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Spontaneous Breaking of Global Gauge Symmetries in the Higgs Mechanism

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Abstract

The Higgs mechanism is invoked to explain how gauge bosons can be massive while Yang-Mills theory describes only massless gauge fields. Central to it is the notion of spontaneous symmetry breaking (SSB), applied to the $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauge symmetry of the electroweak theory. However, over the past two decades, philosophers of physics have challenged the standard narrative of the Higgs mechanism as an instance of gauge symmetry breaking. They have pointed out the apparent contradiction between the status of gauge symmetries as mathematical redundancies and the account of mass generation in the Higgs mechanism by means of gauge symmetry breaking. In addition, they have pointed to Elitzur's theorem, a result from lattice gauge theory forbidding local gauge symmetry breaking. This has led philosophers to the conclusion that there cannot be any SSB in the Higgs mechanism, an idea supported by the dressing field method of gauge symmetry reduction. In this thesis we mitigate this conclusion by showing that global gauge symmetries, i.e. transformations independent of spacetime, are not mere mathematical redundancies but carry direct empirical significance. This can be seen from constrained Hamiltonian analysis by the fact that the Gauss constraint in Yang-Mills theory only generates gauge transformations which asymptotically become the identity. The classical Higgs mechanism can indeed be reformulated as a breaking of only this global gauge symmetry. We subsequently extend this result to quantum field theory by considering SSB in algebraic quantum field theory (AQFT). The Abelian $\mathrm{U}(1)$ Higgs mechanism can be shown to be an instance of SSB in the algebraic sense and we discuss the extent to which this can be generalised to the non-Abelian case. Finally we discuss the implications of our results for the interpretation of the electroweak phase transition and the analogy between the Higgs mechanism and superconductivity.

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Dedicated to Peter Ware Higgs, who passed away on 8 April 2024.
May this thesis serve as a living tribute to his memory.

Contents

1. Introduction

> Space is something absolutely uniform; and, without the things placed in it, one point of space does not absolutely differ in any respect whatsoever from another point of space. Now from hence it follows [...] that 'tis impossible there should be a reason, why God, preserving the same situations of bodies among themselves, should have placed them in space after one certain particular manner, and not otherwise; why everything was not placed the quite contrary way, for instance, by changing East into West.[1]

Symmetries are ubiquitous in modern physics. In particular, gauge symmetries play a central role in our understanding of particle physics and quantum field theory. An essential aspect of the role of symmetries in physics is the idea that they can be *broken*, in the sense that certain states of a physical system do not necessarily exhibit the same symmetry as the equations of motion describing that system. However, the notion of symmetry breaking seems inherently paradoxical: how can symmetric equations ever give rise to an asymmetric state? An intuition of this paradox is expressed in the above quote from 1716 by Leibniz, who wondered why God should place material bodies at any specific point in space, if every point is exactly like any other. In a similar vein, Pierre Curie announced a principle in 1894 stating that "when certain effects show a certain asymmetry, this asymmetry must be found in the causes which gave rise to it" [2]. At first sight, though, Curie's principle seems to fail in modern physics. A striking example of this is the phenomenon of *spontaneous symmetry breaking* (SSB) [3], in which a physical system supposedly evolves from a symmetric state to an asymmetric state *spontaneously*, i.e. by itself, without external stimulus. More specifically, SSB is usually applied to systems in which the ground state is not invariant under some symmetry transformation of the Lagrangian.

The concepts of symmetry and symmetry breaking have been extensively discussed by philosophers of physics since the turn of the century [4]. Part of this discussion has focused on gauge symmetries and the two best known examples of "gauge symmetry breaking": superconductivity and the Higgs mechanism. Conceptually speaking, gauge symmetry breaking is even more paradoxical than "regular" symmetry breaking, since, in addition to the questions surrounding Curie's principle alluded to above, there is the fact that gauge symmetries are often seen as expressions of mathematical redundancy or "descriptive fluff" [5]. As the standard narrative would have it, superconductivity and the Higgs mechanism are instances of spontaneous gauge symmetry breaking. But this immediately raises many questions: how can the breaking of a mathematical redundancy represent a physical process? And what would even be the cause of such a process? What is the gauge-invariant content of superconductivity and the Higgs mechanism?

These and many others are the issues that philosophers of physics have tried to clarify over the past two decades. While a general consensus has been reached that the textbook account of gauge symmetry breaking is conceptually inadequate and that it is impossible to break local gauge symmetries, many questions remain, and it is still not clear how the Higgs mechanism can really be seen as a *mechanism*, i.e. as a "a natural or established process by which something takes place or is brought about."[2]

In this thesis, we focus on the breaking of *global* (i.e. spacetime-independent) gauge symmetries, as we believe that this cuts rights to the core of many of the philosophical mysteries surrounding the Higgs mechanism and provides a unique, rigorous path towards a better conceptual understanding. The rest of this introduction will serve to explain why we should expect global gauge symmetries to play such a central role and to outline a roadmap for studying them.

[1]From Leibniz' third letter in his correspondence with Samuel Clarke [1].

[2]This definition of 'mechanism' was generously provided by Google, based on Oxford Languages.

1.1. History of the Higgs

It is useful to start with a short historical overview of superconductivity and the Higgs mechanism, for this way we can introduce many of the most important ideas in a logical order. Our historical preamble not only serves an introductory purpose though. It also allows us to better understand the analogy between superconductivity and the Higgs mechanism, which will be relevant for some of the later philosophical discussion. In addition, we can scan the history of the Higgs mechanism for hints from the founders as to how to interpret it. Lastly, by providing a historical overview of the origin of the Higgs mechanism, we can do justice to the physicists other than Peter Higgs himself who played a role in its inception.

1.1.1. History of superconductivity

Some of the most important events in the history of superconductivity occurred in Leiden in the Netherlands. In 1908, Heike Kamerlingh Onnes managed to cool down helium to the point that it liquefied. This was a great challenge, because helium boils at 4.2 K. Kamerlingh Onnes had reasons to investigate the behaviour of electrical resistance at very low temperatures, making use of his ultracold, liquid helium, because an important open problem of the time was the question of what would happen to the resistance of metals at absolute zero [6, p. 38]. It was known that electrons were responsible for electrical conductance, and it was known too that electrical resistance generally decreases as a metal is cooled down. Lord Kelvin, however, expected the electrical resistance of metals to reach a minimum at a certain temperature and then become infinite at very low temperatures, as electrons would then no longer be moving and become fixed in the metal in question [6, p. 38].

Thus, Kamerlingh Onnes conducted an experiment on 8 April 1911 in which he determined the electrical resistance of gold and mercury at extremely low temperatures. The experiment started at 07:00 and Kamerlingh Onnes himself arrived at 11:20, when the circulation of liquid helium began [6, p. 41]. At exactly 16:00 the resistance of the gold and mercury was measured, and Kamerlingh Onnes wrote down his legendary words "Kwik nagenoeg nul" ("Mercury nearly zero"). In doing so he was the first person to perceive the phenomenon of superconductivity, a discovery that would earn him the Nobel Prize for physics in 1913.

Theoretical explanations for superconductivity came only decades later, although the German physicists Walther Meissner and Robert Ochsenfeld did discover the so-called Meissner effect in 1933, which is the phenomenon in which a superconductor expels a magnetic field. The Meissner effect can be understood as a consequence of photons obtaining an effective mass inside a superconducting material, in which we first see the analogy between superconductivity and the Higgs mechanism. The brothers Fritz and Heinz London subsequently published a theory in 1935 relating the superconducting current inside a "supraconductor" to the electromagnetic field [7]. Their equations allowed them to derive the Meissner effect as a consequence of the minimisation of the free energy and calculate what is now called the London penetration depth, i.e. the depth to which a magnetic field can penetrate into a superconductor from the outside.

A great breakthrough in the theory of superconductivity came in 1950 with the Ginzburg-Landau (GL) theory, published by the Russian physicists Vitaly Ginzburg and Lev Landau [8]. This was a phenomenological model, meant not to give a microscopic explanation but only a macroscopic description of the transition from a material's normal phase to its superconducting phase [9, p. 76]. Despite its phenomenological nature, the Ginzburg-Landau theory already brings the notion of spontaneous symmetry breaking to the fore. It describes the free energy F in terms of a parameter ψ:

$$F = F_0 + a|\psi|^2 + b|\psi|^4 + |D_i\psi|^2 + \frac{B^2}{8\pi}.$$

Here F_0 stands for the free energy without superconductivity, $D_i = \frac{1}{2m^*}(\nabla - ie^*A)$ is the gauge-covariant derivative for the vector potential A in terms of the effective mass m^* and charge e^*, $B = \nabla \times A$ is the magnetic field and a, b are functions of the temperature T of the material [9, p. 76]. Importantly, the function $b(T)$ is taken to be positive, whereas $a(T)$ is positive only above the critical temperature T_c and negative below it. As a consequence, the minimum of the free energy F for a temperature $T > T_c$ lies at $\psi = 0$, while the minimum of F for $T < T_c$ lies away from $\psi = 0$. Moreover, the free energy possesses a local $U(1)$ gauge symmetry given by $\psi \mapsto e^{ie^*\alpha}\psi$ and $A \mapsto A + \nabla\alpha$ for $\alpha\colon \mathbb{R}^3 \to U(1)$ an arbitrary smooth function.

All in all we can understand the GL theory as follows: above the critical temperature T_c the system's free energy F has its minimum at $\psi = 0$. When the temperature drops below T_c, the free energy in terms of ψ takes on a special shape, visualised as a *mexican hat*. The original local $U(1)$ symmetry is then *broken*, i.e. the system assumes a new minimum at $\psi \neq 0$. More precisely, the minimum $\psi = 0$ for $T > T_c$ exhibits the same $U(1)$ symmetry as the free energy, but for $T < T_c$ the minima away from zero are not invariant under this symmetry transformation. Thus, if the system moves from a ground state $\psi = 0$ to $\psi \neq 0$, the $U(1)$ symmetry transformation under which the free energy is always invariant no longer leaves the ground state invariant. The ground state is said to break the $U(1)$ gauge symmetry. We can calculate the value of the minimum for $T < T_c$ by differentiating:

$$\frac{\partial F}{\partial|\psi|} = 2a|\psi| + 4b|\psi|^3 = 0.$$

This gives:

$$|\psi| = 0 \text{ or } |\psi| = \sqrt{\frac{-a}{2b}}.$$

Ginzburg en Landau assumed that the spontaneous breaking of $U(1)$ gauge symmetry, in which $|\psi|$ moves from a value of 0 to $\sqrt{-a/2b}$, causes the electrons in the system to form a 'superfluid'-like state, which would then have superconductivity as a consequence. However, they were unable to provide an explanation of the underlying causes and mechanism [9, p. 76].

Such a causal explanation came in 1957 with the BCS theory of superconductivity, discovered by American physicists John Bardeen, Leon Cooper and Robert Schrieffer, for which they received the Nobel Prize in 1972. They postulated a wave function which models how electrons in a superconductor form so-called *cooper pairs* and showed how this cooper pair formation indeed makes electrons form a superliquid in the metal, causing superconductivity. Lev Gor'kov derived the macroscopic GL theory from the microscopic BCS theory in 1959 [10], thus completing the picture of (type I) superconductivity. Gor'kov interpreted the field ψ from the GL theory as the "wave function of a cooper pair" [10, p. 1366], such that the effective charge is $e^* = 2e$, where e denotes the electron charge.

Although this already reaches into the 1960s - treated in the next section - we mention here the 1962 discovery by Josephson [11], who showed that when two superconductors A and B are brought close together, electron pairs can tunnel through the barrier separating them. This causes a current to flow, called the Josephson current, which depends on the Josephson phase $\varphi = \varphi_A - \varphi_B$, i.e. the difference between the global phases of the GL parameters ψ_A and ψ_B, interpreted as the wave functions of the Cooper pairs in the superconductors. We therefore see that through the Josephson effect the *global* $U(1)$ symmetry in a superconductor attains empirical significance: if two superconductors break this symmetry in different directions, we can detect that. This suggests that the global $U(1)$ symmetry is different from the local $U(1)$ gauge symmetry - an idea that is central to this thesis and that we will extend to the Higgs mechanism.

1.1.2. The Golden Decade

Where physicists' theoretical understanding of superconductivity was revolutionised in the 1950s, the 1960s can veritably be called the golden decade for the Higgs mechanism and the electroweak theory.

In a beautiful example of cross-fertilisation, ideas from superconductivity were applied to the very different domain of particle physics and used to solve an open problem there: the existence of massive gauge bosons. It was known that the weak interaction is short-ranged and that its associated gauge bosons must therefore be massive, but Yang-Mills theory incorporated only massless bosons. Like in the GL theory, SSB was then applied to gauge symmetries to explain the fact that some gauge bosons are massive.

This discovery, known as the Higgs mechanism, was preceded by invaluable work in the early 1960s. In 1960 Yoichiro Nambu applied techniques from quantum electrodynamics (QED) to the BCS theory [12], thereby first bridging the gap between superconductivity and particle physics. He then suggested in 1961 that elementary particles might obtain their masses in a fashion similar to superconductivity, namely by a vacuum state spontaneously breaking a symmetry of the theory in question [13]. The universe would then, in a sense, be one big superconductor. He published these ideas in a two-part article titled *Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity* [14, 15]. Interestingly, Nambu explicitly mentions the analogy with superconductivity, and especially its dynamical aspect. He states: "it is suggested that the nucleon mass arises largely as a self-energy of some primary fermion field through the same mechanism as the appearance of energy gap in the theory of superconductivity", and stresses that "dynamical treatment of the interaction makes up an essential part of the theory" [14, p. 345].

In that same year, Jeffrey Goldstone published an article in which he predicted the existence of massless bosons from the spontaneous breakdown of symmetry in a superconductor [16]. These particles became known as Goldstone bosons. In 1962 Goldstone, together with Abdus Salam and Steven Weinberg, proved his prediction to be part of a more general theorem [17]. This theorem, known as Goldstone's theorem, states that when a Lagrangian is invariant under a continuous symmetry, either its ground state is also invariant under that same symmetry, or else massless bosons must exist.

Initially the Goldstone theorem caused confusion, for Nambu's suggestion that SSB could be responsible for the masses of gauge bosons would also imply the existence of massless bosons. Other than the photon, such massless bosons were not known at the time [13]. In 1964, three independent groups simultaneously proposed the solution: the original $U(1) \times SU(2)$ symmetry of the electroweak theory is broken down to a $U(1)$ symmetry because there exists another field whose potential has the shape of a mexican hat. This field assumes a vacuum expectation value away from zero, giving rise to three massive bosons: the W^+, W^- and Z bosons which carry the weak force. The three Goldstone bosons arising from the broken symmetry are subsumed into these massive bosons (they "get eaten"). The unbroken $U(1)$ symmetry gives a massless gauge field, namely the electromagnetic field.

The three groups that made the discovery were Robert Brout and François Englert on 31 August [18], Peter Higgs on 19 October [19, 20] and Gerald Guralnik, Carl Richard Hagen and Tom Kibble on 16 November [21]. The full name of the phenomenon is therefore the 'Brout-Englert-Higgs-Guralnik-Hagen-Kibble mechanism', and the term 'BEH effect' is also encountered in the literature.

These groundbreaking papers treated the *Abelian* Higgs mechanism for a $U(1)$ gauge theory, which is also the version widely used as a toy model in the philosophical literature, because it is conceptually adequate but mathematically simpler than the full non-Abelian version from the Standard Model. Its starting point is the Lagrangian (with the metric signature "mostly plus" used throughout this thesis)

$$\mathcal{L} = -\left(D_\mu \varphi\right)^* D^\mu \varphi - V\left(\varphi^* \varphi\right) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{1.1}$$

where φ is a complex scalar field, $D_\mu = \partial_\mu - ieA_\mu$ is the gauge covariant derivative for the $U(1)$ gauge field A_μ with coupling constant e, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength (curvature) of the gauge field and $V(\varphi^*\varphi) = \mu^2 \varphi^* \varphi + \lambda(\varphi^*\varphi)^2$ is the Higgs potential. The Lagrangian exhibits a local $U(1)$ gauge symmetry given by

$$\begin{aligned} \varphi(x) &\to e^{ie\alpha(x)} \varphi(x), \\ A_\mu(x) &\to A_\mu(x) + \partial_\mu \alpha(x), \end{aligned}$$

for any smooth real function on spacetime α. Indeed, we know the field strength $F_{\mu\nu}$ to be gauge-invariant, and the covariant derivative transforms covariantly as $D_\mu\varphi(x) \to e^{ie\alpha(x)}D_\mu\varphi(x)$.

Now, for $\mu^2 > 0$, the Higgs potential is parabola-like, with a minimum at $\varphi = 0$. For $\mu^2 < 0$, however, the potential takes on the shape of the mexican hat, just like the GL-theory in section 1.1.1. Thus, the minimum in the latter case lies away from zero. Indeed, it is at

$$|\varphi| = \sqrt{\frac{-\mu^2}{2\lambda}} =: \frac{\nu}{\sqrt{2}}.$$

Field configurations that minimise the potential energy are therefore given by $\varphi = \nu/\sqrt{2}e^{i\theta}, A_\mu = 0$, where θ is some real number. We can change θ by means of a gauge transformation. Thus, we might gauge fix the vacuum configuration of the fields by choosing $\theta = 0$, giving simply $\varphi = \nu/\sqrt{2}$. We then consider small perturbations around this vacuum configuration by introducing new small fields ρ and θ through

$$\begin{aligned} \varphi(x) &= \frac{1}{\sqrt{2}}(\nu + \rho(x))e^{ie\theta(x)/\nu}, \\ A_\mu(x) &= B_\mu(x) + \frac{1}{\nu}\partial_\mu\theta(x). \end{aligned}$$

Using the quadratic order approximation $\varphi = (\nu + \rho + ie\theta)/\sqrt{2}$, the original Lagrangian then becomes

$$\mathcal{L} \approx -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{2}(e\nu)^2B^\mu B_\mu - \frac{1}{2}(\partial^\mu\rho)\partial_\mu\rho - \mu^2\rho^2,$$

where $B_{\mu\nu}$ is the curvature of B_μ [9, 22]. We see that the gauge field B_μ and the real field ρ have mass terms! This is the result that brought about the metaphor of the gauge field acquiring mass through gauge symmetry "breaking". Moreover, in the Standard Model, leptons also gain masses through the Higgs mechanism. We will present the details of this calculation in the non-Abelian setting in chapter 2.

After 1964, the Golden Decade was far from over. The work mentioned above led to the unification of the electromagnetic and weak forces into electroweak theory in 1967 [23]. This earned Sheldon Glashow, Abdus Salam and Steven Weinberg a Nobel Prize in 1979. In 1971 Gerard 't Hooft and Martinus Veltman proved this electroweak theory to be renormalisable, for which they won the Nobel Prize in 1999. Before 2012 the Higgs boson was the last missing piece of the Standard Model of particle physics, and finding it was one of the main motivations for building the Large Hadron Collider (LHC) at CERN (Centre Européen pour la Recherche Nucléaire). The construction of the LHC - the biggest machine on Earth - took about ten years and cost about five billion euros [24]. It is no wonder, then, that the announcement of the detection of the Higgs boson became world news and was a great triumph for CERN.

Before turning to the philosophical discussion, let us briefly consider whether the founding fathers of the Higgs mechanism made any suggestions on how to interpret it. Englert and Brout did not mention any analogy with superconductivity or any possible interpretation of SSB in their 1964 article. Higgs does, in his second article, state: "this phenomenon is just the relativistic analog of the plasmon phenomenon to which Anderson has drawn attention" [20, p. 508]. Guralnik, Hagen and Kibble similarly say that "preliminary investigations indicate that superconductivity displays an analogous behavior" [21, p. 587]. In 1966, Higgs published an article providing a quantum mechanical consideration of the mechanism [25]. In it, he says that the models are "inspired by the BCS theory" and even repeatedly calls them "superconductor models". It is therefore likely that Higgs thought the analogy between superconductivity and his own mechanism to be very tight. This analogy, however, has been argued against, notably by Doreen Fraser [9]. In a sense this issue of analogy is very fundamental, for we know from experiments that SSB occurs as an actual process in superconductors. By analogy, we may therefore expect the Higgs mechanism in particle physics to also be a temporal process that occured when the early universe cooled below a certain critical temperature. However, such a direct causal interpretation suffers from a number of conceptual problems that have been pointed out by philosophers of physics. Let us now turn to these issues.

1.2. The philosophical discussion

The attention of the philosophy of physics community was drawn to SSB and the Higgs mechanism by Earman, in three articles from 2003 and 2004 [2, 5, 26]. In these articles, Earman identifies several issues and solves some of them within the framework of algebraic quantum field theory (AQFT). Other philosophers subsequently added to the discussion of the Higgs mechanism, answering some of Earman's questions but also raising new points [22, 27–32]. In parallel, other issues were discussed that are not as directly about the interpretation of the Higgs mechanism, but nonetheless have a bearing on this thesis. Among these are the debate between Wallace and Fraser on the right approach to the philosophy of QFT [33–36] and Fraser's study of the analogy between superconductivity and the Higgs mechanism [9]. We come back to the latter in chapter 7.

More recently, work by Fröhlich, Morchio and Strocchi (FMS) from 1981 [37,38] that had been largely ignored by philosophers has been brought to the fore by Maas and Berghoger et al. [39–41]. In addition, Morchio, Strocchi and De Palma have proven several non-perturbative results on the Higgs mechanism as an instance of global gauge symmetry breaking [42–48].

In this section we present an overview of this literature, focusing on the emergence of two seemingly contradictory ideas: on the one hand the idea that the Higgs mechanism is not an instance of SSB at all, and on the other hand the idea that the Higgs mechanism is a case of global gauge symmetry breaking. From this seeming contradiction we will then distill the research question that this thesis aims to answer.

1.2.1. Earman's questions and answers

In his rough guide to SSB [26], Earman identifies two main puzzles related to SSB in QFT, which he subsequently attempts to solve using the formalism of AQFT. The first puzzle is the fact that in cases of SSB such as the Higgs mechanism, vacuum states are not invariant under some symmetry the theory. In fact, the broken symmetry group of the Lagrangian sends one vacuum state to another, i.e. the vacuum is degenerate. But the vacuum in QFT is defined to be the unique state satisfying Poincaré invariance and positivity of energy, so how can this be?

Earman solves this puzzle by recognising that the different vacuum states in a spontaneously broken theory yield unitarily inequivalent Hilbert space representations of the C^*-algebra describing that theory (see appendix A for an introduction to operator algebras). The uniqueness of the vacuum holds only for a particular representation of the C^*-algebra, so the paradox is resolved.

The second of Earman's puzzles arises from the response to the first. Earman has explained that for SSB the relevant symmetry of the theory is not unitarily implementable on a Hilbert space. But how can this be, if Wigner's theorem tells us that symmetries in quantum mechanics always are unitarily or anti-unitarily implementable? The answer, Earman tells us, lies in the crucial fact that in the case of SSB, the C^*-algebra of the theory is not isomorphic to $B(H)$, the algebra of bounded operators on the Hilbert space H (see A.25 for a definition). We know by the Gelfand-Naimark theorem A.28 that every C^*-algebra can be faithfully represented as a subalgebra of $B(H)$, but it does not have to be isomorphic to the whole of $B(H)$. The broken symmetry is modelled as an automorphism of the C^*-algebra in question. Wigner's theorem, however, defines a symmetry as a probability-preserving map on a Hilbert-space, i.e. within $B(H)$. Thus, two different notions of symmetry are at play here, and the second puzzle is also resolved. Earman stresses that these features of SSB can only arise for systems with infinite degrees of freedom, for then the Stone-von Neumann theorem, which states that the canonical commutation relations of a finite quantum system have a unique irreducible representation up to unitary equivalence, is not applicable.

With the two puzzles solved, Earman goes on to discuss some interpretational issues of SSB, among which those surrounding the Higgs mechanism, where SSB supposedly occurs when a gauge symmetry is broken. This, however, seems contradictory, since a choice of gauge is supposed to be unphysical: it represents a descriptive convention. How can SSB be understood as a physical event if all that is "broken" is a gauge symmetry?

In his subsequent paper [5], Earman expounds on the notion of symmetry in physics and on Nozick's theme of relating objectivity to invariance [49]. He argues for the merit of the constrained hamiltonian formalism to find objective, gauge-invariant quantities and applies this theme to the Higgs mechanism. In this light he notoriously remarks that "a genuine property like mass cannot be gained by eating descriptive fluff, which is just what gauge is." Earman thus demands a gauge-invariant account of the Higgs mechanism, but seems to ignore the fact that such accounts had already been provided (even by Higgs himself in 1966 [25]).

The issues raised by Earman in these two papers are treated in more detail still in his other 2004 article [2]. Here Earman starts by sketching a model of SSB in classical mechanics, but explains that the appearance of actually asymmetric states must there be caused by some statistical fluctuations (in other words: symmetry breaking cannot really be *spontaneous*). He moves on to quantum mechanics and stresses that, under some interpretations of the measurement problem, genuinely spontaneous symmetry breaking can occur when a wavefunction collapses.

Earman continues by considering these issues in the framework of algebraic quantum field theory (AQFT) and calls for a programme in which the constrained Hamiltonian formalism is applied to the Higgs mechanism, and says that: "while there are too many what-ifs in this exercise to allow any firm conclusions to be drawn, it does suffice to plant the suspicion that when the veil of gauge is lifted, what is revealed is that the Higgs mechanism has worked its magic of suppressing zero mass modes and giving particles their masses by quashing spontaneous symmetry breaking. However, confirming the suspicion or putting it to rest require detailed calculations, not philosophizing" [2].

From Earman's considerations we can take away three main points: (1) to describe SSB in quantum systems, one needs an infinite number of degrees of freedom so that there can be unitarily inequivalent irreducible representations of the underlying C^*-algebra; (2) since gauge symmetries are (to some extent) a descriptive redundancy, their breaking cannot constitute a physical process and (3) the constrained Hamiltonian formalism should be applied to the Higgs mechanism to find its gauge-invariant content.

1.2.2. Elitzur's theorem

In his 2006 article [27], Smeenk further focuses on the issues raised by Earman. He stresses the fact that the mathematical structure of the Higgs mechanism looks qualitatively different with respect to different gauges [27, p. 496]. Indeed, in the Coulomb gauge, Goldstone's theorem fails to hold, whereas in the Lorentz gauge it does hold and there are Goldstone bosons, but these are part of a larger unphysical Hilbert space (they are *ghost fields*). We explain this in more detail in chapter 6. These considerations aggravate Earman's worries about the gauge dependence of the Higgs mechanism. Indeed, fundamental aspects such as whether there are massless bosons now seem to be gauge-dependent.

Therefore, Smeenk turns to "the problem of extracting gauge-invariant content" of the Higgs mechanism. He notes that what is really done in the Higgs mechanism is to construct an effective potential that includes quantum corrections, such that standard perturbation techniques can be used to expand around the minimum of this effective potential. The problem with this procedure, however, is that the effective potential is itself gauge-dependent. This issue was solved by Nielsen [50], who "proved that the gauge invariance of various quantities, such as the value of the effective potential at its minima, the mass of the Higgs boson, the mass of the vector boson, and so on, follows from the Ward-Takahashi identities for the Abelian Higgs model" [27, p. 496].

Smeenk's also brings attention to Elitzur's theorem from 1975 [51], which at first sight seems to forbid the Higgs mechanism. This theorem was proven in the non-perturbative context of lattice gauge theory and states that local observables cannot exhibit spontaneous breaking of local gauge symmetries. More precisely, it shows that the vacuum expectation value (VEV) of any gauge-dependent quantity vanishes. This seems to threaten the basic narrative of the Higgs mechanism, in which the gauge-dependent Higgs field supposedly assumes a nonzero VEV. Crucially, however, Elitzur's theorem does not apply to global symmetries, which is an important indication that we should examine the role of global gauge symmetry breaking in the Higgs mechanism.

Smeenk concludes by stating that the abuse of terminology in the term 'SSB of local gauge symmetry' is "relatively benign" [27, p. 498], since the consequences of the Higgs mechanism have been rederived within the gauge-invariant Fröhlich-Morchio-Strocchi (FMS) framework [37, 38]. He does, however, pose an important open question about the dynamics of the Higgs mechanism, namely: "what is the status of a semiclassical description of the scalar field rolling down the effective potential toward or tunnelling to the minima during a phase transition, an idea invoked in inflationary cosmology?" We will return to this question of the dynamics of the Higgs mechanism in chapter 7.

1.2.3. Does the Higgs mechanism exist?

In his provocatively titled 2008 article *Does the Higgs mechanism exist?*, Lyre considers the ontic status of the Higgs mechanism. He presents the mechanism as a simple rewriting of a Lagrangian using different choices of variables and gauge, which leads him to the suspicion that the whole mechanism "consists in a mere reshuffling of degrees of freedom" which "eventually undermines the prospect of an ontological picture of the Higgs mechanism" [28, p. 126]. According to Lyre, SSB in a ferromagnet does allow for a dynamical interpretation, whereas SSB in the Higgs mechanism does not. He presents several objections to viewing the Higgs mechanism as a real dynamical process in time, the most important of which is: "whereas in the case of the ferromagnet SO(3) is instantiated by real rotations of the dipoles, quantum gauge transformations possess no such real instantiations. This was already highlighted in the introduction: neither global nor local unitary gauge transformations are observable, the status of gauge symmetries is a non-empirical and merely conventional one" [28, p. 127]. This leads him to the conclusion that "the Higgs mechanism does not exist" [28, p. 128].

However, Lyre's statement that global gauge transformations are unobservable, just like local ones, is incorrect. Indeed, on both a philosophical and mathematical level, global gauge transformations can be argued to exhibit direct empirical significance. Our chapter 4 is entirely devoted to this point.

Lyre continues by stating that the Higgs mechanism does not carry any explanatory value either: the Lagrangian of the Standard Model after SSB could be written down immediately, without going through the steps of the Higgs mechanism. The value of the Higgs mechanism, then, lies purely in its historical context. It is a useful guiding story to get from a Lagrangian that is easier to guess to the more complicated, symmetry-broken Lagrangian of the Standard Model. "But at the end of the day, this is only a matter of mathematical representation" [28, p. 130]. This statement also seems too simplistic to us, and we will argue against it in chapter 2 by showing how the Higgs mechanism solves several problems related to the definition of mass terms in the Standard Model, both for bosons and fermions.

Wüthrich has also argued against Lyre's analysis: "none of Lyre's worries, therefore, gives us reason to doubt that the Higgs mechanism can have the same ontological status as any other mechanism of spontaneous symmetry breaking, which we observe, for instance, in ferromagnets or superconductors" [29, p. 10]. Lyre responded to Wüthrich's objections, stating basically that there is no need nor reason for physicists to postulate a phase transition from an unbroken to a broken phase in the early universe [52] and claiming that Struyve's gauge-invariant accounts [22] and Friederich's analysis of remnant global symmetries [31] support his position. In our view, Lyre's position is incorrect, but can only be properly addressed by clarifying what really is the physical content of the Higgs mechanism.

1.2.4. Gauge-invariant accounts

An important step towards this clarification was made by Struyve [22] by taking seriously Earman's call for a gauge-invariant treatment of the Higgs mechanism, though his work builds on that by Lusanna and Valtancoli [53–55]. Struyve begins by recalling that Higgs himself already provided a gauge-invariant treatment of the Abelian version of the mechanism in 1966 [25] and briefly presents this treatment. He next discusses gauge freedom and the elimination of gauge symmetries through the reduction of phase space, and notes that the gauge group depends on boundary conditions and does not necessarily correspond to the full group of all gauge transformations. Boundary conditions are imposed to ensure finiteness of energy and action and to make variational operations on the action well-defined. Gauge transformations must respect such boundary conditions.

Struyve explores these ideas within the context of the Abelian Higgs mechanism with a scalar field φ and a vector field A_μ. He considers a set of appropriate boundary conditions such that transformations preserving these boundary conditions are of the form $g = e^{i\alpha}$, with α a real function that goes to a constant sufficiently rapidly as infinite distance is approached, i.e. as $r \to \infty$. We denote the group of these transformations by $\mathcal{G}^I$. The unphysical gauge group is $\mathcal{G}^\infty$, the group of local $U(1)$ transformations that go to the identity at spatial infinity. Thus, we find that there is a non-trivial group of residual physical gauge transformations $\mathcal{G}^I/\mathcal{G}^\infty \cong U(1)$. We will work this out in much more detail in chapter 4.

We are therefore led to the suspicion that the Higgs mechanism could be understood as SSB of this residual group of physical symmetry transformations. Indeed, Struyve supports this idea by an analysis in the constrained Hamiltonian formalism. He defines a complete set of gauge-independent fields in order to eliminate local gauge symmetry, but these fields do leave the global $U(1)$ symmetry. There is then a degenerate set of ground states, such that when a vacuum state is chosen and perturbations around it are considered, one obtains a Hamiltonian with a massive vector boson [22, p. 235]. We will present this derivation in detail in section 4.2, but for now the important point is that global gauge symmetries should not be seen as merely a special type of local gauge symmetries and actually have a physical significance that local gauge symmetries do not have.

