On the Viability of Galilean Relationalism

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Abstract

I explore the viability of a Galilean relational theory of spacetime—a theory that includes among its stock of basic relations a three-place collinearity relation. Two formal results are established. First, I prove the existence of a class of dynamically possible models of Newtonian mechanics in which collinearity is uninstantiated. Second, I prove that the dynamical properties of Newtonian systems fail to supervene on their Galilean relations. On the basis of these two results, I argue that Galilean relational spacetime is too weak of a structure to support a relational interpretation of classical mechanics.

1 Introduction

Classical mechanics requires an absolute distinction between two different kinds of motion: inertial motion, which is both uniform and rectilinear, and accelerated motion, which is non-inertial motion. On the four-dimensional spacetime approach, this gets understood geometrically as a constraint on the structure of spacetime. In particular, it gets understood as the requirement that spacetime carry an affine connection. As far as mechanics goes, the function of an affine connection is to provide a 'standard of inertia'. It does so by determining a class of spacetime geodesics, each a straightest possible path through spacetime. With such structure in place, inertial motion becomes motion along a timelike geodesic and accelerated motion becomes deviation from a tangent geodesic.

Surprisingly, a classical connection needn't be flat: classical mechanics can be set in either a flat Galilean spacetime or in a variably curved Newton-Cartan spacetime. In the first case, inertial frames of reference are globally well-defined; in the second, inertial frames of reference are locally well-defined. But flat or curved, in either case the geometry can be said to support an absolute distinction between accelerated motion and inertial motion.[1]

The demand for affine structure has consequences for our theorizing about the ontology of spacetime. Suppose that Newton's laws are true and (to pick a geometry) suppose that Galilean spacetime accurately represents the structure of physical spacetime, so that physical spacetime can be said to instantiate a Galilean geometry. Then ask: in what does such structure inhere? It has seemed to many that the only plausible candidate is a manifold of substantival points.[2] Thus, classical mechanics seems to demand, via the structure of Galilean spacetime, a substantival conception of spacetime—that is, a conception according to which spacetime is something ontologically distinct from whatever particles or fields happen to occupy its parts. Call this the 'argument from physical geometry'. In the context of classical mechanics, the argument from physical geometry is the chief argument for substantivalism.

One way to respond to the argument from physical geometry would be to propose an alternative spacetime framework—one avoiding the ontological costs of Galilean spacetime while at the same time still offering enough structure for classical mechanics. And one way to do this would be to (1) propose a spacetime framework involving only spatiotemporal relations among material bodies, avoiding all reference (whether direct or indirect) to spacetime itself, and then (2) show that this framework is rich enough to support dynamical laws of motion capable of capturing the content of their Newtonian counterparts. A spacetime framework that satisfies (2), I will say, is 'complete' with respect to classical Newtonian mechanics. A framework that satisfies (1) as well, I will say, constitutes a 'relational interpretation' of classical Newtonian mechanics.

One approach along these lines is Galilean relationalism. Since an affine connection on a manifold is uniquely determined by its class of geodesics, and since the class of geodesics in a manifold can be picked out with the aid of a three-place collinearity relation, a natural option to consider is a relational theory of spacetime that includes among its stock of basic relations a collinearity relation. Such a theory was first described by Tim Maudlin in his ([1993]); important discussions of the view include (Huggett [1999]) and (Pooley [2013]).

[1] On the spacetime structure of classical mechanics, see (Friedman [1983]), (Earman [1989]), (Malament [2012]), or (Pooley [2013]). See (Knox [2014]) for an argument for the claim that Newton-Cartan spacetime constitutes the proper spacetime setting for Newtonian gravity. Not everyone agrees that Newton-Cartan spacetime supports an absolute distinction between accelerated motion and inertial motion. For an argument for the claim that it doesn't, see (Norton [1995]). For an explanation of why it does, see (Friedman [1983], pp. 97-99). Not everyone agrees that classical mechanics requires an absolute distinction between inertial and accelerated motion. See (Saunders [2013]) for an argument for the claim that the proper spacetime setting for classical mechanics is Maxwellian spacetime, which includes a standard of absolute rotation, but no general standard of absolute acceleration. In response, see (Weatherall [forthcoming]) where it is argued that once you set Newtonian gravity in Maxwellian spacetime the natural result is a Newton-Cartan spacetime.

[2] The classic statement of this argument is (Earman [1989], p. 125).

My aim in this paper is to show that Galilean relational spacetime constitutes too weak of a structure for classical mechanics. The problem is that the dynamical properties of Newtonian systems fail to supervene on their Galilean relations. As I explain below, this has consequences for the possibility of Galilean dynamical laws: it means that Galilean relationalism is incomplete.

The plan for the paper is as follows. Sections 2 through 4 provide all the necessary background and context. I define 'completeness' in section 2, review some of the traditional problems facing relationalism in section 3, and introduce Galilean relationalism in section 4. Then comes the main argument. In section 5, I prove that a two-body system moving under an inverse cube law will fail to instantiate collinearity. In section 6, I apply this result in a proof of the claim that the dynamical properties of Newtonian systems fail to supervene on their Galilean relations. Finally, in section 7, I consider the consequences of this for the possibility of Galilean dynamical laws. A brief conclusion is offered in section 8.

2 Completeness

Anyone hoping to provide an alternative spacetime framework for classical Newtonian mechanics has a job to do: she must prove, at a minimum, that her theory is 'complete'. My aim in this section is to make this notion of completeness precise. I will define the term twice, first for theories of motion, and then for theories of spacetime. My first definition will be primary, my second derivative.

To start, we distinguish between relational theories of motion and relational theories of spacetime. Call the path of a point-sized material object in spacetime a 'worldline'. Call a point along the worldline of a material object a 'stage point'. Then a relational theory of spacetime is a theory according to which the structure of spacetime is represented by a set of models of the form $\langle D, \{R_i\}\rangle$, where D is a set of stage points and $\{R_i\}$ is a set of spatiotemporal relations on D. This set of models constitutes a set of kinematically, or geometrically, possible models.[3] A relational theory of motion, then, is a theory of motion whose laws are expressible in terms of the relations $\{R_i\}$, including nth-order derivatives of R_i (where these are well-defined). The function of the laws is to pick out a subset of dynamically possible models from within the theory's set of kinematically possible models. Each dynamically possible model, then, is understood to represent a nomologically possible state of affairs.

