Kneebone and Lakatos: at the roots of a dialectical philosophy of mathematics

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Abstract

This paper examines the origins of the dialectical approach to the philosophy of mathematics. While this approach is commonly taken to begin with Imre Lakatos's *Proofs and Refutations*, first published as a series of articles (1963-64), it turns out that it was pre-empted by the British logician G. T. Kneebone in a pair of forgotten articles (1955, 1957) and a chapter of his book (1963). We introduce Kneebone's dialectical approach to mathematics, and compare it to Lakatos's. Furthermore, we give evidence from the LSE Lakatos archives that Kneebone and Lakatos were acquainted, in correspondence, and that Lakatos read and annotated Kneebone's papers. Nonetheless, Kneebone is nowhere mentioned in Lakatos's work. We finish on the question of why this might be.

Keywords

G. T. Kneebone, Imre Lakatos, dialectic, philosophy of mathematics, mathematical practice

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1. Introduction

In recent decades, the philosophy of mathematics has seen a rapid growth of research into mathematical practices. Rather than traditional philosophical questions about foundations, logic, ontology, and idealised epistemology that have dominated since the advent of modern logic, the practical turn in the philosophy of mathematics looks instead at questions about mathematics as an activity carried out by human agents, influenced by social, historical, and other contextual features. Major topics include informal proofs, the use of diagrams, the historical development and change of mathematical concepts, the role of computers in proof-checking and proof generation, the values by which mathematicians make evaluative judgements, and the messy features of real-world mathematical knowledge and understanding.

Imre Lakatos is routinely cast as the founding father of the practical turn in the philosophy of mathematics (Carter 2019, Van Bendegem 2013, Mancosu 2008, Aspray & Kitcher 1988), primarily through his book *Proofs and Refutations* (1976a), which itself was developed from his PhD work from the early 1960s and first published as a series of articles in the *British Journal for the Philosophy of Science* (Lakatos 1963a, 1963b, 1963c, 1964). According to the acknowledgements in his PhD thesis, Lakatos's work blends elements of three "ideological sources": Pólya's work on mathematical problem solving (1945, 1954, 1962); Popper's falsificationism (1959); and Hegel's dialectical method (1807). These jointly inform his account of how mathematical concepts, ideas, knowledge, and methods emerge and grow from mathematical practices. In particular, proofs do not simply establish the truth of theorems, Lakatos argues, but can also shape the concepts in theorems. Likewise, refutations and counterexamples do not merely show that a proof has failed, but are also catalysts for changing mathematical concepts and generating more advanced mathematical ideas in response. In both its form (a dialogue among pupils who represent successive shapes of mathematical reflection drawn from history which is related in footnotes) and its content (the generation of new concepts through the interplay of proofs and refutations), Lakatos's book is unlike any other canonical contribution to the philosophy of mathematics.

In a letter to Marx Wartofsky,[1] Lakatos expressed his desire to become the founder of a dialectical school in the philosophy of mathematics. Lakatos's work from the 1960s on *Proofs and Refutations* was a first step in this direction. Arguably, Lakatos's desire went unfulfilled. Whilst he is widely regarded as a founding figure of the practice turn in the philosophy of mathematics, hardly any works in the philosophy of mathematical practices are dialectical, and even fewer claim to be (Van Bendegem 2013, Larvor 2001). What may be surprising, then, is that Lakatos had a philosophical fellow traveller whom he knew but never acknowledged in print. The mathematical logician Geoffrey T. Kneebone wrote a pair of papers (1955, 1957) and a chapter in his book (1963) describing the outline of a dialectical philosophy of mathematics. Furthermore, Kneebone's philosophical papers seem to have had no acknowledged influence whatsoever.[2] One aim of this paper is to compare Kneebone's and Lakatos's presentations of the various core ideas of dialectical philosophy of mathematics.

A second aim for this paper is to report on two sets of items connecting Lakatos and Kneebone found in the LSE Lakatos archives. First, we report on the correspondence between Lakatos and Kneebone between the years 1966 and 1969, much of which was concerned with concurrent seminars they were running. Second, the archives also contain copies of Kneebone's two papers with annotations by Lakatos. These marginalia indicate some of Lakatos's thoughts on Kneebone's approach to the topic, but also suggest the tantalising possibility of Kneebone's unacknowledged influence on Lakatos's work, and thereby also on the philosophy of mathematical practices as a whole.

The structure of this paper is as follows. In section 2, we will set out the core features of a dialectical philosophy of mathematics and compare Lakatos's and Kneebone's versions of the proposal. In section 3 we will look at the archival materials containing Lakatos's notes on Kneebone's papers and the correspondence between the two figures. In section 4 we will then order the material into a timeline

[1] Lakatos archive, folder 21.1, item 12.

[2] According to Google Scholar, Kneebone's articles have been cited six times, the earliest citation being in 2016. His book has been cited 247 times, but none of these refer to the discussion of dialectics.

and discuss some ramifications. Section 5 concludes. For ease of overview, we offer a timeline of relevant events as an appendix.

2. Dialectical Philosophy of Mathematics

In this section we provide the contours of dialectical philosophies of mathematics (2.1), and then trace these lines in Lakatos's (2.2) and Kneebone's (2.3) concrete proposals. In 2.4 we briefly engage with a different form of dialectical philosophy of mathematics.

2.1 Components of Dialectical Philosophy of Mathematics

In this paper we will follow Larvor (2001) in identifying the following key features of a dialectical philosophy of mathematics:

1) Taking an "inside phenomenological" stance.
2) Focusing on processes, not products.
3) Focusing on mathematical concepts, not propositions.
4) Using a dialectical logic and a dialectical notion of rigour
5) Indifference to the ontology of mathematics.

The "inside phenomenological" stance seeks rationality, stability and coherence in the history and development of mathematics. The idea is that mathematical changes happen for mathematical reasons that are internal to the practice, guided by the internal values and judgements of its practitioners. Rather than thinking about the (sometimes idiosyncratic) judgements of the individual mathematician, the "inside phenomenological" stance takes a larger view by considering the perspective of mathematics itself. This has to be metaphorical, of course, but the underlying point is just to see the overall development of mathematical ideas as being primarily driven by reasons from within mathematics, as opposed to external pressures. There is, however, a difficulty in properly separating internal and external reasons. For example, mathematical advancement is also driven by applications, such as the usual suspects of accounting, trade, measurement, astronomy, shipping, engineering, physics, computing, cryptography etc. The dialectical philosopher of mathematics will believe there is no good way to draw the internal/external boundary for mathematics precisely because there is no rigid and exact model of the nature of mathematics, but they will emphasise that the mathematical parts of these applications develop in line with properly mathematical reasons. Thus, dialectical philosophers articulate the inside phenomenological stance by writing up the history of mathematics as a rational enterprise in which the body of mathematics develops for mathematical reasons. This distinguishes philosophically motivated accounts of the history of mathematics from those of historians, who may focus on individual mathematicians, blend internal and external influences on the direction of mathematical research, privilege material or ideological explanations, and otherwise write the history of mathematics with no ambition to reveal the rationality of mathematics as such. For historians, the philosophers' use of history to exhibit the rationality, stability and coherence of mathematics seems excessively *a priori*, at best.[3]

The focus on processes and concepts rather than products and propositions means that the dialectical philosopher of mathematics is more interested in the dynamics of changing mathematical ideas than the static end products. Questions about truth and validity, for example, are questions about the relationships between static propositions. Instead, the dialectical philosopher is interested in how concepts grow and develop in response to new problem situations, new ideas, new questions, new counterexamples, new applications etc. These changes are driven by mathematics as a human activity, and so are placed in a social and historical context, though these contexts are primarily analysed from the internal stance. A stronger version of this thought is that the products of mathematical activity seem arbitrary, and therefore philosophically unsatisfactory, when viewed in isolation from the processes that produced them.