1.2.5. Remnant global gauge symmetries

This idea has been further developed by Friederich [31, 56], who considers so-called remnant global symmetries at great length. These are global symmetries that are left over after a particular gauge-fixing and evade Elitzur's theorem, since that theorem makes the crucial assumption "that for any finite volume of spacetime local gauge transformations can always be chosen such as to act non-trivially only in that finite volume (and to reduce to the identity transformation everywhere else)" [31, p. 171]. This assumption does not hold for global gauge transformations, which act non-trivially on all of spacetime, since they are spacetime-independent. Friederich argues that the breaking of remnant gauge symmetries is the only type of SSB in gauge theories worth studying at all: "since we do not presently have any notion of a spontaneously broken local gauge symmetry in a gauge quantum field theory, the breaking of these remnant global subgroups is the only sense of gauge symmetry breaking that remains to be elucidated" [31, p. 175].

The question, then, is whether the breaking of remnant gauge symmetries corresponds to some phase transition, and Friederich argues that there is in fact no such correspondence. To support this claim, he refers to a study of an $SU(2)$-symmetric lattice gauge model with fixed-modulus Higgs field [57]. In this study, it was shown that the breaking of global remnant gauge symmetries does not necessarily match with phase transitions, and moreover that the symmetry breaking depends on a choice of gauge: "we show that in an $SU(2)$ gauge-Higgs system such symmetries do indeed break spontaneously, but the location of the breaking in the phase diagram depends on the choice of global subgroup. The implication is that there is no unique broken gauge symmetry, but rather many symmetries which break in different places" [57].

It seems to us, however, that the major weakness in this argument is that it considers the various remnant gauge symmetries on equal footing, instead of just the one particular global gauge group which is the remnant symmetry group of the Coulomb gauge, and whose physical significance we derive in chapter 4 by means of constrained Hamiltonian analysis. It is this physical global gauge symmetry whose breaking should be studied in the context of the Higgs mechanism. Indeed, it should be expected that SSB of remnant symmetries that are not precisely the one singled out by the constrained Hamiltonian formalism yields ambiguous, gauge-dependent results. As the Coulomb gauge is the gauge with the global gauge group as its remnant symmetry group, we will use it throughout chapters 4 and 6.

1.2.6. Higgs without SSB

Recently, however, Berghofer et al. have analysed the Higgs mechanism in a framework inspired by FMS [37, 38], suggesting that SSB does not play a role at all in the Higgs mechanism. More precisely, they implement two programmes: the *dressing field method* (DFM) and the FMS approach. What these methods have in common is that instead of gauge-dependent elementary fields they use gauge-invariant composite fields. The DFM is a purely classical method of rewriting a gauge theory in terms of *dressed* fields which, when applied to the electroweak theory, supposedly shows that "the interpretation of the model in terms of SSB is here superfluous, and indeed impossible" [41, p. 61]. The FMS approach expresses n-point functions of gauge-invariant composite objects such as $\varphi^\dagger\varphi$ in terms of n-point functions of elementary fields and thereby explains how perturbation theory with gauge-dependent fields deviates only slightly from gauge-invariant computations [40]. This is worked out in great detail for the Standard Model by Maas [39], who adopts the term "gauge-invariant perturbation theory." We will present the DFM and FMS approach in chapter 3.

For now, let us consider the statement that "applied to the electroweak model, they [the DFM and FMS approach] converge on the conclusion that the spontaneous breaking of gauge symmetry is not a physical phenomenon in this case, [...] giving rise to a local gauge-invariant description of the massive gauge bosons that renders the SU(2) symmetry an artificial one" [41, p. 80-81]. This alleged complete absence of SSB in the electroweak theory contradicts the idea that global gauge symmetry breaking plays a crucial role in the Higgs mechanism, and a tension arises with the results explained in sections 1.2.4 and 1.2.5. How to resolve this tension will be one of our research questions.

1.2.7. Going non-perturbative

The tension is further aggravated by rigorous non-perturbative results by Morchio and Strocchi, most poignantly Theorem 6.2 in Strocchi's book on non-perturbative QFT [46], which corresponds to Proposition 6.1 in the original paper [42], Theorem 2.8.3 in Strocchi's lecture notes [47] and to Theorem 19.3 in Strocchi's book *Symmetry Breaking* [44]. This result about global gauge symmetries in the Abelian Higgs mechanism basically states that in the Coulomb gauge, the global $U(1)$ gauge symmetry is *unbroken* whenever one has only massless bosons, whereas if the global $U(1)$ symmetry is broken, then there are massive vector bosons but no Goldstone bosons. Landsman calls it "one of the very few rigorous result about the Higgs mechanism (in the continuum)" [58, p. 429], so it is a rare but highly valuable piece of insight into the structure of the Higgs mechanism in QFT and actually connects the Higgs mechanism to the algebraic definition of SSB. We will prove it in detail in chapter 6 (Theorem 6.13).

In addition, De Palma and Strocchi have proven that for general Yang-Mills theory in the BRST gauge, the breaking of the global gauge group gives only Goldstone modes which cannot belong to the physical spectrum, i.e. which are unphysical [43]. This result can be viewed as a step towards the Higgs mechanism in the non-Abelian setting. We discuss this briefly in section 6.6.

1.3. Research questions and outline

Having reviewed how global gauge symmetries come to the fore in the (philosophical) literature on the Higgs mechanism, we are in a position to formulate the main research question of this thesis.

RQ1: *What role does global gauge symmetry breaking play in the Higgs mechanism?*

Answering this question will require us to treat several sub-questions, which we now identify so that we can revisit them throughout this thesis.

RQ1.1: *How can the apparent contradiction be resolved between the implication of the DFM that there is no SSB in the Higgs mechanism, and results presenting the Higgs mechanism as SSB of global gauge symmetry?*

RQ1.2: *Why should the global gauge group not be considered merely as a subgroup of the local gauge group, but rather as having a different physical significance?*

RQ1.3: *To what extent can results on the Abelian Higgs mechanism be used to interpret the complete non-Abelian Higgs mechanism in the Standard Model?*

Even if an answer to all these research questions is provided, it is not necessarily clear how the Higgs mechanism should be understood as a dynamical process that may have occurred in the early universe. We only address this issue directly in chapter 7, but we think can be clarified at least in part by the rest of this thesis.

Before delving into the detailed analyses of gauge-invariant and non-perturbative formulations of the Higgs mechanism that are needed to answer RQ1.1 and RQ1.2, we must familiarise ourselves in as much detail as possible with the Higgs mechanism in the Standard Model of particle physics. We do this in chapter 2. Firstly, this serves the purpose of rigourously introducing the notions that are important to any treatment of the Higgs mechanism: gauge transformations, Yang-Mills theory, vacuum configurations, unitary gauge, electroweak theory etc. Secondly, an elaborate understanding of the Higgs mechanism in the Standard Model is necessary to properly answer RQ1.3, and throughout this thesis we must constantly ask ourselves whether the methods considered can adequately handle the full non-Abelian complexity of the Higgs mechanism as presented in chapter 2.

We then present the DFM and FMS approach in chapter 3, building on definitions and results from chapter 2. As explained in section 1.2.6, these methods suggest that there is no SSB in the Higgs mechanism at all, and they therefore most radically oppose the standard narrative of gauge symmetry breaking. Treating them in much detail allows us to answer RQ1.1, but to do this we must also obtain a deeper understanding of the meaning of global gauge symmetries - in other words: we must answer RQ1.2. This we do in chapter 4 by means of constrained Hamiltonian analysis.

The results from chapter 4 mark the end of our study of classical field theory and form the transition into chapters 5 and 6, where we introduce important notions of AQFT, examine global gauge symmetries in QFT and also address RQ1.3. We think the order thus sketched is most logical: we begin with the standard narrative, then we treat the approaches that aim to completely get rid of SSB, subsequently we mitigate these approaches by carefully considering the significance of global gauge symmetries and finally we support our classical ideas by results from axiomatic QFT.

Lastly, we summarise our results, address the question of global gauge symmetry breaking as a dynamical process in time, discuss the analogy with superconductivity and present our conclusions in chapter 7. There we also suggest promising directions for further philosophical inquiry into the Higgs mechanism.

2. The Higgs Mechanism in the Standard Model

When investigating the Higgs mechanism, philosophers of physics usually focus on the Abelian version. In the Standard Model of particle physics, however, the Higgs field is coupled to a non-Abelian gauge field as well as to fermions through the Yukawa interaction. In other words: the situation in the Standard Model is vastly more involved. Thus, the question arises of whether the philosophical studies hitherto performed really have a bearing on the Higgs mechanism as particle physicists see it. This relates to our RQ1.3, which asks to what extent results on the Abelian Higgs mechanism can be used to make claims about the Standard Model.

To our mind, the only way to address this issue is to familiarise ourselves with the gauge-theoretical foundations of the full Higgs mechanism in the Standard Model, and we aim to do so in this chapter, making heavy use of [59]. We begin by introducing some important definitions and results on gauge transformations in section 2.1. Subsequently, we define Yang-Mills theory and the complete Lagrangian of the Standard Model in section 2.2, and finally we treat the full Higgs mechanism in section 2.3. Defining fermions in the Standard Model requires spinors, to which appendix B provides a self-contained introduction. We show how the Higgs mechanism in the Standard Model solves not one but three issues related to mass terms:

- mass terms for gauge bosons;
- different masses for fermions in the same gauge multiplet;
- mass terms for twisted chiral fermions.

Any adequate perspective on the Higgs mechanism, including the one based on global gauge symmetry breaking put forward in this thesis, must be able to account for all its aspects. Whether we can do this will be a point of discussion in chapter 7.

Before we begin, let us stress that this chapter also serves the purpose of introducing many important concepts that are used in later parts of this thesis, even those parts that deal only with the Abelian Higgs model. Especially sections 2.1 and 2.3 are essential in that regard and should therefore be read carefully, even by someone who is interested in conceptual rather than technical aspects.

2.1. Gauge transformations

First of all, we must understand precisely what is meant by notions such as gauge, gauge transformation and gauge group. This is very important with an eye towards understanding how global gauge transformations differ from local ones, which we will explain in chapter 4. We present gauge transformations in section 2.1.1 and consider how they affect local expressions for connections, curvature and covariant derivatives in section 2.1.2. We assume familiarity with Lie theory and differential geometry.

2.1.1. Gauges and the gauge group

We commence with the mathematical definition of a gauge.

Definition 2.1. Let $\pi\colon P \to M$ be a principal bundle. A *global gauge* for this bundle is a global (smooth) section $s\colon M \to P$. Similarly, a *local gauge* is a local section $s\colon U \to P$ on some open subset $U \subset M$.

Thus, gauges are just sections, but the important point is that sections correspond to trivialisations - this is a well-known fact about principal bundles. More precisely, we have the following basic result, which corresponds to Proposition 4.2.19 in [59].

Proposition 2.2. Let $\pi\colon P \to M$ be a principal G-bundle and $s\colon U \to P$ a local gauge. Then the map $U \times G \to P_U$ given by $(x, g) \mapsto s(x) \cdot g$ is a G-equivariant diffeomorphism (here $P_U = \pi^{-1}(U)$). In particular, if $s\colon M \to P$ is a global gauge then P is trivial with trivialisation given by the inverse of the above map.

If we have a (local) section of a principal bundle, we want to know how this affects the way we describe associated bundles. The following theorem tells us how we can use a (local) gauge to (locally) trivialise any associated vector bundle (cf. Proposition 4.7.6 in [59]).

Theorem 2.3. Let $\pi\colon P \to M$ be a principal G-bundle and $E = P \times_\rho V$ an associated vector bundle through the representation $\rho\colon G \to GL(V)$. Let $s\colon U \to P$ be a local gauge. Then there is a bijective correspondence between local frames $\tau\colon U \to E$ and maps $f\colon U \to V$, given by

$$\tau(x) = [s(x), f(x)], \qquad x \in U.$$

This means that the local gauge defines a preferred isomorphism between V and every fibre E_x for $x \in U$.

Proof. Let $f\colon U \to V$ be a smooth map. Then the map $U \to P \times V$ given by $x \mapsto (s(x), f(x))$ is clearly smooth and hence $\tau\colon U \to E$ is smooth, since it is just the composition of this map with the projection $P \times V \to E$. Moreover, τ is indeed a section since

$$(\pi_E \circ \tau)(x) = \pi_E([s(x), f(x)]) = (\pi \circ s)(x) = x,$$

as s is a local section. Conversely, let $\tau\colon U \to E$ be a smooth section. By definition $E_x = (P_x \times V)/G$ and the action of G on P_x is simply transitive, so there is a unique $f(x)$ such that $\tau(x) = [s(x), f(x)]$. To check that f is smooth we define a bundle chart $\phi_U\colon P_U \to U \times G$ using s by setting $\phi_U^{-1}(x, g) = s(x) \cdot g$. Writing $\phi_U(p) = (\pi(p), \varphi_U(p))$ with $\varphi_U\colon P_U \to G$, we get $\varphi_U(s(x)) = x$ for any $x \in U$ and therefore

$$(\psi_U \circ \tau)(x) = \psi_U([s(x), f(x)]) = (x, \rho(\varphi_U(s(x)))f(x)) = (x, f(x)), \tag{2.1}$$

where $\psi_U\colon E_U \to U \times V$ is the induced associated bundle chart given by

$$[p, v] \mapsto (\pi(p), \rho(\varphi_U(p))v), \qquad p \in P_U, v \in V.$$

This map is a diffeomorphism with inverse $\psi_U^{-1}\colon U \times V \to E_U$ given by $(x, v) \mapsto [\phi_U^{-1}(x, e), v]$. But equation 2.1 shows that f is smooth since ψ_U and τ are smooth. □

Having defined (local) gauges, we want to formalise the notion of a transformation between these gauges. These turn out to just be maps from the principal bundle to itself.

Definition 2.4. Let $\pi\colon P \to M$ be a principal G-bundle. Then a *gauge transformation* is a bundle automorphism of P, i.e. a diffeomorphism $f\colon P \to P$ which is G-equivariant and preserves the fibres: $\pi \circ f = \pi$ and $f(p \cdot g) = f(p) \cdot g$ for all $p \in P, g \in G$. A *local gauge transformation* is a bundle automorphism of the principal bundle $\pi_U\colon P_U \to U$, where $U \subset M$ is an open subset.

The automorphisms of a principal bundle $P \to M$ form a group $\mathcal{G}(P)$ under composition, and this group is called the *gauge group*. However, in physics parlor, gauge transformations usually are not defined as bundle automorphisms, but rather as G-valued maps on spacetime. Let us define $C^\infty(P, G)^G$ to be the set of smooth maps $f\colon P \to G$ satisfying $\sigma(p \cdot g) = g^{-1}\sigma(p)g$ for all $p \in P, g \in G$. This set is a group under pointwise multiplication with its identity element being the constant map on P with value $e \in G$. As a first step to understanding how the two notions of gauge transformations relate we prove the following result.

Proposition 2.5. Let $\pi\colon P \to M$ be a principal G-bundle. Then the map $\mathcal{G}(P) \to C^\infty(P, G)^G$ given by $f \mapsto \sigma_f$, where $f(p) = p \cdot \sigma_f(p)$ for all $p \in P$, is a well-defined group isomorphism.

Proof. To see that $f(p) = p \cdot \sigma_f(p)$ makes σ_f well-defined, note that since $\pi \circ f = \pi$ we know that $f(p)$ is in the fibre of p, which means there is a unique $g \in G$ such that $f(p) = p \cdot g$. We thus define $\sigma_f(p) = g$. We need to check that $\sigma_f \in C^\infty(P, G)^G$. The smoothness of σ_f follows locally from the smoothness of f and the smoothness of the G-action.[1] Moreover, for any $p \in P, g \in G$ we have

$$p \cdot (g\sigma_f(p \cdot g)) = (p \cdot g)\sigma_f(p \cdot g) = f(p \cdot g) = f(p) \cdot g = (p \cdot \sigma_f(p)) \cdot g = p \cdot (\sigma_f(p)g).$$

This shows that $g\sigma_f(p \cdot g) = \sigma_f(p)g$, i.e. $\sigma_f(p \cdot g) = g^{-1}\sigma_f(p)g$, as required.

It is evident that the inverse of the map in the proposition is a map $C^\infty(P, G)^G \to \mathcal{G}(P)$ that sends $\sigma \mapsto f_\sigma$ defined by $f_\sigma(p) = p \cdot \sigma(p)$. We do need to verify that f_σ is a bundle automorphism for any $\sigma \in C^\infty(P, G)^G$. Clearly $f_\sigma(p)$ is in the same fibre as p for any $p \in P$, and also $f_\sigma^{-1} = f_{\sigma^{-1}}$, so f_σ is a diffeomorphism. Lastly, we have

$$f_\sigma(p \cdot g) = (p \cdot g) \cdot \sigma_f(p \cdot g) = (p \cdot g) \cdot g^{-1}\sigma_f(p)g = (p \cdot \sigma_f(p)) \cdot g = f_\sigma(p) \cdot g, \qquad p \in P, g \in G,$$

so f_σ is G-equivariant and thus $f_\sigma \in \mathcal{G}(P)$. We have now shown that the map in the proposition is a bijection, but it remains to check that it respects the group structure. Let $f_1, f_2 \in \mathcal{G}(P)$. We need to show that $\sigma_{f_1 \circ f_2} = \sigma_{f_1}\sigma_{f_2}$. For all $p \in P$ we have

$$p \cdot \sigma_{f_1\circ f_2}(p) = (f_1 \circ f_2)(p) = f_1(p \cdot \sigma_{f_2}(p)) = f_1(p) \cdot \sigma_{f_2}(p) = (p \cdot \sigma_{f_1}(p)) \cdot \sigma_{f_2}(p) = p \cdot \sigma_{f_1}\sigma_{f_2}(p).$$

Thus, the map in the proposition is a bijective group homomorphism. □

We are now in a position to define gauge transformations as it is usually done in the physics literature and to show how this relates to automorphisms of principal bundles. We use the terminology from section 5.3.2 in [59].

Definition 2.6. Let $\pi\colon P \to M$ be a principal G-bundle. A *physical gauge transformation* is a smooth map $\tau\colon U \to G$ defined on some open subset $U \subset M$. A *global* or *rigid* physical gauge transformation is a constant map $\tau\colon U \to G$. The set of all physical gauge transformations on U forms a group $C^\infty(U, G)$ with pointwise multiplication.

Proposition 2.7. Let $s\colon U \to P$ be a local section of a principal G-bundle. Then s defines a group isomorphism $C^\infty(P_U, G)^G \to C^\infty(U, G)$ given by $\sigma \mapsto \tau_\sigma = \sigma \circ s$. The inverse of this map is given by $\tau \mapsto \sigma_\tau$, where $\sigma_\tau(s(x) \cdot g) = g^{-1}\tau(x)g$ for any $x \in U, g \in G$.

Proof. Since any point in P_U can be written uniquely as $s(x) \cdot g$ with $x \in U, g \in G$, we see σ_τ is well-defined, though we do need to verify that is is in $C^\infty(P_U, G)^G$. For $x \in U$ and $g_1, g_2 \in G$ we have

$$\sigma_\tau((s(x) \cdot g_1) \cdot g_2) = \sigma_\tau(s(x) \cdot g_1g_2) = (g_1g_2)^{-1}\tau(x)g_1g_2 = g_2^{-1}g_1^{-1}\tau(x)g_1g_2 = g_2^{-1}\sigma_\tau(s(x) \cdot g_1)g_2.$$

Let us also check that the two maps are actually inverses.

$$\tau_{\sigma_\tau}(x) = \sigma_\tau \circ s(x) = \tau(s(x) \cdot e) = \tau(x), \qquad x \in U,$$
$$\sigma_{\tau_\sigma}(s(x) \cdot g) = g^{-1}\tau_\sigma(x)g = g^{-1}\sigma(s(x))g = \sigma(s(x) \cdot g), \qquad x \in U, g \in G.$$

In the last equality we used that $\sigma \in C^\infty(P_U, G)^G$. We have established that we have a well-defined bijection. Lastly, it is easy to see that $\tau_{\sigma_1\sigma_2} = \tau_{\sigma_1}\tau_{\sigma_2}$. We conclude that the map is a group isomorphism. □

[1] If we consider a trivialisation $P_U \to U \times G$, then $f \in \mathcal{G}(U \times G)$ can just be viewed as a smooth map $f_U\colon U \to G$ via $f(x, g) = (x, f_U(x)g)$. We then see $\sigma_{f_U}\colon U \times G \to G$ is defined by $\sigma_f(x, g) = g^{-1}f_U(x)g$, which is smooth.

Thus, we see that *after choosing a local gauge*, physical gauge transformations correspond to maps in $C^\infty(P_U, G)^G$, which in turn correspond to local gauge transformations, as shown by Proposition 2.5. It is not hard to show that the group of bundle automorphisms $\mathcal{G}(P)$ of a principal G-bundle $\pi\colon P \to M$ acts on an associated vector bundle $\pi_E\colon E = P \times_\rho V \to M$ through bundle isomorphisms (Theorem 5.3.8 in [59]) via

$$f \cdot [p, v] = [f(p), v] = [p \cdot \sigma_f(p), v], \qquad f \in \mathcal{G}(P), p \in P, v \in V. \tag{2.2}$$

More interesting for the relation to physics, however, is the action of *physical* gauge transformations on associated bundles, as detailed in the following result.

Proposition 2.8. Let $\pi\colon P \to M$ be a principal G-bundle with associated bundle $\pi_E\colon E = P \times_\rho V \to M$ and local sections $s\colon U \to P$ and $\Phi\colon U \to E$. Let us write the section Φ with respect to the local gauge s as $\Phi(x) = [s(x), \phi(x)]$, where $\phi\colon U \to V$ is a smooth map. Let $f \in \mathcal{G}(P_U)$ and let $\tau_f\colon U \to G$ be the physical gauge transformation associated to f (cf. Proposition 2.7). Then for all $x \in U$ we have

$$(f \cdot \Phi)(x) = [s(x), \rho(\tau_f(x))\phi(x)],$$

where $f \cdot \Phi$ is defined as in equation 2.2 above.

Proof. We simply calculate.

$$\begin{aligned}(f \cdot \Phi)(x) &= f \cdot [s(x), \phi(x)] = [f(s(x)), \phi(x)] = [s(x) \cdot \sigma_f(s(x)), \phi(x)] = [s(x) \cdot \tau_{\sigma_f}(x), \phi(x)] \\ &= [s(x) \cdot \tau_{\sigma_f}(x), \rho(\tau_{\sigma_f}(x)^{-1})\rho(\tau_{\sigma_f}(x))\phi(x)] = [s(x), \rho\left(\tau_{\sigma_f}(x)\right)\phi(x)].\end{aligned}$$

Writing $\tau_f = \tau_{\sigma_f}$ the claim follows. □

The upshot is that the action of a local bundle automorphism on a local section of an associated bundle is given through the action of the corresponding physical gauge transformation on the vector-valued map $\phi\colon U \to V$. This is what we see in physics, where the action of a physical gauge transformation $\tau\colon U \to G$ on a field $\phi\colon U \to V$ is written as $\phi(x) \mapsto \tau(x)\phi(x)$. The more general treatment in terms of bundle automorphisms has the advantage that it works for topologically non-trivial situations, when considering instantons for instance.

2.1.2. Connections, curvature and covariant derivatives

We have considered the relation between the mathematical definition of gauge transformations and the physics parlor, and in particular we have seen how matter fields (sections of associated bundles) transform. We have, however, not considered the transformation of gauge fields themselves. Since gauge fields are modelled as connection 1-forms on principal bundles, we have to understand how such 1-forms can be described locally on spacetime and see how this local description changes under gauge transformations. The same must be done for curvature 2-forms and covariant derivatives on associated bundles.

Definition 2.9. Let $\pi\colon P \to M$ be a principal G-bundle and $A \in \Omega^1(P, \mathfrak{g})$ a connection 1-form. Let $s\colon U \to P$ be a local gauge on an open subset $U \subset M$. Then the *local connection 1-form* or *local gauge field* determined by s is $A_s = s^*A \in \Omega^1(U, \mathfrak{g})$. If U is a chart on M with local tangent bundle frame ∂_μ then we write $A_\mu = A_s(\partial_\mu) \in C^\infty(U)$. If in addition we choose a basis $\{e_a\}$ of the Lie algebra $\mathfrak{g}$ and expand $A_\mu = A_\mu^a e_a$, then we call the $A_s^a = A_\mu^a dx^\mu$ the *local gauge boson fields*.

Famously, if $P \to M$ is a principal G-bundle with $G \subset GL(n, \mathbb{R})$, and $s_i\colon U_i \to P$, $s_j\colon U_j \to P$ are local gauges with $U_i \cap U_j \neq \emptyset$, and we write $s_i(x) = s_j(x) \cdot g_{ji}(x)$ for all $x \in U_i \cap U_j$, where $g_{ji}\colon U_i \cap U_j \to G$ is the transition function, then on $U_i \cap U_j$:

$$A_i = g_{ji}^{-1} \cdot A_j \cdot g_{ji} + g_{ji}^{-1} \cdot dg_{ji},$$

where $\cdot$ denotes matrix multiplication and dg_{ji} is the componentwise differential. For a proof see Theorem 5.4.2 of [59]. In the Abelian case this reduces to $A_i = A_j + g_{ji}^{-1} \cdot dg_{ji}$. Let us now consider the curvature 2-form or gauge field strength in the same fashion.

Definition 2.10. Let $\pi\colon P \to M$ be a principal G-bundle and $A \in \Omega^1(P, \mathfrak{g})$ a connection 1-form with curvature $F = dA + \frac{1}{2}[A, A]$. Let $s\colon U \to P$ be a local gauge on an open subset $U \subset M$. Then the *local curvature 2-form* or *local field strength* is $F_s = s^*F \in \Omega^2(U, \mathfrak{g})$. If U is a chart with frame ∂_μ then we write $F_{\mu\nu} = F_s(\partial_\mu, \partial_\nu)$, and if e_a is a basis of $\mathfrak{g}$ we expand $F_{\mu\nu} = F^a_{\mu\nu} e_a$.

The following result makes the link to the physics notation entirely clear. Its proof is a matter of simple calculation.

Proposition 2.11. The structure equation holds locally, i.e. $F_s = dA_s + \frac{1}{2}[A_s, A_s]$ in a local gauge s. Moreover, in coordinates we have $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.

Using the same notation as above for $G \subset GL(n, \mathbb{R})$, the local field strength transforms in the well-known way $F_i = g_{ji}^{-1} \cdot F_j g_{ji}$. For a proof see Theorem 5.6.3 in [59]. In the Abelian case this implies that F_s is independent of the local gauge s, so we can construct a global closed 2-form $F_M \in \Omega^2(M, \mathfrak{g})$. Indeed, in that case we locally have $F_s = dA_s$ by Proposition 2.11, which locally implies $dF_s = d^2A_s = 0$. Still, this global closed 2-form F_M need not be globally exact, because the base space might be topologically non-trivial.

Lastly, it remains to see why the covariant derivative is called covariant. Let $\pi\colon P \to M$ be a principal G-bundle with connection $A \in \Omega^1(P, \mathfrak{g})$ and associated bundle $\pi_E\colon E = P \times_\rho V \to M$. Then the induced[2] covariant derivative $\nabla^A\colon \Gamma(E) \to \Omega^1(M, E)$ can be expressed in terms of a local gauge $s\colon U \to P$ by writing a section $\Phi \in \Gamma(E)$ as $\Phi|_U = [s, \phi]$, with $\phi\colon U \to V$, and writing $\nabla^A_X \Phi = [s, \nabla^A_X \phi]$ along a vector field $X \in \mathfrak{X}(M)$. Then

$$\nabla^A_X \phi = d\phi(X) + \rho_* (A_s(X))\, \phi. \tag{2.3}$$

In physics notation, we locally write

$$\nabla^A_\mu \phi = \nabla^A_{\partial_\mu} \phi = \partial_\mu \phi + A_\mu \phi,$$

where we mean $A_\mu \phi = \rho_*(A_\mu)\phi$. Why, then, is the covariant derivative called covariant? From the expression $\nabla^A_\mu = \partial_\mu + A_\mu$ we only see that if the gauge field changes under a gauge transformation, then so does the covariant derivative. But for a section $\Phi \in \Gamma(E)$ of an associated bundle $E = P \times_\rho V$ the object $\nabla^A_X \Phi$ is not yet defined with respect to some gauge. It lives on the untrivialised bundle. However, if we have two local gauges $s_1, s_2\colon U \to P$, then for both we can write

$$[s_1, \nabla^A_X \phi_1] = \nabla^A_X \Phi = [s_2, \nabla^A_X \phi_2].$$

Thus, if we relate the gauges by a gauge transformation $g\colon U \to G$ such that $s_2 = s_1 \cdot g$, then we see

$$[s_1, \nabla^A_X \phi_1] = [s_1 \cdot g, \rho(g)^{-1} \nabla^A_X \phi_1] = [s_2, \rho(g)^{-1} \nabla^A_X \phi_1] = [s_2, \nabla^A_X \phi_2].$$

In other words, we have $\nabla^A_X \phi_1 = \rho(g) \nabla^A_X \phi_2$, which is where the nomenclature *covariant derivative* comes from - the derived field transforms in the same way as the underived field.

2.2. The Standard Model

Having worked our way through the basic notions of gauge theory, we have all the material we need to define the full Lagrangian of the Standard Model. In this section we consider its four parts: the Yang-Mills, Higgs, Dirac and Yukawa terms. We spend some time especially on the details of Yang-Mills theory, as we must understand this deeply for the remainder of this thesis. For an introduction to spinors and the Dirac operator, we refer the reader to appendix B.

[2]The induced covariant derivative is defined on a section of E, viewed as an equivariant map $f\colon P \to V$, by $\nabla^A_X f = df(X^H)$, where $X^H \in \mathfrak{X}(P)$ is the unique horizontal lift of $X \in \mathfrak{X}(M)$.

2.2.1. Yang-Mills theory

Before we can define the Yang-Mills Lagrangian, we need to understand why it is said that gauge bosons transform under the adjoint representation. We recall that a k-form $\omega \in \Omega^k(P, \mathfrak{g})$ is called *horizontal* if it vanishes whenever at least one vector it eats is vertical, i.e. for all $p \in P$ we have $\omega_p(X_1, ..., X_k) = 0$ if $X_i \in V_pP$ for some $1 \leq i \leq k$. Furthermore, we say a k-form is *of type Ad* if $r_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$ for any $g \in G$. We denote the set of horizontal k-forms of type Ad by $\Omega^k_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$. We have the following result.

Proposition 2.12. Let $P \to M$ be a principal G-bundle. If $A, A' \in \Omega^1(P, \mathfrak{g})$ are two connection 1-forms then $A - A' \in \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$ and for any $\omega \in \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$ we have that $A + \omega$ is a connection 1-form. For the curvature we have $F^A \in \Omega^2_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$.

Proof. The fact that $A - A'$ is horizontal follows from the fact that connection 1-forms identically yield $A(\tilde{X}) = A'(\tilde{X}) = X$ when applied to the fundamental vector field of $X \in \mathfrak{g}$, since the fundamental vector fields span the vertical subspace at each point. Since r_g^* is linear it also follows that $A - A'$ is of type Ad. These same observations ensure that $A + \omega$ is a connection 1-form for any $\omega \in \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$.

The horizontality of the curvature 2-form can be most readily seen from the definition $F^A(X, Y) = dA(\pi_H(X), \pi_H(Y))$, where $\pi_H \colon TP \to H$ denotes the projection onto the horizontal subbundle. The fact that curvature 2-forms are of type Ad is well-known and can be checked with a simple calculation. □

The interest of this proposition lies in the following theorem, in which the *adjoint bundle* appears, i.e. the associated real vector bundle $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g}$ constructed through the adjoint representation $\mathrm{Ad} \colon G \to GL(\mathfrak{g})$, defined by $\mathrm{Ad}(g)(X) = gXg^{-1}$ for a matrix Lie group G.

Theorem 2.13. Let $P \to M$ be a principal G-bundle. Then $\Omega^k_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$ and $\Omega^k(M, \mathrm{Ad}(P))$ are canonically isomorphic as vector spaces.

