}{2}\sigma_1, \quad J_y = \frac{1}{2}\sigma_2, \quad J_z = \frac{1}{2}\sigma_3, \quad (12)$$

where the factors of $\frac{1}{2}$ are conventional. The operation of J_z on the doublet states gives the conventional numerical values for labeling spin states: the z -spin up state is label by $+\frac{1}{2}$, and the z -spin down state is label by $-\frac{1}{2}$. That is, the eigenvalues of the diagonalized matrix in the chosen basis for the Lie algebra serve as labels for distinct, *specific* states (z -spin up or z -spin down) of the *general* property (spin- $\frac{1}{2}$) under study. These eigenvalues of the diagonalized Lie algebra element are called *weights*.

Finally, complex sums of the two non-diagonal operators are used to define *raising* and *lowering* operators,

$$J_+ = (J_x + iJ_y), \quad J_- = (J_x - iJ_y) \quad (13)$$

The action of J_+ on a z -spin down state raises it to a z -spin up state, and similarly, the action of J_- on a z -spin up state lowers it to a z -spin down state.

This generalized. Given a Lie algebra in the $\mathfrak{su}(n)$ family, and given a basis for $\mathfrak{su}(n)$, the simultaneous eigenvalues of the diagonalized matrices serve as labels for distinct, specific states instantiating the general property associated with the

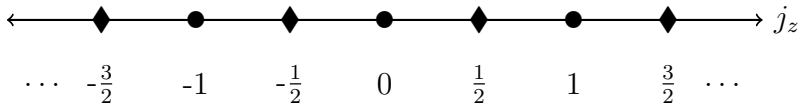


Figure 2: Weight diagram from $SU(2)$. Diamond points indicate possible states for fermions, whereas circle points give possible states for bosons.

symmetry group. In general, the number of diagonalizable matrices is greater than 1, in which case the weights are ordered m -tuples, where $m = n - 1$. In what follows, we will deploy the technical machinery of the weights of $SU(3)$ to develop the physical interpretation of color charge conservation.

Much of the physical interpretation of the $\mathfrak{su}(2)$ representations can be usefully summarized by diagramming the weights of the various spin states. We call such a diagram a “weight diagram” as given in figure 2. Certain subsets of this diagram (e.g. $\{-\frac{1}{2}, \frac{1}{2}\}$ or $\{-1, 0, 1\}$, etc.) correspond to possible spin states of various classes of particles. The highest weight in such a set labels the corresponding class of particles (e.g., spin $\frac{1}{2}$ or spin 1 particles). In this case, the relevant ordering on the weights is obvious. This will not be the case for the weights of $SU(3)$ where some choice of ordering must be made. Additionally, as noted above, for $SU(2)$ no two inequivalent irreducible representations have the same dimension. It is therefore not misleading to use terminology such as “doublet” and “triplet” to name distinct representations of $SU(2)$, since the dimensionality of an irreducible representation suffices to distinguish it from all other irreducible representations. This, also, is not the case for $SU(3)$.

The key lessons, which will launch our study of the relevant $SU(3)$ representations for color charge, are as follows. Points in the weight space label distinct spin states for particles of various kinds of spin. Different kinds of spin correspond to different representations of the group, and each representation corresponds to a subset of points in the weight space. The highest weight in each subset serves as a label for the representation. Analogously, we will be able to construct the weight diagram for $SU(3)$ and use points in that diagram to label distinct color/anti-color charge states. Different classes of particles with different constitutive ways of having color charge—namely, quarks, anti-quarks, and gluons—correspond to different representations of $SU(3)$. These different representations are distinguished by their highest weight. The weight space of $SU(3)$ will be an essential tool for demonstrating the relationship between the quark, anti-quark, and gluon representations of $SU(3)$ and for explicating the conservation of color charge. Unlike $SU(2)$, the weight space for $SU(3)$ is two dimensional.

3.2 The two fundamental representations of $SU(3)$

In its defining representation, elements of $SU(3)$ are written as 3×3 special unitary matrices. The associated defining representation of the Lie algebra $\mathfrak{su}(3)$ is given by the set of 3×3 traceless Hermitian matrices, spanned by the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

By convention, and in explicit analogy to Pauli spin matrices⁵, we adjust each matrix, except for λ_8 by a factor of $\frac{1}{2}$ to redefine these Lie algebra elements as

$$T_i = \frac{1}{2} \lambda_i. \quad (14)$$

For ease of reference later, this gives,

$$T_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (15)$$

Just as before, by writing out these elements as explicit 3×3 matrices, we are already thinking of the Lie algebra *in a representation*. More abstractly, $\mathfrak{su}(3)$ is spanned by any set of eight ordered elements T_a ($a = 1, \dots, 8$) that obey the following commutation relations,

$$[T_a, T_b] = i \sum_c f_{ab}^c T_c \quad (16)$$

where the real numbers f^{abc} are the *structure constants* of $SU(3)$. They are the generalization of ϵ_{ijk} in eq. (10). The constants f^{abc} are completely antisymmetric. A minimal, defining number of non-zero constants are listed in equation 17. The remaining non-zero constants can be determined by antisymmetry, and all others are zero.

$$f_{123} = 1, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}, \quad f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}. \quad (17)$$

The set of constants in eq. 17 is of course dependent upon the choice of basis for $\mathfrak{su}(3)$. Any set of $n \times n$ matrices, for any n , that obey these same commutation relations, forms a basis for a representation $\mathfrak{su}(3)$.