While the static relations of truth and validity are modelled by formal logic, the changes of meaning of mathematical concepts are not. Central to the usual conception of logic is that meanings ought not

[3] In a recent book (Gillies 2023), Donald Gillies credits Lakatos with introducing the historical method to philosophy of mathematics. This is correct, except that there is no single historical method in philosophy. Philosophically motivated narratives are not history as written by historians, and the relation between these two is a question for any philosopher of this sort. Lakatos's answer to this question is found in his remarks about rational reconstructions being a caricature of real history.

change in the course of an argument, for fear of committing the fallacy of equivocation. This is why the picture of dialectics that emphasises the changes of concepts and meanings posits a dialectical notion of reason, which is revealed in the development of concepts. This raises the question: what exactly is the relationship between deductive and dialectical thought? How can dialectical thought be rigorous if the meanings of terms are not fixed? Different dialectical philosophers might have different things to say on this matter.

Finally, the dialectical approach to the philosophy of mathematics sets aside questions of ontology. The dialectical development of mathematics is understood to be propelled by human activities and mathematical values, leaving little room for ontology to play an important part in the account. Part of the motivation for the inside phenomenological stance is that there is, on this sort of view, no position outside of the practice in question (here, mathematics) from which the relation between the enquirer and the object of enquiry might be described. This line originates with Kant, in whose work the thing-in-itself became a sort of grammatical placeholder, and is neatly summarised in Wittgenstein's remark (not about mathematics) that "a nothing would serve just as well as a something about which nothing can be said" (*Philosophical Investigations* §304). Whether mathematical objects exist or not, and in what shape or form, does not play an explanatory role in philosophical accounts of the human activities of mathematicians.[4]

While few philosophers of mathematical practices explicitly declare themselves to be dialectical philosophers of mathematics[5], it is clear that these central ideas have been influential in the practical turn. The shift to focusing on activities and the development of mathematical concepts is fundamental to the study of mathematical practices, and ubiquitous in published works. The rejection of formal logic as adequate to model all of mathematics—rather than just a static slice of it—is seen in ongoing

[4] The dialectical philosopher may be over-zealous here; see Carter (2004), Rittberg (2016, 2020).

[5] This holds even for committed Lakatosians. For example, Celluci (2022) describes his approach as a "heuristic philosophy of mathematics", placing emphasis on Pólya's influence rather than Hegel's. Our hypothesis is that this is likely due to a combination of the unpopularity of canonically Continental ideas in contemporary analytic philosophy, as well as the fact that the primary source for such a dialectical philosophy of mathematics would be Lakatos himself, but is tricky to pin down in the dialogical form of *Proofs and Refutations*.

debates about the nature of informal mathematical proofs. The use of the history of mathematics to understand how concepts have changed and evolved, and as a case-study for the rational development of mathematics, is widely exemplified.

2.2 Lakatos's Dialectical Philosophy of Mathematics

As mentioned, Lakatos had hoped to found a dialectical school in the philosophy of mathematics. This was unsuccessful. Even *Proofs and Refutations*, is (we feel) not read by most readers as a piece of dialectical philosophy of mathematics in the sense just elaborated. We therefore summarise it here to bring out its dialectical character.[6]

In *Proofs and Refutations* Lakatos presents a classroom dialogue between a teacher and students named after Greek letters, about "Euler's conjecture" which connects the vertices (V), edges (E) and faces (F) of a polyhedron by the formula $V - E + F = 2$. The dialogue begins with the teacher proposing a proof that involves removing one face, flattening out a polyhedron into a net, then proving the conjecture using a pair of algorithms operating on this net. The first "triangulates" all of the flattened faces to turn them into triangles, and the second sequentially removes these triangles until just one remains. By tracking the number of vertices, edges, and faces, the reasoning is meant to show that the original polyhedron must have satisfied the Euler formula. However, immediately the students start to raise objections and counterexamples of various kinds. There appear to be "global" counterexamples of polyhedra that don't satisfy the Euler conjecture at all, like the picture frame and a cylinder. They also find "local" counterexamples that one or more steps of the proposed proof do not work for, such as the cylinder failing the triangulation step. Over the course of the book, the class discusses whether these examples are genuine counterexamples or merely mathematical "monsters", and then what strategies there are for responding to these alleged refutations of the proof. Strategies include simply

[6] We are grateful to an anonymous referee who directed us to (Shaffer 2015), the burden of which is that a proper understanding of Lakatos's philosophy of mathematics must take account of Lakatos's posthumously published paper 'A renaissance of empiricism in the recent philosophy of mathematics' (Lakatos 1976b). It is true that this paper is often neglected in the literature on Lakatos, and perhaps unjustly so. However, our aim is not to give a comprehensive account of Lakatos's philosophy of mathematics but rather to explore his similarities and commonalities with Kneebone. For this reason, we focus on *Proofs and Refutations* because that is where Lakatos is at his most dialectical and therefore closest in spirit to Kneebone's philosophy.

giving up (the method of surrender), ruling out the monsters as not being legitimate examples after all (the method of monster-barring), adding extra lemmas as preconditions on the proof (the method of lemma incorporation), amongst others, and builds towards a natural history of dialectical patterns in the growth of mathematical knowledge.

The dialogue is also a “rational reconstruction” of the historical development of the proof attempts, counterexamples, arguments, and concepts relating to this conjecture, where the various students are used as representatives—or maybe caricatures—of various historical positions and reactions. Sometimes the connection is direct, with footnotes to the text indicating that the students’ words are quotations from various mathematicians, such as Euler, Poincaré, Cauchy, Abel, Kepler, while others echo philosophical positions such as Popper’s falsificationism. In this rational reconstruction, we can see one aspect of the dialectical project at work. By showing that in this one quasi-historical case-study the development of mathematical ideas happened in a rational, internally motivated way, Lakatos demonstrated the dialectic of mathematical history at work. The very presentation of the book therefore embodies the inside phenomenological stance and the dialectical approach (point 1). Indeed, in his infamous sardonic style, Lakatos anticipates the objections of historians, joking that the rational reconstruction can distort history to its philosophical ends:

> Thus Pi’s statement, although heuristically correct (i.e. true in a rational history of mathematics), is historically false. (This should not worry us: actual history is frequently a caricature of its rational reconstructions.) (Lakatos 1976a, fn. 140)

It is also clear that Lakatos embraces the other main features of a dialectical philosophy. The book is about the process of how mathematics is done and not its end result (point 2). He shows that flawed proofs and counterexamples are a central part of the process of mathematics. Rather than being dead ends, they are part of creating more mathematics. Likewise, part of mathematical activity is revising and improving mathematical concepts, not merely generating new propositions. In the dialogue, the concepts of polyhedron, vertex, edge, and face, are all challenged and refined in light of the proposed proofs and refutations.[7]

Lakatos opens the *Author's Introduction* to P&R by rejecting the formalist, Euclidean approach that analyses mathematics using formal logic in favour of the problems such an analysis ignores:

> Among these are all problems relating to informal (*inhaltliche*) mathematics and to its growth, and all problems relating to the situational logic of mathematical problem-solving. (Lakatos 1976a, 1)

In contrast, formal logic is seen to govern logical relations between propositions, which are fixed and static units of meaning. Lakatos's philosophical interest, however, concerns the changing of concepts (point 3), which are seen as dynamic entities susceptible to the internal pressures of mathematical rationality (point 4):

> PI: That is right. Heuristic is concerned with language dynamics, while logic is concerned with language statics. (Lakatos 1976a, 93)

Finally, the work is indifferent to the ontology of mathematics and foundational matters (point 5). For example:

> BETA: […] I want to do mathematics and I am not interested in the philosophical difficulties of justifying its foundations. Even if reason fails to provide such justification my natural instinct reassures me. (Lakatos 1976a, 58)

These various themes found concise expression in this remark in the second appendix to *Proofs and Refutations*:

> Mathematical activity is a human activity. Certain aspects of this activity—as of any human activity—can be studied by psychology, others by history. Heuristic is not primarily interested in these aspects. Mathematics, this product of human activity, 'alienates itself' from the human

[7] Tanswell (2018) discusses Lakatos specifically on concept-change at length.