Proof. We define a map $\varphi \colon \Omega^k_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}} \to \Omega^k(M, \mathrm{Ad}(P))$ by $(\varphi(\omega))_x(X_1, ..., X_k) = [p, \omega_p(Y_1, ..., Y_k)]$, where $\omega \in \Omega^k_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$, $x \in M$, $p \in P$ such that $\pi(p) = x$ and $X_i \in T_xM$, $Y_i \in T_pP$ such that $\pi_*(Y_i) = X_i$. Of course, we need to check that this is well-defined. The independence of the choice of vectors Y_i is not hard to see: if $\pi_*(Y_i') = X_i = \pi_*(Y_i)$, then $\pi_*(Y_i - Y_i') = 0$, implying that $Y_i - Y_i'$ is vertical. The horizontality of ω then makes it vanish on this difference $Y_i - Y_i'$ in any slot. As for the independence of the point $p \in P_x$, suppose $p' \in P_x$. By free transitivity there exists a unique $g \in G$ such that $p \cdot g = p'$. We then have

$$\begin{aligned} &[p', \omega_{p'}(Y_1, ..., Y_k)] = [p \cdot g, \omega_{p \cdot g}(Y_1, ..., Y_k)] = [p, \mathrm{Ad}_g \omega_{p \cdot g}(Y_1, ..., Y_k)] \\ =&[p, (r_{g^{-1}}^* \omega)_{p \cdot g}(Y_1, ..., Y_k)] = [p, \omega_p((r_{g^{-1}})_* Y_1, ..., (r_{g^{-1}})_* Y_k)] = [p, \omega_p(Y_1, ..., Y_k)]. \end{aligned}$$

In the last step we have used that $\pi_*((r_{g^{-1}})_* Y_i) = \pi_*(Y_i)$. In a local gauge it is easy to see that $\varphi(\omega)$ is smooth, so indeed $\varphi(\omega) \in \Omega^K(M, \mathrm{Ad}(P))$. The map φ is also linear, but we need to verify its bijectivity. Let $\omega \in \Omega^k(M, \mathrm{Ad}(P))$. Then $\varphi^{-1}(\omega)$ is given by $[p, \varphi^{-1}(\omega)_p(Y_1, ..., Y_k)] = \omega_{\pi(p)}(\pi_* Y_1, ..., \pi_* Y_k)$. Clearly $\varphi(\varphi^{-1}(\omega)) = \omega$. Moreover, $\varphi^{-1}(\omega)$ is horizontal since it is defined through the projected vectors $\pi_* Y_i$ and of type Ad by construction. □

From the above two results it follows that differences of connection 1-forms on a principal bundle can be identified with elements of $\Omega^1(M, \mathrm{Ad}(P))$, and the curvature F^A of a connection 1-form A can be identified with an element $F^A_M \in \Omega^2(M, \mathrm{Ad}(P))$. This can be seen as a generalisation of the Abelian case, in which the curvature defines a 2-form in $\Omega^2(M, \mathfrak{g})$, as explained under Proposition 2.11. Indeed, if G is abelian then $\mathrm{Ad}(P)$ is the trivial vector bundle $M \times \mathfrak{g}$.

Remark 2.14. In gauge quantum field theory, particles are described as excitations of a vacuum field. This vacuum gauge field is described by some connection 1-form A_0. Thus, classically speaking, gauge bosons as excitations of the vacuum field are described by a difference $A - A_0$ of two connection 1-forms. This difference can be identified with an element of $\Omega^1(M, \mathrm{Ad}(P))$ and it is therefore that gauge boson fields are said to transform under the adjoint representation [59, p. 311].

To define the Yang-Mills Lagrangian we now fix an n-dimensional pseudo-Riemannian manifold (M, g) and a principal G-bundle $\pi\colon P \to M$, where G is compact of dimension r, with an Ad-invariant positive definite scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. On simple Lie algebras this scalar product is unique up to a positive factor and is given by the negative Killing form, and for direct sums of simple Lie algebras (like in the Standard model), it is an orthogonal direct sum of such scalar products on each of the terms (cf. Theorems 2.5.3 and 2.5.4 in [59]). The physical constants that determine the Ad-invariant positive scalar product on the compact Lie algebra of the theory are called the *coupling constants*, and for the Standard Model there are three of them.

Now, for any representation $\rho\colon G \to GL(V)$, a G-invariant scalar product $\langle\cdot,\cdot\rangle_V\colon V\times V \to \mathbb{K}$ determines a so-called *bundle metric* $\langle\cdot,\cdot\rangle_E$ on the associated bundle $E = P \times_\rho V$ by

$$\langle [p, v], [p, w]\rangle_{E_x} = \langle v, w\rangle_V \qquad p \in P, v, w \in V.$$

The G-invariance of the scalar product guarantees that this is well-defined. A bundle metric on a vector bundle is just a metric on each fibre which varies smoothly with the basepoint, i.e. it is a section in $\Gamma(E^* \otimes E^*)$, defining a metric at each base point. The bundle metric allows us to define a scalar product of twisted forms

$$\langle\cdot,\cdot\rangle_E\colon \Omega^k(M, E) \times \Omega^k(M, E) \to C^\infty(M, \mathbb{K})$$

by choosing a local frame $e_1, ..., e_l$ for E_U, expanding $F, G \in \Omega^k(M, E)$ locally as $F_U = \sum_{i=1}^{l} F_i \otimes e_i$ and $G_U = \sum_{j=1}^{l} G_j \otimes e_j$, with $F_i, F_j \in \Omega^k(U, \mathbb{K})$, and locally setting

$$\langle F, G\rangle_E(x) = \sum_{i,j=1}^{l} \langle F_i, G_i\rangle(x)\langle e_i, e_j\rangle_E(x), \qquad x \in U,$$

where $\langle\cdot,\cdot\rangle\colon \Omega^k(M, \mathbb{K}) \times \Omega^k(M, \mathbb{K}) \to C^\infty(M, \mathbb{K})$ is the scalar product of forms which for real forms is given locally by

$$\langle \omega, \eta\rangle(x) = \sum_{\mu_1<\cdots<\mu_k} \omega_{\mu_1\ldots\mu_k}(x)\eta^{\mu_1\cdots\mu_k} = \frac{1}{k!}\omega_{\mu_1\ldots\mu_k}\eta^{\mu_1\cdots\mu_k}(x), \qquad x \in U, \tag{2.4}$$

and for complex forms gets a complex conjugate on the $\omega_{\mu_1\ldots\mu_k}$. For the adjoint bundle and the Ad-invariant scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ we thus also get a bundle metric $\langle\cdot,\cdot\rangle_{\mathrm{Ad}(P)} \in \Gamma(\mathrm{Ad}(P)^* \otimes \mathrm{Ad}(P)^*)$ and also a scalar product of twisted forms $\langle\cdot,\cdot\rangle_{\mathrm{Ad}(P)}\colon \Omega^k(M, \mathrm{Ad}(P)) \times \Omega^k(M, \mathrm{Ad}(P)) \to C^\infty(M, \mathbb{R})$. We use this to define Yang-Mills theory.

Definition 2.15. The *Yang-Mills Lagrangian* for a connection 1-form $A \in \Omega^1(P, \mathfrak{g})$ and its associated curvature 2-form $F^A_M \in \Omega^2(M, \mathrm{Ad}(P))$ (cf. Theorem 2.13) is given by

$$\mathcal{L}_{YM}[A] = -\frac{1}{2}\langle F^A_M, F^A_M\rangle_{\mathrm{Ad}(P)} \in C^\infty(M, \mathbb{R}).$$

The Yang-Mills action is then

$$S_{YM}[A] = -\frac{1}{2}\int_M \langle F^A_M, F^A_M\rangle_{\mathrm{Ad}(P)} d\mathrm{vol}_g.$$

Proposition 2.16. The Yang-Mills Lagrangian is gauge-invariant.

Proof. Let $f \in \mathcal{G}(P)$ be a principal bundle automorphism. The well-known transformation behaviour of $F^A \in \Omega^2(P, \mathfrak{g})$ is $F^{f^*A} = \mathrm{Ad}_{\sigma_f} \circ F^A$, where σ_f is defined as in Proposition 2.5. Denoting by $f\cdot$ the action on the associated adjoint bundle as explained around equation 2.2, we have $F^{f^*A}_M = f^{-1} \cdot F^A_M$. From the construction in Theorem 2.13 and the Ad-invariance of $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ it follows that $\langle\cdot,\cdot\rangle_{\mathrm{Ad}(P)}$ is invariant under the action of f^{-1}, and the claim follows. □

Remark 2.17. The *Hodge star operator* $*\colon \Omega^k(M,\mathbb{K}) \to \Omega^{n-k}(M,\mathbb{K})$ is the linear map which for real-valued forms satisfies $\langle \omega, \eta\rangle \mathrm{dvol}_g = \omega \wedge *\eta$ and for complex-valued forms $\langle \omega, \eta\rangle \mathrm{dvol}_g = \bar{\omega} \wedge *\eta$. In the case of twisted forms on a vector bundle E it can be generalised in a local frame $\{e_i\}$ as $*F = \sum_{i=1}^r (*F_i) \otimes e_i$ for $F \in \Omega^k(M, E)$. This allows us to alternatively write the Yang-Mills action for $G = U(1), SU(N)$ as

$$S_{YM}[A] = -\frac{1}{2}\int_M \mathrm{Tr}\, F_M^A \wedge *F_M^A.$$

To describe the Yang-Mills Lagrangian locally, let us choose a $\langle \cdot,\cdot\rangle_{\mathfrak{g}}$-orthonormal basis $T_1, ..., T_r$ of $\mathfrak{g}$ and a local gauge $s\colon U \to P$. We define a scalar product of local forms $\langle \cdot,\cdot\rangle_{\mathfrak{g}}\colon \Omega^k(U,\mathfrak{g}) \times \Omega^k(U,\mathfrak{g}) \to C^\infty(U,\mathfrak{g})$ in the same way as in equation 2.4, i.e.

$$\langle \omega, \eta\rangle_{\mathfrak{g}} = \sum_{\mu_1<\cdots<\mu_k} \langle \omega_{\mu_1\ldots\mu_k}, \eta^{\mu_1\cdots\mu_k}\rangle_{\mathfrak{g}} = \frac{1}{k!}\langle \omega_{\mu_1\ldots\mu_k}, \eta^{\mu_1\cdots\mu_k}\rangle_{\mathfrak{g}}.$$

Writing the local field strength $F_s^A = s^*F^A \in \Omega^2(U,\mathfrak{g})$ in coordinates as $F_{\mu\nu}^A = F_s^A(\partial_\mu, \partial_\nu)$, we expand $F_s^A = F_s^a \otimes T_a$ and $F_{\mu\nu}^A = F_{\mu\nu}^a T_a$, where $F_s^a \in \Omega^2(U), F_{\mu\nu}^a \in C^\infty(U)$. Then the local expression for the Yang-Mills action becomes

$$\mathcal{L}_{YM}[A] = -\frac{1}{2}\langle F_s^A, F_s^A\rangle_{\mathfrak{g}} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu}.$$

Defining the *structure constants* f_{abc} through $[T_a, T_b] = \sum_{c=1}^r f_{abc} T_c$ we can express the local field strength $F_{\mu\nu}^A = \partial_\mu A^\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ in the Lie algebra basis as

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bca} A_\mu^b A_\nu^c.$$

Writing out the entire Lagrangian in these terms yields

$$\mathcal{L}_{YM}[A] = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} = -\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu\right) \tag{2.5}$$

$$-\frac{1}{2} f_{abc}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) A^{b\mu} A^{c\nu} - \frac{1}{4} f_{abc} f_{ade} A_\mu^b A_\nu^c A^{d\mu} A^{e\nu}. \tag{2.6}$$

This shows that in quantum Yang-Mills theory, there are cubic and quartic interactions between gauge bosons, but only in the non-Abelian case in which the structure constants do not vanish.

2.2.2. Scalar fields

To describe scalar fields coupled to a gauge field A, we consider a complex representation written $\rho\colon G \to GL(W)$, where $W = \mathbb{C}^d$. We also need a G-invariant Hermitian inner product $\langle \cdot,\cdot\rangle_W$ which induces a bundle metric on the associated bundle $E = P \times_\rho W \to M$. We again have a scalar product of forms $\langle \cdot,\cdot\rangle_E$ and we denote the covariant derivative by $\nabla^A\colon \Gamma(E) \to \Omega^1(M, E)$.

Definition 2.18. The *Higgs Lagrangian* for a gauge field $A \in \Omega^1(P,\mathfrak{g})$ and a multiplet of scalar fields $\Phi \in \Gamma(E)$ is

$$\mathcal{L}_H[A,\Phi] = \langle \nabla^A\Phi, \nabla^A\Phi\rangle_E - V(\Phi),$$

where $V(\Phi) = V(\langle \Phi, \Phi\rangle_E)$ is a gauge-invariant potential and $V\colon \mathbb{R} \to \mathbb{R}$ is a polynomial. The Yang-Mills-Higgs Lagrangian is $\mathcal{L}_H[A,\Phi] + \mathcal{L}_{YM}[A]$.

The gauge-invariance of the Higgs Lagrangian follows immediately from the covariance of the covariant derivative and the fact that the potential is a polynomial in a gauge-invariant term.

2.2.3. The Dirac term

To describe fermions in physics we use spinors, introduced in detail in appendix B. We thus fix an n-dimensional oriented and time-oriented (Definition B.11) manifold (M, g) of signature (s, t) (for the remainder of this thesis this will be $(1, 3)$), together with a spin structure $\mathrm{Spin}^+(M) \to M$ (Definition B.21) and complex spinor bundle $S \to M$ (Definition B.24). We also need an $\mathbb{R}$-bilinear form denoted $\langle\cdot,\cdot\rangle \colon \Delta \times \Delta \to \mathbb{C}$ on the Dirac spinor space (Definition B.6), which must be invariant under the action of $\mathrm{Spin}^+(s, t)$ and is called a *Dirac form*. The *Dirac conjugate* $\bar{\psi}$ of a Dirac spinor $\psi \in \Delta$ is then defined by $\bar{\psi} = \langle\psi, \cdot\rangle \colon \Delta \to \mathbb{C}$. For Minkowski spacetime we use the Weyl representation of the Clifford algebra $\mathrm{Cl}(1, 3)$, which is given by

$$\Gamma_0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},$$

where σ_i are the Pauli matrices. The Dirac form is then $\langle\psi, \phi\rangle = \psi^\dagger\Gamma_0\phi$. The Dirac form gives us an induced Dirac bundle metric $\langle\cdot,\cdot\rangle_S$, written $\langle\Psi, \Phi\rangle_S = \bar{\Psi}\Phi$.

Definition 2.19. We define the *Dirac Lagrangian* of a free spinor field $\Psi \in \Gamma(S)$ of mass m to be

$$\mathcal{L}_D[\Psi] = \mathrm{Re}(\bar{\Psi}D\Psi) - m\bar{\Psi}\Psi,$$

where $D \colon \Gamma(S) \to \Gamma(S)$ denotes the Dirac operator (Definition B.29).

If we want to couple fermions to a gauge field on a principal G-bundle $P \to M$, we need to consider a complex representation $\rho \colon G \to GL(V)$ with associated bundle $E = P \times_\rho V$ and G-invariant scalar product $\langle\cdot,\cdot\rangle_V$ with induced bundle metric $\langle\cdot,\cdot\rangle_E$. Together with the Dirac bundle metric $\langle\cdot,\cdot\rangle_S$ we then get a Hermitian scalar product $\langle\cdot,\cdot\rangle_{S\otimes E}$ on the twisted spinor bundle $S \otimes E$ (Definition B.30), again abbreviated $\langle\Psi, \Phi\rangle_{S\otimes E} = \bar{\Psi}\Phi$.

Definition 2.20. The Dirac Lagrangian for a twisted spinor field $\Psi \in \Gamma(S \otimes E)$ of mass m coupled to a gauge field $A \in \Omega^1(P, \mathfrak{g})$ is

$$\mathcal{L}_D[A, \Psi] = \mathrm{Re}(\bar{\Psi}D_A\Psi) - m\bar{\Psi}\Psi,$$

where $D_A \colon \Gamma(S \otimes E) \to \Gamma(S \otimes E)$ denotes the twisted Dirac operator (Definition B.32).

Remark 2.21. The Dirac Lagrangian for spinors coupled to a gauge field is gauge-invariant and all components of the gauge multiplet get the same mass from the mass term $m\bar{\Psi}\Psi$. If we would try to give the components of the multiplet different masses by introducing separate mass terms, we would run into the issue of the Lagrangian no longer being gauge-invariant when multiplet components are mixed under gauge transformations [59, p. 433]. We will treat this in more detail in section 2.3, where we will also show how the Higgs mechanism solves this issue.

For $\dim M = n$ even we want to extend this Lagrangian to account for twisted *chiral* fermions. This can straightforwardly be done for the massless Dirac Lagrangian, but it turns out that this is not so easy for massive twisted chiral fermions. Instead, a coupling to the Higgs field is needed through the Yukawa term. We will explain why and how in section 2.3.

2.2.4. Yukawa couplings

Before we get there, we briefly introduce the Yukawa sector of the Standard Model Lagrangian.

Definition 2.22. Let V_L, W, V_R be unitary representation spaces of the compact structure group G. Then a *Yukawa form* is a map $\tau \colon V_L \times W \times V_R \to \mathbb{C}$ which is invariant under the G-action, complex antilinear in V_L, real linear in W and complex linear in V_R.

Definition 2.23. If $\tau\colon V_L \times W \times V_R \to \mathbb{C}$ is a Yukawa form and $g_Y \in \mathbb{R}$ a (coupling) constant then the map $(\Delta_L \otimes V_L) \times W \times (\Delta_R \otimes V_R) \to \mathbb{R}$ given by

$$(\psi_L \otimes v_L, \phi, \psi_R \otimes v_R) \mapsto -2g_Y \mathrm{Re}(\bar{\psi}_L \psi_R \tau(v_L, \phi, v_R))$$

is called the *Yukawa coupling* and defines a gauge-invariant term

$$\mathcal{L}_Y [\Psi_L, \Phi, \Psi_R] = -2g_Y \,\mathrm{Re}\left(\bar{\Psi}_L \Phi \Psi_R\right) = -g_Y \left(\bar{\Psi}_L \Phi \Psi_R\right) - g_Y \left(\bar{\Psi}_L \Phi \Psi_R\right)^*$$

for the fields $\Psi_L \in \Gamma(S_L \otimes E_L), \Phi \in \Gamma(F), \Psi_R \in \Gamma(S_R \otimes E_R)$ where E_L, F, E_R are the associated bundles for the representation spaces V_L, W, V_R and S_L, S_R are the spinor bundles for the Dirac spinor spaces Δ_L, Δ_R.

We have now considered all the basic ingredients of the Standard Model, of which the Lagrangian is the sum of all the terms we have seen, i.e. the Yang-Mills-Higgs-Dirac-Yukawa Lagrangian. It is time, then, to examine how the Higgs mechanism relates to all these terms.

2.3. The full Higgs mechanism

We recall from the preamble to this chapter that the Higgs mechanism solves not only the problem of massive gauge bosons, but three problems related to mass terms:

- the existence of massive gauge bosons;
- different masses for fermions in the same gauge multiplet;
- non-zero masses of twisted chiral fermions.

We begin this section by showing how these problems arise in the first place. We then rigourously define vacuum configurations and the Higgs condensate and examine how these appear in the well-known unitary gauge. Subsequently we present the details of the "generation" of gauge boson masses, and we end with the meaning of the Higgs mechanism for fermions through the Yukawa couplings.

2.3.1. Massive problems

To describe massive particles in QFT we need quadratic terms in the fields. However, adding quadratic terms directly in the Standard Model Lagrangian turns out to be impossible both for bosons as well as for fermions. We will now explain why.

In Yang-Mills theory it is quite easy to see why quadratic terms lead to problems. In a local gauge the Lagrangian is that of equation 2.5, which contains kinetic terms, cubic and quartic terms, but no quadratic terms in the gauge field A. We could therefore try to add a term $\frac{1}{2}m^2 A^a_\mu A^\mu_a$, but it is immediately clear that this is not gauge-invariant. We could try to use the Ad-invariant scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ as in Definition 2.15, but we can only do so if we have a form in $\Omega^1(M, \mathrm{Ad}(P))$, and Theorem 2.13 shows that only differences of connection 1-forms can be interpreted that way. Thus, we need to find a different way to write down a gauge-invariant Lagrangian which contains quadratic terms in the gauge field.

Let us now consider fermions. Here the problem of mass terms seems less pressing, since in Definition 2.20 we have given the Dirac Lagrangian for a massive spinor field coupled to a gauge field. As noted in Remark 2.21 however, in the case of a gauge multiplet, all components of the multiplet get the same mass in the massive Dirac Lagrangian. Indeed, choosing a local section $s\colon U \to P$ and a basis $v_1, ..., v_r$ of the associated bundle vector space V to locally write a twisted spinor $\Psi = (\Psi_1, ..., \Psi_r)$ with $\Psi_i \in \Gamma(S)$ (cf. equation B.30), we have $m\bar{\Psi}\Psi = m\sum_{i=1}^r \bar{\Psi}_i\Psi_i$. In other words, all components have mass m. If we wanted to give different masses to different components, we would need terms $m_1\bar{\Psi}_1\Psi + ... + m_r\bar{\Psi}_r\Psi$.

To see why this could never be gauge-invariant unless all m_i were equal, we need to understand what a gauge transformation does on the spinor components. Since the spinor components are defined in the trivialisation of the associated bundle through the local gauge, as under definition B.30, they transform under gauge transformations as in Proposition 2.2, i.e. by the action of $\rho(\tau)$, where $\tau\colon U \to G$ is a gauge transformation. Thus, if $\rho\colon G \to V$ does not leave the basis vectors $v_1, \dots, v_r$ invariant, gauge transformations mix the spinor components. This will of course be the case for the standard representation, which is irreducible. Consider for instance the simple case of a spinor $\Psi = (\Psi_1, \Psi_2)$ whose components are rotated into each other by some gauge transformation, i.e. $\Psi_1 \to \Psi_2, \Psi_2 \to -\Psi_1$. If we were to have different masses $m_1 \neq m_2$, then this gauge transformation would act by sending $m_1\bar{\Psi}_1\Psi_1 + m_2\bar{\Psi}_2\Psi_2 \to m_1\bar{\Psi}_2\Psi_2 + m_2\bar{\Psi}_1\Psi_1$, so the gauge transformation would exchange the masses of the fields! We must therefore find another way to give fields in the same gauge multiplet different masses. This is crucial in the Standard Model, because at least all left-handed fermions appear in isospin multiplets: left-handed leptons appear as electron-neutrino doublets (plus heavier generations) and the up and down quarks also form a doublet (plus heavier generations). All such particle masses can thus only be differentiated by different Yukawa couplings to the Higgs field.

In addition to the problem of mass in gauge multiplets there is the issue of chirality, making the massive problems for the Standard Model even heavier. Chirality plays a central role in the Standard Model: the left- and right-handed representations of the structure group are 24- and 21-dimensional respectively. The problem arises from the following fact (see Propositions 6.7.13 and 7.6.7 in [59]).

Proposition 2.24. On an even-dimensional oriented and time-oriented Lorentzian spin manifold, any choice of Dirac form gives a bundle metric that vanishes on S_L and S_R separately and thus only pairs left-handed with right-handed spinors.

The same holds for twisted spinors on $S \otimes E = (S_L \otimes E) \oplus (S_R \otimes E)$, so the Dirac Lagrangian for spinors coupled to a gauge field can be written

$$\mathcal{L}_D[\Psi, A] = \mathrm{Re}\left(\bar{\Psi} D_A \Psi\right) - m\bar{\Psi}\Psi = \mathrm{Re}\left(\bar{\Psi}_L D_A \Psi_L + \bar{\Psi}_R D_A \Psi_R\right) - 2m\,\mathrm{Re}\left(\bar{\Psi}_L \Psi_R\right),$$

since the Dirac operator interchanges left- and right-handed spinors. So far there is nothing wrong: this Lagrangian has a mass term pairing left-handed and right-handed spinors. However, the problem arises when we consider twisted chiral spinors, i.e. if we have two representations $\rho_L\colon G \to GL(V_L)$ and $\rho_R\colon G \to GL(V_R)$ and define the twisted chiral spinor bundle $(S_L \otimes E_L) \oplus (S_R \otimes E_R)$ (Definition B.33). The massless Dirac Lagrangian $\mathcal{L}_D[\Psi, A] = \mathrm{Re}\left(\bar{\Psi}_L D_A \Psi_L + \bar{\Psi}_R D_A \Psi_R\right)$ then still works, but the mass term $-2m\,\mathrm{Re}\left(\bar{\Psi}_L \Psi_R\right)$ is not defined. To see this, recall that the Hermitian scalar product $\langle \cdot, \cdot \rangle_{S\otimes E}$ is constructed from the associated bundle metric $\langle \cdot, \cdot \rangle_E$ and the Dirac bundle metric $\langle \cdot, \cdot \rangle_S$. But this construction only makes sense if the left-handed and right-handed spinors have the same representation space, i.e. if $V_L \cong V_R$ [59, p. 435]. Indeed, to construct a Hermitian scalar product for the twisted chiral spinor bundle we would need a G-invariant *mass pairing* $\kappa\colon V_L \times V_R \to \mathbb{C}$ (complex antilinear in the first argument and complex linear in the second) of unitary representations. This mass pairing would then give a form $\kappa_E\colon E_L \times E_R \to \mathbb{C}$ which we could use to define a scalar product and thence a mass term for twisted chiral spinors. Unfortunately, we have the following version of Schur's lemma.

Lemma 2.25. Let $\rho_L\colon G \to GL(V_L)$ and $\rho_R\colon G \to GL(V_R)$ be irreducible, unitary, non-isomorphic representations. Then every mass pairing $\kappa\colon V_L \times V_R \to \mathbb{C}$ is zero.

Proof. We consider the dual $\bar{V}_L^*$ of the complex conjugate of V_L, i.e. $\bar{V}_L^* = \{\alpha\colon V_L \to \mathbb{C} \mid \alpha\ \mathbb{C}\text{-antilinear}\}$. This is also a representation space of G through $(g \cdot \alpha)(v_L) = \alpha(\rho_L(g^{-1})v_L)$ for any $g \in G, \alpha \in \bar{V}_L^*, v_L \in V_L$. Denoting by $\langle \cdot, \cdot \rangle_{V_L}$ the Hermitian form for which ρ_L is unitary, the $\mathbb{C}$-linear isomorphism $V_L \mapsto \bar{V}_L^*$ given by $v_L \to \langle \cdot, v_L \rangle_{V_L}$ is G-equivariant:

$$\rho_L(g)v_L \mapsto \langle \cdot, \rho_L(g)v_L \rangle_{V_L} = \langle \rho_L(g^{-1})\cdot, v_L \rangle_{V_L} = g \cdot \langle \cdot, v_L \rangle_{V_L}, \qquad g \in G, v_L \in V_L.$$

Suppose now that a mass pairing $\kappa \neq 0$ exists. Then we get a $\mathbb{C}$-linear map $V_R \to \bar{V}_L^*$ defined through $v_R \mapsto \kappa(\cdot, v_R)$, and this map is also G-equivariant since κ is G-invariant:

$$\rho_R(g)v_R \mapsto \kappa(\cdot, \rho_R(g)v_R) = \kappa(\rho_R(g^{-1})\cdot, v_R) = g \cdot \kappa(\cdot, v_R), \qquad g \in G, v_R \in V_R.$$

Combining these maps with the inverse of the isomorphism $V_L \to \bar{V}_L^*$, we get a G-equivariant $\mathbb{C}$-linear map $V_R \to V_L$ which is non-zero because $\kappa \neq 0$. But Schur's lemma states that if V_L and V_R are not isomorphic there exist no such non-trivial G-equivariant maps. We arrive at a contradiction. □

In conclusion, directly adding mass terms is highly problematic in any realistic theory of particle physics, which must include massive gauge bosons, gauge multiplets whose components have different masses and massive twisted chiral fermions. We need a different way of thinking about mass terms in a gauge-invariant Lagrangian. This way is the Higgs mechanism, in which mass is thought of not as something intrinsic to a quantum field, but as something arising through the interaction of a field with the Higgs field. This Higgs field must have a potential with a minimum away from zero so that it can function as a non-zero background called the *Higgs condensate*, with which massive particles can interact to varying degrees. Such an *extrinsic* view of mass is also advocated by Rivat [32]. As we will see in section 2.3.5, all massive particles in the Standard Model obtain their masses this way.

2.3.2. Vacuum gauges, vectors and configurations

For the gauge-theoretical definition of the Higgs condensate we consider the Yang-Mills-Higgs Lagrangian from Definition 2.18. Thus, we fix an n-dimensional pseudo-Riemannian manifold (M, g), a principal G-bundle $\pi\colon P \to M$ with compact structure group G of dimension r, a complex representation $\rho\colon G \to GL(W)$ with associated complex vector bundle $\pi_E\colon E = P \times_\rho W \to M$ and a G-invariant Hermitian scalar product $\langle\cdot,\cdot\rangle_W$ with associated bundle metric $\langle\cdot,\cdot\rangle_E$. We assume the Higgs vector space to be $W = \mathbb{C}^n$ with standard Hermitian product $\langle v, w\rangle_W = v^\dagger w$.

Definition 2.26. A *vacuum configuration* for $\mathcal{L}_{YM}[A] + \mathcal{L}_H[A, \Phi]$ is a pair (A_0, Φ_0) such that A_0 is flat (i.e. $F^{A_0} = 0$), Φ_0 is covariantly constant (i.e. $\nabla^{A_0}\Phi_0 = 0$) and Φ_0 is a minimum of V at every point of M.

Definition 2.27. An element $w_0 \in W$ is called a *vacuum vector* if it is a minimum of the potential function $V(w) = V(\langle w, w\rangle_W)$ from W to $\mathbb{R}$. The set of vacuum vectors in W is called the *vacuum manifold* for V.

The following result then shows how vacuum configurations correspond to vacuum vectors when the base space is connected and simply connected and the principal bundle is trivial.

Proposition 2.28. Assume M to be connected and simply connected and $P \to M$ to be trivial. Let (A_0, Φ_0) be a vacuum configuration. Then there exists a global gauge $s_0\colon M \to P$, called the *vacuum gauge*, such that $s_0^*A_0 = 0$ and $\Phi_0 = [s_0, w_0]$, where $w_0 \in W$ is a constant vacuum vector. Conversely, for any fixed global gauge $s_0\colon M \to P$, every vacuum vector $w_0 \in W$ defines a unique vacuum configuration (A_0, Φ_0) of this form, i.e. through $s_0^*A_0 = 0$, $\Phi_0 = [s_0, w_0]$.

Proof. Let (A_0, Φ_0) be a vacuum configuration. We work on $P = M \times G$ and use the well-known result that, on a simply connected base space, for every flat connection on the trivial bundle there is a bundle map that maps this flat connection to the canonical flat connection A^{can} (c.f Corollary II.9.2 in [60]). The canonical flat connection's horizontal subspace at a point $(x, g) \in M \times G$ is the tangent space to $M \times \{e\}$, and it can be written $A_0 = A^{can} = \text{proj}^*\theta$, where $\text{proj}\colon M \times G \to G$ is the projection and $\theta \in \Omega^1(G, \mathfrak{g})$ the Maurer-Cartan form $g^{-1}dg$. Let $s_0\colon M \to M \times G$ denote the section that sends $x \to (x, e)$. Then clearly $s^*A_0 = s^*\text{proj}^*\theta = (\text{proj} \circ s)^*\theta = 0$. Writing $\Phi_0 = [s_0, \phi_0]$ with $\phi_0\colon M \to W$, the local expression for the covariant derivative (equation 2.3) then gives $\nabla^{A_0}\phi_0 = d\phi_0 + \rho_*(s^*A_0)\phi_0 = d\phi_0$. Since Φ_0 is covariantly constant we conclude that ϕ_0 is constant, so that we can write $\Phi_0 = [s_0, w_0]$ for some $w_0 \in W$. But Φ_0 is also a minimum of V at every point, so w_0 must be a vacuum vector.

Conversely, let us fix a global gauge $s_0\colon M \to P$. By the result referred to above, any flat connection $A_0 \in \Omega^1(P, \mathfrak{g})$ is isomorphic to the canonical flat connection $A^{can} \in \Omega^1(M \times G, \mathfrak{g})$ in the sense that there exists a bundle map $f\colon P \to M \times G$ such that $A_0 = f^*A^{can}$. Thus, if we are to find a flat connection $A_0 \in \Omega^1(P, \mathfrak{g})$ such that $s_0^*A_0 = 0$, then we must find a bundle map $f\colon P \to M \times G$ for which it holds that $s_0^*f^*A^{can} = (f \circ s_0)^*A^{can} = 0$. But since the horizontal subspace of A^{can} at $(x, g) \in M \times G$ is the tangent space to $M \times \{e\}$, this just amounts to finding a bundle map $f\colon P \to M \times G$ such that $f \circ s_0(x) = (x, e)$ for any $x \in M$. This requirement uniquely defines f. Again, by the local expression for the covariant derivative it is clear that any vacuum vector $w_0 \in W$ defines a covariantly constant section $\Phi_0 = [s_0, w_0]$. □

We now fix a global vacuum gauge $s_0\colon M \to P$ and a vacuum vector $w_0 \in W$ and we let (A_0, Φ_0) denote the associated vacuum configuration.

Definition 2.29. We call the isotropy group $H = G_{w_0} \subset G$ of the vacuum vector $w_0 \in W$ the *unbroken subgroup*. If H is a proper subgroup of G then we call the gauge theory *spontaneously broken*.

Assuming the Higgs potential $V(w)$ to have a minimum w_0 away from $w = 0$ such that $w_0 \neq 0$, we call the nowhere vanishing field Φ_0 the *Higgs condensate*. In the Standard Model we have the well-known $V(w) = \mu^2 \|w\|^2 + \lambda \|w\|^4$ with $\lambda > 0, \mu^2 < 0$.