In this defining representation, these Gell-mann matrices can act upon a 3 dimensional carrier space. Basis vectors for this space are used to describe the three basic quark colors, *red*, *blue*, and *green*.

$$r = \begin{pmatrix} \mathbf{1} \\ 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \mathbf{1} \\ 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \\ \mathbf{1} \end{pmatrix}. \quad (18)$$

These three basic color are roughly analogous to the properties of z -spin up and z -spin down. The notion of z -spin appeals to the isomorphism between $\mathfrak{su}(2)$ and

$\mathfrak{su}(3)$ to give an external, spacial z direction, whereas the color ‘space’ operative here is internal. Color space—or at least, one of the internal color-valued spaces deserving of the name—is eight-dimensional. A natural extension of the analogy with spin would suggest that gluons can have determinate color values in two dimensions simultaneously.

In the $\mathfrak{su}(2)$ case, only one of the Pauli matrices can be diagonalized at a time. But here in $\mathfrak{su}(3)$, both T_3 and T_8 are diagonal. For any Lie algebra in general, the subspace of simultaneously diagonalizable matrices is called the *Cartan subalgebra*. The diagonal matrices T_3 and T_8 (and their counterparts in other representations) that span $\mathfrak{su}(3)$ ’s Cartan subalgebra have the same special role to play in classifying and distinguishing all of the irreducible representations of $SU(3)$ as did J_z in $SU(2)$. Each basis vector of the carrier space of a given representation ρ is labeled by the pair of its simultaneous eigenvalues (t_3, t_8) of these diagonal matrices. A partial order is placed on these pairs of simultaneous eigenvalues. Finally, the unique highest weight for each irreducible representation according to this partial order distinguishes between inequivalent representations.

Returning, now, to this defining representation of $SU(3)$, direct calculation gives the weights for each state. We thus label the r color state by $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$, the b color state by $(-\frac{1}{2}, \frac{1}{2\sqrt{3}})$, and the green color state by $(0, -\frac{1}{\sqrt{3}})$. These are the weights for the first fundamental representation of $SU(3)$. They are plotted in figure 3 (a).

Next, we need a way of determining the highest weight. In general, one first designates a set of weights of the adjoint representation, called *positive simple roots*, which can serve as a basis for all of the other weights, and these positive simple roots are used to define a partial ordering on the space of weights (see [Hall, 2015] §6.1 - 6.3). We will return to the determination of positive simple roots below at eq. (25). In the meantime, we will accept by fiat the following ordering: $(\frac{1}{2}, \frac{1}{2\sqrt{3}}) \succ (-\frac{1}{2}, \frac{1}{2\sqrt{3}})$ and $(\frac{1}{2}, \frac{1}{2\sqrt{3}}) \succ (0, -\frac{1}{\sqrt{3}})$. Thus the *red* state has the highest weight, which will serve as a suitable label to distinguish this representation for color states from other representations of $SU(3)$. Thus, just as we use the $\frac{1}{2}$ representation of $SU(2)$ for elementary fermionic spin states, we use the $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$ representation of $SU(3)$ for the quark color states.

In further analogy with the case of spin, we have at our disposal a notion of raising and lowering operators that move states of a higher/lower weight to states of a lower/higher weight. As with the Pauli spin matrices, these are given by complex sums of the remaining non-Cartan matrices. But in this case we get three pairs of raising and lowering operators:

$$T_{\pm} = (T_1 \pm iT_2), \quad U_{\pm} = (T_6 \pm iT_7), \quad V_{\pm} = (T_4 \pm iT_5). \quad (19)$$

Each set of raising and lowering operators corresponds to an $SU(2)$ subgroup of $SU(3)$. The T subgroup is spanned by T_1 , T_2 , and T_3 . The U subgroup is spanned by T_6 , T_7 , and $(T_8 - T_3)$. And the V subgroup is spanned by T_4 , T_5 , and $(T_8 + T_3)$. Explicitly, in the defining representation, the raising and lowering

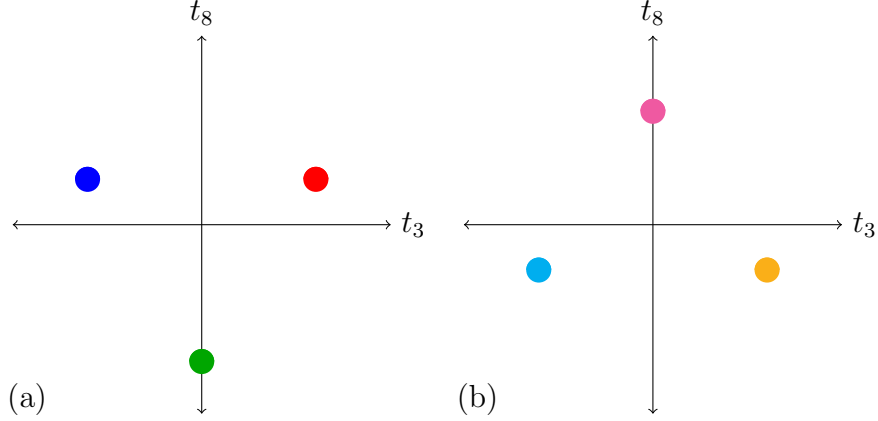


Figure 3: (a) Weight space for the quark color states given by a basis for the carrier space of the first fundamental representation of $SU(3)$. (b) Weight space for the anti-color states of anti-quarks given by a basis for the carrier space of the second fundamental representation of $SU(3)$.

operators are:

$$\begin{aligned}
 T_+ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & U_+ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & V_+ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
 T_- &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & U_- &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & V_- &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
 \end{aligned}$$

One may easily verify that T_{\pm} raises and lowers between r and b states, whereas U_{\pm} raises and lowers between b and g states, and V_{\pm} raises and lowers between g and r . There is an element of conventionality here. We chose an ordering on the space of weights, such that r was the highest weight. Once that is fixed, our choice of sorting operators into those that ‘raise’ and those that ‘lower’ are constrained, in so far as the highest weight must be annihilated by all raising operators.