> activity which has been producing it. It becomes a living, growing organism, it develops its own autonomous laws of growth, its own dialectic. The genuine creative mathematician is just a personification, an incarnation of these laws which can only realise them selves in human action. Their incarnation, however, is rarely perfect. The activity of human mathematicians, as it appears in history, is only a fumbling realisation of the wonderful dialectic of mathematical ideas. But any mathematician, if he has talent, spark, genius, communicates with, feels the sweep of, and obeys this dialectic of ideas.[8]

Overall, then Lakatos's *Proofs and Refutations* exemplifies dialectical philosophy of mathematics in the sense articulated above.

2.3 Kneebone's Dialectical Philosophy of Mathematics

In contrast to Lakatos's influence on the philosophy of mathematical practice, Geoffrey Kneebone's earlier proposal to understand mathematics dialectically seems to have had no notable influence. His earlier philosophical writings (Kneebone 1952, 1955)[9] and his main paper proposing a dialectical philosophy of mathematical practice (Kneebone 1957) had no Google scholar citations at all prior to our own work. Kneebone's book (1963) is widely known[10], but no citations that we have seen engage with the dialectical final chapter, referring instead to the longer parts on mathematical logic.[11] We find this disparity instructive. In this section, we will begin to investigate this peculiar situation by describing Kneebone's dialectical philosophy of mathematics.

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[8] (Lakatos, 1976a, 146). The editors added a note in which they suggest that Lakatos would have retreated from this thoroughly Hegelian outlook and allowed that the direction of mathematical growth owed something to individual ingenuity as well as the impersonal logic of mathematical dialectics. When reading Lakatos's rare mentions of Hegel, it is important to remember that he did his philosophical work in the shadow of two successive orthodoxies about Hegel. First, Marxism maintained the myth of a Hegelian method of thesis-antithesis-synthesis, which is nowhere to be found in Hegel's writing except in a critical remark about Kant (Mueller, 1958). Second, Popper regarded Hegel as a philosophical charlatan and an enemy of freedom (Popper, 1940, 1945).

[9] Kneebone's 1943 PhD thesis is Kantian in spirit and not overtly dialectical, with the only mention of dialectics being "Generally speaking, we may say that the development of science—the evolution of the Unity of Experience—takes place according to the dialectical process." (Kneebone 1943, 54).

[10] See fn. 2.

[11] The only exception to this is the entry on Kneebone in the *Dictionary of Twentieth Century Philosophers* by Wilfrid Hodges (2005).

The starting point for Kneebone is the role of logic in mathematics. Tracing the emphasis on symbolic logic and the formalisation of mathematical reasoning through mathematical history[12], he places particular emphasis on Peano as seeing the formalisation of mathematics as a move away from intuition towards rigour:

> Peano, on the other hand, realized that if complete rigour is to be achieved intuition must be banished completely from mathematical argument. (Kneebone 1957, 206)

Peano's project was of major influence on two of the big schools of thought in the philosophy of mathematics: Logicism and Formalism. The former proposed to define all mathematical concepts in strictly logical terms and thereby reduce mathematical rigour to formal, logical rigour. Its chief representative at the time was the *Principia Mathematica* project by Russell and Whitehead (1910).[13] Hilbertian formalism aimed to formalise a system for all of mathematics, in part in order to prove that it is consistent. The failures of both are well-known: the axiom of infinity Russell needed was not purely logical, the various class-theoretic paradoxes stood in the way of a naïve picture of classes and membership, and Gödel's theorems seemed to undermine the possibility of the kind of consistency proofs that Hilbert's programme required. From this, Kneebone concludes:

> The failure of both undertakings suggests that the relationship between mathematics and logic may perhaps have been wrongly understood, and prompts reconsideration of the nature of this relationship. (Kneebone 1957, 210)

The answer, according to Kneebone, is to acknowledge the divide between formal logic with its notion of logical validity, and dialectical reasoning:

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[12] It is worth remarking that Kneebone's picture of history is shaky at best, for example describing Logicism and Formalism as sequential. In fact, he says "Up to this point in the history of the philosophy of mathematics the idea of rigour had developed naturally and smoothly" (Kneebone 1957, 207) which seems even further from actual historical development than Lakatos's rational reconstruction of history.

[13] Kneebone sets Frege aside because he had little influence on philosophers besides Russell. This is, of course, no longer true in analytic philosophy generally, but also questionable for the 1950s. For example, Frege was also mentioned in the preface to Dedekind's *Was sind und was sollen die Zahlen?* (1887).

> Mathematics, on the face of it, is completely undialectical; but this is only appearance, and it is my thesis in this paper that the basically dialectical character of mathematics is precisely the feature that has been neglected hitherto. (Kneebone 1957, 212)

Taking the list of features of dialectical philosophy of mathematics set out above, we can now run through and see some of them present in Kneebone's work. To begin, the ‘internal stance’ of seeing mathematical developments come about for rational, mathematical reasons is certainly not as fully developed as Lakatos's detailed case studies but is implicitly present in Kneebone's consideration of historical examples that he believes cannot be fully explained on the purely deductivist model. For example, he includes a discussion of Kummer's ideal factors as being of this kind. More explicitly, Kneebone acknowledges that mathematics must not be separated from the human agents who practice it, but then argues that this gives rise to one of the puzzling aspects of a dialectical logic for mathematics: that it is both personal and impersonal. He says:

> [I]n so far as mathematical thinking is dialectical it appears to be both personal and impersonal at the same time. It is personal because thinking is a process that can only take place in the minds of individual mathematicians, and impersonal because it produces a body of mathematical knowledge that is accessible to every individual mathematician, and valid for all alike. (Kneebone 1957, 220-21)

Kneebone’s solution to this puzzle is that the mathematicians who carry out the mathematics are embedded in what he calls ‘cultural traditions’ – “which comprises both existing knowledge and the concepts which provide the articulation of this knowledge” (Kneebone 1957, 221) - and that concepts within these manage to span the personal and impersonal divide:

> The concepts of the cultural tradition are thus part of the mind’s equipment and at the same time part of the structure of the known world, and they are able therefore to be both personal and impersonal together. (Kneebone 1957, 221)

The emphasis on concepts and conceptual development, instead of seeing mathematics as a body of propositions, is also central to Kneebone's dialectical philosophy of mathematics:

> [W]hereas deductive reasoning operates with fixed concepts, which might be represented by the symbols of a logical calculus, dialectical reasoning always brings about development of concepts. (Kneebone 1957, 212)

> Mathematics is usually thought of, by philosophers of the subject no less than by those who take it more for granted, as a body of propositions. But although it is true that the mathematics of the textbooks is such a propositional edifice, propositions are by no means all that the creative mathematician is concerned with. (Kneebone 1957, 215)

As we have already seen, Kneebone separates deductive and dialectical aspects of mathematics, but additionally he is clear that these must come with separate notions of rigour:

> There is rigour of demonstration and also rigour of dialectical development, and the two are by no means the same. (Kneebone 1957, 222)

Indeed, he calls the analysis of dialectical rigour one of the most important issues for philosophy of mathematics.

> Dialectical rigour is almost wholly unanalysed, and its very existence as a kind of rigour different from, but of at least the same importance with, deductive rigour seems to be universally overlooked, at any rate as far as mathematical thinking is concerned. But examination of the nature of dialectical rigour offers the only hope of escape from the impasse reached by both Logistic [i.e. Logicism] and Formalism, and it is therefore the most urgent task facing the philosophy of mathematics. (Kneebone 1957, 223)

With the contemporary practical turn in the philosophy of mathematics in mind, this proposal seems prescient.