Example 2.30. For the *electroweak model* we take M to be 4-dimensional Minkowski spacetime. The structure group is $G = SU(2) \times U(1)$ and the Higgs vector space is $W = \mathbb{C}^2$ with the standard Hermitian scalar product. The representation $\rho\colon SU(2) \times U(1) \to GL(W)$ is unitary and given by

$$\left(A, e^{i\alpha}\right) \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = A \begin{pmatrix} e^{in_Y\alpha} & 0 \\ 0 & e^{in_Y\alpha} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where n_Y is non-zero natural number that we fix to to $n_Y = 3$, following [59]. Vacuum vectors for the Higgs potential satisfy $\|w_0\| = \sqrt{-\mu^2/2\lambda} = v/\sqrt{2}$, so the vacuum manifold is a 3-sphere in $\mathbb{C}^2$ with radius $v/\sqrt{2}$, and G acts transitively on this vacuum manifold. Choosing $w_0 = (0, v/\sqrt{2})$ we get

$$H \cong U(1) \cong \left\{ \left(\begin{pmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix}, e^{i\delta/(2n_Y)} \right) \mid \delta \in \mathbb{R} \right\} \subset G.$$

Crucially, the unbroken subgroup $H \cong U(1)$ is not just the second component of $G = SU(2) \times U(1)$ but lies diagonally in it. This leads to the famous Weinberg mixing angle, as we will explain in chapter 3.

2.3.3. Nambu-Goldstone and Higgs bosons

In the vacuum gauge $s_0\colon M \to P$ we can write the Higgs field as $\Phi = [s_0, \phi]$ with $\phi\colon M \to W$, where ϕ is also called the Higgs field. If we then shift $\phi = w_0 + \Delta\phi$ with respect to the vacuum vector $w_0 \in W$, we uncreatively call $\Delta\phi$ the *shifted Higgs field*. Denoting by $O_{w_0} = G \cdot w_0 \subset W$ the orbit of w_0, we have that O_{w_0} is an embedded submanifold of W isomorphic to G/H (see Corollary 3.8.10 in [59]). We split

$$W \cong T_{w_0}W = T_{w_0}O_{w_0} \oplus (T_{w_0}O_{w_0})^\perp,$$

with respect to the positive definite scalar product $\mathrm{Re}\langle\cdot,\cdot\rangle_W$. We will now consider the Hessian $\mathrm{Hess}(V)$ (the symmetric matrix of second derivatives) of the potential, which at every point $w \in W$ is a map $\mathrm{Hess}(V)_w\colon T_wW \to T_wW$. More generally, the Hessian can be defined on a Riemannian manifold with Levi-Civita connection ∇ by $\mathrm{Hess}(V)(X) = \nabla_X \mathrm{grad}\, V$, and it is symmetric in the sense that

$$\mathrm{Re}\langle \mathrm{Hess}(V)(X), Y\rangle_W = \mathrm{Re}\langle X, \mathrm{Hess}(V)(Y)\rangle_W, \qquad X, Y \in \mathfrak{X}(M).$$

Proposition 2.31. The Hessian $\mathrm{Hess}(V)_{w_0}$ preserves the splitting $W \cong T_{w_0}O_{w_0} \oplus (T_{w_0}O_{w_0})^\perp$.

Proof. The potential V is minimal on the entire orbit $O_{w_0} = G \cdot w_0$, so the gradient grad V is zero along the orbit, i.e. for all $X_{w_0} \in T_{w_0}O_{w_0}$ we have $\mathrm{Hess}(V)_{w_0}(X_{w_0}) = 0 \in T_{w_0}O_{w_0}$. In addition, for any $X_{w_0} \in (T_{w_0}O_{w_0})^\perp, Y_{w_0} \in T_{w_0}O_{w_0}$ we have by symmetry that

$$\mathrm{Re}\langle \mathrm{Hess}(V)_{w_0}(X_{w_0}), Y_{w_0}\rangle_W = \mathrm{Re}\langle X_{w_0}, \mathrm{Hess}(V)_{w_0}(Y_{w_0})\rangle_W = \mathrm{Re}\langle X_{w_0}, 0\rangle_W = 0,$$

so indeed $\mathrm{Hess}(V)_{w_0}(X_{w_0}) \in (T_{w_0}O_{w_0})^\perp$, which proves the result. □

Definition 2.32. Setting $d = \dim O_{w_0} = \dim G - \dim H$, by Proposition 2.31 there exist real orthonormal bases $e_1, ..., e_d$ of $T_{w_0}O_{w_0}$ and $f_1, ..., f_{2n-d}$ of $(T_{w_0}O_{w_0})^\perp$ (where $2n = \dim W = \dim \mathbb{C}^n$) consisting of eigenvectors of $\mathrm{Hess}(V)_{w_0}$, such that the e_i have eigenvalue 0 and the f_j have non-negative (since w_0 is a minimum the second derivatives are non-negative) eigenvalues $2m_j^2$ with $m_j \geq 0$. Through the isomorphism $W \cong T_{w_0}W$ we expand the shifted Higgs fields in these bases:

$$\Delta\phi = \frac{1}{\sqrt{2}} \sum_{i=1}^{d} \xi_i e_i + \frac{1}{\sqrt{2}} \sum_{j=1}^{2n-d} \eta_j f_j,$$

where the ξ_i and η_j are real scalar fields called *Nambu-Goldstone bosons* and *Higgs bosons* respectively. There are as many Nambu-Goldstone bosons as there are broken degrees of freedom and as many Higgs bosons as the real dimension of the Higgs vector space minus the number of Nambu-Goldstone bosons.

A Taylor expansion up to second order then gives $V(\phi) \approx V(w_0) + \frac{1}{2}\sum_{j=1}^{2n-d} m_{f_j}^2 \eta_j^2$ for the Higgs potential [59, p. 454]. Since the Lagrangian contains kinetic terms for the scalar fields, we can interpret the Higgs fields as scalar fields with masses m_{f_j} and the Nambu-Goldstone fields as massless bosons.

Example 2.33. For the electroweak theory from Example 2.30, in which the 4-dimensional structure group $G = SU(2) \times U(1)$ is broken to the 1-dimensional $U(1)$, there are three Nambu-Goldstone bosons ξ_1, ξ_2, ξ_3 and one Higgs boson η (also written H), since the Higgs vector space $\mathbb{C}^2$ has four real dimensions. Again choosing $w_0 = (0, \sqrt{-\mu^2/2\lambda}) = (0, v/\sqrt{2})$, we see that $T_{w_0}O_{w_0}$ is spanned (as a real vector space) by the vectors $(1,0), (i,0), (0,i)$ and $(T_{w_0}O_{w_0})^\perp$ is spanned by $(0,1)$. In this basis we decompose the Higgs field as

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 + i\xi_2 \\ i\xi_3 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H \end{pmatrix},$$

where H is a real scalar field. In the standard coordinates $x_1 + ix_2, x_3 + ix_4$ of $W = \mathbb{C}^2$ the potential is $V(x) = \mu^2 \|x\|^2 + \lambda \|x\|^4$, of which the second derivatives (i.e. the Hessian matrix) are given by

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = 2\mu^2 \delta_{ij} + 4\lambda \left(2x_i x_j + \|x\|^2 \delta_{ij}\right).$$

We take $e_1 = (1,0), e_2 = (i,0), e_3 = (0,i), f = (0,1)$ for the bases described in Definition 2.32. To find the eigenvalue $2m_f^2$ of the eigenvector $f \in (T_{w_0}O_{w_0})^\perp$ we calculate the Hessian's x_4x_4-component in w_0:

$$\frac{\partial^2 V}{\partial x_4 \partial x_4}\left(0, \frac{v}{\sqrt{2}}\right) = 2\mu^2 + 4\lambda\left(2\left(\frac{1}{\sqrt{2}}v\right)^2 + \frac{v^2}{2}\right) = 2\mu^2 + 6\lambda v^2 = 2\mu^2 - 6\mu^2 = -4\mu^2.$$

So $2m_H^2 = -4\mu^2$, i.e. $m_H = \sqrt{-2\mu^2} = v\sqrt{2\lambda}$. This shows how the Higgs mass is itself dependent on the parameters of the potential.

2.3.4. Unitary gauge

The Nambu-Goldstone bosons are unphysical in the Higgs mechanism. Indeed, their appearance is gauge-dependent: gauges can be chosen in which they vanish, as we will now show. Still denoting by $s_0 \colon M \to P$ the vacuum gauge and writing the Higgs field as $\Phi = [s_0, \phi]$, we consider gauge transformations $\tau \colon M \to G$ with respect to the vacuum gauge as in Proposition 2.7, such that $\phi(x) \mapsto \rho(\tau(x))\phi(x)$.

Definition 2.34. For a Higgs field $\phi \colon M \to W$ a physical gauge transformation $\tau \colon M \to G$ is called a *unitary gauge* with respect to a vacuum vector $w_0 \in W$ if all Nambu-Goldstone bosons of the transformed field $\rho(\tau)\phi$ with respect to w_0 vanish identically on M. We then say the transformed field $\phi' = \rho(\tau)\phi$ is in unitary gauge with respect to w_0.

Theorem 2.35. Consider the electroweak theory (Examples 2.30 and 2.33) with $G = SU(2) \times U(1)$ and Higgs field $\phi = (\phi_1, \phi_2)$, where $\phi_1, \phi_2 \colon M \to \mathbb{C}$. Assume $\phi_2(x) \neq 0$ for all $x \in M$. Then there exists a gauge transformation $\tau \colon M \to G$ such that $\rho(\tau)\phi = (0, \psi)$ for some $\psi \colon M \to \mathbb{R}$ and $\rho(\tau)\phi$ is in unitary gauge with respect to the vacuum vector $w_0 = (0, \nu/\sqrt{2})$.

Proof. We perform an $SU(2)$ gauge transformation $\tau_1 \colon M \to SU(2) \subset G$ given by

$$\tau(x) = \frac{1}{\sqrt{|\phi_1(x)|^2 + |\phi_2(x)|^2}} \begin{pmatrix} \phi_2(x) & -\phi_1(x) \\ \phi_1^*(x) & \phi_2^*(x) \end{pmatrix}.$$

This is well-defined since ϕ_2 is nowhere zero, and gives the transformed fields $(\rho(\tau)\phi)_1 = 0$ and $(\rho(\tau)\phi)_2(x) = \sqrt{|\phi_1(x)|^2 + |\phi_2(x)|^2}$. Clearly $\rho(\tau)\phi$ is in unitary gauge with respect to w_0 because the transformed field can be expressed purely in the Higgs basis vector $f = (0, 1)$ from Example 2.33. ☐

2.3.5. Mass generation

Let us now finally consider how particle masses are "generated" through the Higgs mechanism in the Standard Model, both for the weak gauge bosons as well as the leptons and quarks. We continue with the Yang-Mills-Higgs Lagrangian as in the previous section, with unbroken subgroup $H = G_{w_0} \subset G$ for the vacuum vector $w_0 \in W$. We denote by $\mathfrak{h} \subset \mathfrak{g}$ the Lie algebra of H and let $\mathfrak{h}^\perp \cong \mathfrak{g}/\mathfrak{h}$ denote its orthogonal complement with respect to the Ad-invariant positive definite scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. We define a mass form such that we can distinguish between broken and unbroken gauge bosons.

Definition 2.36. For the vacuum vector $w_0 \in W$ we define the positive semi-definite symmetric bilinear *mass form* $m \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by $(A, B) \to \langle \rho_*(A)w_0, \rho_*(B)w_0 \rangle_{\mathfrak{g}}$, where ρ_* is the Lie algebra representation induced from the representation $\rho \colon G \to GL(W)$.

Note that the kernel of the map $\mathfrak{g} \to W$ sending $A \mapsto \rho_*(A)w_0$ is precisely $\mathfrak{h}$ and that this map is injective on $\mathfrak{h}^\perp$. Thus, we can diagonalise the symmetric mass form m and find $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$-orthonormal bases $\alpha_1, ..., \alpha_d$ of $\mathfrak{h}^\perp$ and $\alpha_{d+1}, ..., \alpha_r$ of $\mathfrak{h}$ such that m is diagonal in this basis and satisfies $m(\alpha_a, \alpha_a) = \frac{1}{2}M_a^2$ with $M_a > 0$ for $1 \leq a \leq d$ and $M_a = 0$ for $d + 1 \leq a \leq r$. The $\alpha_1, ..., \alpha_d$ are called the *broken generators* and the $\alpha_{d+1}, ..., \alpha_r$ are called the *unbroken generators*. We can then expand a local gauge field $A_\mu = \sum_{a=1}^r A_\mu^a \alpha_a$ into *broken* and *unbroken gauge bosons*, such that the broken gauge bosons have masses M_a. These masses depend on $\|w_0\|$ and on the coupling constants used to define $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. Let us then express the local Yang-Mills-Higgs Lagrangian $\mathcal{L}_{\text{YMH}} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} - V(\phi) - (\nabla_\mu^A \phi)^\dagger \nabla^{A\mu}\phi$ in terms of the shifted Higgs field $\Delta\phi = \phi - w_0$ up to second order:

$$\mathcal{L}_{\text{YMH}} \approx -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} - V(\phi) - (\partial_\mu \Delta\phi)^\dagger(\partial^\mu \Delta\phi) - 2\text{Re}((\partial_\mu \Delta\phi)^\dagger(\rho_*(A_\mu)w_0)) - (\rho_*(A_\mu)w_0))^\dagger(\rho_*(A^\mu)w_0)).$$

This expression can be obtained by writing $\nabla_\mu^A \phi = \partial_\mu \Delta\phi + \rho_*(A_\mu)(w_0 + \Delta\phi)$ and using that

$$2\text{Re}((\partial_\mu \Delta\phi)^\dagger(\rho_*(A_\mu)w_0)) = (\partial_\mu \Delta\phi)^\dagger(\rho_*(A_\mu)w_0) + (\rho_*(A_\mu)w_0)^\dagger(\partial_\mu \Delta\phi).$$

We now assume ϕ to be in unitary gauge with respect to w_0, such that the Nambu-Goldstone bosons vanish. We saw in Theorem 2.35 that the unitary gauge always exists for the electroweak theory if the second component of the Higgs field is non-zero. This is the case if the fluctuations of the Higgs field around the vacuum vector w_0 are not too large. In unitary gauge, the term $2\text{Re}((\partial_\mu \Delta\phi)^\dagger(\rho_*(A_\mu)w_0))$ vanishes because $\Delta\phi$ is orthogonal to the orbit of w_0 by assumption, whereas $\rho_*(A_\mu)w_0$ is tangential to this orbit. We can thus also write $(\partial_\mu \Delta\phi)^\dagger(\partial^\mu \Delta\phi) = \frac{1}{2}\sum_{j=1}^{2n-d} \partial_\mu \eta_j \partial^\mu \eta_j$ purely in terms of the real Higgs fields η_j from Definition 2.32 (we have $f_j^\dagger f_j = 1$ by orthonormality). We also saw that up to second order $V(\phi) \approx V(w_0) + \frac{1}{2}\sum_{j=1}^{2n-d} m_{f_j}^2 \eta_j^2$. Lastly, we get

$$(\rho_*(A_\mu)w_0))^\dagger(\rho_*(A^\mu)w_0)) = m(A_\mu, A^\mu) = \frac{1}{2}\sum_{a=1}^{d} M_a^2 A_\mu^a A_a^\mu,$$

since the mass form vanishes for the unbroken gauge bosons. Putting this all together and ignoring the constant $V(w_0)$ we get the Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{YMH}} &\approx -\frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu}_{a} - \frac{1}{2}\sum_{j=1}^{2n-d} m_{f_j}^2\eta_j^2 - \frac{1}{2}\sum_{j=1}^{2n-d}\partial_\mu\eta_j\partial^\mu\eta_j - \frac{1}{2}\sum_{a=1}^{d} M_a^2 A^a_\mu A^\mu_a \\ &= \sum_{a=1}^{d}\left(-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a) - \frac{1}{2}M_a^2 A^a_\mu A^\mu_a\right) - \sum_{b=d+1}^{r}\frac{1}{4}(\partial_\mu A^b_\nu - \partial_\nu A^b_\mu)(\partial^\mu A^\nu_b - \partial^\nu A^\mu_b) \\ &\quad - \sum_{j=1}^{2n-d}\frac{1}{2}\left(\partial_\mu\eta_j\partial^\mu\eta_j + m_{f_j}^2\eta_j^2\right).\end{aligned}$$

In other words, in unitary gauge we have d broken gauge bosons with masses M_a, $r-d$ unbroken, massless gauge bosons and $2n-d$ scalar fields with masses m_{f_j}. For the electroweak theory specifically we have three broken gauge bosons (the W^+, W^- and Z), one unbroken gauge boson (the photon) and one Higgs field.

Mass generation for fermions occurs through the Yukawa couplings. We focus on leptons because this is what we will apply the dressing field method to in chapter 3, but we also briefly consider quarks for the sake of completeness. Leptons exist in three generations $i = e, \mu, \tau$ with $SU(2)\times U(1)$ representation spaces $L^i_L = \mathbb{C}^2\otimes\mathbb{C}_{-1}$, $L^i_R = \mathbb{C}\otimes\mathbb{C}_{-2}$ and $W = \mathbb{C}^2\otimes\mathbb{C}_1$, where $\mathbb{C}^2$ denotes the standard representation of $SU(2)$, $\mathbb{C}$ denotes the trivial 1-dimensional representation of $SU(2)$ and $\mathbb{C}_y$ denotes the 1-dimensional representation in which $\alpha\in U(1)$ acts by $z\mapsto\alpha^{3y}z$. It is not hard to see then that $\tau^i_l\colon L^i_L\times W\times L^i_R\to\mathbb{C}$ defined by $(l_L,\phi,l_R)\mapsto g_i l_L^\dagger\phi l_R$ is an $SU(2)\times U(1)$-invariant Yukawa form, where g_i is a coupling constant.

Writing the first generation left-handed $SU(2)$-doublet as $l_L = (\nu_{1L}, e_L)$ and $\phi = (0, v+H)/\sqrt{2}$ in unitary gauge and denoting $\nu_R = e_R$, we get $\tau^1_L(\nu_L,\phi,\nu_R) = g_e\bar{e}_L(v+H)e_R/\sqrt{2}$ and similarly for the other generations. Here ν_{iL} denotes the left-handed neutrino. For the first generation, the Yukawa term from Definition 2.23 then becomes

$$\mathcal{L}^e_Y = -2m_e\mathrm{Re}(\bar{e}_L e_R) - \frac{2}{v}m_e\mathrm{Re}(\bar{e}_L e_R)H, \tag{2.7}$$

and similarly for the other generations. Here $m_i = g_i v/\sqrt{2}$ are the lepton masses, which depend on the coupling constant and Higgs field vacuum value. We thus have massive leptons which can interact with the Higgs field by changing handedness through the cubic term.

Mass generation for quarks is analogous, with $SU(2)\times U(1)$ representation spaces $Q_L = \mathbb{C}^2\otimes\mathbb{C}_{1/3}$ and $Q_R = (\mathbb{C}\otimes\mathbb{C}_{4/3})\oplus(\mathbb{C}\otimes\mathbb{C}_{-2/3})$ for every quark generation. The strong force is not taken into account here. A Yukawa form is then also defined for the quarks (see Lemma 8.8.4 in [59]) such that by expanding the Higgs field around a vacuum vector the quarks obtain masses $m_i = g_i v/\sqrt{2}$ for every flavour $i = u, d, c, s, t, b$ and interact with the Higgs field by changing handedness. The Lagrangian for every quark generation (up-down, charm-strange, top-bottom) then looks exactly like 2.7.

Thus, it has become clear that every massive particle in the Standard Model gets its mass by interacting with the Higgs field. Even if right-handed neutrinos are added in order to account for neutrino masses, this happens by Yukawa coupling to the Higgs field in a similar way as for quarks [59, p. 532]. Philosophically speaking we may therefore be inclined to say that the Higgs mechanism has explanatory value even if it is not viewed as an instance of dynamical symmetry breaking. It explains that mass terms in the Standard Model cannot be added naively, but must always come from interaction with some other field. It supports an *extrinsic* instead of *intrinsic* perspective on the property of mass.

3. The Dressing Field Method

As announced in section 1.3, we begin our journey through the landscape of alternative accounts of the Higgs mechanism with the dressing field method (DFM), since it provides the most radical viewpoint, namely, that there is no SSB at all. The DFM traces back to work by Dirac from 1955, in which he recognised that "the requirement of manifest gauge invariance prevents one from using the concept of an electron separated from its Coulomb field" [61]. Its basic idea is that one should not consider gauge-dependent elementary fields, but rather gauge-invariant *dressed* fields. This is what Dirac means when he says one cannot separate an electron from the Coulomb field it generates, i.e. from its "dress". To create dressed fields one in turn uses a *dressing* field. This approach is claimed to provide "an alternative interpretation of the BEHGHK mechanism that is more in line with the conclusions of the community of philosophers of physics" [62, p. 4]. In particular "the DFM approach to the electroweak model is consistent with Elitzur's theorem stating that in lattice gauge theory a gauge symmetry cannot be spontaneously broken" [41, p. 66]. François' article even contains a section titled "there is no SSB in the electroweak model and we long suspected it" [63].

In this chapter we present and criticise the DFM from the perspective of mathematical gauge theory. We introduce it in section 3.1, based on [41, 62–66]. We then apply it to the Abelian Higgs model and the electroweak theory in section 3.2 and discuss how the FMS approach allows us to transfer DFM-like ideas into the context of perturbative QFT in section 3.3. We end with some philosophical reflections on the DFM in section 3.4.

3.1. Dressing fields and dressed fields

In the following we work with a principal G-bundle $\pi\colon P \to M$, where $G \subset GL(n, \mathbb{R})$, and we let $H \subset G$ denote a closed subgroup. Let us define the fundamental object of the DFM.

Definition 3.1. A map $u\colon P \to H$ satisfying $u(ph) = h^{-1}u(p)$ for all $h \in H$ is called an H-*dressing field*.

Taking inspiration from Proposition 2.5, an H-dressing field u allows us to define a map $f_u\colon P \to P$ by $f_u(p) = pu(p)$. This map, however, is not a bundle automorphism (i.e. a gauge transformation), since

$$f_u(ph) = (ph)u(ph) = phh^{-1}u(p) = pu(p) = f_u(p), \qquad p \in P, h \in H.$$

In other words, f_u is not H-equivariant and therefore also not G-equivariant if H is non-trivial. This equation shows that f_u is constant along the orbits of the action of H, which means that it factors through $P \to P/H$ [64]. In the case where $H = G$ this means that f_u defines a global section, since in that case $P/H = P/G \cong M$, implying that P is trivial. Thus, we can think of dressing fields as trivialisations in the direction of the subgroup H [65]. This is formalised in the following result.

Proposition 3.2. A dressing field $u\colon P \to H$ exists if and only if there is an isomorphism of H-spaces $P \cong P/H \times H$, where the action of H on P/H is trivial.

Proof. If $P \cong P/H \times H$ as H-spaces, then we can find an isomorphism of H-spaces $f\colon P \to P/H \times H$ that we can write as $f(p) = ([p], \tilde{f}(p))$ with $\tilde{f}\colon P \to H$. We then define the dressing field $u(p) = \tilde{f}^{-1}(p)$. Indeed, we then have $u(ph) = \tilde{f}^{-1}(ph) = (\tilde{f}(p)h)^{-1} = h^{-1}\tilde{f}^{-1}(p) = h^{-1}u(p)$.

Conversely, suppose a dressing field $u\colon P \to H$ exists. Then u is surjective since for any $h \in H$ we can take any $p \in P$, and then $u(pu(p)h^{-1}) = hu(p)^{-1}u(p) = h$. We now consider $Q = u^{-1}(\{e\}) \subset P$

with the trivial action of H. Then we have an isomorphism of H-spaces $P \cong Q \times H$ defined through $p \mapsto (pu(p), u(p)^{-1})$, with inverse $(q,h) \mapsto qh$. But since the map $P \to Q$ given by $p \mapsto pu(p)$ satisfies $ph \mapsto phu(ph) = phh^{-1}u(p) = pu(p)$ for any $h \in H$, it factors through the quotient $P \to P/H$. In fact, $P/H \to Q$ given by $[p] \mapsto pu(p)$ is an isomorphism of (trivial) H-spaces with inverse $q \mapsto [q]$. Indeed, for any $q \in Q$ we have $[q] \mapsto qu(q) = qe = q$ and for any $[p] \in P/H$ we have $[pu(p)] = [p]$ since $u(p) \in H$. With the two isomorphisms of H-spaces $P \cong Q \times H$ and $Q \cong P/H$ we thus get an isormorphism $P \cong P/H \times H$. □

Proposition 3.2 tells us how dressing fields trivialise the principal bundle, but it does not say anything about the fields on that bundle. The point of dressing fields, however, is that they can dress other fields.

Definition 3.3. Let u be an H-dressing field and $A \in \Omega^1(P, \mathfrak{g})$ a connection 1-form with curvature F. Let $\rho\colon G \to V$ be a representation giving an associated bundle $E = P \times_\rho V$ and $\phi\colon P \to V$ a G-equivariant map (equivalently a section of E). Then we define the *dressed fields*

$$\begin{aligned} A^u &= f_u^* A = u^{-1}Au + u^{-1}du, \\ \phi^u &= f_u^*\phi = \rho(u^{-1})\phi, \\ F^u &= f_u^* F = u^{-1}Fu = dA^u + \frac{1}{2}[A^u, A^u]. \end{aligned}$$

We note that A^u is not itself a connection 1-form. Similarly we define a dressed covariant derivative

$$D^u\phi^u = f_u^*(D\phi) = \rho(u^{-1})D\phi = d\phi^u + \rho_*(A^u)\phi^u.$$

These equalities follow from the transformation behaviour of the respective fields under gauge transformations, although the dressing field is of course not itself a gauge transformation.

The idea behind this definition is that the dressed fields have been rendered invariant under the H-valued gauge group $C^\infty(P,H)^H$. To see this, note that in the same way as in Proposition 2.5, an H-gauge transformation $\gamma \in C^\infty(P,H)^H$ defines $f_\gamma\colon P \to P$ by $f_\gamma(p) = p\gamma(p)$, such that we have

$$f_u \circ f_\gamma(p) = f_u(p\gamma(p)) = p\gamma(p)u(p\gamma(p)) = p\gamma(p)\gamma(p)^{-1}u(p) = pu(p) = f_u(p).$$

This shows that $f_\gamma^* f_u^* = (f_u \circ f_\gamma)^* = f_u^*$, so that all the dressed fields are invariant under the action of any $\gamma \in C^\infty(P,H)^H$, which works on the dressed fields via the pullback f_γ^*. Now, we would like to formalise the idea that a dressed connection is trivial in the direction of the dressing field, but can at the same time be viewed as a new connection 1-form on the reduced bundle with the residual group G/H as its structure group. This can indeed be done in the case where $H \subset G$ is a normal subgroup, such as for product groups $G = H \times J$ like the $SU(2) \times U(1)$ structure group of the electroweak theory.

Corollary 3.4. Suppose $u\colon P \to H$ is a dressing field and $G = H \times J$, where $H, J \subset G$ are closed subgroups. We split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{j}$ accordingly. Then $Q = u^{-1}(\{e\}) \subset P$ is a principal J-bundle and we can write any dressed connection $A^u \in \Omega^1(P, \mathfrak{g})$ as a direct sum $A^u|_Q = A_Q^{\mathfrak{h}} \oplus A_Q^{\mathfrak{j}}$ on Q, where $A_Q^{\mathfrak{h}} \in \Omega^1(Q, \mathfrak{h})$ is an H-gauge-invariant 1-form and $A_Q^{\mathfrak{j}} \in \Omega^1(Q, \mathfrak{j})$ is a J-connection 1-form.

Proof. From the proof of Proposition 3.2 we know $Q \cong P/H$, which is the quotient bundle with structure group $G/H \cong J$. Writing a connection 1-form $A \in \Omega^1(P, \mathfrak{g})$ as $A = A^{\mathfrak{h}} \oplus A^{\mathfrak{j}}$, we have $A^u = (A^{\mathfrak{h}})^u \oplus A^{\mathfrak{j}}$, since u is H-valued. Denoting by $\iota\colon Q \to P$ the inclusion, we get $\iota^* A^u = \iota^*(A^{\mathfrak{h}})^u \oplus \iota^* A^{\mathfrak{j}}$. Here $A_Q^{\mathfrak{j}} := \iota^* A^{\mathfrak{j}}$ is a J-connection 1-form (this follows from the properties of A) and

$$A_Q^{\mathfrak{h}} := \iota^*(A^{\mathfrak{h}})^u$$

is H-gauge invariant since it is dressed by $u\colon P \to H$. □

Thus, when reducing a gauge symmetry using a dressing field, the dressed connection fields can be viewed as gauge fields for the residual structure group. Similarly, a dressed matter field $\phi^u \colon P \to V$ reduces to a J-equivariant map $Q \to V$, i.e. a section of $Q \times_J V$, and the dressed covariant derivative becomes a proper covariant derivative for this reduced associated bundle [62]. Accordingly, we define a gauge symmetry to be *artificial* if we can find a local dressing field for the corresponding structure group. We then replace the fields in the Lagrangian by their dressed versions, such that we can reinterpret the Lagrangian as a functional of gauge-invariant fields. However, a theory might display a trade-off between locality and gauge-invariance in the sense that it can either be written in local gauge-dependent variables or in non-local gauge-invariant variables, but not in local gauge-invariant variables. Then we can only find a non-local dressing field, and in that case we call the gauge symmetry *substantial* [41].

3.2. The Higgs mechanism

Having introduced the DFM, we consider our two main theories of interest: the Abelian Higgs model and the electroweak theory. We present and criticise the argument that the DFM shows that the respective $U(1)$ and $SU(2)$ gauge symmetries are actually artificial, implying that SSB of these symmetries cannot represent anything physical. The derivations we present come from [41].

3.2.1. Abelian Higgs model

For the Abelian Higgs model we take the Yang-Mills-Higgs Lagrangian from Definition 2.18, with structure group $G = U(1)$ and standard representation $\rho \colon U(1) \to GL(\mathbb{C})$. A complex scalar field $\phi \in \Gamma(E)$ with $E = P \times_\rho \mathbb{C}$ is equivalently a G-equivariant map $\phi \colon P \to \mathbb{C}$, and we can use such a scalar field to define a dressing field. We do so by considering the polar decomposition $\phi = u\sqrt{\phi^*\phi}$ and taking $u \colon P \to U(1)$ to be the $U(1)$-dressing field [41]. To do this we must assume ϕ to be nowhere vanishing - we will shortly return to this issue. The G-equivariance of ϕ precisely guarantees that u is a dressing field. We then replace the Lagrangian $\mathcal{L}_{\text{YMH}}[A, \phi]$ as a function of the gauge field A and scalar field ϕ by the same Lagrangian but in terms of the dressed fields, i.e.

$$\mathcal{L}_{\text{YMH}}[A^u, \phi^u] = -\frac{1}{2}\langle F^u_M, F^u_M\rangle_{\text{Ad}(P)} + \langle D^u\phi^u, D^u\phi^u\rangle_E - V(\phi^u).$$

The fact that $\mathcal{L}_{\text{YMH}}[A, \phi] = \mathcal{L}_{\text{YMH}}[A^u, \phi^u]$ follows from the expressions $F^u = u^{-1}Fu$ and $\phi^u = \rho(u^{-1})\phi$, the Ad-invariance of the scalar product $\langle\cdot,\cdot\rangle_{\text{Ad}(P)}$ in the Yang-Mills Lagrangian, as well as from the G-invariance of the Hermitian inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$, which is here of course just given by $\langle\phi, \phi\rangle_{\mathbb{C}} = \phi^*\phi$. The dressed scalar field is actually just $\phi^u = \rho(u^{-1})\phi = \sqrt{\phi^*\phi}$, i.e. the modulus of ϕ.

Since we can apparently find a local dressing field through the polar decomposition, Berghofer et al. conclude that the $U(1)$ symmetry in the Abelian Higgs model is artificial and should therefore be reduced by means of the dressing field. But what are the implications of this for the standard account of the Higgs mechanism as an instance of SSB? Clearly, there is no more SSB to occur, since the dressed fields A^u and ϕ^u are $U(1)$-invariant. The potential $V \colon \mathbb{R}^+ \to \mathbb{R}$ is now a function of the positive real field $\phi^u = |\phi|$ and therefore has a *unique* ground state for any values of the parameters μ^2 and λ. For $\mu^2 > 0$ this is just $\phi^u = 0$ and for $\mu^2 < 0$ it is $\phi^u = \sqrt{-\mu^2/2\lambda}$.