For $SU(3)$, this defining representation is not the only fundamental representation. In general for $SU(n)$, there are $1 - n$ many fundamental representations, so called because all other irreducible representations of $SU(n)$ may be systematically constructed from these fundamental representations. Thus, $SU(3)$ has a two fundamental representations. What we have been calling the defining representation is one of them, and it is used for quark color charge states. The other is used to describe the anti-color states of anti-quarks.⁶ This second fundamental representation of $SU(3)$ is dual to the first. Thus, the carrier space for this representation is \mathbb{C}^{3*} , the dual space of \mathbb{C}^3 . We select the following basis⁷ for \mathbb{C}^{3*} for the anti-colors states:

$$\bar{r} = \begin{pmatrix} \mathbf{1} \\ 0 \\ 0 \end{pmatrix}, \quad \bar{b} = -\begin{pmatrix} 0 \\ \mathbf{1} \\ 0 \end{pmatrix}, \quad \bar{g} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{1} \end{pmatrix}. \quad (20)$$

We further exploit the analogy with the colors of ordinary light using complementary hues to denote anti-colors: thus anti-red is red’s compliment, cyan, and similarly anti-blue is yellow, and finally anti-green is magenta.

Next we need to define the action of $\mathfrak{su}(3)$ elements on \mathbb{C}^{3*} to complete the definition of this dual representation. This action is given by,

$$\bar{X} = -\rho(X)^{tr} \quad (21)$$

for all $X \in \mathfrak{su}(3)$, where ρ is the mapping for the first fundamental representation. Consequently,

$$\bar{T}_3 = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{T}_8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad (22)$$

The simultaneous eigenvalues of \bar{T}_3 and \bar{T}_8 give the weights for this second fundamental representation. Each weight designates a different anti-color state. These are plotted in 3 (b). Notice the way in which the weight diagram captures a sense of a relation of ‘opposites’ between the color and anti-color states: they differ by a minus sign in both their t_3 and t_8 entries.

In this second fundamental representation we again have instances of—or better yet, ‘representatives’ of—the raising and lowering operators. Thus, for example,

$$\bar{T}_- = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{U}_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{V}_- = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One may verify that \bar{T}_- lowers \bar{b} states into \bar{r} states, while \bar{U}_- lowers \bar{g} into \bar{b} , and \bar{V}_- lowers \bar{g} into $-\bar{r}$. The partial ordering for these weights is $(0, \frac{1}{\sqrt{3}}) \succ (\frac{1}{2}, -\frac{1}{2\sqrt{3}})$, $(0, \frac{1}{\sqrt{3}}) \succ (-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$, and $(\frac{1}{2}, -\frac{1}{2\sqrt{3}}) \succ (-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$. The weight $(0, \frac{1}{\sqrt{3}})$ of the \bar{g} state is the highest weight of this representation.

3.3 Diagrammatic view of the $SU(3)$ adjoint representation

So far we have distinguished between the two fundamental representations of $SU(3)$, and we have identified their carrier spaces with the spaces of color and anti-color states of quarks and anti-quarks, respectively. We now turn to the adjoint representation, recalling our aim of showing the relationship between this adjoint representation (and the Noether charge which transforms according to it) and the two fundamental representations.

As we have already seen, a key feature of a representation is its carrier space. In general, the carrier space may be any vector space. The adjoint representation uses the Lie algebra *itself* as its underlying carrier space. In this case, then,

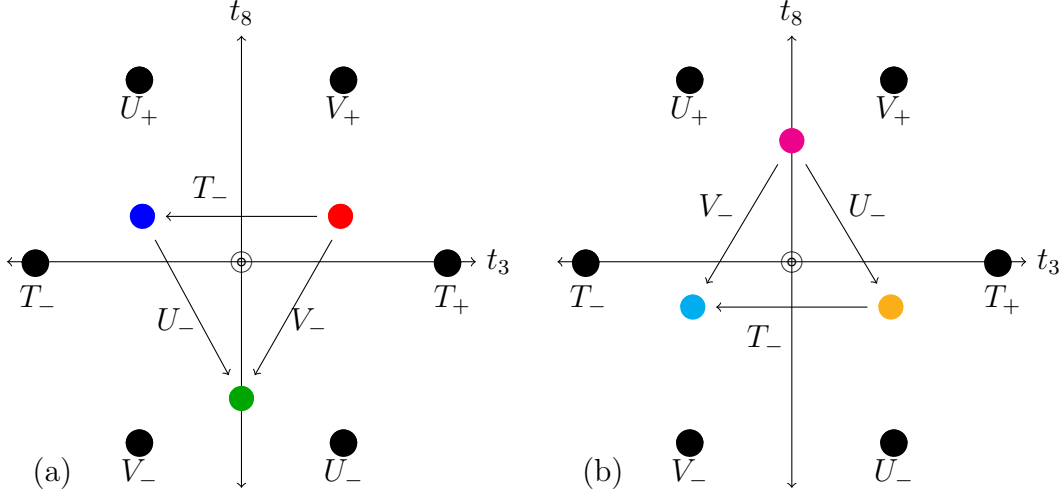


Figure 4: Action of lowering operators in (a) first fundamental representation and (b) second fundamental representation.

the Lie algebra's role becomes twofold: it is both the generators for the group elements, and it is the carrier space on which those group elements act. For $\mathfrak{su}(3)$, the adjoint representation is therefore eight-dimensional.