Intuitively, a relational theory of motion is complete with respect to some particular Newtonian dynamical theory of motion just in case it captures the

[3] It is an interesting question how the relationalist ought to understand this modality. See (Belot [2012]) for a discussion of the issue as it arises in connection with the geometry of space.

content of that theory, where the content of a dynamical theory of motion is understood to be identifiable with its set of dynamically possible models. Thus, a relational theory of motion RT will capture the content of a Newtonian theory of motion NT just in case RT's laws determine a set of dynamical models that 'matches up' with the set of dynamical models of NT. Matching up can then be understood in terms of the existence of a bijective function between sets of models that preserves physically relevant properties and relations, so that models that we might intuitively recognize as having the same physical content get paired up.[4] Here, then, is my first definition of completeness.[5]

> Completeness (Relational Theories of Motion): Let RT be a relational theory of motion. Let $\mathcal{R}$ be the set of dynamical models determined by the laws of RT. Let NT be some particular Newtonian dynamical theory of motion (e.g. Newtonian gravity). Let $\mathcal{N}$ be the set of dynamical models determined by the laws of NT. Assume that members of $\mathcal{N}$ are characterized up to diffeomorphism. Then RT is 'complete' with respect to NT just in case there exists a bijective function $\varphi : \mathcal{R} \rightarrow \mathcal{N}$ such that,
>
> (i) for each model $m \in \mathcal{R}$, m and $\varphi(m)$ represent the same number of material objects, and
>
> (ii) φ preserves the masses and charges of those objects, as well as the spatiotemporal relations between them.[6]

There are a couple of points here worth emphasizing. First, my definition requires that $\mathcal{R}$ be a set of models of some relational theory, which I take to mean that $\mathcal{R}$ is determined by some set of relational laws. It would be uninteresting if one could prove the existence of some miscellaneous set of models that matches up with the models of NT. In many cases, it is trivial that such a set exists—simply start with NT and use φ to build $\mathcal{R}$. The interesting question

[4]There is an interesting question whether spacetime models related by a diffeomorphism represent the same possibility or not. Nothing in this paper will hinge on this question. We can say either, so let's pick one: all models in this paper will be characterized up to diffeomorphism. In particular, Newtonian models that differ by a symmetry of the laws will be identified. This will allow us to think in terms of bijective maps between sets of models. Note that this is out of step with the more usual approach according to which the relation between Newtonian models and relational models is many-one.

[5]A similar notion can be found in (Huggett [1999]) and in (Skow [2007]), as well as in Oliver Pooley's remark that, for a particular kind of relational theory, 'the relationalist succeeds so long as they can identify, in a relationally respectable manner, a set of relational DPMs that correspond to the full set of Newtonian DPMs' ([2013], 547). I thank an anonymous referee for suggestions leading to improvements in the final formulation of this definition.

[6]Let a and b be (representations of) material objects in model m. Then φ preserves masses, charges, and relations just in case for each pair of models $\langle m, \varphi(m)\rangle$ there exists a bijective function f from the set of material objects in m to the set of material objects in $\varphi(m)$ such that (i) a and $f(a)$ have the same mass and charge and (ii) if Rab then $Rf(a)f(b)$.

is whether there exists a relational theory capable of doing the work of NT. My definition speaks to this. Second, given the constraints that I have put on φ, each element $m \in \mathcal{R}$ represents a 'dynamically possible relational history'. Indeed, doubly so—each element of $\mathcal{R}$ is dynamically possible with respect to the laws of NT as well as with respect to the laws of RT. Finally, although I require that $\mathcal{R}$ be determined by some set of relational laws, I do not require that those laws look anything like their Newtonian counterparts. Of course, one way to capture the content of a Newtonian theory would be to capture that theory's laws—in other words, to provide something like a relational foundation for its laws (or a relational interpretation of its laws, or a relational understanding of its laws). But my definition requires nothing like this; it is at least open to the possibility of a relational theory whose laws differ from the laws with which we are familiar from Newtonian mechanics.

A relational theory of motion needn't be complete to be interesting. Julian Barbour and Bruno Bertotti have developed relational theories of motion which are incomplete but which are just as empirically adequate as their Newtonian counterparts.[7] Of course, this makes their theory extremely interesting. My main interest, however, is in whether classical mechanics admits a relational interpretation, and the theory developed by Barbour and Bertotti does not constitute a relational interpretation of classical mechanics—theirs is a rival theory representing new physics. Fans of their approach are welcome to take my arguments against Galilean relationalism as further, indirect evidence for the need for new physics.

So far, I have defined completeness for relational theories of motion. Having done so, a definition for relational theories of spacetime is not far off. The purpose of a relational theory of spacetime is to provide a framework for relational theories of motion. One can always ask, then, whether a proposed framework is rich enough for its intended purpose. And in particular, one can ask whether a proposed framework is rich enough to support relational theories of motion that are complete with respect to their Newtonian counterparts. If it is, then we will say that the theory is complete with respect to classical Newtonian mechanics.

> Completeness (Relational Theories of Spacetime): Let S be a relational theory of spacetime according to which the structure of spacetime is faithfully represented by a set of models of the form $\langle D, \{R_i\}\rangle$. Then S is 'complete' with respect to classical Newtonian mechanics just in case for any Newtonian dynamical theory of motion NT, there exists some relational theory of motion RT such that,
>
> (i) RT's laws are expressed entirely in terms of the relations $\{R_i\}$, and
>
> (ii) RT is complete with respect to NT.

[7] See (Barbour & Bertotti [1982]). See also (Barbour [2001]).

Galilean relationalism is a relational theory of spacetime. So when I ask whether Galilean relationalism is complete with respect to classical Newtonian mechanics, it is this second definition that I have in mind.