In the phase $\mu^2 < 0$ an expansion around the unique vacuum configuration away from zero still gives a mass term for the dressed gauge field A^u, and in the $\mu^2 > 0$ phase the dressed gauge field is still massless. The mass of the gauge field is therefore still generated through a vacuum phase transition, but one in which SSB plays no role [41, p. 60]. However, something does not seem right in this analysis. If there really is a local dressing field for the Abelian Higgs model, then by Proposition 3.2 the underlying principal bundle is trivial. This is absurd, for we can very well define the Abelian Higgs model on a non-trivial principal $U(1)$ bundle with non-trivial associated bundle. The reason that we thought ourselves able to construct a dressing field through $\phi = u\sqrt{\phi^*\phi}$ is that we ignored the fact that u is ill-defined if $\phi = 0$. In other words, the DFM does not allow the dressed field ϕ^u to be zero: the dressed field space excludes field configurations in which the field is zero somewhere and the potential is now a map $V\colon \mathbb{R}^+ \to \mathbb{R}$ [41]. This is rather problematic, since in the massless phase we supposedly have a vanishing vacuum value $\phi^u = 0$, which is a field configuration that is not included in the field space of the DFM. On the other hand, when directly writing $\phi^u = \sqrt{\phi^*\phi}$ this expression is actually always well-defined and for $\phi = 0$ simply vanishes itself. It just cannot be considered to arise through a dressing field u, but must instead be viewed as a gauge-invariant composite objects which we might try to use to detect SSB. More on this in section 3.3.

In the massive phase the problem of a vanishing scalar field seems less pertinent, for there the vacuum configuration is away from zero. We could therefore consider small fluctuations around this vacuum configuration without running into the issue. We will return to this point shortly, for it similarly appears when the DFM is applied to the electroweak theory.

3.2.2. Electroweak theory with leptons

The picture of the DFM for the electroweak theory is similar to that of the Abelian Higgs model, albeit more involved, since in this case a residual $U(1)$ symmetry remains when the $SU(2)$ symmetry is reduced through a dressing field. In other words, we are in the situation described in Proposition 3.2 and its corollary. Moreover, we also add in leptons as in [41, 66].

We work in a gauge $s\colon M \to P$ (assuming P to be trivial), where M is Minkowski spacetime, as in Example 2.30. We split the gauge field $A \in \Omega^1(P, \mathfrak{u}(1) \oplus \mathfrak{su}(2))$ into $a + b$, where $a \in \Omega^1(M, \mathfrak{u}(1)$ and $b \in \Omega^1(M, \mathfrak{su}(2))$, and write the Higgs field $\varphi\colon M \to \mathbb{C}^2$ as $\varphi = (\varphi_1, \varphi_2)$, with the action of $SU(2) \times U(1)$ as in Example 2.30. In addition, we consider one generation of leptons consisting of a left-handed doublet $\psi_L = (\nu_L, e_L)$ and a right-handed singlet $\psi_R = e_R$ (see equation B.3 for the definition of a gauge multiplet). We recall from section 2.3.5 that the respective representation spaces of $SU(2) \times U(1)$ are $\mathbb{C}^2 \otimes \mathbb{C}_{-1}$ and $\mathbb{C} \otimes \mathbb{C}_{-2}$. Following [41] we therefore write the covariant derivatives as

$$\begin{aligned} D\varphi &= d\varphi + (gb + g'a)\varphi, \\ D\psi_L &= d\psi_L + (gb - g'a)\psi_L, \\ D\psi_R &= d\psi_R - 2g'a\psi_R. \end{aligned}$$

Here g and g' are coupling constants and the factor 2 for ψ_R comes from the representation $\mathbb{C}_{-2}$ of $U(1)$ for the right-handed lepton. Of course this is physics notation, and we should keep in mind that the gauge fields a and b only work on the fermion fields through their respective induced Lie algebra representations, as in equation 2.3. The Lagrangian of the theory is [41]

$$\begin{aligned} \mathcal{L}_{EW}[a, b, \varphi, \psi_L, \psi_R] = &-\frac{1}{2}\mathrm{Tr}\, F \wedge *F - \frac{1}{2}\mathrm{Tr}\, G \wedge *G + \langle D\varphi, *D\varphi\rangle - V(\varphi)\mathrm{dvol}_g \\ &+ \langle \psi_L, \not{D}\psi_L\rangle + \langle \psi_R, \not{D}\psi_R\rangle + g_l\langle \psi_L, *\varphi\rangle\psi_R + g_l\bar{\psi}_R\langle \varphi, *\psi_L\rangle, \end{aligned}$$

where F and G are the curvatures of a and b and g_l denotes the lepton Yukawa coupling.

Since we have included the coupling constants in the covariant derivatives, these must also appear in the action of gauge transformations $\alpha\colon M \to U(1)$ and $\beta\colon M \to SU(2)$. Using the notation from [41], the transformation behaviour of the fields is

$$\begin{aligned} a &\to a + \alpha^{-1}d\alpha/g', & a &\to a, \\ b &\to b, & b &\to \beta^{-1}b\beta + \beta^{-1}d\beta/g, \\ \varphi &\to \alpha^{-1}\varphi, & \varphi &\to \beta^{-1}\varphi, \\ \psi_L &\to \alpha\psi_L, & \psi_L &\to \beta^{-1}\psi_L, \\ \psi_R &\to \alpha^2\psi_R, & \psi_R &\to \psi_R. \end{aligned}$$

Again, these transformation rules follow from the definitions of the representation spaces $\mathbb{C}^2$, $\mathbb{C}^2 \otimes \mathbb{C}_{-1}$ and $\mathbb{C} \otimes \mathbb{C}_{-2}$ for the Higgs field and left- and right-handed leptons.

Just as for the Abelian Higgs model, we try to find an $SU(2)$ dressing field through the polar decomposition of the Higgs field. Since we are now working in a global gauge, a dressing field is equivalently a map $u\colon M \to SU(2)$ that transforms as $u \to \beta^{-1}u$ under the action of a gauge transformation $\beta\colon M \to SU(2)$. We decompose the Higgs field into $\varphi = u\rho$, where $\rho = (0, \|\varphi\|) \in \mathbb{C}^2$ and

$$u(\varphi) = \frac{1}{\rho}\begin{pmatrix} \varphi_2^* & \varphi_1 \\ -\varphi_1^* & \varphi_2 \end{pmatrix} \in SU(2),$$

where $\rho = \|\varphi\|$ is viewed as $\mathbb{R}$-valued. This should remind the attentive reader of Theorem 2.35 on the existence of the unitary gauge for the electroweak theory. The gauge transformation used there serves as inspiration for the dressing field here, though a dressing field is of course not a gauge transformation.

We must check that u defined this way is actually a dressing field. Now, ρ is clearly gauge-invariant, so under a gauge transformation $\beta\colon M \to SU(2)$ sending $\varphi = u\rho \to \beta^{-1}\varphi = \beta^{-1}u\rho$, we must have $u \to \beta^{-1}u$, as required. Thus, we can use u to dress all the fields in the theory, and since u is local, we call the $SU(2)$ gauge symmetry artificial. The Higgs mechanism in terms of SSB is then no longer possible, since there is no symmetry left to break. Indeed, the dressed Higgs field is $\varphi^u = u^{-1}\varphi = u^{-1}u\rho = \rho$, like in the Abelian Higgs model.

But what of the residual $U(1)$ symmetry? The dressing field u is not invariant under it: for a gauge transformation $\alpha\colon M \to U(1)$ we have

$$u(\varphi) = \frac{1}{\rho}\begin{pmatrix} \varphi_2^* & \varphi_1 \\ -\varphi_1^* & \varphi_2 \end{pmatrix} \to u(\alpha^{-1}\varphi) = \frac{1}{\rho}\begin{pmatrix} \alpha\varphi_2^* & \alpha^*\varphi_1 \\ -\alpha\varphi_1^* & \alpha^*\varphi_2 \end{pmatrix} = \frac{1}{\rho}\begin{pmatrix} \varphi_2^* & \varphi_1 \\ -\varphi_1^* & \varphi_2 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix} =: u(\varphi)\tilde{\alpha}.$$

Moreover, it follows from this that the dressed fields transform as

$$\begin{aligned}
&b^u = u^{-1}bu + u^{-1}du/g \to \tilde{\alpha}^{-1}b^u\tilde{\alpha} + \tilde{\alpha}^{-1}d\tilde{\alpha}/g,\\
&G^u = u^{-1}Gu \to \tilde{\alpha}^{-1}G^u\tilde{\alpha},\\
&\psi_L^u = (\nu_L^u, e_L^u) = u^{-1}\psi_L \to \tilde{\alpha}^{-1}u^{-1}\alpha\psi_L = \alpha\tilde{\alpha}^{-1}\psi_L^u.
\end{aligned}$$

Two observations are in order here. Firstly, since the dressed field b^u still transforms like a connection 1-form under $U(1)$ gauge transformations, it seems the problem of defining gauge-invariant mass terms remains. Secondly, we have

$$\alpha\tilde{\alpha}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^2 \end{pmatrix},$$

which means that the top component ν_L^u of the dressed left-handed doublet is $U(1)$-invariant, whereas the bottom component e_L^u transforms in the same fashion as ψ_R, which allows us to easily pair them in the Yukawa term. As for the first observation, let us write out $b^u = b_a^u\sigma_a$ in terms of the Pauli matrices and define

$$b^u = b_a^u\sigma^a = \begin{pmatrix} b_3^u & b_1 - ib_2^u \\ b_1^u + ib_2^u & -b_3^u \end{pmatrix} = \begin{pmatrix} b_3^u & W^- \\ W^+ & -b_3^u \end{pmatrix}.$$

If we now consider the $U(1)$ transformation behaviour of these fields, we find

$$\begin{aligned}
b^u \to \tilde{\alpha}^{-1}b^u\tilde{\alpha} + \tilde{\alpha}^{-1}d\tilde{\alpha}/g &= \begin{pmatrix} \alpha^* & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} b_3^u & W^- \\ W^+ & -b_3^u \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix} + \frac{1}{g}\begin{pmatrix} \alpha^* & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} d\alpha & 0 \\ 0 & d\alpha^{-1} \end{pmatrix}\\
&= \begin{pmatrix} b_3^u + \alpha^{-1}d\alpha/g & \alpha^{-2}W^- \\ \alpha^2W^+ & -b_3^u - \alpha^{-1}d\alpha/g \end{pmatrix}.
\end{aligned}$$

Thus, the $W^{\pm}$ fields transform tensorially and can be massive [62]. In addition, the combination term $gb_3^u - g'a$ is $U(1)$-invariant, whereas $g'b_3^u + ga$ transforms as a $U(1)$ connection 1-form. These observations lead us to the introduction of the famous *weak mixing angle* $\tan\theta_W = \frac{g'}{g}$, so that we can rotate

$$\begin{pmatrix} A \\ Z^0 \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} a \\ b_3^u \end{pmatrix} = \frac{1}{\sqrt{g^2 + g'^2}} \begin{pmatrix} ga + g'b_3^u \\ gb_3^u - g'a \end{pmatrix}.$$

Finally, the Lagrangian then becomes [41, p. 65]

$$\mathcal{L}_{EW}[A, W^{\pm}, Z^0, \rho, e_L^u, e_R, \nu_L^u] = -\frac{1}{2}\mathrm{Tr}\, F \wedge *F - \frac{1}{2}\mathrm{Tr}\, G^u \wedge *G^u + d\rho \wedge *d\rho - g^2\rho^2 W^+ \wedge *W^-$$
$$- \left(g^2 + g^2\right)\rho^2 Z^0 \wedge *Z^0 - V(\rho)\mathrm{dvol}_g + \left\langle \psi_L^u, \not{D}^u \psi_L^u \right\rangle + \left\langle \psi_R, \not{D}\psi_R \right\rangle + g_l\left(\bar{e}_L^u \rho e_R + \bar{e}_R \rho e_L^u\right)\mathrm{dvol}_g\,.$$

It contains the massive $W^{\pm}$ and Z bosons, the massless photon, and massive left-handed and right-handed leptons. What, then, has become of the idea that SSB is responsible for the generation of mass terms for the weak gauge fields $W^{\pm}$ and Z^0, as well as for the leptons e_L and e_R? Like in the Abelian case, the potential $V(\rho)$ has a unique minimum $\rho_0 = \sqrt{-\mu^2/2\lambda}$, and it is not hard to see from the Lagrangian that a split $\rho = \rho_0 + H$ gives masses $m_{W^{\pm}} = \sqrt{2}g\rho_0$ and $m_{Z^0} = \sqrt{2}\rho_0\sqrt{g^2 + g'^2}$ for the weak gauge bosons, $m_H = 2\rho_0\sqrt{\lambda}$ for the Higgs field via its self-interaction and $m_l = g_l\rho_0$ for the leptons e_L and e_R through the Yukawa coupling. Yet no breaking of gauge symmetry can occur since the fields are gauge-invariant, though we can imagine that there is a vacuum phase transition from $\mu^2 > 0$ to $\mu^2 < 0$ in which the fields "slide down the potential" without any SSB. However, the problem that $\rho = 0$ is not an available configuration because of ill-definedness of the polar decomposition at that value persists. It is argued in [41] that this is not necessarily a severe issue, at least in the perturbative regime. However, not admitting field configurations in which the Higgs field vanishes somewhere amounts to changing the very field space one is working on. As we will see in chapter 4, this means that one is really avoiding the central difficulty of the empirical significance of gauge symmetries, namely the fact that the gauge group does not act freely on infinite-dimensional field space. It is precisely on the configurations with vanishing scalar fields that this happens. We will also return to this point in chapter 7.

3.3. The Fröhlich-Morchio-Strocchi approach

So far, we have not seen how the DFM could relate to perturbative QFT. Here, the Fröhlich-Morchio-Strocchi (FMS) approach can help. Its basic idea is that one can expand n-point functions of gauge-invariant composite fields in terms of n-point functions of gauge-dependent elementary fields, such that when one performs perturbation theory one finds that quantities like particle masses match up on both sides of the equality. The FMS approach is therefore not a quantised version of the DFM, but rather it carries over the idea of using gauge-invariant composite objects to the perturbative domain.

Indeed, FMS originally used the scalar field φ to create gauge-invariant objects in the electroweak theory, such as $\varphi^{\dagger}\psi_L$ and $\varphi^{\dagger}\varphi$ [38], similar to the way in which we used the polar decomposition of the Higgs field as a dressing field in section 3.2 above. This has led François to remark that "as far as I know the first to give a fully $SU(2)$-gauge invariant formulation of the electroweak theory were Fröhlich, Morchio and Strocchi in 1981. Their account is actually fully equivalent to ours, but much less synthetic and systematic: They are working on individual scalar components of all the fields involved!" [63]. Compared to the DFM, however, the FMS approach has the advantage that no polar decomposition is necessary so that one does not run into the issue of it being ill-defined at $\varphi = 0$ [41].

Let us make our exposition of the FMS approach more precise by considering an example, namely the propagator $\langle(\varphi^{\dagger}\varphi)(x)(\varphi^{\dagger}\varphi)(y)\rangle$. We wish to expand it by splitting the Higgs field into a vacuum expectation value and a field fluctuating around it. However, this cannot be done in a gauge-invariant way, as is shown by the following heuristic path integral argument, which can be viewed as a generalisation of

Elitzur's theorem [40]. Let O be any expression transforming as a representation of the structure group G, $\mathcal{D}\mu$ a gauge-invariant measure and e^{iS} an invariant action as weight measure. Then if g denotes a gauge transformation, we have

$$\langle O \rangle = \int \mathcal{D}\mu O e^{iS} = \int \mathcal{D}\mu^{g^{-1}} O e^{iS} = \int \mathcal{D}\mu O^{g} e^{iS} = \langle O^{g} \rangle . \tag{3.1}$$

But if this is to hold for an arbitrary transformation g, we must conclude that

$$\int \mathcal{D}\mu O e^{iS} = 0.$$

This result can be thought of as the idea that "if all possible gauge transformations are included in the path integral, no particular direction reachable by a gauge transformation can survive" [39, p. 156]. The Higgs field vacuum expectation value (VEV) is just a particular instance of this result. Thus, we need to fix a gauge, so that we can split the scalar field as

$$\varphi(x) = \frac{v}{\sqrt{2}}\varphi_0 + \Delta\varphi(x),$$

where v is the non-zero VEV in our gauge and φ_0 is the unit vector describing the VEV direction, for which $\varphi_0 = (0, 1)$ is the common choice [41]. Defining the Higgs field $h = \sqrt{2}\mathrm{Re}(\varphi_0^{\dagger}\Delta\varphi)$ to be the radial component of the fluctuation field $\Delta\varphi$ in the direction of the VEV, we get the following expression for the connected part of the propagator [41]

$$\left\langle \left(\varphi^{\dagger}\varphi\right)(x)\left(\varphi^{\dagger}\varphi\right)(y)\right\rangle = v^2\langle h(x)h(y)\rangle + 2v\left\langle h(x)\left(\Delta\varphi^{\dagger}\Delta\varphi\right)(y)\right\rangle + \left\langle \left(\Delta\varphi^{\dagger}\Delta\varphi\right)(x)\left(\Delta\varphi^{\dagger}\Delta\varphi\right)(y)\right\rangle .$$

We note that this is an exact rewriting which holds both perturbatively and non-perturbatively. The right hand side has been suggestively ordered by the order of the fluctuation field $\Delta\varphi$. Indeed, when performing perturbation theory we consider only the first term on the right hand side, i.e. the Higgs field propagator $v^2\langle h(x)h(y)\rangle$. If the fluctuation field $\Delta\varphi$ is very small, then this gives a good approximation to the gauge-invariant composite object on the left hand side. This way, we can extract properties of the gauge-invariant composite propagator from the gauge-dependent elementary Higgs propagator. Berghofer et al. succinctly summarise this idea:

> For instance, let us consider the mass and decay width of the state generated by $\varphi^{\dagger}\varphi$. These properties are encoded in the pole structure of its propagator. Ignoring for a moment the higher-order terms of the FMS expansion, we obtain that the pole of the gauge-invariant bound state propagator coincides with the pole structure of the elementary Higgs propagator. In addition, it can be shown to all orders in a perturbative expansion of the n-point functions that the higher-order terms of the FMS expansion do not alter the pole structure on the right-hand side. Therefore, the on-shell properties of $\varphi^{\dagger}\varphi$ are well described by the propagator $\langle h(x)h(y)\rangle$. [41, p. 69]

The FMS approach has been applied in great detail to the Standard Model [39], but we are mostly interested in its philosophical implications for SSB in the Higgs mechanism. Now, the basic point about this made by Berghofer et al. is the same as for the DFM. That is, it is claimed that the FMS approach shows that there is no SSB in the Higgs mechanism, because gauge-invariant composite objects do not exhibit any gauge symmetry at all, not even a global one. The FMS approach shows that, conceptually speaking, the notion of SSB is not necessary for obtaining the empirically highly succesful perturbative results, because quantities that depend on the gauge-dependent Higgs VEV have can be related to gauge-independent objects. These composite objects are then taken to constitute the fundamental ontology of the theory: "the only physical degrees of freedom would be those that correspond to hadrons, the electroweak objects of the FMS approach, and photon-cloud dressed QED states. The conventional notions of quarks, electrons etc. would need to be regarded as mere auxiliaries that are technically useful but do not have any physical reality" [41, p. 83].

3.4. Reflections on the DFM

But how justified is this idea that gauge-invariant composite fields are fundamental and that therefore there can be no SSB, including SSB of global gauge symmetries? And for the DFM specifically: why should we take the existence of a local dressing field to be the criterion for distinguishing artificial and substantial gauge symmetries? In Proposition 3.2 and its corollary we have formalised the idea that a dressing field trivialises the principal bundle in the direction of the subgroup it takes values in, but trivialisability is not the same as artificiality. On Minkowski spacetime every principal bundle is trivialisable, but it is not evident that every gauge symmetry on Minkowski spacetime is artificial, especially when boundary conditions are taken into consideration, as we do in chapter 4.

One might say that the point of using a dressing field or composite field is to find a rewriting of a theory in terms of local gauge-invariant fields. It is then not claimed that the existence of a local dressing field is a necessary criterion for artificiality, but it does demonstrate that we can rewrite the Lagrangian in terms of local gauge-invariant fields. The existence of a local dressing field is therefore a sufficient condition for concluding that a gauge symmetry is artificial. But are we sure that rewriting a Lagrangian in terms of dressed fields does not discard valuable information? The configuration space of dressed fields is of course smaller than that of elementary fields, so do we know for certain that the part of field configuration space that is eliminated by the DFM is entirely unphysical? To our mind, this is not the case. Indeed, a proper examination of this issue must occur by means of a constrained Hamiltonian analysis or phase space reduction, as will be done in the next chapter.

A concrete indication that the DFM goes too far in its reduction of gauge symmetry is the example of complex scalar electromagnetism, which is very similar to the Abelian Higgs model: the polar decomposition of the scalar field can be used to find a local dressing field which removes the $U(1)$ gauge symmetry. In this framework, the Aharonov-Bohm effect supposedly "loses its puzzling edge [...] since it can be interpreted as resulting from the local interaction of the gauge-invariant local fields A^u and φ^u outside the cylinder" [41, p. 59]. But in this application we once again run into the problem that the polar decomposition does not exist for $\varphi = 0$. Thus, it seems that the principal bundle describing the Aharonov-Bohm effect is trivial, whereas in reality the fact that is not is precisely the central point (at least according to one mainstream interpretation of the effect). A naive application of the DFM in which this issue is discarded therefore leads to error.

All of these problems relate to the general question of the physical significance of gauge symmetries, which has been studied extensively by philosophers of physics. It seems that the DFM is too drastic a method, in the sense that it aims to completely reduce certain gauge symmetries, even though it is not at all clear that this should be our aim. If gauge symmetries are completely reduced, then we also ignore the associated conserved Noether currents, superselection sectors, SSB and other aspects, whether they have direct or indirect empirical significance.

As for the theme of this thesis, i.e. global gauge symmetry breaking, the DFM and FMS approach suggest that it does not play a role in the Higgs mechanism because global gauge symmetries are also removed when working with composite fields. This seems to be some sort of "collateral damage" that comes with the reduction of local gauge symmetries. To examine whether the DFM and FMS approach are really justified in doing this, we must carefully consider the status of global gauge symmetries in relation to local gauge symmetries. We turn to this now.

4. Constraints and Global Gauge Symmetries

The overarching aim of this thesis is to show that spontaneous breaking of global gauge symmetry can serve as the physical content of the Higgs mechanism. If we are to achieve this, we must understand in what way global gauge symmetries differ from their local counterparts. In particular, we need to elucidate their empirical significance, an issue which has been debated over the past twenty years [67–74], in parallel with the philosophical discussion on the Higgs mechanism. Building on this debate, Gomes and others have recently developed sophisticated techniques for singling out global (rigid) symmetries as the ones with direct empirical signifance [75–80]. The aim of this chapter is to bring together these ideas to illuminate the significance of global gauge symmetries for the Higgs mechanism.

A useful starting point for this formidable task is the constrained Hamiltonian analysis, which Struyve uses in his treatment of the Abelian Higgs mechanism [22], though this had already been done for both the Abelian and non-Abelian cases by Lusanna and Valtancoli [53–55]. From Struyve's account it becomes clear that the Abelian Higgs model can be reformulated in terms of fields which are invariant under all transformations except global ones, such that the electromagnetic field gains mass only when that remaining global symmetry is broken. However, this breaking of global gauge symmetry by fields which are invariant under local gauge symmetries can only be interpreted as a physical explanation of the Higgs mechanism if global gauge symmetries are *not* mere "descriptive fluff" or "mathematical redundancy". To understand why global gauge symmetries are physical we must delve into the philosophical discussion referred to in the above paragraph. This, then, we shall do in section 4.3, after having introduced the constrained Hamiltonian formalism in section 4.1 and having applied it to the Higgs mechanism in section 4.2. We end this chapter with a reflection on what we have achieved so far and what is yet to be done.

4.1. Constrained Hamiltonian analysis

In the Hamiltonian formulation of a theory we go from a Lagrangian, defined in terms of coordinates q^i which parametrise the configuration space Q, and their velocities $\dot{q}^i$, to a Hamiltonian, which is a function of the coordinates and their conjugate momenta p_i (or the fields and their conjugate momentum fields). The Hamiltonian equivalent of the Euler-Lagrange equations are Hamilton's equations, and the evolution of a system is represented by a trajectory in phase space solving those equations. For a field theory phase space is infinite-dimensional.

More precisely, a Lagrangian is a function $\mathcal{L}\colon TQ \to \mathbb{R}$ on the tangent bundle, whereas a Hamiltonian is a function $H\colon T^*Q \to \mathbb{R}$ on the cotangent bundle. The Legendre transform $TQ \to T^*Q$ takes us from one to the other, sending a generalised velocity $\dot{q}^i$ to its canonical conjugate momentum $p_i = \partial\mathcal{L}/\partial\dot{q}^i$ and yielding the Hamiltonian $\mathcal{H} = \sum_i \dot{q}^i p_i - \mathcal{L}$. It might be, however, that the conjugate momenta are not invertible as functions of the generalised velocities, in which case there are *constraints* among the momenta [81]. Mathematically speaking, we say that the Legendre transform is not *hyperregular*, i.e. not a diffeomorphism [82]. This is the case for gauge theories and we shall explain why in section 4.1.1. We then consider electromagnetism as a constrained system in section 4.1.2, before we continue to the application to the Higgs mechanism in section 4.2.

4.1.1. Gauge theories as constrained systems

In a gauge theory the evolution of a system is prima facie indeterministic, in the sense that the equations of motion do not uniquely determine the future values of all dynamical variables from given initial conditions [83]. This is because we can always perform a gauge transformation that maps a point in phase space to another, such that a system can evolve to different phase space points from the same initial conditions. This indeterminism manifests itself in the fact that the solution to the equations of motion of a gauge theory contains arbitrary functions of time [83, p. 3]. The presence of arbitrary functions of time in turn results in constraint relations between the canonical variables, so we say that gauge systems are constrained systems.

If we have a Lagrangian $\mathcal{L}(q^i, \dot{q}^i)$ in terms of coordinates q^i with conjugate momenta p_i, and a canonical Hamiltonian density $\mathcal{H} = \dot{q}^i p_i - \mathcal{L}$, then the Lagrangian is called *singular* if the matrix $\left(\partial p_i / \partial \dot{q}^i\right)$ is not invertible, for then one cannot uniquely get back the velocities $\dot{q}^i$ from the fields and their conjugate momenta. This means that there are momenta in the system which do not depend on the time derivatives of the fields and are subject to *primary constraints*: not all conjugate momenta are independent, but there are relations

$$c_m(q^i, p_j) = 0, \qquad m = 1, ..., M$$

between them (as many as there are zero eigenvalues of the Hessian matrix [84]). These relations define the *primary constraint set* $C \subset T^*Q$, which is the image of the Legendre transform (provided the Lagrangian $\mathcal{L}$ is *regular* [82, p. 237]). Primary constraints are called primary to emphasise that the equations of motion are not used to obtain them [83]. Secondary constraints arise as the requirement that the primary constraints be preserved in time and the equations of motion are used to define them. This can be iterated to obtain tertiary constraints etc. [85]. The space that is defined by requiring all constraints to be satisfied is called the *constraint surface*. It is assumed to be a submanifold smoothly embedded in phase space. Two variables F and G are then said to be weakly equal, written $F \approx G$, if they coincide on the constraint surface [83, p. 13]. A strong equality $F = G$ instead implies that two quantities are equal on all of phase space T^*Q. Imposing some regularity conditions on the constraints (see [83, p. 7]), we have the following result (Theorem 1.1 in [83]).

Theorem 4.1. If a smooth phase space function F vanishes on the constraint surface defined by $c_m = 0$, then $F = f^m c_m$ for some functions f^m.

It follows immediately from this theorem that

$$F \approx G \iff F - G = f^m c_m.$$

Now, the classification of constraints as primary or secondary is not actually so important, although it is useful as an introduction. There is, however, a different classification of constraints and more generally of functions on phase space that is crucial.

Definition 4.2. A function F on phase space is called *first class* if its Poisson bracket with every constraint vanishes weakly, i.e. for all $m = 1, .., M$ we have $\{F, c_m\} \approx 0$. Otherwise it is called *second class*.

Let us now consider the underlying symplectic formalism. The cotangent bundle T^*Q is a symplectic manifold, i.e. a manifold carrying a closed non-degenerate 2-form $\omega \in \Omega^2(T^*Q)$. It is the canonical one, expressed locally as

$$\omega = \sum_i dq^i \wedge dp_i.$$

On a symplectic manifold $(\mathcal{M}, \omega)$, every smooth function $H \in C^\infty(\mathcal{M})$ defines a Hamiltonian vector field $X_H \in \mathfrak{X}(\mathcal{M})$ through

$$dH = \omega(X_H, \cdot).$$

The Poisson bracket between $f, g \in C^\infty(\mathcal{M})$ is then defined as

$$\{f, g\} = \omega(X_f, X_g).$$

It is not difficult to see that for the canonical 2-form on the cotangent bundle this locally gives the familiar Poisson bracket

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}.$$

The idea behind Hamiltonian vector fields is that their integral curves are precisely the trajectories through phase space satisfying Hamilton's equations. Similarly, we can consider the vector fields X_{c_m} coming from the constraints $c_m \colon \mathcal{M} \to \mathbb{R}$. If all constraints are first class, then these vector fields flow tangentially to the constraint surface. If not all constraints are first class, then one can use the Dirac algorithm to guarantee so. We will not be concerned with that possibility since in our case of interest all constraints are first class.

If all constraints are first class then their associated vector fields are null directions of the symplectic form *on the constraint surface*, i.e. $\omega(X_{c_m}, X_{c_n}) = \{c_m, c_n\} \approx 0$. It is for this reason that we call these directions *gauge* and their integral curves *gauge orbits*. Points within one orbit are interpreted as physically equivalent, so any physical quantity must be gauge-invariant in the sense of being constant on every gauge orbit. As a consequence of the existence of null directions, the Hamiltonian flow is not unique on the constraint surface. That is: X_H and $X_H + a^m X_{c_m}$, with a^m arbitrary functions of time, give the same dynamics on the constraint surface. To see this, let $i \colon \Gamma \to \mathcal{M}$ denote the inclusion of the constraint surface into $\mathcal{M}$. Then we have $i^*\omega(X_H, \cdot) = d(H|_\Gamma)$, and since the constraints vanish on Γ we know

$$i^*\omega(X_{c_m}, \cdot) = d(c_m|_\Gamma) = 0.$$

This implies that

$$i^*\omega(X_H + a^m X_{c_m}, \cdot) = i^*\omega(X_H, \cdot) = d(H|_\Gamma),$$

i.e. the Hamiltonian vector field X_H is not unique. This is possible because $i^*\omega$ is degenerate on Γ, whereas ω is non-degenerate on the full phase space $\mathcal{M}$ [81]. However, the non-uniqueness of the Hamiltonian flow is not a detrimental form of indeterminism, because, whether we evolve a point in phase space according to H or $H + a^m c_m$, we always end up in the same gauge orbit. We thus arrive at the familiar slogan that "first class constraints generate gauge transformations", although there are many subtleties that we do not consider here, see e.g. [86]. In fact, Pitts has objected to this slogan [87], but this objection has again been objected to, vindicating orthodoxy [88].

4.1.2. Electromagnetism

Let us now work out what constraints appear in electromagnetism. Following [81] we also show how the constrained Hamiltonian formalism for Maxwell theory naturally leads to the Coulomb gauge, which is interesting in light of both Friederich's discussion of remnant gauge symmetries from section 1.2.5, and will be very important for the results in sections 6.4 and 6.5.