To determine the weights of the adjoint representation, we need the simultaneous eigenvalues of the Cartan subalgebra. At this stage it is more convenient to use the ordered basis $T_+, T_-, T_3, U_+, U_-, V_+, V_-$, and T_8 , rather than the T_i of eq. (14). The Cartan subalgebra of $\mathfrak{su}(3)$ in the adjoint representation is spanned by,

$$\rho_{adj}(T_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (23)$$

$$\rho_{adj}(T_8) = \frac{\sqrt{3}}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Plotting their simultaneous eigenvalues gives the weights for the chosen basis of the Lie algebra in its role as carrier space (the black dots in figures 4 and 5).

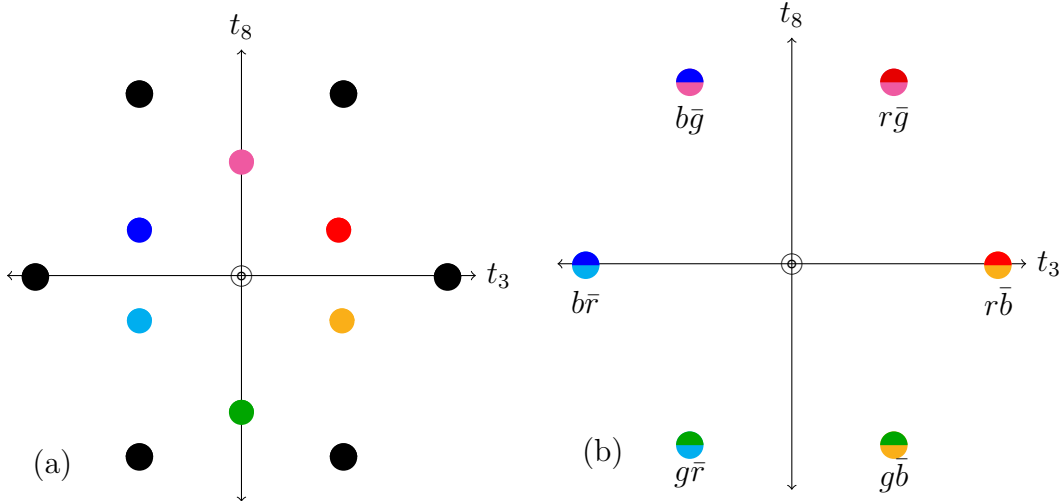


Figure 5: Color states in weight space of first fundamental representation (*red*, *blue*, and *green*), the second fundamental representation (*magenta*, *yellow*, and *cyan*), and of the adjoint representation (*black*) of $SU(3)$.

We may now compare the weights for each of our three representations. For ease of comparison the weights of the two fundamental representations are included in figure 5. *This diagrammatic comparison of weights among the three representations is the first step in answering our question concerning the physical interpretation of the Noether current $J_a^{\mathcal{Q}}$.* The comparison shows that each non-zero weight of the adjoint representation is the vector sum of a weight from the first fundamental representation and a weight from the second fundamental representation. This suggests an interpretation of each state of the adjoint representation as precisely those combinations of color and anti-color given by the appropriate vector sum. This is noted in figure 5 (b). For example, consider the suggestion that the T_- lowering operator should be identified with the $b\bar{r}$ state. This fits nicely with our Feynman diagram view of charge conservation and the action of T_- on quark and anti-quark states. T_- lowers r quarks into b quarks by annihilating the r , and replacing it with b . And T_- lowers \bar{b} anti-quarks into \bar{r} anti-quarks by annihilating the \bar{b} and replacing it with \bar{r} . In this way, the actions of lowering operators noted in figure 4 supports the attributions of color/anti-color combinations to adjoint representation states in figure 5.

While suggestive, this diagrammatic explanation of the relationship between our three representations is insufficient in two ways. First, it needs to be clarified how it is that these vector sums are not mere coincidence: *why* is it that adding r with \bar{b} within the weight space results in an adjoint-representation state? We need a more principled reason to identify elements of $\mathfrak{su}(3)$ with combinations of color and anti-color. Second, there are two weight $(0,0)$ states of the adjoint representation whose interpretation is underdetermined by the diagram alone. To remedy these two insufficiencies, we turn in the next section to the formal construction of the adjoint representation out of the two fundamental representations.

4 Construction of the Adjoint Representation

In the previous section, we saw that, after first determining the weights of the adjoint representation of $\mathfrak{su}(3)$, it turned out that these weights could be associated with pairs of weights from the two fundamental representations. In this section, we reverse directions. We start instead with pairs of states from the two fundamental representations, and we ultimately arrive at a principled way of identifying these pairs with elements of $\mathfrak{su}(3)$. The process begins by building a basis for the carrier space for the adjoint representation, and it then proceeds by defining the action of the group on that space. Next, we identify the state of highest weight. Finally, successive application of the lowering operators to this state produces all the remaining states of the adjoint representation.

To build a basis for the carrier space for the adjoint representation, we start with pairs of states from the two fundamental representations, understood as elements of the space $\mathbb{C}^3 \otimes \mathbb{C}^{3*}$. This is a nine dimensional vector space, one dimension too big. The group acts on this space by

$$\rho_{adj}(X) = X \otimes I + I \otimes \bar{X}, \quad (24)$$

for all X in $\mathfrak{su}(3)$. For example, $\rho_{adj}(T_-)(r \otimes \bar{g}) = b \otimes \bar{g}$.