3 Leibnizian Relationalism

Relational theories of spacetime differ in terms of the relations they allow. The traditional view, Leibnizian relationalism, allows just two relations: a two-place relation of temporal separation $t(x, y)$ defined for any two stage points whatsoever, and a two-place relation of spatial separation $r(x, y)$ defined for simultaneous stage points, where x and y are simultaneous just in case $t(x, y) = 0$. A brief survey of some of the problems facing this view will help to set up the discussion of Galilean relationalism below.

The chief problem with Leibnizian relationalism is that it lacks sufficient structure to support an absolute distinction between accelerated motion and inertial motion. The problem is clearest around the physics of rotation. And here the problem can be developed in different directions. In one direction, one can argue that because it lacks the resources to distinguish between accelerated motion and inertial motion, the Leibnizian framework lacks the resources necessary to account for the inertial effects exhibited by rotating systems. The standard example here is a pair of rigid globes strung together with a rigid cord. Newton's theory recognizes many distinct dynamic possibilities for such a system. In particular, the system may be uniformly rotating about its center of mass, with each rate of rotation corresponding to a unique amount of tension in the cord. The problem is that the Leibnizian cannot distinguish between these different states of rotation—rigid, global rotations fail to supervene on their Leibnizian relations. Consequently, the Leibnizian lacks the tools to explain the presence and amount of tension in the cord.[8]

In another direction, one can argue that because the Leibnizian lacks sufficient structure to support an absolute distinction between accelerated motion and inertial motion, a Leibnizian theory of motion that captures the content of Newton's theory will lack a well-posed initial value problem. To see the problem, consider a pair of equal mass point particles moving under Newtonian gravity. Here, the problem is that we can generate different future motions from the

[8]See (Maudlin [1993]) and (Huggett [1999]). At times (e.g. ([1999], pp. 20-23)), Huggett seems to attribute the above 'globes argument' to Newton himself, who discusses a rigid globes system in the *Principia*. See (Rynasiewicz [1995]) for a more historically accurate reading of this part of the *Principia*. It is not always clear how seriously to take the globes argument. It might be felt, for example, that rigid systems are too 'unphysical' to take seriously. In response, one might point to the fact that rigid systems are compatible with Newton's laws, and then insist that the relationalist contend with all systems compatible with Newton's laws. In any case, none of the arguments in this paper will rely upon the existence of rigid bodies.

same Leibnizian relational initial state by varying the system's rate of rotation. If the particles start off one meter apart with zero relative velocity and zero rotation, then eventually the system will collapse and the particles will collide. But if the rotation of the system is set just so, then the system will remain stable and the particles will maintain a relative distance of one meter.

In either direction, one is able to generate problems for the Leibnizian by playing with a system's rate of rotation, its total angular momentum. If a relational theory of spacetime is going to capture the content of Newton's theory, then it is going to need to make angular momentum relationally tractable.

4 Galilean Relationalism

Since the problem with Leibnizian relationalism comes down to a lack of structure, a natural response is to help oneself to more relations.[9] This is the strategy behind the move to Galilean relationalism. Since the structure that we need is affine structure, and since the function of an affine connection is to pick out a class of geodesics, a natural relation to try is a three-place collinearity relation (Maudlin [1993], pp. 193-4). Thus, the Galilean relationalist accepts all of the same relations that the Leibnizian accepts, plus one more: a three-place relation $col(x, y, z)$ that holds for stage points x, y, and z just in case x, y, and z lie along a spacetime geodesic.[10] The hope is that this expanded set of resources will provide a spacetime framework rich enough for classical mechanics by allowing the relationalist to pick out a class of inertial trajectories so that accelerations in general, and rotations in particular, are well-defined—an acceleration being, roughly, a departure from an inertial trajectory. In this respect, the collinearity relation is meant to mimic, as far as possible, the affine structure of a classical spacetime. In keeping with her ontological commitments, however, the relationalist is forced to limit the domain of *col* to the set of stage points at a world.

Certainly, the collinearity relation allows the Galilean relationalist to draw distinctions that the Leibnizian cannot. For starters, for any model containing an inertial body α, the Galilean relationalist will have no trouble identifying it as such—α is inertial iff for any triple of points x, y, and z along its worldline, $col(x, y, z)$. If enough such bodies exist, then the Galilean relationalist will be able to construct an inertial frame of reference. And if spacetime is full of matter, then the Galilean relationalist will have access to a spacetime structure that is

[9] See (Huggett, [1999]) and (Pooley, [2013]) for discussion of other strategies.

[10] I assume that x, y, and z are ordered so that $col(x, y, z)$ entails that y is between x and z. When Maudlin introduces the relation, he restricts its domain to non-simultaneous points. But you can shed the restriction: if x, y, and z are simultaneous, then $col(x, y, z)$ iff $r(x, y) + r(y, z) = r(x, z)$. See (Pooley [2013], note 52).

every bit as strong as the structure of Galilean spacetime (or even Newton-Cartan spacetime).[11]

Things get more complicated when we start considering models in which there exist no inertial bodies. Consider, for example, a pair of particles in circular orbit about their common center of mass. Suppose that the system rotates at a rate of one revolution per second. On the one hand, the system's rate of rotation is uniquely determined by its Galilean relations. In this case, we have that for every point x there exist points y and z such that $col(x, y, z)$ with $t(x, z) = 1$, and that there exist no u and v such that $col(x, u, v)$ with $t(x, v) < 1$. On the other hand, as Tim Maudlin ([1993], p. 194) has pointed out, the Galilean relations instantiated by a system such as this will fail to determine a unique embedding into Galilean spacetime. The problem is that there are myriad different ways for a system to rotate at a rate of one revolution per second—from simple, uniform rotation to any one of many different complex, non-uniform rotations—and no two of these will be distinguishable in terms of their Galilean relations.

On its face, this looks like an analog of the globes argument: it looks like the dynamical properties of Newtonian systems fail to supervene on their Galilean relations. But as Nick Huggett ([1999], p. 22) has stressed, of the many different ways in which a system can rotate, in the case of an isolated system only uniform rotation is dynamically possible. Models representing non-uniform rotation may be kinematically possible, but such models will run foul of the law of conservation of angular momentum. Consequently, though the Galilean relations associated with our two-particle system will fail to pick out a unique model from among a set of kinematically possible models, they will succeed in picking out a unique model from among a set of dynamically possible models (assuming, as we are throughout, that Newtonian models are characterized up to diffeomorphism).