On Larvor's last point, ontological neutrality, we don't find anything explicit, although the desire to see beyond the limitations of logicism, formalism and intuitionism (which take mathematical ontology to reside in the world, in language and in the mind respectively) to a dialectical form of justification seems to indicate sentiments in this direction.

3. Kneebone in the Lakatos Archives

We have established that both Lakatos and Kneebone proposed dialectical philosophies of mathematics, with Kneebone publishing his in the 1950s and 60s and Lakatos in the 1960s until his early death. Not only did they take the same approach, but they were also working in the same city: Kneebone was at Bedford College in London, and Lakatos at the London School of Economics. A natural question to ask is: was Lakatos's work influenced by Kneebone's earlier contributions? Since Kneebone's dialectical philosophy of mathematics received next to no citations for decades, and certainly no mentions in Lakatos's works, the answer would seem to be that it did not, nor did it influence anybody else for that matter. However, we will argue that this may not be the case, and that there is an open possibility that Kneebone did have a direct and unacknowledged influence on Lakatos.

The Lakatos archive, held at the London School of Economics archives, contains two interesting sets of connections between Lakatos and Kneebone. The first are copies of Kneebone's (1955) and (1957) papers on dialectical philosophy of mathematics, with a few hand-written notes, annotations, and marginalia by Lakatos in English and Hungarian. The second is correspondence between Lakatos and Kneebone, primarily concerning logic courses they ran in the late 1960s. Let us look at these in turn.

3.1 Lakatos's Notes on Kneebone

Two of Kneebone's papers with hand-written notes and annotations can be found in the LSE Lakatos archives: 'Abstract logic and concrete thought' (Kneebone 1955) and 'The philosophical basis of mathematical rigour' (Kneebone 1957). Here we will just highlight the most interesting notes and underlined text.

Let us begin with the first paper, Kneebone (1955), which is 20 pages long. There are 18 instances of underlined text, five lines made next to text in the marginalia, one note in Hungarian, and one note in English made by Lakatos. The underlining begins with the first line for the thesis statement, then continues at the foot of the first page, where Lakatos further underlined "formal logic provides an adequate representation of the process of thinking only when concepts are used in a severely limited

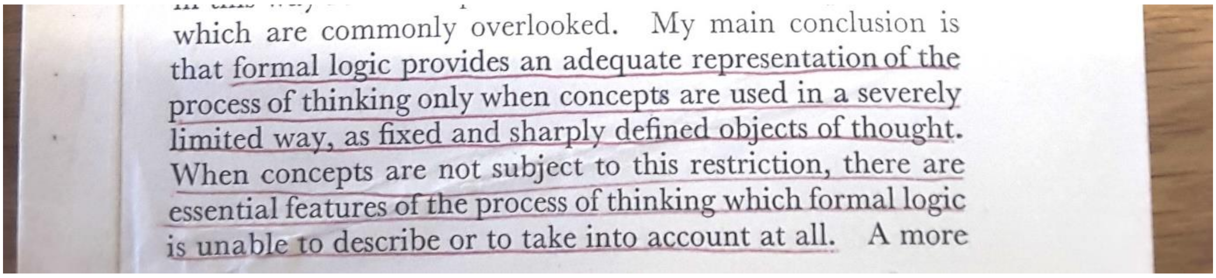
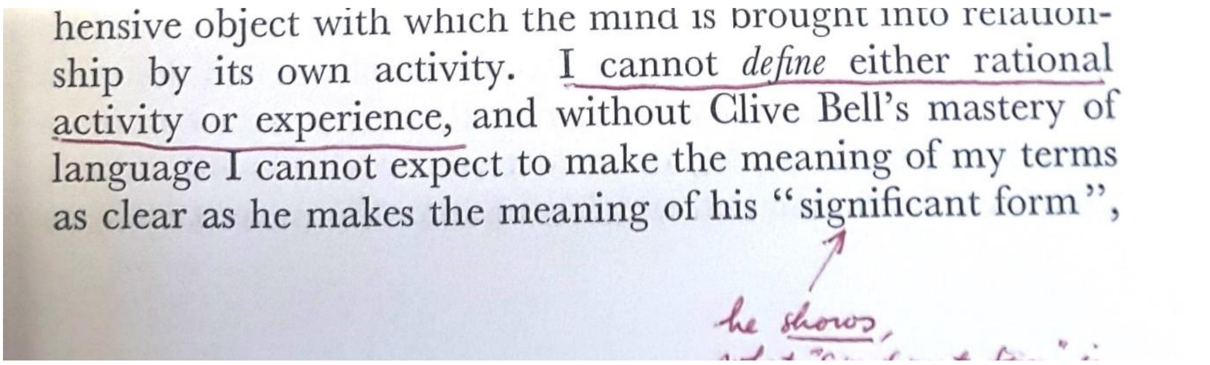

which are commonly overlooked. My main conclusion is that formal logic provides an adequate representation of the process of thinking only when concepts are used in a severely limited way, as fixed and sharply defined objects of thought. When concepts are not subject to this restriction, there are essential features of the process of thinking which formal logic is unable to describe or to take into account at all. A more

Figure 1. Lakatos's underlining on the first page (p. 25) of Kneebone (1955).

way, as fixed and sharply defined objects of thought. When concepts are not subject to this restriction, there are essential features of the process of thinking which formal logic is unable to describe or take into account at all." (p. 25), as in Fig. 1. This would be a clear point of agreement between Lakatos and Kneebone.

On page 31 we meet our first marginal note, as shown in Fig. 2. Lakatos underlined, "I cannot define either rational activity or experience". Kneebone continues, "…and without Clive Bell's mastery of language I cannot expect to make the meaning of my terms as clear as he makes the meaning of his 'significant forms'" (p. 31). Lakatos drew an arrow to 'significant' and wrote (in English) "he shows what significant form is" with 'shows' underlined as here. Given Lakatos's published views on Wittgenstein,

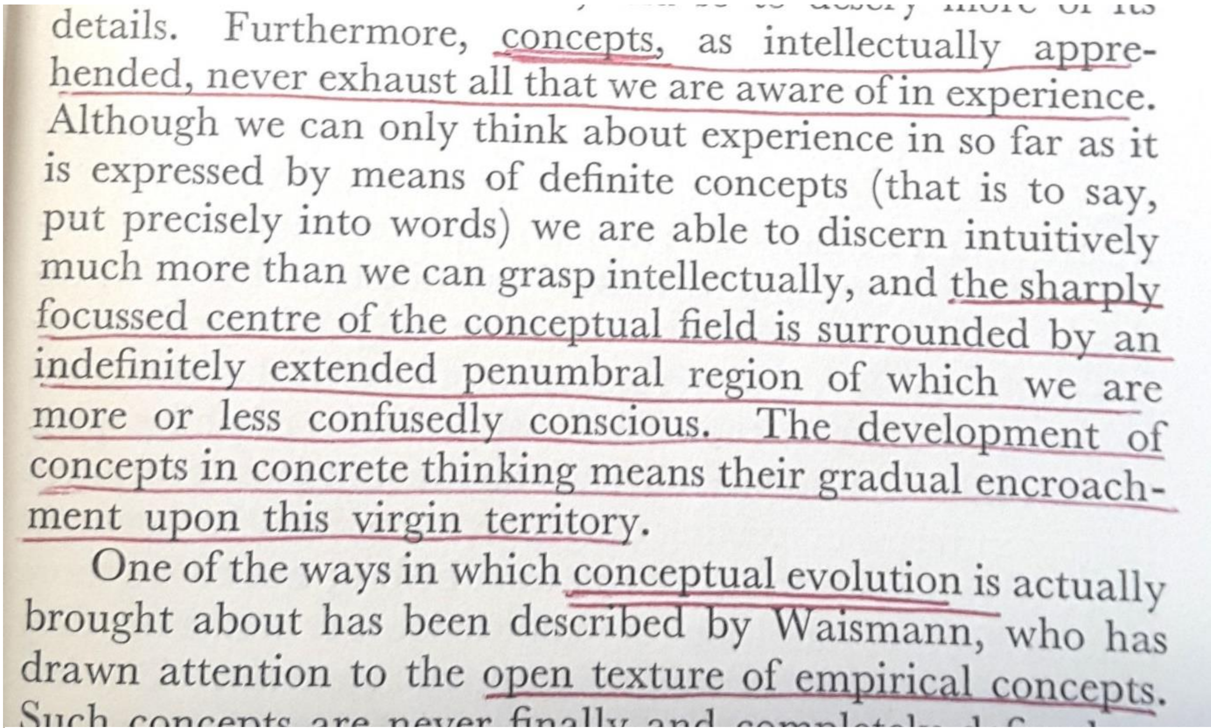

hensive object with which the mind is brought into relationship by its own activity. I cannot *define* either rational activity or experience, and without Clive Bell's mastery of language I cannot expect to make the meaning of my terms as clear as he makes the meaning of his "significant form",

he shows,

Figure 2. Lakatos's underlinings and marginal notes on p. 31 of Kneebone (1955).