It can be shown that, in a higher dimensional representation built out of lower dimensional representations, the state of the higher dimensional representation of the highest weight is that state built from the tensor product of the states of highest weight from the lower dimensional representations (see [Hall, 2015] prop. 6.17). Thus, the state of highest weight in the adjoint representation is $r \otimes \bar{g}$. Successive application of the lowering operators to this $r \otimes \bar{g}$ state produces each of the other basis elements of the adjoint representation's carrier space. It suffices to consider just two lowering operators since $[U_-, T_-] = V_-$.

We can summarize the results of this process in the diagram given in figure 6. To save space, we omit the tensor product symbol and simply write the states as, e.g. $r\bar{b}$. Arrows to the left indicate the action of U_- on the previous state, and arrows to the right indicate the action of T_- . From the diagram, we see that the resulting eight states are: $r\bar{g}$, $r\bar{b}$, $b\bar{g}$, $(b\bar{b} + r\bar{r})$, $(g\bar{g} + b\bar{b})$, $g\bar{b}$, $b\bar{r}$, and $g\bar{r}$. Note that the $(r\bar{r} + b\bar{b} + g\bar{g})$ dimension of $\mathbb{C}^3 \otimes \mathbb{C}^{3*}$ is excluded.

At this stage we have eight linearly independent combinations of color and anti-color identified with elements of $\mathbb{C}^3 \otimes \mathbb{C}^{3*}$. We now need a principled way of associating each of these eight states with elements of $\mathfrak{su}(3)$. More precisely, we have a way of thinking of basis vectors for the *carrier space* of an eight-dimensional representation of $SU(3)$ as combinations of color and anti-color, but it remains to be seen how these combinations can be associated with elements of $\mathfrak{su}(3)$ understood as *operators* on that space. So we next need to say more about how $\mathfrak{su}(3)$ can act on itself.

In general, a Lie-algebra acts on itself via the adjoint map: $Ad_X(\cdot) = [X, \cdot]$ for all Lie-algebra elements X . This adjoint action corresponds to the foundational commutation relations, such as eq. (10) for $\mathfrak{su}(2)$ and eq. (16) for $SU(3)$. These commutations relations for $\mathfrak{su}(3)$ are recorded in table 1. The next step is to

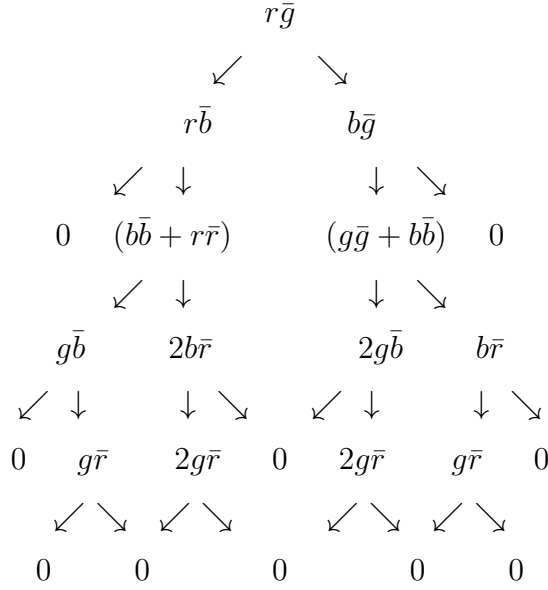


Figure 6: Construction of the adjoint representation's carrier space in color/anti-color states. Arrows to the left indicate the action of U_- on the previous state, and arrows to the right indicate the action of T_-

match the operator within our basis for $\mathfrak{su}(3)$ with the highest weight to the state of the highest weight, $r\bar{g}$. To do so, we need to give the principled reason behind our ordering on the space of weights.

We now introduce some additional terminology to explain how we arrive at an ordering on the space of weights.⁸ We follow [Hall, 2015] closely. First, we call all non-zero weights of the adjoint representation *roots*. These are recorded in table 2. The next step is to select two roots such that all of the roots can be expressed as linear combinations thereof with, crucially, integer coefficients. Furthermore, these coefficients are, for each root, either both greater than or equal to zero, or both less than or equal to zero. We call these the *simple positive roots*. For our purposes, we select the eigenvalues of T_+ and U_+ to be our simple positive roots, and label these as α_1 and α_2 . Some straightforward if tedious calculation confirms that these satisfy the requirements for simple positive roots.

$$\alpha_1 = (1, 0) \quad \alpha_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \quad (25)$$

We then define our ordering as follows. For any two weights μ_1 and μ_2 , we say that $\mu_2 \succ \mu_1$ provided that the following equation holds,

$$\mu_1 - \mu_2 = a\alpha_1 + b\alpha_2 \quad (26)$$

with the coefficients $a \geq 0$ and $b \geq 0$. Given a collection of weights for a specified representation, the highest weight is that which is higher than each of the other weights. Further calculation confirms that V_+ has the highest weight of the adjoint representation. One may also apply the same calculation to the