All this prompts Huggett to ask a more focused question: do the dynamical properties of *dynamically possible* Newtonian systems supervene on their Galilean relations? Ultimately, he leaves the question open, adding that 'I have been able neither to prove that they do, nor find models which demonstrate a failure of supervenience…' ([1999], p. 25). But despite this, he is mildly optimistic about the view, and his final assessment is that '…Galilean relationalism still seems a viable option…' ([1999], p. 26).[12]

I will answer Huggett's question in section 6: the dynamical properties of dynamically possible Newtonian systems do not supervene on their Galilean relations. Before I do though, I want to comment quickly on Huggett's question.

[11] This follows from the fact that the class of geodesics in a manifold determines a unique connection on the manifold. See (Malament [2012], prop. 1.7.8).

[12] Not that Huggett endorses the view. His preferred response to the argument from physical geometry is presented in his ([2006]).

Huggett's question might give the impression that the chief issue at stake in the substantival-relational debate is an issue of supervenience. But I do not think that this is the right way to frame the debate. Really, what we want are relational laws of motion, and establishing that the dynamical properties of Newtonian systems supervene (in this case) on their Galilean relations does not guarantee Galilean laws.[13] That said, Huggett's question is still worth pursuing: though establishing supervenience might not convince us of the possibility of Galilean laws, establishing a failure of supervenience might convince us of the impossibility of Galilean laws. After all, if the dynamical properties of Newtonian systems fail to supervene on their Galilean relations—if the dynamical properties of Newtonian systems can vary independent of their Galilean relations—then it is difficult to see that the Galilean is in any position to provide a lawful treatment of those properties. Or at least, this is how I will argue in section 7. But first, we need to see how supervenience fails.

5 Inverse Cube Force Laws

The point in introducing the collinearity relation is that it is supposed to enable an analysis of absolute acceleration in general, and absolute rotation in particular. But I doubt that any such analysis is possible. The problem is that there exist dynamically possible models of Newtonian mechanics in which collinearity is simply uninstantiated. In this section, I prove that a two-body system moving under an inverse cube law will, if set up a certain way, fail to instantiate collinearity. I prove it first for systems with zero angular momentum and then for systems with non-zero angular momentum. Later, in section 6, I prove that this same class of models witnesses a failure of supervenience.

5.1 Warm-up

Let me start with a rough, qualitative description of the kinds of models that I want to talk about. Model 1 will consist of a pair of point-sized particles of equal mass and equal charge in four-dimensional Galilean spacetime. In the center of mass frame, the setup is as follows:

> **Initial Positions**. At $t = -\infty$, draw a spacelike line l. Place both particles on the line l, one at $+\infty$ and the other at $-\infty$.

[13] Compare, for example, (Skow [2007]), where it is argued that although the dynamical properties of Newtonian systems may supervene on their Leibnizian relations plus a distribution of 'sklarations' (primitive, intrinsic properties of bodies which are supposed to correspond to Newtonian absolute accelerations), there can be no relational dynamical laws of motion involving sklarations.

> **Initial Velocities**. At $t = -\infty$, put the particles in motion so that they have the same initial speed, move along the line l, move in directions opposite one another, and move in such a way that the distance between them is initially decreasing.
>
> **Dynamics**. Suppose that the system moves under an inverse cube law with potential
> $$U(r) = \frac{kq_1q_2}{r^2}$$
> where q_1 is the charge on particle 1, q_2 is the charge on particle 2, r is the relative distance between the particles, and k is a positive constant. Since we are supposing particles of equal charge, $U(r)$ indicates a repulsive force.

At this point, I am allowing myself certain liberties. Technically, it makes no sense to talk about initial conditions at infinity. But the goal right now is just a qualitative description, and it simplifies things to imagine the particles approaching from infinity. In the next section, I will make it all precise. For now, it is enough to observe that the particles in model 1 will start off moving in an approximately inertial fashion. As they draw near one another, the repulsive force between them will build. Eventually, the particles will slow, stop, and turn, before moving back out to spatial infinity. Given our setup, all motion will take place along the line l. Consequently, there is no angular momentum in model 1.

Model 2 is similar to model 1. The basic setup is the same: a pair of point-sized particles of equal mass and equal charge in four-dimensional Galilean spacetime. The dynamics, too, are the same. The initial conditions, however, are slightly different.

> **Initial Positions**. At $t = -\infty$, draw three parallel, spacelike, coplanar lines, l, m, and n, with l in the middle and m and n equidistant from l. Place one particle at $+\infty$ on m and the other at $-\infty$ on n. See Figure 1.
>
> **Initial Velocities**. At $t = -\infty$, put the particles in motion so that they have the same initial speed, move along lines parallel to l, move in directions opposite one another, and move in such a way that the distance between them is initially decreasing.

Here, too, the particles will start off moving in an approximately inertial fashion. Here, too, as they draw close the force between them will build. In this case, however, the particles will 'scatter' one another. Since the system's motion in this case is not along a fixed line, there is angular momentum in model 2.

My claim is that neither model 1 nor model 2 instantiates collinearity. To prove it, I need to recast each model in more formal terms.

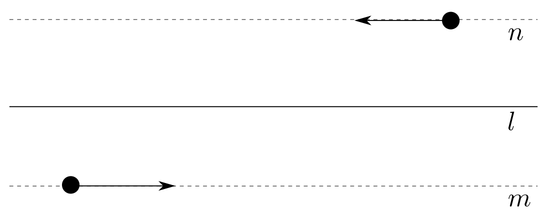

Figure 1: The initial conditions of model 2.

5.2 Model 1: zero angular momentum

Model 1 is a two-body system in four-dimensional Galilean spacetime. The physics here is familiar, and my analysis will follow standard presentations.[14] What will be unique in my discussion is just the fact that I am working with an inverse cube force law.