We work on the trivial bundle $P = M \times U(1)$, where M denotes Minkowski spacetime. We consider a space-time split $M = \Sigma \times \mathbb{R}$ with Σ a Cauchy surface, such that the configuration space consists of all spatial potentials A_i with $i = 1, 2, 3$, plus the possible matter configurations on Σ. To guarantee that $\mathbf{A}$ falls of sufficiently rapidly towards spatial infinity, let us assume it to be compactly supported, i.e. $A_i \in C_c^\infty(\Sigma)$. We will return to this crucial issue of asymptotic behaviour in the next sections. From the kinetic term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ of the Lagrangian we find the conjugate momenta

$$\Pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -\frac{1}{2}F^{\mu\nu}\frac{\partial}{\partial \dot{A}_i}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = -\frac{1}{2}F^{\mu\nu}(\delta^0_\mu \delta^i_\nu - \delta^0_\nu \delta^i_\mu) = -\frac{1}{2}(F^{0i} - F^{i0}) = F^{i0} = -E^i.$$

We also take $E^i \in C_c^\infty(\Sigma)$. These momenta, however, are not independent, for one of the four Maxwell equations is the Gauss law $\nabla \cdot \mathbf{E} = \partial_i E^i = \rho$, where $\rho = j_0$ is the charge density of the current j_μ. In other words, the Gauss law gives the constraint $c(\mathbf{A}, \mathbf{E}, j) = \partial_i E^i - \rho$. If we use a function $\lambda\colon \Sigma \to \mathbb{R}$, also called a Lagrange multiplier, to integrate the Gauss law, we obtain what is known as the *smeared* Gauss constraint

$$G_\lambda = \int_\Sigma dx^3 \lambda(\nabla \cdot \mathbf{E} - \rho),$$

which is a function on phase space, which consists of pairs $(\mathbf{A}, \mathbf{E})$. In section 4.3.3 we will show how to understand this phase space as the cotangent bundle of the infinite-dimensional configuration space of connection 1-forms on Σ. We can now calculate how the smeared Gauss constraint acts on the fields, i.e. what its Poisson brackets with A_i and E^i are. The symplectic form is

$$\Omega = \int_\Sigma dx^3 ⅆA_i \wedge ⅆE^i, \tag{4.1}$$

where, following [81], we have used the double struck ⅆ to indicate that this is the differential operator on the infinite-dimensional phase space coordinatised by the fields A_i and E^i, and not on the three-dimensional Cauchy surface Σ, for which we use the regular d. From the expression for the symplectic form Ω it follows that the Poisson bracket of two functionals F and G of the fields is given by [81]

$$\{F, G\} = \int_\Sigma dx^3 \left(\frac{\delta F}{\delta A_i(\mathbf{x})} \frac{\delta G}{\delta E^i(\mathbf{x})} - \frac{\delta F}{\delta E^i(\mathbf{x})} \frac{\delta G}{\delta A_i(\mathbf{x})} \right),$$

where $\delta/\delta A_i(\mathbf{x})$ and $\delta/\delta E^i(\mathbf{x})$ denote the functional derivatives of phase space functions with respect to the fields $A_i(\mathbf{x}), E^i(\mathbf{x})$, which are themselves functions on Σ. The poisson bracket of the smeared Gauss constraint with the gauge potential thus becomes

$$\begin{aligned} \{G_\lambda, A_i(\mathbf{x})\} &= -\frac{\delta G_\lambda}{\delta E^i(\mathbf{x})} = -\frac{\delta}{\delta E^i(\mathbf{x})} \int_\Sigma dy^3 \lambda(\mathbf{y})(\nabla \cdot \mathbf{E}(\mathbf{y}) - \rho) \\ &= -\int_\Sigma dy^3 \lambda(\mathbf{y}) \partial_i \delta(\mathbf{x} - \mathbf{y}) = \int_\Sigma dy^3 \partial_i \lambda(\mathbf{y}) \delta(\mathbf{x} - \mathbf{y}) = \partial_i \lambda(\mathbf{x}). \end{aligned}$$

Here we have performed partial integration and assumed the boundary term to vanish. However, we shall shortly return to the issue of the behaviour of λ "at infinity", which is the central point of this chapter and possibly of our understanding of the Higgs mechanism more generally. For now we ignore these details, because the important result is that G_λ generates the familiar gauge transformation of A_i: when we apply G_λ to A_i by taking the Poisson bracket we get the pure gauge term $\partial_i \lambda$. In addition, it is not hard to see that

$$\{G_\lambda, E^i(\mathbf{x})\} = \frac{\delta}{\delta A_i(\mathbf{x})} \int_\Sigma dy^3 \lambda(\mathbf{y})(\nabla \cdot \mathbf{E} - \rho) = 0.$$

In other words: the smeared Gauss constraint G_λ leaves the electric field invariant, as expected since the electric field is observable and must therefore be gauge-invariant.

Now, the Gauss law equates the divergence of the electric field with the distribution of charge in space. It is natural, then, to seek for a decomposition of the electric field into a *Coulombic* part, whose divergence equals the charge distribution, i.e. which automatically satisfies the Gauss constraint, and a *transverse* part, which carries no divergence but all the curl. In other words, we seek the *Helmholtz decomposition* $E^i = E^i_L + E^i_T$ of the electric field into longitudinal (irrotational, i.e. curl-free) and transverse (solenoidal, i.e. divergence-free) components. Since the longitudinal part E^i_L is curl-free, it can be written as the gradient of a scalar function (Poincaré lemma), i.e. $E^i_L = \partial^i \phi$ with $\phi \in C_c^\infty(\Sigma)$. We then generate the Coulombic electric field coordinates in phase space by the vector field [81]

$$\mathbb{E}^L = \int_\Sigma dx^3 E^L_i \frac{\delta}{\delta E_i(\mathbf{x})} = \int_\Sigma dx^3 \partial_i \phi \frac{\delta}{\delta E_i(\mathbf{x})},$$

where again we have used the double struck notation to stress that this vector field lives on infinite-dimensional phase space. We now extend the Coulombic-radiative split to the vector potential $\mathbf{A}$, in the sense that we look for the component $\mathbf{A}^T$ of $\mathbf{A}$ that is symplectically orthogonal to $\mathbf{E}^L$. To find this radiative component we define another vector field in phase space

$$\mathbb{A}^T = \int_\Sigma dx^3 A_i^T \frac{\delta}{\delta A_i(\mathbf{x})},$$

and we require

$$0 = \Omega(\mathbb{A}^T, \mathbb{E}^L) = \int_\Sigma dx^3 A_i^T (E^L)^i = \int_\Sigma dx^3 A_i^T \partial^i \phi = -\int_\Sigma dx^3 \phi \partial^i A_i^T \tag{4.2}$$

for all $\phi \in C_c^\infty(\Sigma)$, again using partial integration and assuming the boundary term to vanish. This equation must hold for any $\phi \in C_c^\infty(\Sigma)$, so we find $\partial^i A_i^T = 0$, which is of course the Coulomb gauge condition. The projection onto the component of $\mathbf{A}$ satisfying this condition is called *radiative projection* and is given by [81]

$$A_i^T(\mathbf{A}) = A_i - \partial_i(\Delta^{-1}\partial^j A_j), \tag{4.3}$$

where $\Delta^{-1} = \nabla^{-2}$ is the inverse of the Laplacian with Green's function $-\frac{1}{4\pi r}$, i.e.

$$\Delta^{-1} f(\mathbf{x}) = -\int_\Sigma dy^3 \frac{f(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}.$$

The radiatively projected vector potential is indeed gauge-invariant: under a gauge transformation $A_i \to A_i + \partial_i\lambda$ we have

$$\begin{aligned} A_i^T &\to A_i + \partial_i\lambda - \partial_i(\Delta^{-1}(\partial^j A_j + \partial^j\partial_j\lambda)) = A_i + \partial_i\lambda - \partial_i(\Delta^{-1}\partial^j A_j) - \partial_i\Delta^{-1}\Delta\lambda \\ &= A_i + \partial_i\lambda - \partial_i(\Delta^{-1}\partial^j A_j) - \partial_i\lambda = A_i - \partial_i(\Delta^{-1}\partial^j A_j) = A_i^T. \end{aligned}$$

Just like for the electric field, the vector field in phase space generating the *pure gauge* part $A_i^L = \partial_i\lambda$ of the potential is

$$\mathbb{A}^L = \int_\Sigma dx^3 A_i^L \frac{\delta}{\delta A_i(\mathbf{x})} = \int_\Sigma dx^3 \partial_i\lambda \frac{\delta}{\delta A_i(\mathbf{x})},$$

for $\lambda \in C^\infty(\Sigma)$ with appropriate asymptotic behaviour. In analogy to equation 4.2 we can then require

$$0 = \Omega(\mathbb{A}^L, \mathbb{E}^T) = \int_\Sigma dx^3 E_i^T \partial^i\lambda = -\int_\Sigma dx^3 \lambda \partial^i E_i^T,$$

giving $\partial^i E_i^T = 0$, i.e. precisely the divergenceless component of $\mathbf{E}$. In summary, the radiative part of the electric field is symplectically orthogonal to the pure gauge part of the gauge potential, while the Coulombic part of the electric field is symplectically orthogonal to the radiative part of the potential. As a corollary, the symplectic form in these coordinates reduces to [81]

$$\Omega = \int_\Sigma dx^3 \left(\mathbb{d}A_i^T \wedge \mathbb{d}E_i^T + \mathbb{d}A_i^L \wedge \mathbb{d}E_i^L \right).$$

4.2. Application to the Higgs mechanism

We shall now put the constrained formalism to use by applying it to the Higgs mechanism, following Struyve's treatment of the Abelian Higgs model [22], which is in turn based on the work of Lusanna and Valtancoli [53], who also applied the Hamiltonian formalism to a non-Abelian SU(2) Higgs model [54] and the full Standard Model [55]. We will not consider the non-Abelian Higgs mechanism in this section because we would get bogged down in lengthy expressions, losing sight of the conceptual points, but of course we must ask ourselves to what extent results from the Abelian case generalise to the full electroweak model (our RQ1.3). We take up this issue in section 4.3 and chapter 7.

4.2.1. Hamiltonian formulation

In order to get a proper grasp of Struyve's presentation, it is important to understand his view on gauge symmetries and their physical significance. He recalls a definition according to which gauge transformations map solutions of the equations of motion to other solutions and preserve the initial data. In addition, boundary conditions are imposed to ensure finiteness of energy and action, and gauge transformations need to preserve these boundary conditions too. There is thus room for a residual group of physical transformations, namely those that do preserve boundary conditions but not the initial data. In the next section, we will make these statements more precise by showing that the residual group of physical transformations consists of those gauge transformations that preserve the boundary conditions but are *not* generated by the Gauss constraint.

Struyve considers the Abelian Higgs mechanism in Minkowski spacetime with scalar field φ and vector field A_μ on a trivial bundle, with boundary conditions

$$A_\mu \to 0 + \mathcal{O}\left(r^{-2}\right), \quad \partial_\mu A_\nu \to 0 + \mathcal{O}\left(r^{-2}\right), \quad \varphi \to \frac{1}{\sqrt{2}} v e^{i\theta} + \mathcal{O}\left(r^{-2}\right), \quad \partial_\mu \varphi \to 0 + \mathcal{O}\left(r^{-2}\right),$$

as $r \to \infty$ and where $v/\sqrt{2} = \sqrt{-\mu^2/2\lambda}$ is the positive minimum of the Mexican hat potential $V(\varphi)$ and $\theta \in [0, 2\pi]$ a constant [22, p. 231]. In other words, these boundary conditions require the gauge field to vanish sufficiently quickly and the scalar field to become a constant minimum of the potential at infinity. The transformations preserving the boundary conditions are of the form $g = e^{i\lambda}$, with λ a real function that goes to a constant sufficiently rapidly as infinite distance is approached, i.e. as $r \to \infty$. We denote the group of these transformations by $\mathcal{G}^I$. The unphysical gauge group is the subgroup $\mathcal{G}^\infty$, consisting of the local $U(1)$ transformations that go to the identity at spatial infinity. Clearly, a transformation that goes to the identity at infinity is in particular a transformation of the form $e^{i\lambda}$ with λ vanishing asymptotically. Thus, we find that there is a non-trivial group of residual physical symmetries $\mathcal{G}^I/\mathcal{G}^\infty \cong U(1)$ (see Proposition 4.4 for a precise statement). This latter identification is made by noting that the requirement in $\mathcal{G}^I$ that transformations become asymptotically constant is equivalent to adding one point at infinity on which the transformations take their asymptotic value. The transformations in $\mathcal{G}^\infty$ are required to be the identity at this point at infinity, and so an element in the quotient $\mathcal{G}^I/\mathcal{G}^\infty$ is just a choice of element of $U(1)$ for the point at infinity.

Of course, to even talk about the identity at infinity we need a section of the principal bundle at infinity. Since we are working on the trivial bundle $P = M \times U(1)$ we already have a canonical such section, but in general this is not the case. We will come back to this in section 4.3. For now, the central point is that Struyve thinks of global gauge transformations as carrying a physical significance that the local transformations $\mathcal{G}^\infty$ do not have, and that the group of these global transformations $e^{i\lambda}$ with λ constant is isomorphic to the structure group $U(1)$.

With this idea in mind we seek to present the Higgs mechanism as a breaking of the group of global gauge symmetries *only*, using the constrained Hamiltonian formalism. The Lagrangian is as always the one from equation 1.1, with conjugate momenta

$$\Pi_{A_0} = 0, \quad \Pi_{A_i} = F^{i0} = -E^i = \partial^i A^0 - \partial^0 A^i, \quad \Pi_\varphi = (D^0\varphi)^*, \quad \Pi_{\varphi^*} = D^0\varphi.$$

The canonical Hamiltonian density is then

$$\begin{aligned}
\mathcal{H}_L &= \dot{A}_i E^i + \dot{\varphi}(D^0\varphi)^* + \dot{\varphi}^* D^0\varphi - \mathcal{L} \\
&= \dot{A}_i E^i + D_0\varphi(D^0\varphi)^* + ieA_0\varphi(D^0\varphi)^* + (D_0\varphi)^* D^0\varphi - ieA_0\varphi^* D^0\varphi + D_\mu\varphi(D^\mu\varphi)^* + V(\varphi) + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\
&= (\partial_i A^0 + E_i)E^i + D_0\varphi(D^0\varphi)^* + A_0 j^0 + D_i\varphi(D^i\varphi)^* + V(\varphi) - \frac{1}{2}E^i E_i + \frac{1}{4}F^{ij}F_{ij} \\
&= A^0(j_0 - \partial_i E^i) + \partial_i(A^0 E^i) + D_0\varphi(D^0\varphi)^* + D_i\varphi(D^i\varphi)^* + V(\varphi) + \frac{1}{2}|\mathbf{E}|^2 + \frac{1}{2}|\mathbf{B}|^2 \\
&= A^0(j_0 + \partial_i \Pi_{A_i}) - \partial_i(A^0 \Pi_{A_i}) + \Pi_{\varphi^*}\Pi_\varphi + D_i\varphi(D^i\varphi)^* + V(\varphi) + \frac{1}{2}\Pi_{A_i}\Pi_{A_i} + \frac{1}{4}F^{ij}F_{ij},
\end{aligned}$$

subject to two first class constraints $c_1, c_2 = 0$, namely

$$c_1 = \Pi_{A_0}, \quad c_2 = j_0 + \partial_i \Pi_{A_i} = ie(\varphi(D_0\varphi)^* - \varphi^* D_0 \varphi) - \partial_i E^i.$$

We recognise the second constraint as the Gauss law, since $j_0 = \rho$. In order to obtain unconstrained gauge-invariant fields we use the radiative projection:

$$\begin{aligned}
\bar{\varphi} &= e^{-ie\Delta^{-1}\partial^i A_i}\varphi, & \Pi_{\bar{\varphi}} &= e^{ie\Delta^{-1}\partial^i A_i}\Pi_\varphi, \\
\bar{\varphi}^* &= e^{ie\Delta^{-1}\partial^i A_i}\varphi^*, & \Pi_{\bar{\varphi}^*} &= e^{-ie\Delta^{-1}\partial^i A_i}\Pi_{\varphi^*}, \\
A_i^T &= A_i - \partial_i \Delta^{-1}\partial^j A_j, & \Pi_{A_i^T} &= \Pi_{A_i} - \partial^i \Delta^{-1}\partial_j \Pi_{A_j}, \\
A_i^L &= \partial^i \Delta^{-1}\partial^j A_j, & \Pi_{A_i^L} &= \partial^i \Delta^{-1}\left(j_0 + \partial_j \Pi_{A_j}\right).
\end{aligned}$$

Here the subscripts T and L stand for 'transverse' and 'longitudinal', as in section 4.1.2. The Gauss constraint has become $\Pi_{A_i^L} = 0$ in terms of the new fields, so the two constraints are now both of the form of a vanishing momentum, the other being $\Pi_{A_0} = 0$. Thus, on the constraint surface the momenta of A_0 and A_i^L are zero and a complete set of gauge-independent fields is $\bar{\varphi}, \Pi_{\bar{\varphi}}, \bar{\varphi}^*, \Pi_{\bar{\varphi}^*}, A_i^T, \Pi_{A_i^T}$. Dropping the total derivative $\partial_i(A^0 \Pi_{A_i})$ (we come back to whether this is permitted in section 4.3), the Hamiltonian can be expressed on the constraint surface in terms of these gauge-independent variables as [22, p. 234]

$$\mathcal{H} = \Pi_{\bar{\varphi}^*}\Pi_{\bar{\varphi}} + D_i^T \bar{\varphi}(D_T^i \bar{\varphi})^* + V(\bar{\varphi}) + \frac{1}{2}\Pi_{A_i^T}\Pi_{A_i^T} + \frac{1}{2}\bar{j}_0 \Delta^{-1}\bar{j}_0 - \frac{1}{2}A_i^T \Delta A_T^i,$$

where $D_i^T = \partial_i - ieA_i^T$ and $\bar{j}_0 = ie(\bar{\varphi}\Pi_{\bar{\varphi}} - \bar{\varphi}^*\Pi_{\bar{\varphi}^*})$. We get the current term $\frac{1}{2}\bar{j}_0\Delta^{-1}\bar{j}_0$ from writing out $\Pi_{A_i} = \Pi_{A_i^T} - \partial^i \Delta^{-1} j_0$ and the last term from $\frac{1}{4}F^{ij}F_{ij}$, using that A_i^T is divergence-free. We note that neither the scalar field nor the gauge field has a mass term in this Hamiltonian.

4.2.2. Global gauge symmetry breaking

Crucially, although in terms of the new fields the local gauge group $\mathcal{G}^\infty$ has been eliminated, in the sense that the new fields are invariant under the action of $\mathcal{G}^\infty$, there is still the physical $U(1)$ symmetry of transformations in the quotient $e^{i\lambda} \in \mathcal{G}^I/\mathcal{G}^\infty$ (with λ constant), whose action on the new fields is

$$\bar{\varphi} \to e^{i\lambda}\bar{\varphi}, \quad \bar{\varphi}^* \to e^{-i\lambda}\bar{\varphi}^*, \quad \Pi_{\bar{\varphi}} \to e^{-i\lambda}\Pi_{\bar{\varphi}}, \quad \Pi_{\bar{\varphi}^*} \to e^{i\lambda}\Pi_{\bar{\varphi}^*}.$$

Thus, the reduction of gauge symmetry here is less radical than that of the DFM in chapter 3, where the fields were made invariant under *all* transformations, including the global ones. Like in the standard account of the Higgs mechanism, there is a degenerate set of ground states $\bar{\varphi} = ve^{i\theta}/\sqrt{2}$, $\bar{\varphi}^* = ve^{-i\theta}/\sqrt{2}$ with θ constant and [22, p. 234]

$$\Pi_{\bar{\varphi}} = \Pi_{\bar{\varphi}^*} = A_i^T = \Pi_{A_i^T} = 0.$$

The residual $\mathcal{G}^I/\mathcal{G}^\infty \cong U(1)$ symmetry may now be spontaneously broken by choosing a vacuum value $\bar{\varphi} = v/\sqrt{2}$ and performing a field split around this choice of vacuum:

$$\begin{aligned}
\bar{\varphi} &= \frac{1}{\sqrt{2}}(v + \eta + i\xi), \quad \Pi_{\bar{\varphi}} = \frac{1}{\sqrt{2}}\left(\Pi_\eta - i\Pi_\xi\right), \\
\bar{\varphi}^* &= \frac{1}{\sqrt{2}}(v + \eta - i\xi), \quad \Pi_{\bar{\varphi}^*} = \frac{1}{\sqrt{2}}\left(\Pi_\eta + i\Pi_\xi\right),
\end{aligned}$$

where $\eta, \xi, \Pi_\eta, \Pi_\xi$ are real and η, ξ are small. Then we can approximate the Hamiltonian as [22, p. 234]

$$\mathcal{H} = \frac{1}{2}\Pi_\eta^2 + \frac{1}{2}\partial^i\eta\partial_i\eta + \mu^2\eta^2 + \frac{1}{2}\Pi_\xi^2 - \frac{e^2v^2}{2}\Pi_\xi\Delta^{-1}\Pi_\xi + \frac{1}{2}\partial^i\xi\partial_i\xi + \frac{1}{2}\Pi_{A_i^T}\Pi_{A_i^T} + \frac{e^2v^2}{2}A_T^i A_i^T - \frac{1}{2}A_T^i \Delta A_i^T.$$

Viewing $\partial_i \xi$ as the longitudinal (Coulombic) component of the vector potential, i.e.

$$A_i^L = -\frac{1}{ev}\partial^i \xi, \quad \Pi_{A_i^L} = ev\partial^i \Delta^{-1}\Pi_\xi,$$

such that we have the canonical pairs

$$A_i = A_i^T + A_i^L, \quad \Pi_{A_i} = \Pi_{A_i^T} + \Pi_{A_i^L},$$

we finally obtain the Hamiltonian [22, p. 235]

$$\mathcal{H} = \frac{1}{2}\Pi_\eta^2 + \frac{1}{2}\partial^i\eta\partial_i\eta + \mu^2\eta^2 + \frac{1}{2}\Pi_{A_i}\Pi_{A_i} + \frac{1}{2e^2v^2}(\partial_i\Pi_{A_i})^2 + \frac{e^2v^2}{2}A_iA^i + \frac{1}{4}F_{ij}F_{ij}.$$

We recognise a scalar field η with mass $\mu\sqrt{2}$ and a vector field with mass ev. In other words: we have formulated the Higgs mechanism in terms of the breaking of the global $U(1)$ gauge symmetry. The Hamiltonian before the field split did not contain massive particles, so we do indeed attribute the mass generation to the breaking of the global gauge symmetry. Whether we can truly interpret this as mass *generation* in a causal, temporal sense is a question that we will come back to in chapter 7. One may also be worried about the fact that our result here is perturbative: we assumed ξ and η to be small when expanding around a specific vacuum value. In chapter 6 we shall assuage this worry by considering non-perturbative results.

The derivation just presented is a major piece of evidence for our argument that the physical content of the Higgs mechanism can be understood as the spontaneous breaking of global gauge symmetry. However, the question of the empirical significance of gauge symmetries remains: our argument only works if we can also argue that global gauge symmetries should indeed be understood to carry a physical meaning that local gauge transformations do not have.

As a transition into the philosophical discussion on this question we briefly consider some remarks by Lusanna and Valtancoli on their treatment of the Higgs mechanism for $U(1)$ and $SU(2)$. In both cases, the theory is reformulated such that the original spacetime-dependent gauge symmetry is reduced to only a global symmetry [53,54]. By Noether's first theorem, this means that there are conserved charges Q_a associated to this global symmetry, one for every symmetry generator T_a. By the Gauss theorem, these charges are equal to the total flux Q_a^V (notation of Lusanna and Valtancoli) of the electric field at spatial infinity. In the broken symmetry phase, however, the electric field decays exponentially towards spatial infinity, implying that $Q_a^V = 0$ [54, p. 25]. In this sense, the Gauss theorem "breaks down" in the broken phase: "the electric charge in the Higgs phase is a Noether constant of motion (first Noether theorem) but one cannot measure it by means of the electric flux at space infinity (as in the case of exact, not broken, local gauge symmetry; second Noether theorem). This fact may be taken as a gauge-invariant signal of gauge symmetry breaking, rather than the non-gauge-invariant quantum statement $\langle\varphi\rangle = \varphi_0$" [53, p. 27]. We will come back to these ideas in chapter 6, where the Higgs mechanism will be explained as a failure of the electric charge to generate the global $U(1)$ symmetry.

4.3. Empirical significance

Having shown that the Higgs mechanism can be formulated as the breaking of a global gauge symmetry exhibited by fields which are invariant under local gauge transformations, we turn to the question of the physical significance of these global symmetries. If the conceptual problem of mass generation is to be solved along this path, then we must show global gauge symmetries are not "descriptive fluff". A first hint that they are indeed physical is the aforementioned fact that global symmetries give conservation laws through Noether's first theorem. However, this sort of physical meaning is usually called *indirect empirical significance* (IES) by philosophers of physics, in contrast to the *direct empirical significance* (DES) of, say, a spatial translation which literally moves an object from one place to another. In this section we shall show that global gauge symmetries exhibit DES and not only IES.

4.3.1. Galileo's ship

To exemplify DES, philosophers refer back to a thought experiment from Galileo Galilei's 1632 book *Dialogue Concerning the Two Chief World Systems*. In this thought experiment, Galilei imagines himself inside the cabin of a ship at sea, together with several flying animals [72]. Within the cabin, there is no noticeable difference between the ship standing still and it moving with a constant speed: the animals do not fly differently in the two cases, as long as the ship does not accelerate. Boosting the ship with a constant velocity has no physical consequences inside the cabin, but only in *relation* to the environment: someone on the shore will notice whether the ship stands still or sails. The question that philosophers have focused on is whether gauge symmetries can exhibit such *relational* DES.

At the turn of the millenium, Kosso identified two criteria for empirical significance of a symmetry: "observation of a symmetry will always require two components: one must observe that the specified transformation has taken place, and one must observe that the specified invariant property is in fact the same, before and after" [67, p. 86]. He subsequently applied this criterion to four cases, corresponding to the combinations of *external* (acting on spacetime points) versus *internal* (not acting on spacetime) *global* (spacetime-independent) versus *local* (spacetime-dependent). According to Kosso, gauge symmetries fall in the category of internal, local symmetries. Contra Kosso, we also consider global, internal gauge symmetries, because we adhere to the abstract definition of a gauge symmetry as in Definition 2.6, which includes global symmetries.

Building on this work, Brading and Brown make Kosso's two components of DES more concise by stressing the role of *subsystems*: "we require that two conditions are met in order for a symmetry to have direct empirical significance:

1. *Transformation Condition*: the transformation of a subsystem of the universe with respect to a reference system must yield an empirically distinguishable scenario; and
2. *Symmetry Condition*: the internal evolution of the untransformed and transformed subsystems must be empirically indistinguishable" [68].

They then argue that all gauge symmetries, both local and global, fail to satisfy (1): one cannot observe the effect of gauge symmetries on subsystems, and there is no Galileo's ship scenario for gauge symmetries [72]. In short, they claim that gauge symmetries do not exhibit DES, and this claim has become widespread in the philosophy of physics literature, thus leading to expressions such as "descriptive fluff". After all, it is the conventional wisdom that states related by a gauge symmetry represent the same physical situation. This has led philosophers to often equate the notion of gauge symmetry with mathematical redundancy. In section 4.3.2 we act against this common idea by carefully distinguishing between the mathematical and redundant aspects of gauge transformations.

Greaves and Wallace (GW) challenged the conventional wisdom by developing a framework for treating symmetries and using it to construct an analog of Galileo's ship for electromagnetism [70]. Their framework starts from the idea that a symmetry of the entire universe cannot exhibit any DES, an intuition that is already present in the quote by Leibniz with which we opened this thesis. If we translate the *entire* universe, or change East into West in all of space, then this effects no physical difference. GW assume the same holds for any type of symmetry, including gauge symmetry.

The framework of GW assumes that we can define sets of states of the universe, a subsystem and its environment (which together compose the universe), denoted $\mathcal{U}$, $\mathcal{S}$ and $\mathcal{E}$ respectively. They also assume the existence of a "partial function" $*$ from $\mathcal{S}\times\mathcal{E}$ to $\mathcal{U}$. This function is defined for $s \in \mathcal{S}, e \in \mathcal{E}$ by $s*e = u$ if there exists some $u \in \mathcal{U}$ such that we can project $\pi_{\mathcal{S}}(u) = s, \pi_{\mathcal{E}}(u) = e$, where it is also assumed that there exist projections $\pi_{\mathcal{S}}\colon \mathcal{U} \to \mathcal{S}$ and $\pi_{\mathcal{E}}\colon \mathcal{U} \to \mathcal{E}$. If there is no such u then $s * e$ is undefined, which is why GW call $*$ "partial". The idea is that the $*$-operation glues a subsystem state and an environment state together to form a universe state, but only if there actually exists a universe state such that its restrictions to subsystem and environment give back the states we started with. Denoting by $\Sigma_{\mathcal{U}}$ the group of symmetries $\sigma\colon \mathcal{U} \to \mathcal{U}$ of the universe, we can project a symmetry σ to a subsystem symmetry

$\sigma_{\mathcal{S}}$ or an environment symmetry $\sigma_{\mathcal{E}}$ through $\sigma_{\mathcal{S}}(s) = \pi_{\mathcal{S}}(\sigma(u))$ or $\sigma_{\mathcal{E}}(e) = \pi_{\mathcal{E}}(\sigma(u))$ respectively, where $u \in \mathcal{U}$ such that $\pi_{\mathcal{S}}(u) = s$ and $\pi_{\mathcal{E}}(u) = e$. We can then extend $*$ to a map of symmetries via

$$\sigma_{\mathcal{S}} * \sigma_{\mathcal{E}}(s * e) = \sigma_{\mathcal{S}}(s) * \sigma_{\mathcal{E}}(e).$$

This allows one to define two characterisations of symmetries [72].

1. An *interior* symmetry $\sigma_{\mathcal{S}} \in \Sigma_{\mathcal{S}}$ is one such that for all $s \in \mathcal{S}$, $e \in \mathcal{E}$ for which $s * e$ is defined, we have that the map

$$\sigma_{\mathcal{S}} * 1_{\mathcal{E}}(s * e) = \sigma_{\mathcal{S}}(s) * e$$

is a universe symmetry (i.e. an element of $\Sigma_{\mathcal{U}}$), where $1_{\mathcal{E}}$ is the identity on $\mathcal{E}$.

2. A *boundary-preserving* symmetry $\sigma_{\mathcal{S}} \in \Sigma_{\mathcal{S}}$ of a specific state $s \in \mathcal{S}$ is a symmetry such that for all $e \in \mathcal{E}$ compatible with s, we have that s and $\sigma_{\mathcal{S}}(s)$ satisfy (i.e. are elements of) the same boundary condition C_e. Here C_e denotes the set of all $s' \in \mathcal{S}$ such that $s' * e$ exists.

We interpret the second characterisation as follows: a symmetry $\sigma_{\mathcal{S}}$ is called boundary-preserving if it does not spoil the compatibility of the subsystem with its environment.

Now, according to GW, interior symmetries are precisely the ones that do not have DES because they can be extended to universe symmetries by combining them with the identity $1_{\mathcal{E}}$ on the environment, and universe symmetries were assumed to have no empirical consequences. Moreover, GW argue that the essence of scenarios such as Galileo's ship is captured by non-interior, boundary-preserving symmetries [72]. Indeed, if a non-interior symmetry $\sigma_{\mathcal{S}}$ is boundary-preserving for a state $s \in \mathcal{S}$, then it can be combined with $1_{\mathcal{E}}$, as for any $e \in \mathcal{E}$ compatible with s we know $\sigma_{\mathcal{S}}(s) * e$ exists since $\sigma_{\mathcal{S}}(s) \in C_e$. However, since $\sigma_{\mathcal{S}}$ is non-interior this combination $\sigma_{\mathcal{S}} * 1_{\mathcal{E}}$ is a map of the universe to itself that is *not* a universe symmetry, thus effecting a physical difference on the entire universe that is not visible from the subsystem only, as is the case for Galileo's ship.

Let us now consider how GW's method can be applied to gauge symmetries. This amounts to the following question: how can we circumvent the central problem with DES of gauge symmetries, namely that a gauge transformation applied to any region of spacetime seems to just be a particular instance of a gauge transformation on all of spacetime? If this is so, then any gauge transformation applied to a region is in fact a universe symmetry and therefore does not exhibit DES. The key point stressed by GW, however, is that this is *not* so. Indeed, consider the usual scalar electrodynamics with fields A_μ, φ. Suppose we divide space into two regions A and B with overlap $A \cap B \neq \emptyset$, such that the matter field φ vanishes on this overlap. If we then perform a constant gauge transformation $\varphi|_A \to e^{i\lambda_A}\varphi|_A$, with $\lambda \notin 2\pi\mathbb{Z}$, on A only (which leaves A_μ invariant) while doing nothing on B, then the transformation on A is boundary-preserving since $\varphi|_{A\cap B} = 0$. However, this constant transformation on A cannot be combined with the identity transformation on B to form a gauge transformation of the entire universe $A \cup B$, since there is a discontinuity where the region A ends, and gauge transformations are assumed to be smooth. Thus, the transformation on A is boundary-preserving but not interior and exhibits DES.

4.3.2. Galileo's gauge

In *Galileo's Gauge* [72], Teh further develops the ideas of GW by using the constrained Hamiltonian formalism that we introduced earlier in this chapter. His results are highly interesting for our purposes, as they explicitly single out from the total gauge group $\mathcal{G}$ the subgroup of global gauge transformations as the ones carrying DES. Teh's work is based on an argument by Balachandran [89] about Poisson brackets. We shall examine this argument in detail and generalise and formalise it in the language of momentum maps in symplectic geometry.

To understand why Teh sees the need to develop GW's work by recourse to Hamiltonian analysis, we need to highlight what he calls the "logical puzzle about gauge." This puzzle is the aforementioned confusion between two notions of gauge [72, p. 97].

(Formal) The gauge group $\mathcal{G}$ is the group of maps $M \to G$ from spacetime to the structure group, as in Definition 2.6.

(Redundant) A necessary condition on gauge transformations is that they connect different descriptions of the same physical state of affairs, where a 'description' is taken to be a (mathematical) state of a well-defined dynamical system (i.e., where we have fixed the type of solution of interest, the initial conditions, and the boundary conditions) [72, p. 97-98].

Of course, the Formal characterisation of gauge symmetry requires a trivialisation of the principle bundle. In this thesis we take 'gauge' to refer to the Formal notion, except in our presentation of the constrained Hamiltonian formalism in section 4.1, where we defined gauge directions as null directions of the symplectic form on the constraint surface, which aligns with the Redundant notion.