	T_+	T_-	T_3	U_+	U_-	V_+	V_-	T_8
T_+	0	$2T_3$	$-T_+$	V_+	0	0	$-U_-$	0
T_-	$-2T_3$	0	T_-	0	$-V_-$	U_+	0	0
T_3	T_+	$-T_-$	0	$-\frac{1}{2}U_+$	$\frac{1}{2}U_-$	$\frac{1}{2}V_+$	$-\frac{1}{2}V_-$	0
U_+	$-V_+$	0	$\frac{1}{2}U_+$	0	$\sqrt{3}T_8 - T_3$	0	T_-	$-\frac{\sqrt{3}}{2}U_+$
U_-	0	V_-	$-\frac{1}{2}U_-$	$T_3 - \sqrt{3}T_8$	0	$-T_+$	0	$\frac{\sqrt{3}}{2}U_-$
V_+	0	$-U_+$	$-\frac{1}{2}V_+$	0	T_+	0	$T_3 + \sqrt{3}T_8$	$-\frac{\sqrt{3}}{2}V_+$
V_-	U_-	0	$\frac{1}{2}V_-$	$-T_-$	0	$-T_3 - \sqrt{3}T_8$	0	$\frac{\sqrt{3}}{2}V_-$
T_8	0	0	0	$\frac{\sqrt{3}}{2}U_+$	$-\frac{\sqrt{3}}{2}U_+$	$\frac{\sqrt{3}}{2}V_+$	$-\frac{\sqrt{3}}{2}V_-$	0

Table 1: Commutation Relations for $\mathfrak{su}(3)$ in the raising/lowering basis. The label on the row give the first entry in commutator, and the column label gives the second entry. For example, $[T_+, T_-] = 2T_3$.

T_+	T_-	U_+	U_-	V_+	V_-
$(1, 0)$	$(-1, 0)$	$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$	$(\frac{1}{2}, -\frac{\sqrt{3}}{2})$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$	$(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

Table 2: Non-zero weights of $\mathfrak{su}(3)$ basis elements. These are called roots. We select T_+ and U_+ to be our simple positive roots.

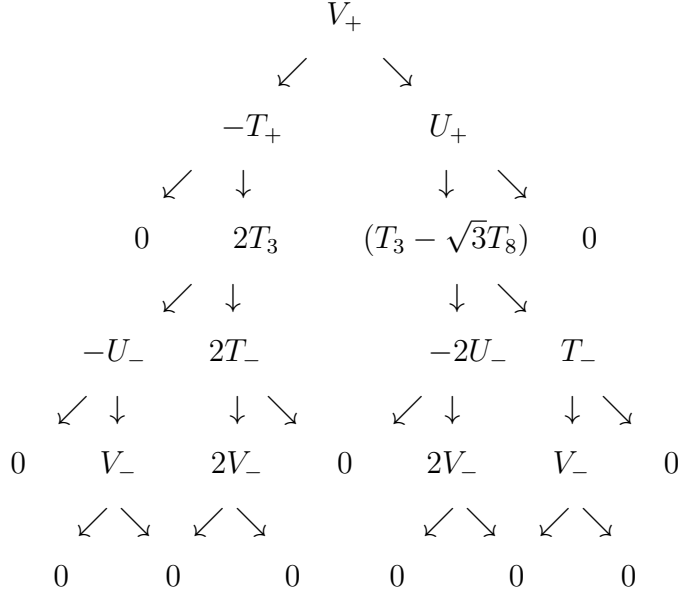


Figure 7: Construction of the adjoint representation in operators.

weights of the fundamental representations to verify that the r and \bar{g} states are the highest in their respective representations.

We have already seen that the state of the highest weight for the adjoint representation is $r\bar{b}$. Thus, the raising operator V_+ corresponds to the $r\bar{b}$ state in the adjoint representation. Successive application of the lowering operators on V_+ will produce the remaining states, as recorded in figure 7. Again, arrows to the left indicate the action of $\rho_{adj}(U_-)$ on the previous state, and arrows to the right indicate the action of $\rho_{adj}(T_-)$. For example, $\rho_{adj}(U_-)(V_+) = [U_-, V_+] = -T_+$. The resulting basis operators may then be assigned their color/anti-color content by matching the diagram in figure 7 to the diagram in figure 6. Those results are recorded in table 3. Reassuringly, these calculations confirm the suggestive interpretations of the weight space diagrams.

This suffices to address the first insufficiency in the diagrammatic interpretation of the adjoint states given at the end of section 3.3. The second insufficiency is remedied with a simple calculation to associate a color-anticolor state with T_3 and T_8 operators which each have weight $(0,0)$. Since we have that $2T_3 = (b\bar{b} + r\bar{r})$ and $(T_3 - \sqrt{3}T_8) = (g\bar{g} + b\bar{b})$, we find that $T_3 = \frac{1}{2}(b\bar{b} + r\bar{r})$ and $T_8 = \frac{\sqrt{3}}{2}(r\bar{r} - b\bar{b} - 2g\bar{g})$. Any element of the Cartan subalgebra will have weight $(0,0)$, and so will be ‘white’ in that it has some combination of $r\bar{r}$, $b\bar{b}$, and $g\bar{g}$. But individual elements of the Cartan subalgebra can be distinguished from each other by the specific amounts of $r\bar{r}$, $b\bar{b}$, and $g\bar{g}$ present in that state.

Lowering Op.	Color/Anti-color	Raising Op.	Color/Anti-color
T_-	$b\bar{r}$	T_+	$-r\bar{b}$
U_-	$-g\bar{b}$	U_+	$b\bar{g}$
V_-	$g\bar{r}$	V_+	$r\bar{g}$

Table 3: Physical interpretation of $\mathfrak{su}(3)$ raising and lowering operators

5 Noether’s Group Theoretic Ledger

Section 3 developed the mathematical machinery needed to interpret basis elements of the Lie algebra $\mathfrak{su}(3)$ in terms of the interpretation given for the basis elements of carrier spaces for the two fundamental representations of $SU(3)$. The construction of the adjoint representation in section 4 took the color basis elements tensored together with the anti-color basis elements. In this section, we discuss how this construction explains the relationship between Noether’s color charge current J^a —which takes values in $\mathfrak{su}(3)$ —and the properties $(r, b, g; \bar{r}, \bar{b}, \bar{g})$ which we so readily think of as *the* color charges.