Consider, then, a two-body system consisting of particles 1 and 2 with masses m_1 and m_2 and charges q_1 and q_2. Suppose that $m_1 = m_2$, that $q_1 = q_2$, and that their interaction is via a central force depending only upon the relative distance r between the particles. In the center of mass frame all motion will take place in a plane and the Lagrangian can be written

$$\mathscr{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

where

$$m = \frac{m_1 m_2}{m_1 + m_2}.$$

Here we have reduced our two-body problem to an equivalent one-body problem: remarkably, everything that we wish to know about the motion of 1 and 2 can be worked out by studying the motion of a fictitious particle with mass m in a potential $U(r)$, where r is now the distance from the center of the potential to m.

Running $\mathscr{L}$ through the Euler-Lagrange equations for r gives us the equation of motion

$$m\ddot{r} - \frac{l^2}{mr^3} + \frac{\partial U}{\partial r} = 0$$

where we have used the fact that $mr^2\dot{\theta} = l$ to write everything in terms of r. Here l represents the angular momentum of the system, a constant of motion.

The relative motion of a system such as this is described by the equation

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r) - \frac{l^2}{2mr^2})}}$$

[14]See, for example, (Goldstein [2001]) or (Thornton and Marion [2003]).

where E is the system's total energy. Where there is no angular momentum, this reduces to

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r))}}.$$

Of course, we are supposing that our particles repel one another via an inverse-cube law with potential $U(r) = kq_1q_2r^{-2}$. Thus,

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}(E - \frac{kq_1q_2}{r^2})}}.$$

There are several constants of motion packed into this equation: E, k, q, and m. To simplify, we set E and k equal to one. We set the individual masses of the particles equal to one, from which it follows that m equals one-half. And we set the charge of each particle equal to the square root of two. This leaves us with

$$t = \int \frac{dr}{\sqrt{4 - \frac{8}{r^2}}}$$

from which it follows that

$$t(r) = \sqrt{\frac{r^2}{4} - \frac{1}{2}}.$$

This tells us how the relative distance between the particles changes with time. To get the actual worldlines of the particles, we need two lines whose separation at each time is given by $t(r)$. Given the symmetry of the problem, this is easy enough to do—the worldlines will be given by the function $t(2x)$. See figure 2.

To prove, now, that collinearity is uninstantiated, we need only note that

$$t(2x) = \sqrt{\frac{(2x)^2}{4} - \frac{1}{2}}$$

is equivalent to

$$2x^2 - 2t^2 = 1.$$

This, it will be recognized, is the equation of a hyperbola. Since no line intersects a hyperbola more than twice, it follows that collinearity is uninstantiated in model 1.

5.3 Model 2: non-zero angular momentum

Model 2 is similar to model 1 except that there is angular momentum in model 2. This makes it more difficult to prove that collinearity is uninstantiated since the particle worldlines are multivariable functions $t(r, \theta)$. Fortunately, it turns out that there is no need to look at the particle worldlines—we can prove that

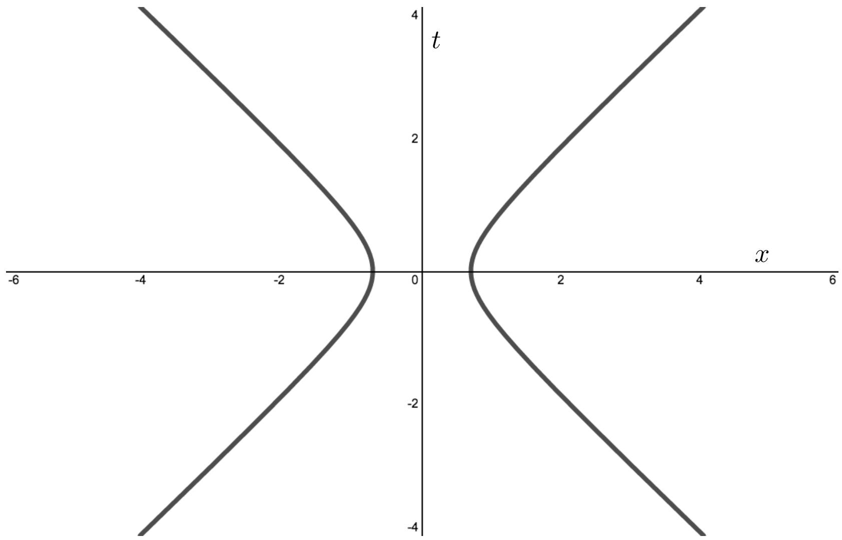

Figure 2: The worldlines of the particles in model 1.

collinearity is uninstantiated through an analysis of the spatial paths of the particles.

It is well-established, if not well-known, that the spatial motion of a particle under an inverse cube law will be along a curve known as a 'Cotes spiral'.[15] There are three different such spirals: the epispiral, the Poinsot spiral, and the hyperbolic spiral. Given our setup, the spatial motion of the particles will be along an epispiral, which is given by the equation

$$r = \frac{r_0}{2} \sec\left(\frac{\theta - \theta_0}{\mu}\right)$$

where r_0 is the distance of closest approach, θ_0 is the angle of closest approach, and $\mu = (1 - mkl^{-2})^{1/2}$. Once again, we set m equal to one-half and k equal to one. This gives us

$$\mu = \sqrt{1 - \frac{1}{2l^2}}.$$

As we dial up the angular momentum in our system, μ will go to 1, $\varphi = 180/\mu$ will go to 180°, and the equation of the epispiral will approach the equation of a line. (In polar coordinates, $r = a\sec(\theta - \theta_0)$ is the equation of a line). Conversely, as we lower the amount of angular momentum in the system (being careful not to go below $\sqrt{mk}$, at which point we get a negative under the radical), φ will go to 0 and we will approach the case of one-dimensional spatial motion. For the special case in which $l = \sqrt{2k/3}$, we get the paths in figure 3. Here we

[15]See, for example, either (Whittaker [1944]) or (Danby [1988]).