the tractarian flavour of this remark is either misleading or combines with some later hints to suggest that Lakatos was more open to Wittgensteinian thought than he liked to admit.

The next Lakatosinalia of note come on p. 37, where Kneebone develops his distinction between the fluidity of experience and the rigidity of abstract thought. As shown in Fig. 3, Lakatos underlined "concepts, as intellectually apprehended, never exhaust all that we are aware of in experience." ('concepts' here underlined repeatedly). On the same page, "the sharply focussed centre of the conceptual field is surrounded by an indefinitely extended penumbral region of which we are more or less confusedly conscious. The development of concepts in concrete thinking means their gradual encroachment upon this virgin territory." (p. 37). In the next paragraph, Lakatos underlined 'conceptual evolution' (twice) and 'open texture of empirical concepts' in the sentence, "One of the ways in which conceptual evolution is actually brought about has been described by Waismann, who has drawn attention to the open texture of empirical concepts." A reason this is of particular note is that, in the recent literature on conceptual engineering and mathematics (Shapiro 2006, 2013, Tanswell 2018, Vecht 2020, Shapiro and Roberts 2021, Zayton 2021), one line of discussion is about interpreting Lakatos's work on conceptual change as demonstrating that mathematical concepts can


details. Furthermore, concepts, as intellectually apprehended, never exhaust all that we are aware of in experience. Although we can only think about experience in so far as it is expressed by means of definite concepts (that is to say, put precisely into words) we are able to discern intuitively much more than we can grasp intellectually, and the sharply focussed centre of the conceptual field is surrounded by an indefinitely extended penumbral region of which we are more or less confusedly conscious. The development of concepts in concrete thinking means their gradual encroachment upon this virgin territory.

One of the ways in which conceptual evolution is actually brought about has been described by Waismann, who has drawn attention to the open texture of empirical concepts. Such concepts are never finally and

Figure 3. Lakatos's underlinings on p. 37 of Kneebone (1955).

be open textured, a term drawn from Friedrich Waismann (1945) who himself explicitly identifies mathematical concepts as closed textured. While the literature uses Lakatos's work to show how mathematical concepts can be open textured, we are not aware of any previously documented direct links from Waismann to Lakatos. However, as just noted, Kneebone (1955, 37) discusses Waismann's open texture of concepts, and this annotated paper thus establishes that Lakatos was aware of Waismann's notion. We will discuss below (section 4) the issue of acknowledgement of Kneebone by Lakatos, but this link also adds the possibility of unacknowledged influence of Waismann, and indirectly Wittgenstein, on Lakatos.

The second article, Kneebone (1957) is 19 pages long. There are 38 instances of underlined text, 4 lines made next to text in the marginalia, 3 notes in Hungarian, one note in English, and 10 punctuation marks in the marginalia made by Lakatos.

Once again, we find Lakatos underlining passages where he is in agreement with Kneebone, such as this dialectical version of (what would later become) the rallying-cry of the philosophy mathematical practices, "The philosophical theories of mathematics, whether Logistic, Formalist or Intuitionist, are perhaps concerned more with ideal conceptions of what mathematics ought to be than with what it actually is; and we should therefore look at mathematics itself in order to see if it is wholly deductive, or if any considerable dialectical element is to be found in it." (p. 215).

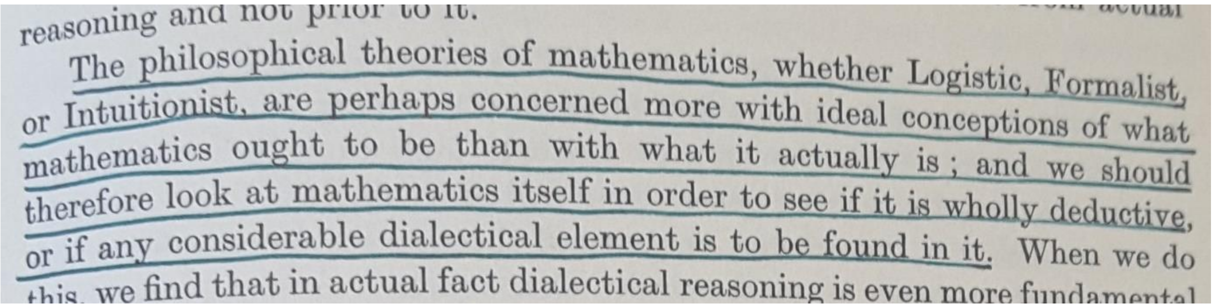

reasoning and not prior to it.
The philosophical theories of mathematics, whether Logistic, Formalist, or Intuitionist, are perhaps concerned more with ideal conceptions of what mathematics ought to be than with what it actually is; and we should therefore look at mathematics itself in order to see if it is wholly deductive, or if any considerable dialectical element is to be found in it. When we do this, we find that in actual fact dialectical reasoning is even more fundamental

Figure 4. Lakatos's underlinings on p. 215 of Kneebone (1957).

We find more notes when Kneebone is discussing the dialectical approach to mathematics. For example, as shown in Fig. 5, Lakatos underlined, "the concepts of our present cultural tradition enable us to recognize a certain complex of features that we associate with demonstrative thinking and a different complex to which I have attached the name 'dialectical'." (p. 219, 'demonstrative thinking' and 'dialectical' underlined twice). Two small question-marks in the margin indicate Lakatos's feelings about this claim.