Contributions to J^a are combinations of the three basic colors and their anti-color counterparts. In this way, the conservation of color charge involves a conceptually complex process of accounting for both color and anti-color—more complex, that is, than the conservation of electric charge. To account for a conserved amount of electric charge in some physical process, it suffices to simply add the initial positive and negative charges and ensure that this is equivalent to the sum of the final positive and negative charges. Positive electric charge and negative electric charge take values within the same mathematical space; we are always dealing with quantities in \mathbb{R} . The accounting work of conservation for electric charge is, therefore, no more sophisticated than arithmetic with real numbers. In contrast, the analogous sense of ‘adding up’ color and anti-color charge contributions to J^a requires vector addition within the space $\mathbb{C}^3 \otimes \mathbb{C}^{3*}$. That is, we needed to construct a new mathematical space specifically designed to put color states together with anti-color states. Neither of the two fundamental representations of $SU(3)$ would suffice.

Notice the difference between the relevant mathematical structures for *opposite* charges in the electric and color cases. In the electric case, the structure of opposite charge is given by the opposite direction with the same vector space. In the color charge case, the structure of *opposite* is that of matching one basis element within a vector space to its dual basis element within the dual vector space. The quantity $\bar{r} \in \mathbb{C}^{3*}$ is not the same as $-r \in \mathbb{C}^3$.

Thus Noether’s theorem does not imply the conservation of the charge of ordinary matter on its own. Rather, the conserved quantity is a union of charge and anti-charge. This foundational role of anti-charge, in concert with ordinary charge, is obscured in the case of electrodynamics. We can explicate this point more fully by investigating the relevant group representation theory for electric

charge. In this case, the group is the Abelian group $U(1)$, which is the set of numbers in the complex plane with unit modulus, $e^{i\theta}$ for $\theta \in [0, 2\pi]$. Its complex irreducible representations are all of the form

$$\rho_n(e^{i\theta}) = e^{in\theta} \tag{27}$$

where n is an integer. The value of n labeling the representation gives the *amount* of electric charge had by particles which transform according to that representation. Thus electrons transform in ρ_{-1} , neutrinos in ρ_0 , positrons in ρ_1 , etc. This labeling of the irreducible representations is the $U(1)$ analog of the weights used to distinguish representations in $SU(n)$.

The ρ_{-n} representation is dual to the ρ_n representation, just as the anti-quark color representation of $SU(3)$ is dual to the quark color representation. Thus, in electrodynamics, the negative numbers correspond to anti-charge while positive numbers correspond to ordinary charge. But this is the point in the analogy between color charge and electric charge where the conceptual structures for electric charge are a drastic simplification of those for color charge. There is no analog of r , b , and g in the electric case. Each irreducible representation of $U(1)$ is one-dimensional, and so there is, literally, no space within a $U(1)$ irrep to capture different directions within electric charge space, in the way that r , b , and g are different directions within matter field color space.

To account for a conserved amount of electric charge in some physical process, it suffices to simply add the initial positive and negative charges and ensure that this is equivalent to the sum of the final positive and negative charges. So in electrodynamics, conservation of charge is also a law regarding a sort of union of charge and anti-charge. And yet the nature of this union in the electric case is far simpler than in the color case. For color charge, the union of colors with anti-colors in the adjoint representation resulted in a novel eight-dimensional, internal color space. For electric charge, the union of charge with anti-charge does not generate a novel internal charge space. Although, in general, the addition of electric charges results in a different irreducible representation of $U(1)$ than the representations used for the component charges, the internal charge space never changes from \mathbb{C} . Thus electric charge conservation needs no more than garden variety arithmetic with real numbers.

6 Conclusion

We began with puzzle of how to make sense of the eight-dimensional Noether color charge current given that there are three colors and three anti-colors. How come the eight-dimensional $\mathfrak{su}(3)$ is the right space for the accounting work of the conservation of color charge? Why is Noether's ledger Lie algebra valued? We saw how to answer these questions through the explicit construction of the $SU(3)$ adjoint representation out of the two fundamental representations of $SU(3)$. This construction revealed principled ways to assign different specific combinations of color and anti-color quantities to different states of the adjoint representation.

Finally, the physical difference of electric charge and electric anti-charge, captured in the mathematical difference between positive and negative real numbers, is far weaker than the physical difference of color charge and color anti-charge, captured in the mathematical difference between the two fundamental representations of $SU(3)$. Thus, Noether’s theorem in non-Abelian gauge theories reveals that the general notion of charge conservation is not simply the conservation of ordinary charge: rather, it is the conservation of a union of ordinary charge and anti-charge. In general, ordinary charge and anti-charge quantities ‘live’ in different mathematical spaces. Thus, in order to accomplish the conceptual accounting work of charge conservation, we generally need to construct a new mathematical space specifically designed to put units of charge and anti-charge together. The Lie algebra, as the carrier space for the adjoint representation, is the natural mathematical home for the work of charge conservation. Noether’s ledger is Lie algebra valued because conserved charge quantities are unions of charge and anti-charge (and *vice versa*).

Notes

¹These colors are related to the macroscopic colors of electromagnetic radiation only by way of analogy: just as white light is a combination of all the different colors of light, so too are certain ‘white’ states of neutral color charge a combination of ‘red,’ ‘green,’ and ‘blue’ color states. These color charge states of quarks have no physical bearing on macroscopic color.