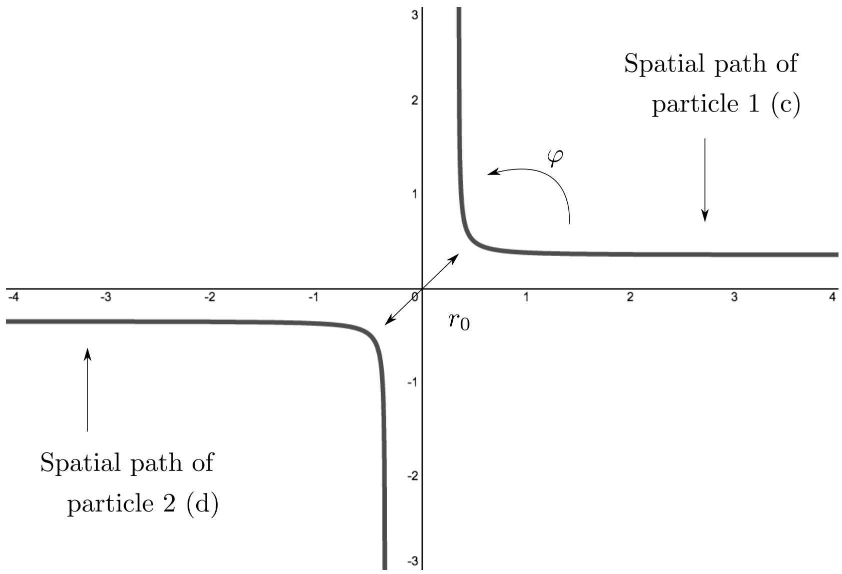

Figure 3: The spatial paths of the particles in model 2.

can imagine particle 1 moving along the path in quadrant 1 (the upper right-hand quadrant of the plane) and particle 2 moving along the path in quadrant 3 (the lower left-hand quadrant of the plane).

Now that we know what the spatial motion of 1 and 2 look like, we can see that collinearity is uninstantiated in model 2. Here is a nice way to see it. Let c denote the spatial path of particle 1. Let d denote the spatial path of particle 2. Let S denote the set of lines that intersect c. First observation: no line in S intersects c more than twice.[16] This, in conjunction with the fact that c does not cross itself, entails that the worldline of particle 1 fails to instantiate collinearity. By symmetry, then, the worldline of particle 2 also fails to instantiate collinearity. Second observation: no line in S intersects d.[17] But then no line containing two points along the worldline of particle 1 will intersect the worldline of particle 2 (and vice versa). This is a straightforward consequence of the fact that you cannot intersect a worldline without passing

[16]**Proof**. The parametric form of an epispiral is $[x(t), y(t)] = [n \cos t \csc at, n \csc at \sin t]$ where the parameter t is an angular measure. The curve c is defined over the interval $0 < t < 90$. It is possible to show that the second derivatives $x''(t)$ and $y''(t)$ are both positive on $0 < t < 90$. This entails that c is everywhere concave up, and from this it follows that no line in S intersects c more than twice.

[17]**Proof**. The curve c is bound by asymptotes at $x = \sqrt{2}/4$ and $y = \sqrt{2}/4$. The curve d is bound by asymptotes at $x = -\sqrt{2}/4$ and $y = -\sqrt{2}/4$. This, in conjunction with the fact that c is everywhere concave up and d is everywhere concave down entails that no line in S intersects d.

through a spatial point along that line. But then neither the worldline of 1, nor the worldline of 2, nor the worldlines of 1 and 2 together instantiate collinearity. It follows that collinearity is uninstantiated in model 2.[18]

Let's take stock. We have proven that a two-body system moving under an inverse cube law, if set up in the way described, will fail to instantiate collinearity. This is true both in the case where the system rotates as well as in the case where it does not. If the purpose of the collinearity relation was to provide a standard of inertia and to support a distinction between systems with angular momentum and systems without, then here is a class of models in which we are missing precisely the tool that we are alleged to need.

6 Supervenience

Nick Huggett has asked ([1999], p. 25) whether the dynamical properties of dynamically possible Newtonian systems supervene on their Galilean relations. They do not. In this section, I prove the existence of a class of dynamically possible models that differ in angular momentum but that agree with respect to their Galilean relations.

In order to prove a failure of supervenience, we need two dynamically possible models that agree with respect to their Galilean relations. To start, let's first build a pair of models that agree with respect to their Leibnizian relations. Go back to the equation of motion

$$m\ddot{r} - \frac{l^2}{mr^3} + \frac{\partial U}{\partial r} = 0.$$

It turns out that there is an especially nice way to read this equation. If we let

$$U'(r) = \frac{l^2}{2mr^2} + U(r)$$

then we can write

$$m\ddot{r} + \frac{\partial U'}{\partial r} = 0,$$

and this will allow us to treat the motion of the fictitious particle m as if it were determined by a potential $U'(r)$. This is the system's 'effective potential'. In our case, the effective potential is

$$U'(r) = \frac{l^2}{2mr^2} + \frac{kq_1q_2}{r^2}.$$

Here is why this is relevant: in the case of a two-body system, the relative motion of the system is completely determined by the system's effective potential. That is to say, the effective potential of a two-body system will completely

[18] It is worth adding that this proof holds for all allowed values of l, not just $l = \sqrt{2k/3}$.

determine its history of Leibnizian relations. Consequently, if our aim is to build a pair of models with the same history of Leibnizian relations, then what we need is a pair of models with the same effective potential.

Start, then, with model 1. Let p_1 and p_2 denote the charges of the particles in model 1. Since there is no angular momentum in model 1, the effective potential for model 1 is $U_1' = kp_1p_2/r^2$. Now model 2. Let q_1 and q_2 denote the charges of the particles in model 2. Since there is angular momentum in model 2, the effective potential for model 2 is $U_2' = l^2/2mr^2 + kq_1q_2/r^2$. Now set

$$p_1 = p_2 = \sqrt{\frac{l^2}{2mk} + q_1q_2}.$$

Doing so will ensure that $U_1' = U_2'$ and that the relative spatial separations of the particles in model 1 are, at each time, identical to the relative spatial separations of the particles in model 2.