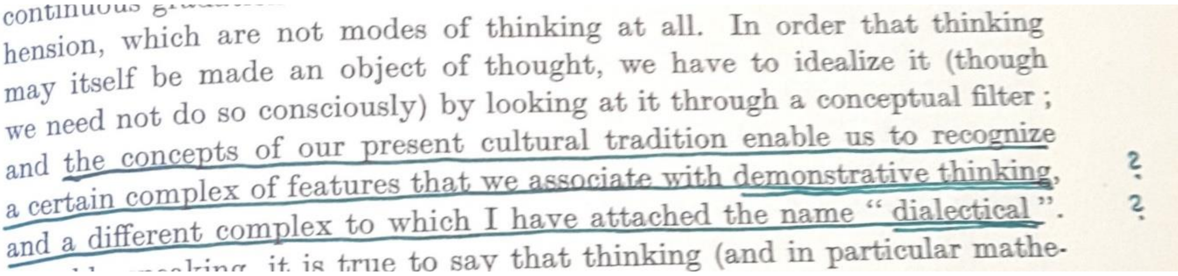
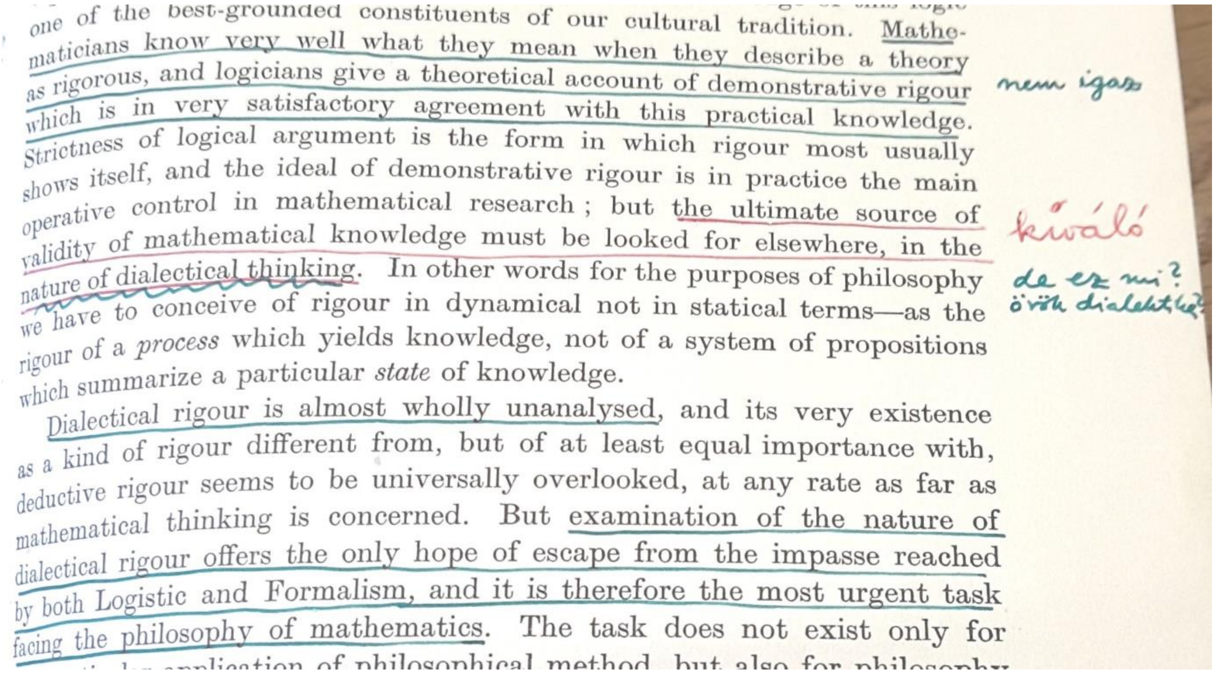

continuous
hension, which are not modes of thinking at all. In order that thinking
may itself be made an object of thought, we have to idealize it (though
we need not do so consciously) by looking at it through a conceptual filter ;
and the concepts of our present cultural tradition enable us to recognize
a certain complex of features that we associate with demonstrative thinking, ?
and a different complex to which I have attached the name " dialectical ". ?
it is true to say that thinking (and in particular mathe-

Figure 5. Lakatos's notes on p. 219 of Kneebone (1957).

Later, Lakatos also underlined the summary of Kneebone's thought, "Mathematics, then, is a process of thinking which is locally deductive but dialectical when we take a sufficiently wide view of it." (p. 221, underlined by Lakatos as shown here). Similarly, Lakatos underlined, "There is rigour of demonstration and also rigour of dialectical development, and the two are by no means the same." (p. 222). Both claims would fit with the dialectical approach of Lakatos's *Proofs and Refutations.*

The final page of Kneebone's article is the most interesting in terms of its marginalia. Two pens are used, one the same green as seen in the rest of the markings in this paper; the other red, perhaps the same red as used to mark up (Kneebone 1955). In green, Lakatos underlined, "Mathematicians know very well what they mean when they describe a theory as rigorous, and logicians give a theoretical account of demonstrative rigour which is in very satisfactory agreement with this practical knowledge." and wrote in the margin "nem igaz"—*not true*. In the following sentence, Kneebone develops this thought about formal rigour into a contrast, the second side of which Lakatos underlined in red, "the ultimate source of validity of mathematical knowledge must be looked for elsewhere, in the nature of dialectical thinking" and wrote in the margin, "kiváló"—*excellent*. However, the final phrase 'nature of dialectical thinking' is underlined again, in a wiggling green line (presumably later than the red underlining) and a less favourable marginal note, "de ez mi? örök dialektika?"—*but what is this? forever* *dialectics*? In the next paragraph, the penultimate paragraph of Kneebone's paper, Lakatos underlined, "Dialectical rigour is almost wholly unanalysed… examination of the nature of dialectical rigour offer the only hope of escape from the impasse reached by both Logistic and Formalism, and it is therefore the most urgent task facing the philosophy of mathematics." (p. 223)


one of the best-grounded constituents of our cultural tradition. Mathematicians know very well what they mean when they describe a theory as rigorous, and logicians give a theoretical account of demonstrative rigour which is in very satisfactory agreement with this practical knowledge. Strictness of logical argument is the form in which rigour most usually shows itself, and the ideal of demonstrative rigour is in practice the main operative control in mathematical research; but the ultimate source of validity of mathematical knowledge must be looked for elsewhere, in the nature of dialectical thinking. In other words for the purposes of philosophy we have to conceive of rigour in dynamical not in statical terms—as the rigour of a *process* which yields knowledge, not of a system of propositions which summarize a particular *state* of knowledge.

Dialectical rigour is almost wholly unanalysed, and its very existence as a kind of rigour different from, but of at least equal importance with, deductive rigour seems to be universally overlooked, at any rate as far as mathematical thinking is concerned. But examination of the nature of dialectical rigour offers the only hope of escape from the impasse reached by both Logistic and Formalism, and it is therefore the most urgent task facing the philosophy of mathematics. The task does not exist only for

nem igaz

kiváló

de ez mi?
övék dialektika?

Figure 6. Lakatos's notes on p. 223 of Kneebone (1957).

Given that we are interested in the possible influence of Kneebone on Lakatos, there is a noteworthy similarity here between the two authors:

> In other words for the purposes of philosophy we have to conceive of rigour in dynamical not in statical terms—as the rigour of a *process* which yields knowledge, not of a system of propositions which summarize a particular *state* of knowledge. (Kneebone 1957, 223)
>
> *PI*: That's right. Heuristic is concerned with language-dynamics, while logic is concerned with language-statics. (Lakatos 1976a, 93)

3.2 The Lakatos-Kneebone correspondence

The correspondence between Kneebone and Lakatos archived at the LSE primarily concerns the organisation of seminars run by Lakatos at the LSE and Kneebone at Bedford College (also based in London). The idea was that students could attend both seminars. Furthermore, Kneebone and Lakatos discussed exam questions and markings. The seminars suffered from problems that arose from their interdisciplinary set-up, primarily philosophy students who struggled with the demanding mathematical material. The LSE archive provides evidence that the seminars ran between 1966 and 1969.

The Lakatos archive at the LSE hosts seventeen items of correspondence between Kneebone and Lakatos, one exam, and one hand-written score sheet of an exam. Whether the score sheet is for the exam present in the archive is unclear. The correspondence dates range from 1962 to 1969, with substantial gaps in between. The first archived letter date is dated June 1962. The chronologically next archived letters were written four years later, in June 1966. The subsequent archived letters were written half a year later, in January and February 1967, followed by another time-gap of half a year until two letters from February 1968 and three from June 1968. Most archived correspondence occurred in 1969 with six letters written between March to June.[14]

The letter with the most philosophical content is the first in the archive, from June 1962, by Kneebone to Lakatos. Kneebone opens the letter by thanking Lakatos for agreeing to chair his upcoming Joint Session presentation. Most of the letter (1.5 out of 2 pages) Kneebone dedicates to comments on a draft of Lakatos's 'Infinite Regress and the Foundations of Mathematics'. Kneebone criticizes Lakatos's use of the term 'trivialization'; Kneebone denies that Fraenkel is a distinguished logician; Kneebone takes issue with the claim that finitary mathematics has axioms; worries that Lakatos's separation of Empiricism and Inductivism is too sharp; and finds Lakatos's section 3 'somewhat thin and scrappy'.