Teh is convinced that, although GW's distinction between interior/non-interior symmetries corresponds to non-DES/DES, the logical puzzle about gauge has not been solved. "That is, how can the Redundant aspect of gauge transformations be squared with the claim that such transformations exhibit DES?" [72, p. 111]. To resolve the puzzle, we need to identify what elements of the gauge group $\mathcal{G}$ are actually Redundant, and this is where the Hamiltonian formalism comes to play, which identifies Redundant gauge transformations as those generated by the primary constraints.

For Maxwell theory the primary constraint is the Gauss law. To guarantee that the Poisson brackets of the smeared Gauss constraint with physical quantities are well-defined, one needs to impose asymptotic conditions on the gauge transformations generated by that constraint [72, p. 113]. Teh simply quotes Balachandran's result stating that the simplest such condition on $g \in \mathcal{G}$ is [89, p. 22]

$$g(\mathbf{x}) \to 1 \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{4.4}$$

We denote the subgroup of maps satisfying this condition by $\mathcal{G}^\infty$ (we will shortly discuss the appropriate rate of convergence). It is important to note that, in order to even be able to speak of the identity at infinity, we need a section or frame, i.e. a trivialisation, of the bundle at infinity. If we assume we are working on a trivial bundle anyway then we need not worry about this.

In addition, the transformations generated by the smeared Gauss constraint through the Poisson bracket are *small* (homotopic to the identity map $x \mapsto 1 \in \mathcal{G}$), and we write $\mathcal{G}_0^\infty$ for the subgroup of small transformations satisfying condition 4.4. To see this, note that the Gauss constraint works on elements of the infinite-dimensional Lie algebra $\mathrm{Lie}(\mathcal{G})$, and these must be exponentiated to get an element of the Lie group $\mathcal{G}$. It is a well-known fact that the image of the exponential map lies in the connected component of the identity of the Lie group, so this explains why the Gauss constraint generates only small gauge transformations. Smallness is an empty condition for electromagnetism in three spatial dimensions, but it does play a role for the non-Abelian structure group $SU(2)$, as we shall see below.

Denoting by $\mathcal{G}^I$ the subgroup that leaves invariant whatever (asymptotic) boundary conditions are imposed on the fields, we get a hierarchy of subgroups [72, p. 115]:

$$\mathcal{G}_0^\infty \subset \mathcal{G}^\infty \subset \mathcal{G}^I \subset \mathcal{G}.$$

If, then, we improve GW's definition of symmetries exhibiting DES, i.e. those that are boundary-preserving but non-interior, to those that are boundary-preserving but non-Redundant (in the sense of being generated by the primary constraints), we obtain [72, p. 116]

$$\mathcal{G}_{\mathrm{DES}} = \mathcal{G}^I / \mathcal{G}_0^\infty.$$

For Maxwell theory with a scalar field, the transformations preserving the asymptotic boundary condition of the fields going to zero are those that tend towards a constant value asymptotically at an appropriate rate. Thus, the quotient $\mathcal{G}_{\mathrm{DES}}$ can be identified as the possible values a constant transformation could take at infinity, i.e. with $U(1)$ implemented through the global gauge transformations. In the non-Abelian case we get topological contributions: for $G = SU(2)$ on flat space, for instance, we can view $\mathcal{G}^I$ as the smooth maps $\mathbb{R}^3 \cup \{\infty\} \to SU(2)$, which equivalently are the maps $S^3 \to S^3$. Since $\pi_3(S^3) \cong \mathbb{Z}$ we have different homotopy classes of gauge transformations labelled by integers [22, 89]. The class of $0 \in \pi_3(S^3)$ then corresponds to $\mathcal{G}_0^\infty$. For $U(1)$ there are no such homotopy classes since $\pi_3(S^1)$ is trivial.

Now, this required asymptotic behaviour of gauge transformations comes rather out of the blue. Let us therefore closely examine Balachandran's argument on which Teh's analysis is based. We consider the Hamiltonian formulation of electromagnetism on a Cauchy surface Σ as in section 4.1.2. For a function $\lambda \in C^\infty(\Sigma)$ the smeared Gauss constraint $G_\lambda = \int_\Sigma dx^3 \lambda \partial^i E_i$ generates the gauge tranformations, i.e. $\{G_\lambda, A_i\} = \partial_i \lambda$. Balachandran then considers the generators of rotations [89, p. 21]

$$J_i = \int_\Sigma dx^3 E^j \left((\mathbf{x} \times \nabla)_i \delta_{jk} + \epsilon_{ijk}\right) A^k = \int_\Sigma dx^3 (\epsilon_{ijn} x^j E^k \partial^n A_k + \epsilon_{ijk} E^j A_k) = \int_\Sigma dx^3 \epsilon_{ijk} (x^j E^n \partial^k A_n + E^j A_k).$$

The Poisson bracket of G_λ with the J_i should be well-defined, but there are two orders of integration, yielding different results. Indeed, by calculating

$$\begin{aligned}
\{\nabla \cdot \mathbf{E}(\mathbf{x}), J_i\} = & \int_\Sigma dy^3 \left(\frac{\delta(\nabla \cdot \mathbf{E}(\mathbf{x}))}{\delta A_m(\mathbf{y})} \frac{\delta J_i}{\delta E^m(\mathbf{y})} - \frac{\delta(\nabla \cdot \mathbf{E}(\mathbf{x}))}{\delta E^m(\mathbf{y})} \frac{\delta J_i}{\delta A_m(\mathbf{y})} \right) \\
&= -\int_\Sigma dy^3 \frac{\delta(\nabla \cdot \mathbf{E}(\mathbf{x}))}{\delta E^m(\mathbf{y})} \frac{\delta J_i}{\delta A_m(\mathbf{y})} = -\int_\Sigma dy^3 \partial_m \delta(\mathbf{x}-\mathbf{y}) \frac{\delta J_i}{\delta A_m(\mathbf{y})} \\
&= -\int_\Sigma dy^3 \partial_m \delta(\mathbf{x}-\mathbf{y}) \int_\Sigma dz^3 (\epsilon_{iln} z^l E_k(\mathbf{z}) \partial_z^n \delta(\mathbf{z}-\mathbf{y}) \delta^{km} + \epsilon_{ijk} E^j(\mathbf{z}) \delta(\mathbf{z}-\mathbf{y}) \delta^{km}) \\
&= -\int_\Sigma dy^3 \partial_m \delta(\mathbf{x}-\mathbf{y}) \int_\Sigma dz^3 \delta(\mathbf{z}-\mathbf{y}) (-\epsilon_{iln} \partial_z^n (z^l E^m(\mathbf{z})) + \epsilon_{ij}^{\ \ m} E^j(\mathbf{z})) \\
&= -\int_\Sigma dy^3 \partial_m \delta(\mathbf{x}-\mathbf{y}) \int_\Sigma dz^3 \delta(\mathbf{z}-\mathbf{y}) (-\epsilon_{iln} (\delta^{nl} E^m(\mathbf{z}) + z^l \partial_z^n E^m(\mathbf{z})) + \epsilon_{ij}^{\ \ m} E^j(\mathbf{z})) \\
&= -\int_\Sigma dy^3 \partial_m \delta(\mathbf{x}-\mathbf{y}) (-\epsilon_{iln} y^l \partial^n E^m(\mathbf{y}) + \epsilon_{ij}^{\ \ m} E^j(\mathbf{y})) \\
&= -\int_\Sigma dy^3 \delta(\mathbf{x}-\mathbf{y}) (-\epsilon_{iln} \partial^m (y^l \partial^n E^m(\mathbf{y})) + \epsilon_{ijm} \partial^m E^j(\mathbf{y})) \\
&= -(-\epsilon_{iln} (\delta^{ml} \partial^n E_m(\mathbf{x}) + x^l \partial^m \partial^n E_m(\mathbf{x})) + \epsilon_{ijm} \partial^m E^j(\mathbf{x})) \\
&= \epsilon_{iln} x^l \partial^n \nabla \cdot \mathbf{E}(\mathbf{x}) = (\mathbf{x} \times \nabla)_i \nabla \cdot \mathbf{E}(\mathbf{x}),
\end{aligned}$$

we can find the Poisson bracket of the Gauss constraint with J_i by first evaluating on $\nabla \cdot \mathbf{E}$, that is:

$$\begin{aligned}
\{G_\lambda, J_i\} = & \int_\Sigma dx^3 \lambda \{\nabla \cdot \mathbf{E}(\mathbf{x}), J_i\} = \int_\Sigma dx^3 \lambda (\mathbf{x} \times \nabla)_i \nabla \cdot \mathbf{E}(\mathbf{x}) \\
&= -\int_\Sigma dx^3 (\mathbf{x} \times \nabla)_i \lambda \nabla \cdot \mathbf{E} = -G_{(\mathbf{x} \times \nabla)_i \lambda}.
\end{aligned}$$

On the other hand, we could also calculate the Poisson bracket by first evaluating with the integrand of J_i as follows:

$$\begin{aligned}
\{G_\lambda, J_i\} = & \int_\Sigma dx^3 \{G_\lambda, E_j \left((\mathbf{x} \times \nabla)_i \delta^{jk} + \epsilon^{ijk}\right) A_k\} = \int_\Sigma dx^3 E_j \left((\mathbf{x} \times \nabla)_i \delta^{jk} + \epsilon^{ijk}\right) \{G_\lambda, A_k\} \\
= & \int_\Sigma dx^3 E_j \left((\mathbf{x} \times \nabla)_i \delta^{jk} + \epsilon^{ijk}\right) \partial_k \lambda = -\int_\Sigma dx^3 \partial_k E_j \left((\mathbf{x} \times \nabla)_i \delta^{jk} + \epsilon^{ijk}\right) \lambda \\
= & -\int_\Sigma dx^3 (\nabla \cdot \mathbf{E}(\mathbf{x} \times \nabla)_i \lambda - (\nabla \times \mathbf{E})_i \lambda) = -G_{(\mathbf{x} \times \nabla)_i \lambda} + \int_\Sigma dx^3 \lambda (\nabla \times \mathbf{E})_i.
\end{aligned}$$

We see that the two methods of integration only agree up to the term $\int_\Sigma dx^3\lambda(\nabla \times \mathbf{E})_i$, which can be re-expressed as a boundary term through Stokes theorem as [89, p. 22]

$$\int_\Sigma dx^3\lambda(\nabla \times \mathbf{E})_i = \int_{|\mathbf{x}|\to\infty} d\Omega|\mathbf{x}|^2 \frac{\mathbf{x}\cdot\mathbf{E}}{|\mathbf{x}|}(\mathbf{x}\times\nabla)_i\lambda = \int_{|\mathbf{x}|\to\infty} d\Omega|\mathbf{x}|(\mathbf{x}\cdot\mathbf{E})(\mathbf{x}\times\nabla)_i\lambda. \tag{4.5}$$

Balachandran then argues that in order to make sure that the Poisson brackets are well-defined, we need this term to vanish and therefore require that $\lambda \to 0$ sufficiently rapidly. Since λ is a gauge parameter that needs to be exponentiated as $e^{i\lambda}$ to get an element of the gauge group, the asymptotic condition on λ yields the asymptotic condition 4.4 on gauge transformations.

4.3.3. Symplectic underpinnings

However, Balachandran's argument looks rather ad hoc. Its deeper meaning is not immediately clear: why really are the Poisson brackets ill-defined if we do not impose the condition 4.4? Why, apparently, can the Gauss constraint not generate all gauge transformations, but only those that become the identity at spatial infinity? And more specifically: in Balachandran's derivations partial integration is used repeatedly and the boundary terms are always assumed to vanish. But why should this be so? And if those boundary terms vanish, does the RHS of 4.5 not vanish automatically too, regardless of the asymptotic behaviour of λ? To understand these issues, let us take the more general approach of examining the momentum map for the action of the gauge group on the space of Yang-Mills connections.

Following [90, 91] we consider a four-dimensional, connected, oriented and time-oriented (see Definition B.11) Lorentzian manifold (M, g) with a compact Cauchy surface Σ and let $\mathfrak{a}$ denote the affine space of $\mathfrak{g}$-valued connection 1-forms on Σ (assuming the bundle to be trivial), as in Theorem 2.13. The phase space is the cotangent bundle $T^*\mathfrak{a}$. We understand this phase space to consists of of pairs $(A, E) \in \Omega^1(\Sigma, \mathfrak{g}) \times \Omega^2(\Sigma, \mathfrak{g})$, such that $E(A) = \langle A, E\rangle$ is defined through the conjugate pairing

$$\langle A, E\rangle = \int_\Sigma \mathrm{Tr}\, A \wedge E.$$

The constraint equations for Yang-Mills theory are the Gauss law [91, p. 364]

$$D_A E := dE + [A \wedge E] = 0.$$

For electromagnetism these reduce to $dE = 0$. The action of the gauge group $\mathcal{G}$ lifts to phase space:

$$g \cdot (A, E) = (g^{-1}Ag + g^{-1}dg, g^{-1}Eg), \qquad g \in \mathcal{G}.$$

The Lie algebra Lie($\mathcal{G}$) is isomorphic to $C^\infty(\Sigma, \mathfrak{g})$. We equip $T^*\mathfrak{a}$ with the canonical symplectic form $\omega = \int_\Sigma \mathbb{d}A \wedge \mathbb{d}E$. We should now like to check that, with this symplectic form, the Gauss constraint actually generates gauge transformations. To this end we recall the definition of a momentum map.

Definition 4.3. Let (N, ω) be a symplectic manifold and H a Lie group that acts on N by symplectomorphisms. Let $\mathfrak{h}$ denote the Lie algebra of H with dual $\mathfrak{h}^*$, and write $\langle\cdot,\cdot\rangle\colon \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{R}$ for the pairing. Then a *momentum map* for the H-action on N is a map $\mu\colon N \to \mathfrak{h}^*$ such that for all $\xi \in \mathfrak{h}$ we have

$$d\langle\mu, \xi\rangle = \iota_{X_\xi}\omega = \omega(X_\xi, \cdot).$$

Here X_ξ denotes the fundamental vector field generated by ξ and $\langle\mu, \xi\rangle$ is understood as a function $M \to \mathbb{R}$ through $\langle\mu, \xi\rangle(x) = \langle\mu(x), \xi\rangle$.

The momentum map $\mu\colon T^*\mathfrak{a} \to \Omega^3(\Sigma, \mathfrak{g})$ for the action of the gauge group $\mathcal{G}$ on $T^*\mathfrak{a}$ is the Gauss constraint $\mu(A, E) = D_A E$ [90, p. 364]. Here we identify $\eta \in \Omega^3(\Sigma, \mathfrak{g})$ as an element in the dual $\mathfrak{g}^*$ through the pairing $\langle\eta, \xi\rangle = \int_\Sigma \mathrm{Tr}\, \xi \wedge \eta$. Thus, for some $\xi \in C^\infty(\Sigma, \mathfrak{g})$ we have

$$\langle\mu, \xi\rangle(A, E) = \int_\Sigma \mathrm{Tr}\, D_A E \wedge \xi = -\int_\Sigma \mathrm{Tr}\, E \wedge D_A\xi + \int_{\partial\Sigma} \mathrm{Tr}\, E \wedge \xi, \tag{4.6}$$

where we can also understand the boundary term asymptotically as a limit $\mathbf{x} \to \infty$ in the case of Minkowski spacetime - we return to this shortly. Evidently, the momentum map μ applied to an element $\xi \in C^\infty(\Sigma, \mathfrak{g})$ is just the Gauss constraint smeared with ξ. But, if $\mu(A, E) = D_A E$ really is to define the momentum map for the action of $\mathcal{G}$, then by definition it must satisfy, for every $\xi \in C^\infty(\Sigma, \mathfrak{g})$:

$$\mathbb{d}\langle \mu, \xi \rangle = \iota_{\mathbb{X}_\xi} \omega := \omega(\mathbb{X}_\xi, \cdot), \tag{4.7}$$

where $\mathbb{X}_\xi \in \mathfrak{X}(T^*\mathfrak{a})$ denotes the fundamental vector field on $T^*\mathfrak{a}$ generated by the Lie algebra element ξ. To check 4.7, let us first calculate $\omega(\mathbb{X}_\xi, \cdot)$. By definition, for any function $\mathbb{F} \in C^\infty(T^*\mathfrak{a})$ we have

$$\mathbb{X}_\xi(\mathbb{F})(A, E) = \frac{d}{dt}\bigg|_{t=0} \mathbb{F}\left(e^{t\xi} \cdot (A, E)\right) = \frac{d}{dt}\bigg|_{t=0} \mathbb{F}\left(e^{-t\xi} A e^{t\xi} + e^{-t\xi} d(e^{t\xi}), e^{-t\xi} E e^{t\xi}\right).$$

Now, for the functions $\mathbb{F} = A, E$ this simply gives

$$\begin{aligned} \mathbb{X}_\xi(A) &= \frac{d}{dt}\bigg|_{t=0} \left(e^{-t\xi} A e^{t\xi} + e^{-t\xi} d(e^{t\xi})\right) = -\xi A + A\xi + d\xi = [A, \xi] + d\xi = D_A \xi, \\ \mathbb{X}_\xi(E) &= \frac{d}{dt}\bigg|_{t=0} \left(e^{-t\xi} E e^{t\xi}\right) = -\xi E + E\xi = [E, \xi] \end{aligned}$$

Thus, if we plug $\mathbb{X}_\xi$ into the first slot of the symplectic form $\omega = \int_\Sigma \mathrm{Tr}\, \mathbb{d}A \wedge \mathbb{d}E$ we obtain

$$\begin{aligned} \omega_{(A,E)}(\mathbb{X}_\xi, \cdot) &= \int_\Sigma \mathrm{Tr}\ (\mathbb{d}A(\mathbb{X}_\xi) \wedge \mathbb{d}E - \mathbb{d}E(\mathbb{X}_\xi) \wedge \mathbb{d}A) = \int_\Sigma \mathrm{Tr}\ (\mathbb{X}_\xi(A) \wedge \mathbb{d}E - \mathbb{X}_\xi(E) \wedge \mathbb{d}A) \\ &= \int_\Sigma \mathrm{Tr}\ (([A, \xi] + d\xi) \wedge \mathbb{d}E - [E, \xi] \wedge \mathbb{d}A) = \int_\Sigma \mathrm{Tr}\ (D_A \xi \wedge \mathbb{d}E - [E, \xi] \wedge \mathbb{d}A). \end{aligned} \tag{4.8}$$

For $G = U(1)$ this reduces to the simpler expression $\iota_{\mathbb{X}_\xi}\omega = \int_\Sigma D_A \xi \wedge \mathbb{d}E = \int_\Sigma d\xi \wedge \mathbb{d}E$. Now, the left hand side of 4.7 gives

$$\mathbb{d}\langle \mu, \xi \rangle = \mathbb{d} \int_\Sigma \mathrm{Tr}\, D_A E \wedge \xi = \int_\Sigma \mathrm{Tr}\ (\mathbb{d}(D_A E) \wedge \xi - D_A E \wedge \mathbb{d}\xi).$$

However, we cannot immediately see how this agrees with 4.8, because 4.8 is an expression in $D_A\xi$, whereas our last result is an expression in ξ. We need to perform the partial integration from 4.6, giving

$$\mathbb{d}\langle \mu, \xi \rangle = -\int_\Sigma \mathrm{Tr}\ (\mathbb{d}E \wedge D_A \xi - E \wedge \mathbb{d}(D_A \xi)) + \mathbb{d} \int_{\partial\Sigma} \mathrm{Tr}\ E \wedge \xi. \tag{4.9}$$

The second term in the first integral can be rewritten as

$$\begin{aligned} E \wedge \mathbb{d}(D_A \xi) &= E \wedge \mathbb{d}(d\xi + [A, \xi]) = E \wedge \mathbb{d}[A, \xi] = E \wedge [\mathbb{d}A, \xi] = E \wedge \mathbb{d}\mathbb{A}\xi - E \wedge \xi \mathbb{d}A \\ &= -E\xi \wedge \mathbb{d}A + \xi E \wedge \mathbb{d}A - \xi E \wedge \mathbb{d}A + E \wedge \mathbb{d}\mathbb{A}\xi = -[E, \xi] \wedge \mathbb{d}A + [E \wedge \mathbb{d}A, \xi]. \end{aligned}$$

Thus, the first integral in 4.9 equals

$$\int_\Sigma \mathrm{Tr}\ (D_A \xi \wedge \mathbb{d}E + E \wedge \mathbb{d}(D_A \xi)) = \int_\Sigma \mathrm{Tr}\ (D_A \xi \wedge \mathbb{d}E - [E, \xi] \wedge \mathbb{d}A + [E \wedge \mathbb{d}A, \xi]).$$

But the trace of the full commutator term gives zero (or, since the trace is Ad-invariant, we could also immediately have rewritten $\mathrm{Tr}\ E \wedge [\mathbb{d}A, \xi] = -\mathrm{Tr}\ [E, \xi] \wedge \mathbb{d}A$), so we obtain precisely 4.8! This implies that, if the Gauss constraint $\mu(A, E) = D_A E$ is to be the momentum map for the action of the gauge group, then the boundary term in 4.9 must be zero. In order to guarantee this, we require that ξ vanishes on the boundary.[1]

But what if we are working on Minkowski spacetime with Cauchy surface $\Sigma \cong \mathbb{R}^3$? Then we cannot speak of an actual boundary, but we must have some asymptotic condition such as 4.4. To understand what happens in this case we need to consider the asymptotic fall-off behaviour of A and E. Indeed, to make integrals like $\int_\Sigma A \wedge E$ or $\int_\Sigma dE$ well-defined, we should require $A \to 0 + \mathcal{O}(r^{-2})$ and $E \to 0 + \mathcal{O}(r^{-2})$ as $r \to \infty$. After all, in spherical coordinates the integral over Σ picks up a factor r^2, meaning that the rest of the integrand should fall of with order $\mathcal{O}(r^{-3-\epsilon})$ at least. The requirement that A vanish asymptotically also puts restrictions on the elements $g \in \mathcal{G}$ of the gauge group. If the gauge group action $A \mapsto g^{-1}Ag + g^{-1}dg$ is to respect the asymptotic condition on A, we should at least require $dg \to 0$, i.e. that g asymptotically tends to a constant value. If we can write $g = e^\xi$ this just means $\xi \to \text{const}$. We want $g^{-1}dg$ to fall off at the same rate as A, so we then need $\xi \to \text{const} + \mathcal{O}(r^{-1})$. As in section 4.3.2 we write $\mathcal{G}^I$ for the group of gauge transformations that become constant asymptotically.

However, if we want to define the momentum map $\mu(A, E) = D_A E$ as above, then further restrictions on the asymptotic behaviour of $\xi \in C^\infty(\Sigma, \mathfrak{g})$ are in order. After all, we need integrals like $\int_\Sigma D_A E \wedge \xi$ and $\int_\Sigma E \wedge D_A \xi$ to exist and the boundary term in 4.9 to vanish. Considering that we have already assumed $E \to 0 + \mathcal{O}(r^{-2})$, it suffices to then also require $\xi \to 0 + \mathcal{O}(r^{-1})$. Exponentiating these Lie algebra elements gives the unphysical gauge group $\mathcal{G}_0^\infty$. We thus recover the quotient $\mathcal{G}^I/\mathcal{G}_0^\infty$ of the well-defined gauge transformations whose momentum map is not the Gauss constraint, i.e. which are not generated by it. Now, this thesis is about global gauge transformations because they are elements of that quotient. Indeed, we have the following result.

Proposition 4.4. Let $\mathcal{G} = C^\infty(\Sigma, G)$ denote the gauge group of the trivial G-bundle over Σ with Lie algebra $\text{Lie}(\mathcal{G}) \cong C^\infty(\Sigma, \mathfrak{g})$. Write $\mathcal{G}_0^I = \{e^{\xi(\mathbf{x})} : \xi(\mathbf{x}) \to \text{const} + \mathcal{O}(r^{-1})\}$ for the group of small gauge tranformations respecting the boundary conditions and $\mathcal{G}_0^\infty = \{e^{\xi(\mathbf{x})} : \xi(\mathbf{x}) \to 0 + \mathcal{O}(r^{-1})\}$. Then

$$\mathcal{G}_0^I/\mathcal{G}_0^\infty \cong G_0 = \{e^\zeta : \zeta \in \mathfrak{g}\}.$$

Proof. It is clear that $G_0 \subset \mathcal{G}_0^I/\mathcal{G}_0^\infty$, since the global gauge transformations are already constant everywhere. Conversely, suppose $g = e^{\xi(\mathbf{x})} \in \mathcal{G}_0^I/\mathcal{G}_0^\infty$. We need to show that we can write g as the product of a global gauge transformation e^ζ with an element of $\mathcal{G}_0^\infty$. If G is Abelian this is immediate: we then have $e^{\xi(\mathbf{x})} = e^{\xi(\mathbf{x})-L}e^L$, where $L = \lim_{|\mathbf{x}|\to\infty} \xi(\mathbf{x})$. For the general case, however, we need to use the Baker-Campbell-Hausdorff formula. We need to find $\rho \colon \mathbb{R}^3 \to \mathfrak{g}$ such that $\rho(\mathbf{x}) \to 0 + \mathcal{O}(r^{-1})$ and $e^{\xi(\mathbf{x})} = e^{\rho(\mathbf{x})}e^L$, i.e. $e^{\rho(\mathbf{x})} = e^{\xi(\mathbf{x})}e^{-L}$. The BCH formula then gives

$$\rho(\mathbf{x}) = \xi(\mathbf{x}) - L - \frac{1}{2}[\xi(\mathbf{x}), L] - \frac{1}{12}[\xi(\mathbf{x}), [\xi(\mathbf{x}), L]] - \frac{1}{12}[L, [\xi(\mathbf{x}), L]] + ...$$

It needs to be checked that $\rho(\mathbf{x})$ defined this way is actually an element of $\mathcal{G}_0^\infty$. Now, it is clear that $\rho(\mathbf{x}) \to 0$, since in the limit $\xi(\mathbf{x}) \to L$, meaning that the commutators vanish in the limit and that $\rho(\mathbf{x}) \to \xi(\mathbf{x}) - L = 0$. Additionally, since $\xi(\mathbf{x})$ attains L with a fall-off behaviour of $\mathcal{O}(r^{-1})$, any product of $\xi(\mathbf{x})$ with L appearing in the nested commutators also has at least this fall-off behaviour. Thus, we find that indeed $\rho(\mathbf{x}) \to 0 + \mathcal{O}(r^{-1})$. □

Of course, G_0 does not equal all of G if the exponential map is not surjective, but we need not worry about that for $U(1)$ or $SU(N)$ since these groups are connected and compact. In addition, we generally will not have that $\mathcal{G}^I$ equals $\mathcal{G}_0^I$ because there might be topological contributions, such as the winding numbers we have met before. For $SU(2)$, for instance, we have seen that the homotopy classes of maps

[1] Another possibility would be to consider the group $\mathcal{G}_*$ of *pointed* gauge transformations, i.e. those transformations which are the identity at some arbitary fixed point $x_0 \in \Sigma$. Then the only global transformation is the trivial one and the action of $\mathcal{G}_*$ is free, such that the symplectic reduction $T^*\mathfrak{a}//\mathcal{G}_*$ is well-defined. This approach is pursued by Belot [92]. We could also consider so-called *irreducible connections*, i.e. connections for which the holonomy group acts irreducibly. The gauge group does act freely on the space of irreducible connections [93].

$\mathbb{R}^3 \cup \{\infty\} \to SU(2)$ are labelled by $\mathbb{Z}$. In such cases, then, the physical quotient of gauge transformations is given not only by the global gauge transformations, but by a copy of G_0 for every homotopy class. For $G = U(1)$ we do not have topological contributions since $\pi_3(S^1) = 0$, so in that case

$$\mathcal{G}^I/\mathcal{G}_0^\infty = \mathcal{G}_0^I/\mathcal{G}_0^\infty \cong U(1)_0 = U(1).$$

4.3.4. Holism and horizontal symplectic geometry

There remains a body of work that is of interest to our purposes with which we have not yet engaged: the articles by Gomes, Riello and Hopfmüller cited in the introduction to this chapter.[2] In these articles, a "unified geometric framework" [75,76] is developed which encompasses and refines notions from this chapter such as DES, boundaries, subsystems and the transverse/longitudinal split and even relates to the DFM from chapter 3. The approach is based on the introduction of a horizontal 1-form ϖ on field space, considered as a principal fibre bundle with the gauge group as its structure group. Here "horizontal directions are essentially a choice of non-gauge directions in field space transforming covariantly along the fiber" [78, p. 18].

Before we introduce the field space formalism, let us consider the less technical exposition of these ideas which focuses on the notion of *holism* as the empirical significance of gauge symmetries. More precisely, in [78] Gomes shows how DES of gauge symmetries can be understood as a failure of *global supervenience on subsystems* (GSS). That is, there are cases in which the states of a number of subsystems do not uniquely define the state of the entire universe consisting of those subsystems (the parts do not completely determine the whole, hence the term 'holism'). In those cases additional relational data, i.e. data on how the various subsystems relate to each other, is needed. A very elementary example of such a failure of GSS would be two separated collections of particles. We can describe each individual collection of particles by specifying only the relative distances between the particles. However, if we want to describe the entire system consisting of the two collections, we need the additional datum of the distance between the respective centers of mass of the particle collections.

For gauge symmetries, Gomes works out how the gauge-invariant data describing subsystems do not suffice to uniquely determine the gauge-invariant description of the composition, a possibility that GW do not even consider. This leads to the following result for the case of electromagnetism (cf. Theorem 1 in [78]), which is similar to the situation pictured at the end of section 4.3.1.

Theorem 4.5 (Rigid variety for $U(1)$)**.** For electromagnetism as coupled to a Klein-Gordon scalar field in a simply-connected universe: given the physical content of two regions, for matter vanishing at the

[2] Others have also continued the discussion of the work of GW and Teh. Using the very framework developed by GW, Friederich has argued that gauge symmetries cannot in fact exhibit DES [71, 94]. This led Ramirez to identify a "puzzle concering local symmetries and their empirical significance" [73]. Ramirez singles out a particular premise in Friederich's articles as the point of conflict between GW and Teh on the one hand and Friederich on the other and argues against it, thereby defending the results of GW and Teh and restoring the view that gauge symmetries can indeed have DES. Ramirez and Teh subsequently decided to "abandon Galileo's ship" in order to look for *non-relational* empirical significance of gauge symmetries [95]. Wallace has also returned to the debate [96,97], considering the Abelian Higgs Lagrangian and the relation between symmetry breaking and asymptotic boundary conditions. He claims the broken and unbroken phases of the Higgs model have different boundary conditions: in the unbroken phase the boundary condition on the Higgs field is supposedly $\varphi(x) \to 0$ as $x \to \infty$, such that the condition for a gauge symmetry $g(x)$ to be boundary-preserving is $g(x) \to e^{i\theta}$, with θ fixed [97, p. 22]. In the broken phase, however, he claims that the asymptotic condition on the gauge symmetries is the more stringent $g(x) \to 1$, since the minimum of V is now some non-zero value which transforms non-trivially even under constant gauge transformations. We have developed a different view in this chapter, namely that the asymptotic boundary condition for the gauge field is $\mathbf{A} \to 0$, but the Higgs field φ need only become constant at infinity. Thus, the gauge transformations $g = e^{i\xi} \in \mathcal{G}$ preserving these boundary conditions are those that become constant asymptotically, for then the pure gauge term $g^{-1}dg$ vanishes so that $\mathbf{A}$ remains zero at infinity, and the Higgs field is only changed by a constant factor at infinity and therefore still respects the required boundary condition. These boundary conditions do not change when we go form the unbroken to the broken phase and it is not the case that in the broken phase the boundary-preserving gauge transformations are only those that become the identity asymptotically. Rather, we identified the transformations satisfying $g \to 1$ as those that are *redundant*, i.e. unphysical.

boundary but not in the bulk of the regions, the universal state is undetermined, resulting in a residual variety parametrised by an element of $U(1)$. Here the particular action of $U(1)$ is that which leaves the gauge-fields invariant, but not the matter fields.

Here *rigid* is used in our sense of 'global'. Now, although this result singles out the global gauge transformations as those exhibiting DES through a failure of GSS, two issues come up: how can we interpret this for *asymptotic* boundary conditions and how does this generalise to the non-Abelian setting? As for the first, Gomes is aware of this and refers to [98] for a "recovery of the asymptotic results [...] using the present framework" [78, p. 17]. Secondly, Gomes treats the non-Abelian case in the appendix and states that "one is able to retain, for the non-Abelian context, all the interesting results obtained" [78, p. 14].