Our trick, then, is to build the angular momentum of model 2, which is just a constant value, into the charges of the particles in model 1. That this is possible to do is highly nontrivial. In fact, it is possible only with an inverse cube force law—the same trick will not work, for example, with an inverse square law. To see why, consider a two-body system with angular momentum $l \neq 0$ moving under an inverse-square law with potential $kq_1q_2r^{-1}$. In this case, the effective potential is $U_1' = l^2/2mr^2 + kq_1q_2r^{-1}$. Consider now a second, similar system where $l = 0$. In this second case, the effective potential is $U_2' = kp_1p_2r^{-1}$. If, now, we set $U_1' = U_2'$ and solve for p_1, we get that $p_1 = (l^2/2mkr + q_1q_2)^{1/2}$. And now we have a problem—the r on the right hand side of this last equation tells us that in order to get the relational histories of these two models to match, the charge on 1 (and by symmetry the charge on 2) will have to vary with changes in r. But this will violate charge conservation, and so such models will fail to represent genuine dynamic possibilities.

I claim that the class of models described in this section confirms that the dynamical properties of Newtonian systems fail to supervene on their Galilean relations. To be concrete, consider the following pair of models.[19]

Model 1. Set $l = 0$ and $p_1 = p_2 = \sqrt{2}$

Model 2. Set $l = 1$ and $q_1 = q_2 = 1$

First, models 1 and 2 agree with respect to their t-relations: for any pair of stage points x and y in model 1 there exist stage points u and v in model 2 such that $t_1(x, y) = t_2(u, v)$. Second, models 1 and 2 agree with respect to their r-relations: by construction, $U_1' = U_2'$ and so $r_1(t) = r_2(t)$. Third, models 1 and 2 agree with respect to their *col*-relations: because these are inverse cube

[19]Just to be clear, here and throughout I am assuming that the total energy E is equal to one.

models, *col* is uninstantiated. It follows that models 1 and 2 agree with respect to their Galilean relations. But, of course, models 1 and 2 disagree with respect to their dynamical properties insofar as there is angular momentum in one but not the other. It follows, then, that the dynamical properties of Newtonian systems fail to supervene on their Galilean relations.

Does it follow straightaway that Galilean relationalism is incomplete with respect to classical Newtonian mechanics? No—not without further argument. The argument in this section does not present us with a perfect analog to the globes argument. A perfect analog would show that the dynamical properties of Newtonian systems fail to supervene on their relational histories, where such histories include information concerning relations *plus* information concerning the intrinsic properties of the bodies in the system. The argument in this section does not show this—it shows a failure of supervenience on relations alone: though models 1 and 2 agree with respect to their Galilean relations, they differ with respect to their charge values. Consequently, the Galilean is able to mark a distinction between models 1 and 2. In the next section, I argue that this is of no help: the distinction marked is the wrong kind of distinction.

7 Incompleteness

Earlier, I claimed that anyone hoping to push Galilean relationalism as an alternative spacetime framework must show that their view is complete with respect to classical Newtonian mechanics. So let's ask: Is Galilean relationalism complete with respect to classical Newtonian mechanics?

There are at least two different ways to argue for incompleteness. The first is to argue for the nonexistence of a bijection of the sort required by our first definition of completeness. This is the respect in which Barbour and Bertotti's theory is incomplete. But one can argue like this only if the theory under consideration posits a set of relational laws of motion. If it doesn't, then it makes no sense to ask about the set of histories permitted by the laws and whether they match up with the set of dynamical models of Newtonian mechanics.

The second way to argue for incompleteness is to *assume* the existence of a bijection and then argue that the resulting set of models is too miscellaneous to cover under a set of laws. This style of argument works best in situations like the one here, where what we have is a relational theory of spacetime but no relational laws of motion. In other words, we need something like the following approach. Start with a Newtonian dynamical theory of motion NT and consider the set $\mathcal{N}$ of its dynamical models. Next, assume the existence of a set of relational models $\mathcal{R}$ that matches up with $\mathcal{N}$ in the way required by our first definition of completeness. This set $\mathcal{R}$ comprises the relational histories allowed by NT. Now ask: do the elements of $\mathcal{R}$ instantiate sufficient law-like patterns and regularities so that the set can be covered under Galilean laws?

The reference here to patterns and regularities is important. Unless restrictions are put in place, covering a set of models with a set of laws is a trivial task—especially if, for example, disjunctive laws are permitted. But such laws hardly merit the title. Generally, we expect that physical laws will track natural patterns and regularities so that a broad set of phenomena are covered under a few simple rules. Consequently, when judging the viability of Galilean relationalism as a framework for dynamical laws of motion, what we ought to be looking for in $\mathcal{R}$ are robust and somewhat natural patterns or regularities.

I think that an argument of this second sort can be made against Galilean relationalism. Consider again the pair of models that I used to argue for a failure of supervenience:

Model 1. Set $l = 0$ and $q_1 = q_2 = \sqrt{2}$

Model 2. Set $l = 1$ and $q_1 = q_2 = 1$

Despite the fact that the particles in models 1 and 2 disagree with respect to charge, they instantiate the same Galilean relations. Consequently, the relative motions of the particles in model 1 are the same as the relative motions of the particles in model 2—in each case, the relative motion is just the motion depicted in figure 2. What is odd about such models is that they give the impression that charge is a dynamically idle quantity, having no influence upon the evolution of the system, for we have, in effect, dialed up the charge on the bodies from 1 to $\sqrt{2}$ with no effect on the particles' relative motion. And of course there are many more models that seem to confirm the same thing. Assuming that $m = 1/2$, $k = 1$, and that $q_1 = q_2$, any triple satisfying the equation $l^2 + q_1 q_2 = 2$ will produce a model relationally isomorphic to models 1 and 2. This means that if we assume the existence of a set $\mathcal{R}$ of relational histories matching up with the set $\mathcal{N}$ of dynamically possible models of our Newtonian theory, then within $\mathcal{R}$ will be an infinitely large subset of models suggesting that charge has no effect whatsoever upon the relative motion of the bodies in the system.

A natural response might be to deny that models 1 and 2 really differ with respect to charge. Empirically, we come to know the charge on a body by observing its motion. If models 1 and 2 agree on motion, then perhaps we ought to say that they agree on charge. The problem with this response is that it means giving up completeness; if we identify models 1 and 2 then we lose the bijection between $\mathcal{N}$ and $\mathcal{R}$.