[14] In chronological order, the LSE archive contains letters dating 18.6.1962; 13.6.1966; 16.6.1966; 22.6.1966 (secretary note); 6.1.1967; 13.2.1967; 6.2.1968; 8.2.1968; 10.6.1968; 16.6.1968; 17.6.1968; 24.3.1969; 26.3.1969; 28.4.1969; 6.6.1969; 9.6.1969; 12.6.1969.

Lakatos' paper appeared one month later later (July 15) in the *Aristotelian Society Supplementary Volume,* still containing the terminology Kneebone found troublesome and without reference to Kneebone in the acknowledgements or elsewhere.

Overall, this correspondence shows that there was friendly and respectful contact between Lakatos and Kneebone, addressing one another with "Dear Lakatos" in the earlier paper, but using "Dear Geoffrey" and "Dear Imre" in the later letters. They did not, however, discuss matters of dialectical philosophy in the letters archived at the LSE. In particular, they did not collaborate to found a school of dialectical philosophy by running suitable seminars to enculturate students in this way of thinking.

4. Intellectual interactions

Despite their intellectual alignment, neither Lakatos nor Kneebone ever referred to the other in their published writing. This section explores why this might be.

Kneebone had published his two papers by 1957, five years before his first (documented) contact with Lakatos. In the 1962 letter from Kneebone to Lakatos, Kneebone mentions that he is busy with the final stages of writing his book, which appeared in 1963. Lakatos's first publication was in 1962.[15] It is therefore likely that Kneebone could not have referred to Lakatos's work even if he had wanted to.

Why Lakatos never acknowledged Kneebone is less clear. Kneebone published his philosophical articles before Lakatos began writing his PhD thesis in 1958-9; cf. the timeline in the appendix. Lakatos could have read Kneebone's two papers during his research for his PhD. The first evidence that Lakatos was aware of Kneebone, however, is the June 1962 letter, well after Lakatos had defended his PhD (in 1961).

As described above, Kneebone opens the June 1962 letter by thanking Lakatos for agreeing to chair and open a discussion for a presentation Kneebone is about to give. We can therefore expect that, as preparation, Lakatos had read at least some of Kneebone's philosophical work by 1962. Most of the letter (1.5 out of 2 pages) Kneebone dedicates to comments on a draft of Lakatos's 'Infinite Regress and

[15] Lakatos had published already before 1962 whilst working in Hungary. The 1962 piece is his first publication whilst based in the UK.

the Foundations of Mathematics'. This paper appears one month later (July 15) in the *Aristotelian Society Supplementary Volume* with substantial acknowledgements (mentioning six scholars by name), but without any reference to Kneebone. This might be warranted, as Lakatos does not seem to have incorporated any of Kneebone's suggestions into the final paper. This does not seem to have negatively impacted the relationship between Kneebone and Lakatos, who continued to communicate in a friendly tone and started to set up their conjunct seminars. It may also be that these comments arrived too late for Lakatos to incorporate them.

Lakatos published his PhD thesis as a four-paper series of articles in the *British Journal for the Philosophy of Science* (BJPS) between 1963 and 1964. The first appeared in May 1963 (Vol 14, No. 53), i.e. nearly one year after the first recorded contact between Lakatos and Kneebone. The BJPS gives the paper as 'received 3.x.60', but changes were clearly made after, as evidenced by the full bibliographic detail given of Popper's book *Conjectures and Refutations* as well as his article 'Science: Problems, Aims, Responsibilities', both of which appeared only in 1963. The latest bibliographical reference given in Lakatos's May 1963 BJPS paper is to his own 'Infinite Regress and Foundations of Mathematics', which appeared in July 1962, i.e. about two years after the BJPS received Lakatos's article. That is, Lakatos made changes to his BJPS articles even after the first archived interaction between Lakatos and Kneebone in June 1962.

One plausible explanation of this timeline is that the core of Lakatos' work for the essays that would become *Proofs and Refutations* was done before Lakatos became aware of Kneebone's work in 1962. Engagement with Kneebone's work might simply not have left a strong enough impression on Lakatos to acknowledge Kneebone in the 1963-64 BJPS articles. Looking at the annotations and marginalia in Lakatos's copies of Kneebone's papers, he found ideas that he was already aware of (the distinction between the formal logic of fixed propositions and the dialectic of developing concepts is already there in Hegel and was retained in Marxism), weak or underdeveloped examples or a respect for orthodox logical theory that Lakatos did not share. Recall his cry of "nem igaz" (*not true*) at Kneebone's claim that:

> Mathematicians know very well what they mean when they describe a theory as rigorous, and logicians give a theoretical account of demonstrative rigour which is in very satisfactory agreement with this practical knowledge. (Kneebone 1957, 223)

It's not clear from the marginal comment whether Lakatos denied one or both sides of this conjunction. It does show, however, that Lakatos had in mind a more radical challenge to the philosophy of mathematics mainstream than Kneebone envisaged.

Another, somewhat speculative, interpretation of the events relies on Kadvany's insights about "Lakatos's terrifying and morally compromised Hungarian past" (Kadvany 2001, xvii). As mentioned above, Lakatos expressed his desire to become the founder of a dialectical school in the philosophy of mathematics. Such a desire provides political incentives for Lakatos to remain silent about any influence Kneebone might have had on him, in order not to share the credit for founding a school or movement. Lakatos' savvy in the politics of reason, well-documented in (Kadvany 2001; Larvor 1998), may have well led him to act on such incentives. And it may have worked. Whilst not being regarded as a founder of a dialectical school, Lakatos is widely seen as the founding father of the philosophy of mathematical practices, which comprises of some, but not only, dialectical approaches to the philosophy of mathematics (Van Bendegem 2013, section 1).[16] However, the evidence from the archives does not settle *when* Lakatos read and annotated Kneebone's papers, so cannot tell us whether he left out reference (deliberately or inadvertently) to Kneebone despite it having an influence, or whether he read Kneebone's papers after he had already written *Proofs and Refutations*. Of course, Lakatos was still tinkering with the book version of his work when he died, so it is possible that Kneebone would have made an appearance if he had completed it himself. Indeed, Gillies (2023, fn. 8) points out that between 1966 and 1973 Lakatos was not actively working on the philosophy of mathematics, having moved to the philosophy of science and probability, but that shortly before his death he gave a talk

[16] Gillies (2023, 18) makes a similar point that Lakatos was not a guru with a faithful school of Lakatosian disciples, but was very influential in producing new ideas that stimulated later researchers to take them up and develop them, even in directions that Lakatos himself would not have done.

applying his methodology of scientific research programmes to the philosophy of mathematics, signalling a desire to return to the case of mathematics.

We will not conclusively settle these matters of priority in this paper, nor are we able to establish why Lakatos never referred to or acknowledged Kneebone in his works. Instead, we have highlighted the roots of a dialectical philosophy of mathematics. In the next section, we finish this paper by elaborating on how this helps to demystify the history of the practice turn in the philosophy of mathematical practices.

5. The origin myth of the practice turn in the philosophy of mathematics

What is there to learn from this analysis of Kneebone's dialectical approach to the philosophy of mathematics, and its possible influence on Lakatos? Besides the philosophical implications, there is also a *sociological* one about our citation practices and the historical storytelling about our disciplines that often occurs at the start of papers, which simplifies and distorts that very history. The process of fixating on specific individuals, such as Lakatos, acts to mythologise the influence that individual has had on mathematical practice as a field of research.

It was not long after *Proofs and Refutations* was written that Lakatos was identified as the starting point of a new movement in the philosophy of mathematics:

> If the mainstream began with Frege, then the origin of the maverick tradition is a series of four papers by Lakatos, published in 1963-64 and later collected into a book. (Aspray & Kitcher 1988, 17).