²The Feynman diagram version of color charge conservation is an important heuristic in QCD calculations. But, as with other aspects of Feynman diagrams, it is best to be extremely cautious against reading them as literal depictions of physical processes.

³There are alternative characterizations of the Lie algebra associated with a given Lie group. The Lie algebra can also be defined as the space of left-invariant vector fields on G , and it can be shown that this space is canonically isomorphic to T_eG . See, for instance, [Kobayashi and Nomizu, 1969] or [Hamilton, 2017].

⁴Many physics books use an alternative convention such that group elements correspond to e^{itX} for real numbers t .

⁵While this convention will be somewhat cumbersome in the calculations that follow, it is standard throughout the relevant physics. In particular, this choice effects the normalization of the QCD coupling g_s , as well as the values for the two QCD Casimir operators and the familiar form of the structure constants. We therefore accept the calculational burden, in order to facilitate cross-referencing physics sources.

⁶Similarly, when we use $SU(3)$ for the inexact flavor symmetry, we use one copy of the first fundamental representation is use for up, down, and strange quarks; a second copy of the first fundamental representation for the top, bottom, and charmed quarks; and two copies of the second fundamental representation for the corresponding anti-quarks.

⁷We might usually use row vector instead in order to emphasize that the dual space of some vector space V is the collection of linear forms on V . But in this context, we wish instead to emphasize the vector space structure of \mathbb{C}^{3*} and its suitability as a carrier space for a new representation of $\mathfrak{su}(3)$. Column vectors are better suited to this purpose.

⁸These technicalities are more useful to the mathematician working in group theory than to the physicist working in particle physics phenomenology. We devote space to it here because it clarifies where our hitherto *fiat* ordering on the space of weights comes from. And for our interpretive purposes, it is worthwhile to see why our raising and lowering operators, applied to various color and anti-color matter states, are sensibly said to ‘raise’ or ‘lower’ in accordance with this ordering.

References

- [Baez and Huerta, 2010] Baez, J. C. and Huerta, J. (2010). The Algebra of Grand Unified Theories. *Bull. Am. Math. Soc.*, 47:483–552. 2
- [Bleecker, 2013] Bleecker, D. (2013). *Gauge Theory and Variational Principles*. Dover Books on Mathematics. Dover Publications. 4, 5
- [Brading and Brown, 2000] Brading, K. and Brown, H. R. (2000). Noether’s theorems and gauge symmetries. *arXiv preprint hep-th/0009058*. 3
- [Brading and Brown, 2003] Brading, K. and Brown, H. R. (2003). Symmetries and noethers theorems. In *Symmetries in physics: Philosophical reflections*, pages 89–109. Cambridge University Press. 3
- [Brading, 2002] Brading, K. A. (2002). Which symmetry? noether, weyl, and conservation of electric charge. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 33(1):3–22. 3
- [Butterfield, 2006] Butterfield, J. (2006). On symmetry and conserved quantities in classical mechanics. In *Physical theory and its interpretation: Essays in honor of Jeffrey bub*, pages 43–100. Springer. 3
- [Cahn, 2014] Cahn, R. N. (2014). *Semi-simple Lie algebras and their representations*. Courier Corporation. 2
- [Georgi, 1999] Georgi, H. (1999). *Lie algebras in particle physics: from isospin to unified theories*, volume 54. Westview Press. 2
- [Gomes, 2022] Gomes, H. (2022). Noether charges, gauge-invariance, and non-locality. In *The Philosophy and Physics of Noether’s Theorems: A Centenary Volume*. Cambridge University Press. 3, 6
- [Hall, 2015] Hall, B. C. (2015). *Lie groups, Lie algebras, and representations: an elementary introduction*, volume 222. Springer. 9, 13, 18, 19
- [Hamilton, 2017] Hamilton, M. J. (2017). *Mathematical Gauge Theory: With Applications to the Standard Model of Particle Physics*. Springer. 4, 5, 24
- [Kobayashi and Nomizu, 1969] Kobayashi, S. and Nomizu, K. (1969). *Foundations of differential geometry*, volume 1 and 2. Wiley New York. 4, 24
- [Kosmann-Schwarzbach, 2011] Kosmann-Schwarzbach, Y. (2011). The noether theorems. In *The Noether Theorems*, pages 55–64. Springer. 3
- [Lichtenberg, 1978] Lichtenberg, D. (1978). *Unitary symmetry and elementary particles*. Academic Press, New York, 2 edition. 2
- [Nakahara, 2003] Nakahara, M. (2003). *Geometry, topology and physics*. CRC Press. 4
- [Noether, 1918] Noether, E. (1918). Invarianten beliebiger differentialausdrücke. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, mathematisch-physikalische Klasse*, 1918:37–44. 1

- [Read and Teh, 2022] Read, J. and Teh, N. J. (2022). *The Philosophy and Physics of Noether's Theorems: A Centenary Volume*. Cambridge University Press. [3](#)
- [Ryder, 1996] Ryder, L. H. (1996). *Quantum field theory*. Cambridge university press. [3](#)
- [Sardanashvily, 2016] Sardanashvily, G. (2016). *Noether's Theorems*. Springer. [3](#)
- [Wald, 1984] Wald, R. M. (1984). *General relativity*. Chicago Univ. Press, Chicago, IL. [4](#)
- [Weatherall, 2016] Weatherall, J. O. (2016). Fiber bundles, yang–mills theory, and general relativity. *Synthese*, 193(8):2389–2425. [4](#)
- [Woit, 2017] Woit, P. (2017). *Quantum Theory, Groups and Representations: An Introduction*. Springer. [3](#)