A more interesting possibility might be for the Galilean to simply deny that charge exists. The Galilean can say that charge is something that Newtonians invoke in an effort to understand the motions of bodies, but that really there is no such property. This kind of response is not without precedent. There are other dynamically relevant variables that show up in Newtonian equations that

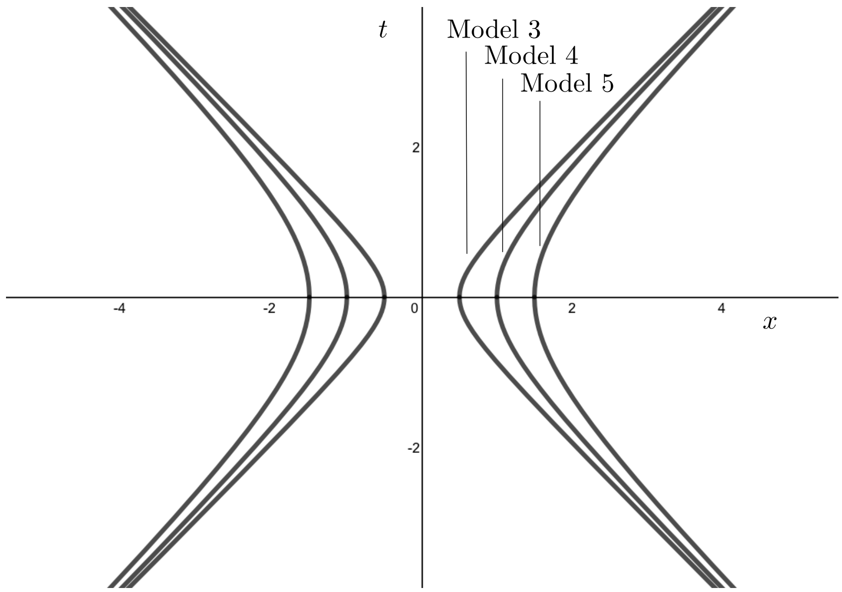

Figure 4: The worldlines of the particles in models 3-5.

relationalists sometimes ignore. Julian Barbour, for example, has argued that there is no property of global angular momentum. Of course, Newtonians talk about systems with global angular momentum, and they invoke the angular momentum of such systems to explain the motions of their parts, but really, says Barbour, there is no such thing. Perhaps Galilean relationalists ought to say something similar about charge.

The problem with this response is that within $\mathcal{R}$ is an infinitely large subset of models that suggest just the opposite—that charge does have an effect on the relative motions of bodies. Thus, consider the following group of models:

Model 3. Set $l = 0$ and $q_1 = q_2 = 1$

Model 4. Set $l = 0$ and $q_1 = q_2 = 2$

Model 5. Set $l = 0$ and $q_1 = q_2 = 3$

As figure 4 makes clear, this set of models points to a clear connection between charge and relative motion. Indeed, the connection in this case is describable in terms of $U(r) = kq_1q_2r^{-2}$.

So assuming the existence of a set $\mathcal{R}$ of relational histories matching up with the set $\mathcal{N}$ of Newtonian models of our electro-dynamical theory, we have that (i) $\mathcal{R}$ contains infinitely many models in which charge has no influence on motion, and (ii) $\mathcal{R}$ contains infinitely many models in which charge has a regular, law-like influence on motion. Taken together, we have a case for the claim that there exist no regular, law-like patterns among the models in $\mathcal{R}$. The problem is that the intrinsic properties of the bodies in these models have effectively been 'screened off' from their relative motions. This is why I say that the distinction that the Galilean is able to mark between, for example, models 1 and 2, is the wrong kind of difference. What she needs is to mark a spatiotemporal difference. It doesn't help her any that she is able to mark a difference in intrinsic properties.

It is pretty clear what is going on here. When working with an inverse cube law, we have an extra variable to play with that the Galilean relationalist cannot "see"—namely, total angular momentum. Part of the point in introducing the collinearity relation was that it was supposed to allow the relationalist to get a handle on angular momentum. What is interesting about the models that I have been discussing here, however, is that, once again, we are missing precisely the tool that we are alleged to need.

8 Conclusion

I have argued that the collinearity relation cannot be used to provide a standard of inertia; that it cannot be used to support an absolute distinction between accelerated motion and inertial motion. The problem is that there exist dynamically possible models of Newtonian mechanics in which the relation is simply uninstantiated. This is true, in particular, for a pair of charged particles moving under an inverse cube law. Of course, actual charged bodies move under an inverse square law. Why care, then, that Galilean relationalism founders on a non-actual theory?

I can think of at least two different ways to parse the worry expressed here. First, the worry might be that because we are working with a non-actual law, our models fail to represent genuine dynamic possibilities. The problem with this is that it misunderstands what it means to say that a state of affairs is dynamically possible. In Newtonian mechanics, a dynamically possible state of affairs is one that evolves in accordance with Newton's laws (and all that follow from them, including the conservation laws). My models do this. In particular, they evolve in accordance with $\vec{F} = m\vec{a}$. Of course, my models assume a non-actual force law. But here there are no grounds for objection: the class of Newtonian worlds is wider than the class of worlds whose force laws match those of the actual world (assuming for the moment that the actual world is Newtonian). And included within this class are worlds with all manner of force laws.

Second, the worry might be not that my models fail to represent genuine possibilities, but that the possibilities represented are remote, and so therefore uninteresting. You might think this if, for example, you thought that Galilean relationalism was a thesis about the structure of the actual world alone (again, assuming for the moment that the actual world is Newtonian). Were this the case, then it would do no good to show that the view founders on some non-actual theory. The problem with this worry, however, is that it misunderstands either Galilean relationalism or Newtonian mechanics or both. Galilean relationalism has been put forth as an alternative spacetime framework for Newtonian mechanics. Newtonian mechanics, in turn, is a meta-theory describing the relation between force and motion, but saying nothing about how forces arise between bodies. Galilean relationalism, then, ought to be judged, in part, according to whether it can capture the meta-theoretical character of Newton's theory. Of course, this is all built into my second definition of completeness, which requires that for every Newtonian theory there exist a proper relational counterpart.

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