In other relatively early mathematical practice work, David Corfield (2003) talks extensively about Lakatos and *Proofs and Refutation*. Likewise, Paolo Mancosu in the introduction to his well-known collection of essays says:

> For it cannot be disputed that already in the 1960s, first with Lakatos and later through a group of `maverick' philosophers of mathematics (Kitcher, Tymoczko, and others), a strong reaction set in against philosophy of mathematics conceived as foundations of mathematics. (Mancosu 2008, 3).

An overview paper by Van Bendegem sets out the Lakatosian, "maverick" approach to philosophy of mathematics as one of eight approaches to mathematical practice. Van Bendegem is careful to point out that choosing a starting point in Lakatos is only a partial reflection of history:

> **Lakatos as the Starting Point.** Taking into account that almost any historical outline is partially a reconstruction, one wants nevertheless to find a historical starting or turning point of a sufficiently symbolic nature. Or, if you like, if the philosophy of mathematics underwent a similar change as the one that "shook" the philosophy of science (at least, according to some), then it seems fair to take Imre Lakatos' 1976 seminal *Proofs and Refutations* (1976), a work that was in part inspired by Georg Pólya's *How to Solve It* (1945), which discusses heuristics and problem-solving techniques in an educational setting. Its focus on mathematical practice was clear by the mere fact that it boldly presented nothing less than a "logic" of mathematical discovery. (Van Bendegem 2013, 215-16)

More recent overviews by Carter (2019), and more so Hamami & Morris (2020), are more careful about the historical placement of Lakatos:

> Perhaps the first steps towards a philosophy of mathematical practice were taken by Wilder (1950, 1981), who argued that mathematics is a cultural system, and Pólya (1945, 1954, 1962) who investigated mathematical problem solving, heuristics, and discovery. Lakatos (1976) dedicated his famous dialogue Proofs and Refutations to Pólya, as well as to Popper, the philosopher of science. In this work, he argued that mathematical knowledge grows not via the continuous production of formal derivations but by a dynamic process which involves proposing "proofs" which are then refuted and subsequently refined. Kitcher (1984) was also

> concerned with the growth of mathematical knowledge, arguing that modern mathematics evolved via a series of rational transitions from earlier practices. (Hamami & Morris 2020, 1113).

> For one thing philosophers were critical towards the one-sided focus on foundational questions, that is, focus on logic, set theory, and the three foundational schools, logicism, formalism and intuitionism. […] Among these critics Imre Lakatos is often mentioned as the first to engage with the philosophy of mathematical practice. (Carter 2019, 3)

While not directly about mathematical *practice*, Gillies (2023) credits Lakatos with the novel introduction of the historical approach to the philosophy of mathematics, which is a significant part of the mathematical practice approach:

> My main thesis is that Lakatos' very important contribution consisted in his introduction in his 1963–4 paper "Proofs and Refutations" of the historical approach to the philosophy of mathematics. The striking nature of this paper and its interesting results led other researchers in philosophy of mathematics to take up the historical approach and it was in subsequent years strongly developed, although it had never been used by philosophers of mathematics before Lakatos. (Gillies 2023, 2)

Undoubtedly, *Proofs and Refutations* has been extremely influential, especially on philosophers interested in mathematical practices, so the high praise and central positioning as a founding influence is clearly justified. However, our study of Kneebone as having set out a similar, earlier dialectical mathematical project to Lakatos, also reveals something about the origin of this movement. There are two ways to interpret influential figures like Lakatos. The first is as *discontinuous* with prior work, taking a radical new step, unanticipated by what came before. Taking Lakatos as the singular originator of the philosophy of mathematical practices would be such an interpretation. But the other interpretation is as *continuous* with other scholars, building on and integrating other work to produce fresh insights. Here we might see *Proofs and Refutations* as being a particularly masterful bringing together of other ideas, but nonetheless one that builds on the work of others. While explicitly these others include Pólya, Popper, and Hegel, we have argued that it may have also been shaped by scholars such as Kneebone and Waismann.[17]

In wanting to found a philosophical school of thought around dialectical philosophy of mathematics, Lakatos seems to want to be seen as doing something discontinuous with previous work. Indeed, this is one potential explanation of the failure to reference Kneebone. However, we believe that the very existence of Kneebone's writing shows that Lakatos's work was actually more continuous with the ideas of others. Ironically, that continuity is clearly more fitting with the Lakatosian, dialectical line of thought that philosophical ideas also arise for rational reasons internal to their practice.

The most substantial difference between Lakatos and Kneebone, in the end, is that Kneebone envisioned a dialectical philosophy of mathematics, but Lakatos actually began to carry it out. *Proofs and Refutations* is not merely a vision for what the philosophy of mathematics could be like, but a demonstration.

[17] This is to say nothing of another possible influence, uncited by Lakatos in *Proofs and Refutations*: Wittgenstein. Lakatos does, however, cite a paper by Alice Ambrose (1959), which is explicitly Wittgensteinian. Indeed, Ambrose was a former student of Wittgenstein's.

Appendix 1: Timeline

1918	Kneebone born.
1922	Lakatos born.
1943	Kneebone awarded PhD on “The Foundations of Mathematics: A Critical Examination of the Foundations of the Mathematical Theory of Probability and Statistics”. Contains some of his early, mainly Kantian, philosophical ideas.
1947	Kneebone publishes “Philosophy and Mathematics”.
1952	Kneebone publishes “Mathematical Formalisms and their Realizations”.
1955	Kneebone presents “Abstract Logic and Concrete Thought” at the Aristotelean Society (Nov).
1956	Lakatos leaves Hungary and arrives in Cambridge to write his PhD.
1957	Kneebone publishes “The Philosophical Basis of Mathematical Rigour”.
1958	Lakatos meets Pólya, who suggests the central case study of P&R; (Larvor 1998, 6).
1958-59	Lakatos writes *Proofs and Refutations* (according to the BJPS foreword).
1959	March. Lakatos first presents his *Proofs and Refutations* at Popper’s seminar in London.
1960	Lakatos becomes an Assistant Professor at the LSE, working alongside Popper.
1961	Lakatos is awarded his PhD for his thesis entitled “Essays in the Logic of Mathematical Discovery”. Chapters 1-3 of this thesis reappear in the posthumously published book *Proofs and Refutations*, edited by J. Worrall and E. Zahar.
1962	June 18. First archived letter from Kneebone to Lakatos. Lakatos is to chair for and open the discussion for Kneebone “on Friday”. Kneebone writes a few comments on Lakatos’s Joint Session piece.
1962	July 14. Lakatos gives a Symposium on “Foundations of Mathematics” at the Joint Session with R. L. Goodstein.
1962	July 15. Lakatos’s ‘Infinite Regress and the Foundations of Mathematics’ appears in the *Aristotelian Society Supplementary Volume*.

1963	Kneebone publishes his book *Mathematical Logic and the Foundations of Mathematics*
1963-4	Lakatos publishes “Proofs and Refutations I-IV” in the BJPS.
1965	July 14. Kneebone chairs Kalmar’s presentation at Lakatos’ *International Colloquium in the philosophy of Science* (July 11-17, London); (Lakatos 1967, VIII).
1966	Kneebone gives a Symposium at the Joint Session with J. L. Mackie on “Proof”.
1966	Lakatos presents his paper “Cauchy and the Continuum” in Hanover (though he had it accepted at the BJPS he withheld it, so it wasn’t published until 1978).
1966-69	Kneebone and Lakatos work together on teaching philosophy of mathematics, and are in regular correspondence; LSE archive, see section 3 for details.
1971	Kneebone gives a Symposium at the Joint Session with Cavendish on “The Use of Formal Logic”.
1974	Lakatos dies suddenly.
1976	Lakatos’s *Proofs and Refutations* published posthumously, edited by Worrall & Zahar.
2003	September 30. Kneebone dies.

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