Let us now introduce the field-space formalism used by Gomes and Riello to derive the above theorem. Following [75] we write Φ_{YM} for the space of Yang-Mills gauge fields A with matter fields φ on a Cauchy surface Σ, with gauge group $\mathcal{G}$ whose infinitesimal action on the fields is, for a Lie algebra element $\xi \in \mathrm{Lie}(\mathcal{G})$

$$\delta_\xi A := D_A \xi = d\xi + [A, \xi], \qquad \delta_\xi \varphi = \xi \varphi.$$

We have the usual fundamental vector field map $\mathrm{Lie}(\mathcal{G}) \to \mathfrak{X}(\Phi_{YM})$ given by

$$\mathbb{X}_\xi(F) = \int \delta_\xi F \frac{\mathbb{d}}{\mathbb{d}F}, \qquad F \in \Phi_{YM}.$$

The flows of the vector fields $\mathbb{X}_\xi$ generate gauge orbits in Φ_{YM}, which can be interpreted as the fibres of an infinite-dimensional principal $\mathcal{G}$-bundle $\pi\colon \Phi_{YM} \to \Phi_{YM}/\mathcal{G}$ if the action of $\mathcal{G}$ is free and proper (for which $\mathcal{G}$ must be equipped with the structure of a Banach-Lie group). If this is not the case, $\Phi_{YM}/\mathcal{G}$ is a so-called *stratified space*, meaning that the space is not a manifold but admits a decomposition into *strata*, which are themselves manifolds. For $\Phi_{YM}/\mathcal{G}$ the strata are related to the conserved global charges [75]. Vector fields $\mathbb{X} \in \mathfrak{X}(\Phi_{YM})$ tangential to the $\mathbb{X}_\xi$ are called *vertical* and they span the vertical tangent space $V_F \subset T_F\Phi_{YM}$ at every point $F \in \Phi_{YM}$. There is no canonical horizontal complement such that $T\Phi_{YM} \cong H \oplus V$ (where V denotes the vertical subbundle), so by analogy to the finite-dimensional case we are led to the introduction of a connection-like 1-form, which is Definition 2.1 in [79].

Definition 4.6. Let $\varpi \in \Omega^1(\Phi_{YM}, \mathrm{Lie}(\mathcal{G}))$. Then ϖ is called a $\mathcal{G}$-compatible functional connection form on Φ_{YM}, or simply a *functional connection*, if it satisfies the following properties for all field-dependent gauge transformations $\xi\colon \Phi_{YM} \to \mathrm{Lie}(\mathcal{G})$:

$$\begin{aligned} \varpi(\mathbb{X}_\xi) &= \xi \\ \mathbb{L}_{\mathbb{X}_\xi} \varpi &= [\varpi, \xi] + \mathbb{d}\xi. \end{aligned}$$

Here $\mathbb{L}$ denotes the Lie derivative on Φ_{YM}.

The horizontal vector fields are those that lie in the kernel of ϖ. Writing $\hat{H}\colon \mathfrak{X}(\Phi_{YM}) \to \mathfrak{X}(\Phi_{YM})$ for the horizontal projection of vector fields we can define a horizontal exterior derivative $\mathbb{d}_H = \mathbb{d}(\hat{H}(\cdot))$ [77]. On the gauge and matter fields it is given by $\mathbb{d}_H A = \mathbb{d}A - D_A \varpi$ and $\mathbb{d}_H \varphi = \mathbb{d}\varphi + \varpi\varphi$ [76].

But what is the point of introducing ϖ? Where does its unifying power lie? How does it help us understand the significance of global gauge symmetries? To understand this, let us consider Maxwell theory. By a derivation much like that in section 4.1.2 (which was also based on work by Gomes), one tries to find the orthogonal horizontal complement of vectors $\int \delta_\xi A \frac{\mathbb{d}}{\mathbb{d}A}$ and it is found that, for electromagnetism on $\Sigma \cong \mathbb{R}^3$ with fast-decaying boundary conditions [75]:

$$\varpi(x) = \left(\nabla^{-2} \operatorname{div}(\mathbb{d}A)\right)(x) = \int_\Sigma \frac{d^3y}{4\pi} \frac{\partial^i \mathbb{d}A_i(y)}{|x - y|}.$$

This is of course precisely the expression for the Coulombic component $\partial_i(\Delta^{-1}\partial^j A_j)$ of the electromagnetic field that we found in section 4.1.2 and thought of as the pure gauge part of A. We call this functional connection ϖ the *Dirac-Singer-De Witt* connection for electromagnetism, or the SdW connection more generally [79]. Since the functional connection is horizontal, i.e. leaves alone the vertical vector fields $\mathbb{X}_\xi$ which are the gauge directions, we would think it cannot be used to pick out the physical transformations in $\mathcal{G}$. Such physical transformations or "actual symmetries" [75] should lie in the kernel of ϖ, but we have assumed that $\varpi(\mathbb{X}_\xi) = \xi$, i.e. that ϖ leaves alone the vertical vectors. However, it has been implicitly assumed that there is a 1-1 correspondence between the Lie algebra elements $\xi \in \text{Lie}(\mathcal{G})$ and the horizontal subbundle V, but such a 1-1 correspondence holds only pointwise [75]. Indeed, for the Dirac-Singer-De Witt connection the constant maps in $\xi_0 \in \text{Lie}(\mathcal{G})$ are precisely the ones that lie in the kernel of ϖ, as can be seen from the fact that

$$\nabla^{-2}\text{div}(\mathbb{d}A(\mathbb{X}_{\xi_0})) = \nabla^{-2}\text{div}(\mathbb{X}_{\xi_0}(A)) = \nabla^{-2}\text{div}(0) = 0.$$

Thus, "we see that the Dirac-Singer-DeWitt connection automatically picks out global gauge transformations in electromagnetism as being physically distinct from local ones" [75]. In the non-Abelian case, things are less straightforward, for then gauge transformations are field-dependent, leading to a field-space curvature $\mathbb{F} = \mathbb{d}_H\varpi \neq 0$. One is then forced to resort to perturbation theory [76].

Besides the Singer-De Witt connection, a *Higgs connection* is also considered in section 7 of [76] for the Yang-Mills-Higgs theory. To our mind, however, it does not tell us very much about the Higgs mechanism that we do not already know. This Higgs connection is flat and exists only if the Higgs field is everywhere non-vanishing, i.e. in the symmetry-broken phase. This reminds us of the dressing fields from chapter 3, which were also definable through the polar decomposition of the Higgs field only for a non-vanishing field. In fact, there is an intimate connection between the formalism of horizontal symplectic geometry and the DFM. The following result is Proposition 2.23 in [79].

Proposition 4.7. A dressing field on field space, i.e. a function $h\colon \Phi_{YM} \to \mathcal{G}$, such that $R_g^* h = hg$ for any $g \in \mathcal{G}$, exists if and only if a flat connection $\varpi = h^{-1}\mathbb{d}h$ exists. In that case the dressed fields are gauge-invariant and their differentials are related to the horizontal differential and the original fields:

$$\begin{aligned}
\hat{A} &= hAh^{-1} + hdh^{-1}, & \mathbb{d}\hat{A} &= h(\mathbb{d}_H A)h^{-1};\\
\hat{E} &= hEh^{-1}, & \mathbb{d}\hat{E} &= h(\mathbb{d}_H E)h^{-1};\\
\hat{\varphi} &= h\varphi, & \mathbb{d}\hat{\varphi} &= h\mathbb{d}_H\varphi.
\end{aligned}$$

This should not surprise us, since what we did in this chapter is to dress the Higgs field in the Coulomb gauge with the dressing field $\exp(-ie\Delta^{-1}\partial^i A_i)$, where the longitudinal component $\Delta^{-1}\partial^i A_i$ is provided by the SdW connection. However, this is a dressing field only for the local gauge group $\mathcal{G}^\infty$ and not for the global gauge transformations. After all, the global gauge transformations lie in the kernel of the functional connection and should therefore not be thought of as gauge orbits in the Redundant sense of Teh. Thus, we should not even want to dress them away.

All in all, we see that the horizontal formalism really has a unifying power: it connects the DES of global gauge symmetries, the transverse/longitudinal split and the DFM. It clearly supports our analysis in this chapter by singling out global (rigid) gauge transformations as the ones that carry DES in rigourous theorems. Yet, for the Higgs mechanism, the fact that a field space connection exists only if the Higgs field is everywhere non-vanishing, just like for the DFM, remains problematic. If the Higgs field can vanish nowhere we are able only to consider small variations around the Higgs condensate, i.e. stay at the perturbative level of analysis. It is high time, therefore, that we examine Strocchi's non-perturbative results on the Higgs mechanism in chapter 6. But before we do so, we briefly reflect on where we stand now.

4.4. Discussion

We are approximately halfway through this thesis. In chapter 1 we gave an overview of the philosophical discussion on the Higgs mechanism and explained the main conceptual issue leveled against the standard narrative: how can gauge symmetry breaking lead to something physical like mass if gauge symmetry is mere "descriptive fluff" and if Elitzur's theorem tells us that gauge symmetries cannot be broken at all (a result that can heuristically be understood via the path integral argument 3.1)? In chapter 2 we then presented the full Higgs mechanism in the Standard Model and in chapter 3 we explained how we can reformulate it without invoking SSB at all, thereby avoiding Elitzur's theorem and the dubious idea of gauge symmetry breaking altogether.

However, Elitzur's theorem does allow for the breaking of *global* gauge symmetries, and in section 4.2 we have seen that, in the Abelian case, the Higgs mechanism can indeed be understood that way. We can rewrite the Abelian Higgs model in terms of massless fields that are invariant under all but global gauge transformations, such that when we do break this global symmetry we get massive vector bosons. How to think of this breaking as a temporal *process*, and in what sense the masses are really *generated* and not the result of a mere reshuffling of degrees of freedom, are important questions that we address in chapter 7.

But, the problem of "descriptive fluff" is only solved if we can show that global gauge transformations are physical. To support this claim was the second aim of this chapter, and we have used the language of symplectic geometry to do so. In a nutshell, we have shown that the gauge transformations that exhibit *redundancy*, i.e. those that are generated by the first class constraints - the Gauss constraint for Yang-Mills theory, do not fully exhaust the group of permissible gauge transformations. This group of permissible gauge transformations consists of those transformations that respect boundary conditions, which for fields on $\mathbb{R}^3$ are asymptotic fall-off requirements, and we have shown it to consist of the transformations that become asymptotically constant. The transformations generated by the Gauss constraint, however, become the *identity* asymptotically. The quotient of these two groups is therefore non-trivial, and for electromagnetism it is isomorphic to $\mathrm{U}(1)$ implemented as the group of global (also called rigid, or constant) gauge transformations. All these results are further supported still by the formalism of horizontal symplectic geometry presented in section 4.3.4. In that formalism, however, it also becomes yet more clear that not all results on electromagnetism can automatically be generalised to the non-Abelian setting.

Looking forward, we will see in chapter 6 that the mass generation through the Abelian Higgs mechanism in QFT can indeed be understood as the breaking of global gauge symmetry. The main result presented there, Theorem 6.13, is derived in the Coulomb gauge. In this chapter we have seen why that is not surprising: the Coulomb gauge is designed precisely to remove the Redundant part of the gauge group and leave the global gauge transformations intact. Before we present these exciting results by Morchio and Strocchi, however, we introduce AQFT to get a general perspective on the role of the global gauge group in QFT.

5. Algebraic Quantum Field Theory

So far we have been almost exclusively concerned with classical field theory as a context for the Higgs mechanism. In the end, however, the Higgs mechanism is a phenomenon in quantum field theory, and we cannot expect to fully unravel it if we confine ourselves to the classical world. It is time, then, to consider Hilbert spaces, operators and algebras.

The overarching research question of this thesis is about the role of SSB in the Higgs mechanism, and there is in fact a precise definition of SSB for quantum systems (sketched in section 1.2.1), in terms of the unitary inequivalence of GNS representations (Definition A.26). This definition is formulated in the algebraic approach to quantum theory, i.e. the description of quantum systems in terms of C^*-algebras (see Definition A.16). Thus, if we wish to study SSB in QFT this way, then we must first formulate QFT algebraically: we must describe a QFT as some C^*-algebra on which we can define states. This is what the approach called *algebraic quantum field theory* (AQFT) does, and in this chapter we shall introduce it. AQFT abstracts away from the concrete Hilbert spaces normally encountered in quantum theory and focuses on the abstract algebra of operators containing the observables of the theory. In that sense its axioms, which we present in section 5.1, are more abstract than the well-known Wightman axioms [99], which we introduce in chapter 6. In section 5.2 we discuss important concepts such as unitarily inequivalent representations (UIR's), folia and superselection rules, such that in section 5.3 we are able to understand a beautiful result which shows that, under basic physical assumptions, there necessarily arises a global gauge group in AQFT. It is of course precisely in terms of the breaking of this global gauge that we are aiming to understand the Higgs mechanism. Thus, we end this chapter by considering SSB in section 5.4.

Our aim is to present the general framework in which SSB of global gauge symmetry can be understood. In the next chapter we concentrate on the details of the Higgs mechanism, basing our work on the background of ideas developed in this chapter.

5.1. Haag-Kastler axioms

In the algebraic formulation of the axioms of QFT, we consider operator algebras $\mathcal{U}(\mathcal{O})$ on regions of spacetime $\mathcal{O}$. In terms of the Wightman axioms (section 6.1), we think of these algebras as being generated by all operator-valued distributions $\Phi(f)$ with f having support in $\mathcal{O}$. However, the Haag-Kastler axioms [100] do not require us to think about fields as operator-valued distributions: they mention only the abstract assignment $\mathcal{O} \to \mathcal{U}(\mathcal{O})$, called a *net* (a map on a directed set, i.e. a non-empty set carrying a partial order). In fact, the fields in the Wightman sense are unbounded operators, whereas the Haag-Kastler axioms consider algebras of bounded operators. The question of whether the Wightman axioms can actually yield a net of bounded operators and whether, vice versa, a net in the Haag-Kastler sense can be understood as being generated by operator-valued distributions, is highly involved [101, p. 106].

The purely algebraic axioms, then, are as follows [101–103]. We define a QFT as a net $\mathcal{O} \mapsto \mathcal{U}(\mathcal{O})$ of C^*-algebras on the directed set of open, bounded regions of Minkowski spacetime M, with inclusion as the partial order. We call $\mathcal{U}(M) = \overline{\bigcup_{\mathcal{O}} \mathcal{U}(\mathcal{O})}$ the *quasilocal* algebra, where the closure is taken in the norm topology, and require the net to satisfy the following axioms:

- **Isotony:** for $\mathcal{O}_1 \subset \mathcal{O}_2$ we have $\mathcal{U}(\mathcal{O}_1) \subset \mathcal{U}(\mathcal{O}_2)$.
- **Locality:** if $\mathcal{O}_1, \mathcal{O}_2$ are spacelike separated, then $[A, B] = 0$ for all $A \in \mathcal{O}_1, B \in \mathcal{O}_2$.

- **Covariance:** the Poincaré group is implemented as a group of automorphisms of the net. That is, we have an assignment $g \in \mathcal{P} \to \alpha_g$ such that

$$\alpha_g \mathcal{U}(\mathcal{O}) = \mathcal{U}(g\mathcal{O}).$$

- **Time slice axiom:** if $\hat{\mathcal{O}}$ is the causal completion of the region $\mathcal{O}$, then $\mathcal{U}(\hat{\mathcal{O}}) = \mathcal{U}(\mathcal{O})$. In particular, if $\mathcal{O}_\Sigma = \Sigma \times (-\epsilon, \epsilon)$ is an open neighbourhood of a Cauchy surface $\Sigma \subset M$, then $\mathcal{U}(M) = \mathcal{U}(\mathcal{O}_\Sigma)$.

- **Existence of vacuum:** the net $\mathcal{U}$ possesses an irreducible representation π_0 called the *vacuum sector*, in which α_g is implementable, i.e. there exists a strongly continuous family $U(g)$ of unitary operators such that

$$U(g)\pi_0(a)U^{-1}(g) = \pi_0(\alpha_g a).$$

- **Spectrum condition:** the joint spectrum of the generators P_μ of translations, i.e. those operators satisfying $e^{x \cdot P} = U(x)$, where $U(x)$ denotes the unitary operator implementing a translation by x, is contained in the closed future light cone $\bar{V}^+$.

These axioms may be formulated slightly differently across different sources. The **locality** axiom is also called **Einstein causality** or **microcausality**, and we refer the reader to [104] for a philosophical discussion of it.

We could go into many details concerning these axioms, such as the structure of the vacuum sector in terms of the GNS representation of a vacuum state ω_0 (III.4 in [101]) and smoothness properties of the net (III.3.1 in [101]). However, these details are not very relevant for our purposes, since in this chapter we only aim to explain the general structure of the algebraic approach to quantum (field) theory and the insights it can bring for SSB. It is only in chapter 6 that we actually prove results on the Higgs mechanism, but based on the Wightman axioms - not the Haag-Kastler axioms.

5.2. Representations, folia and all that

The most distinctive feature of the algebraic approach to quantum theory, as opposed to the Hilbert space approach, is probably the fact that we can represent a C^*-algebra $\mathcal{A}$ on a Hilbert space H in a wide variety of ways. Indeed, every state $\omega\colon \mathcal{A} \to \mathbb{C}$ gives a representation $\pi_\omega\colon \mathcal{A} \to B(H_\omega)$, but there is also the universal representation, which by Theorem A.28 is faithful, and many other representations are conceivable. This leads to conceptual questions: should we attempt to represent the abstract algebra on a concrete Hilbert space at all? If so, then through which representation? When can we say that two representations are equivalent? If two inequivalent representations are both physical, then how do states in the two representations relate to each other? Or do they live in "separate worlds"? These and many more issues are addressed by Ruetsche [103], on which we base much of the following expostion.

5.2.1. Unitary equivalence and quasi-equivalence

The mathematically most natural characterisation of equivalence of representations of C^*-algebras is the unitary equivalence that we have mentioned several times already.

Definition 5.1. We call two representations $\pi_1\colon \mathcal{A} \to B(H_1)$ and $\pi_2\colon \mathcal{A} \to B(H_2)$ of a C^*-algebra *unitarily equivalent* if there is a unitary map $U\colon H_1 \to H_2$ intertwining them, i.e. statisfying $U^{-1}\pi_2(a)U = \pi_1(a)$ for all $a \in \mathcal{A}$.

Through the GNS representation (Definition A.26) we can extend this to unitary equivalence of states.

Definition 5.2. We call two states ω_1, ω_2 on $\mathcal{A}$ unitarily equivalent if their GNS representations π_{ω_1} and π_{ω_2} are unitarily equivalent.

Now, from any representation $\pi\colon \mathcal{A} \to B(H)$ of a C^*-algebra, we can obtain a von Neumann algebra (Definition A.18), since in $B(H)$ we can take the closure in the weak operator topology (Definition A.9). By the double commutant theorem (Theorem A.20) this von Neumann algebra is $\pi(\mathcal{A})''$. We can then use this von Neumann algebra affiliated with a representation to define another notion of equivalence of representations.

Definition 5.3. Two representations π_1, π_2 of $\mathcal{A}$ are called *quasi-equivalent* if there is a $*$-morphism $\alpha\colon \pi_1(\mathcal{A})'' \to \pi_2(\mathcal{A})''$ such that $\alpha(\pi_1(a)) = \pi_2(a)$ for all $a \in \mathcal{A}$. Similarly we can say two states are quasi-equivalent if their GNS representations are.

It is not hard to see that unitarily equivalent representations are also quasi-equivalent: a unitary map $U\colon H_1 \to H_2$ intertwining π_1 and π_2 gives a unitarily implemented $*$-morphism defined through $A \mapsto UAU^*$. However, quasi-equivalent representations need not be unitarily equivalent [103, p. 87]. Indeed, only irreducible quasi-equivalent representations are unitarily equivalent [105], and therefore quasi-equivalence can also be characterised as "unitary equivalence up to multiplicity" [106, p. 5357], as in the following equivalent definition [101, p. 126] [58, p. 319].

Definition 5.4. Two representations $\pi_1, \pi_2\colon \mathcal{A} \to B(H)$ are *quasi-equivalent* if every subrepresentation of π_1 has a subrepresentation that is unitarily equivalent to some subrepresentation of π_2, and vice versa.

5.2.2. Folia

Given a representation $\pi\colon \mathcal{A} \to B(H)$ of a C^*-algebra $\mathcal{A}$, we can ask ourselves what states on the algebra can be expressed as density operators in that representation. This leads to the following definition [101, 103], which turns out to be intimately related to the notion of quasi-equivalence.

Definition 5.5. Let $\omega\colon \mathcal{A} \to \mathbb{C}$ be a state. Then the *folium* $\mathcal{F}_\omega$ of ω is the set of all states on $\mathcal{A}$ which are expressible as density matrices in the GNS representation of ω, i.e. the states ω_ρ such that

$$\omega_\rho(a) = \operatorname{tr} \rho\pi_\omega(a),$$

with ρ a density operator.

We think of the folium of a state as the set of all the states that "live in the same world" and which can be superposed. Folia are a division of the state space of an algebra into sets of states that can be expressed as density operators on the same Hilbert space. The states that are expressible as density operators in a representation π are also called π-*normal* states. This refers to the definition of a state $\omega\colon B(H) \to \mathbb{C}$ as normal if for each orthogonal family $\{e_i\}$ of projections we have

$$\omega\left(\sum_i e_i\right) = \sum_i \omega(e_i). \tag{5.1}$$

The convergence of $\sum_i e_i$ is in the SOT (Definition A.9) [103, p.80]. A well-known result (Theorem 4.12 in [58]) states that a state $\omega\colon B(H) \to \mathbb{C}$ is normal if and only if it takes the form $\omega(a) = \operatorname{tr} \rho a$, with ρ a density operator. Intuitively it seems logical that if two states are unitarily equivalent, then we should be able to express one as a density operator in the GNS representations of the other. Indeed:

Proposition 5.6. If two states ω_1, ω_2 are unitarily equivalent then their folia coincide, i.e. $\mathcal{F}_{\omega_1} = \mathcal{F}_{\omega_2}$.

Proof. Let $\pi_{\omega_i}\colon \mathcal{A} \to B(H_{\omega_i})$ denote the GNS representations of ω_i with unit cyclic vectors Ω_i, where $i = 1, 2$. Suppose $\omega_\rho \in \mathcal{F}_{\omega_1}$, i.e. $\omega_\rho(a) = \operatorname{tr} \rho\pi_{\omega_1}(a)$. Let $U\colon H_{\omega_1} \to H_{\omega_2}$ denote the unitary operator implementing the unitary equivalence of ω_1 and ω_2. Then

$$\omega_\rho(a) = \operatorname{tr} \rho\pi_{\omega_1}(a) = \operatorname{tr} \rho U^*\pi_{\omega_2}(a)U = \operatorname{tr} U\rho U^*\pi_{\omega_2}(a).$$

Here we used the cyclicity of the trace. But since U is unitary $U\rho U^*$ is a density operator on H_{ω_2}, so $\omega_\rho \in \mathcal{F}_{\omega_2}$. This shows that $\mathcal{F}_{\omega_1} \subset \mathcal{F}_{\omega_2}$ and by the same argument we also get $\mathcal{F}_{\omega_2} \subset \mathcal{F}_{\omega_1}$. □

It might also be, however, that two states on an algebra cannot "talk to each other" at all.

Definition 5.7. Two states ω_1, ω_2 on a C*-algebra $\mathcal{A}$ are *disjoint* if their folia are, i.e. if $\mathcal{F}_{\omega_1} \cap \mathcal{F}_{\omega_2} = 0$.

As Ruetsche explains, disjointness can be thought of as an algebraic "radicalisation" of the notion of orthogonality for Hilbert spaces. If two states in a Hilbert space are orthogonal, then they are impossible relative to one another in the sense that the transition probability from one to the other is zero. Yet, there might be a third state which overlaps with both of the orthogonal states. This is not the case for folia. If two states are disjoint, then any third state will either be disjoint to both states, or lie in the folium of one of them, in which case it cannot be superposed with the other. There is no middle way. It is for this reason that we think of folia as "separate worlds" in which states cannot talk to each other.

Now, we have seen that if two states are unitarily equivalent, then their folia coincide. But the converse does not hold: if two states have coinciding folia they need not be unitarily equivalent. Thus, it is not immediately clear how the classification of states by their folia relates to the equivalence of their representations. At the beginning of this subsection, however, we already hinted at a connection between quasi-equivalence and folia, and indeed [103, p. 98] [58, p. 319]:

Theorem 5.8. Two states are quasi-equivalent if and only if their folia coincide.

Since by Proposition 5.6 above unitarily equivalent representations have coinciding folia, and thinking of quasi-equivalence as unitary equivalence up to multiplicity as in Definition 5.4, it is not too hard to see that under the weaker condition of quasi-equivalence folia still coincide. As for the converse, recall that if representations have coinciding folia, any state that is normal relative to one representation is normal relative to the other. But normality is the condition of countable additivity with respect to the SOT, so we expect that representations with coinciding folia generate the same von Neumann algebras (which are closed in the SOT). We will leave it at this intuition since, again, this chapter serves to expose the general structures of the algebraic approach to quantum theory, whereas our precise results about the Higgs mechanism in chapter 6 are based on the Wightman axioms.

We have now come at a classification of states by their quasi-equivalence or equivalently their folia, but this is not a neat division of the space of states, in the sense that states need not be either quasi-equivalent or disjoint [107]. In other words: states might have overlapping folia that do not completely coincide [103]. For irreducible representations (pure states by Theorem A.31) this dichotomy clearly does hold. But it can be generalised:

Definition 5.9. A representation is called *primary* if it is quasi-equivalent to each of its nonzero subrepresentations. Similarly a state is called primary if its GNS representation is primary.

Clearly a pure state is primary since its GNS representation has no proper subrepresentations. Primary representations are also called *factor representations*, which comes from the following definition:

Definition 5.10. A von Neumann algebra $\mathcal{W}$ is said to be a *factor* if its center $W' \cap W$ contains only multiples of the identity. A *factor representation* $\pi\colon \mathcal{A} \to \mathbb{C}$ of a C*-algebra $\mathcal{A}$ is a representation whose von Neumann algebra $\pi(\mathcal{A})''$ is a factor.

A primary state $\omega\colon \mathcal{A} \to \mathbb{C}$ can alternatively be defined as a factor state, i.e. by the triviality of $\pi_\omega(\mathcal{A})' \cap \pi_\omega(\mathcal{A})'' = \mathbb{C}I$ [58, p. 319]. We will not prove the equivalence of the definitions.

For primary states we do have the neat division we have been after (cf. Corollary 8.22 in [58]).

Theorem 5.11. Primary states are either quasi-equivalent or disjoint.

Proof. Let $\omega_1, \omega_2\colon \mathcal{A} \to \mathbb{C}$ be primary states which are not quasi-equivalent. Clearly quasi-equivalence is an equivalence relation: this follows most easily from its characterisation by means of a $*$-isomorphism of the von Neumann algebras generated by the representations in question. But this means that if two primary representations π_1, π_2 are quasi-equivalent, then if any subrepresenation of π_1 is quasi-equivalent to a subrepresentation of π_2, *every* subrepresentation of π_1 is quasi-equivalent to *every* subrepresentation of π_2. After all, every subrepresentation of π_1 is quasi-equivalent to π_1 itself, and the same holds for π_2 and its subrepresentations. Thus, if some subrepresentation π_1' of π_1 is quasi-equivalent to some subrepresentation π_2' of π_2, then for any other subrepresentations π_1'' of π_1 and π_2'' of π_2 we have that π_1'' is quasi-equivalent to π_1, which is quasi-equivalent to π_1', which is quasi-equivalent to π_2', which is quasi-equivalent to π_2, which is quasi-equivalent to π_2''. So by transitivity π_1'' is quasi-equivalent to π_2''. This implies that if π_{ω_1} and π_{ω_2} are not quasi-equivalent, they do not have any quasi-equivalent subrepresentations either. But by Theorem 5.8 this amounts to the states ω_1, ω_2 being disjoint, i.e. $\mathcal{F}_{\omega_1} \cap \mathcal{F}_{\omega_2} = 0$. □

5.2.3. Superselection sectors

Thus, whereas if we would consider quasi-equivalence classes of the entire set of states $S(\mathcal{A})$ of the C*-algebra $\mathcal{A}$, it would not be guaranteed that states belonging to different classes are disjoint, we do have this guarantee if we take quasi-equivalence classes of the set of primary states. This leads to:

Definition 5.12. A primary folium (a folium of a primary state) is called a *superselection sector*.

Superselection sectors give rise to so-called *superselection rules*, which prohibit transitions between different superselection sectors [108]. Such superselection rules are often thought of as forbidding coherent superpositions of states from different sectors - a common example being a superposition of a fermion and a boson. On a decomposable Hilbert space $H = H_1 \oplus H_2$ this is formalised as follows [107].

Definition 5.13. We say $H = H_1 \oplus H_2$ exhibits a superselection rule if for any observable $A \in B(H)$ and states $\psi_1 \in H_1, \psi_2 \in H_2$ we have $\langle \psi_1, A\psi_2 \rangle = 0$.

However, this is only one possible definition of a superselection rule. It is what Earman calls a *weak* superselection rule, of which he considers five equivalent conditions [108]. One of these defines a superselection rule as the impossibility to observe relative phases between states, as is the case for superpositions of states carrying different charges, since gauge transformations then change each charged state by a different phase. Formalised in terms of a von Neumann algebra $\mathcal{W}$ of observables acting on a Hilbert space H, Earman's five characterisations of weak superselection rules are all equivalent to:

Definition 5.14. A von Neumann algebra $\mathcal{W} \subset B(H)$ is said to exhibit a weak superselection rule if the commutant $\mathcal{W}' \subset B(H)$ is non-trivial, i.e. not just $\mathbb{C}I$.

Indeed, since a von Neumann algebra is the closed linear span of its projections (Theorem 4.1.11 in [109]), a non-trivial center $\mathcal{W}'$ contains a projection P that is not just a multiple of the identity. Then PH is a proper $\mathcal{W}$-invariant subspace, and for any $A \in \mathcal{W}, \psi_1 \in PH, \psi_2 \in (PH)^\perp$ we have

$$\langle \psi_1, A\psi_2 \rangle = \langle P\psi_1, A\psi_2 \rangle = \langle \psi_1, PA\psi_2 \rangle = \langle \psi_1, AP\psi_2 \rangle = 0.$$

The situation in which $\mathcal{W}'$ is non-trivial might obtain while $\mathcal{Z}(\mathcal{W}) = \mathcal{W} \cap \mathcal{W}' = \mathbb{C}I$, i.e. while $\mathcal{W}$ is a factor. Earman's *strong* sense of a superselection rule requires this not to be the case:

Definition 5.15. A von Neumann algebra $W \subset B(H)$ exhibits a strong superselection rule if the center $\mathcal{Z} = \mathcal{W} \cap \mathcal{W}'$ is non-trivial. In particular this means the commutant $\mathcal{W}'$ is non-trivial.

This means that there are non-trivial central elements in the algebra $\mathcal{W}$ itself. The *very strong* definition of superselection rules goes further still.

Definition 5.16. A von Neumann algebra $W \subset B(H)$ exhibits a very strong superselection rule if the commutant $\mathcal{W}'$ is non-trivial and $\mathcal{W}' \subset \mathcal{W}$. This automatically implies the commutativity of superselection rules, i.e. $\mathcal{W}' \subset \mathcal{W} = \mathcal{W}'' = (\mathcal{W}')'$ (so $\mathcal{W}'$ is Abelian).

Let us now consider how our discussion of superselection sectors in terms of folia (Definition 5.12) relates to Earman's characterisation of superselection rules. We have the following [108, p. 393].

Theorem 5.17. Let $\mathcal{A}$ be a C*-algebra with representations $\pi_1 \colon \mathcal{A} \to B(H_1), \pi_2 \colon \mathcal{A} \to B(H_2)$. We always have $(\pi_1 \oplus \pi_2)(\mathcal{A})'' \subset \pi_1(\mathcal{A})'' \oplus \pi_2(\mathcal{A})''$, but for disjoint representations the inclusion is an identity. In other words, the following are equivalent:

(i) $(\pi_1 \oplus \pi_2)(\mathcal{A})'' = \pi_1(\mathcal{A})'' \oplus \pi_2(\mathcal{A})''$;

(ii) π_1 and π_2 are disjoint, i.e. $\mathcal{F}_{\pi_1} \cap \mathcal{F}_{\pi_2} = 0$.

These conditions hold if and only if for the projectors E_j from $H = H_1 \oplus H_2$ to H_j for $j = 1, 2$ we have $E_j \in (\pi_1 \oplus \pi_2)(\mathcal{A})''$, and thus $E_j \in \mathcal{Z}((\pi_1 \oplus \pi_2)(\mathcal{A})'')$.

With this theorem we can see how a C*-algebra $\mathcal{A}$ "generates its own superselection rules". We first identify a class of physically admissible states and then consider the primary states among these (here Earman considers only pure states [2, p. 398]). We partition these primary states into their quasi-equivalence classes, which are mutually disjoint. Assuming a countable number of classes and choosing representatives ω_i from every quasi-equivalence class we posit the von Neumann algebra of observables $(\oplus_i \pi_{\omega_i})(\mathcal{A})'' = \oplus_i \pi_{\omega_i}(\mathcal{A})''$ acting on $\oplus_i H_{\omega_i}$. This gives at least weak superselection rules, but it might give (very) strong ones too, as when one considers only GNS representations of pure states [108].

5.3. The global gauge group

The relevance of the notions from the preceding section lies in the fact that, as discovered by Doplicher, Haag and Roberts (DHR), there is a deep connection between superselection and global gauge symmetries. In fact, in Theorem 6.13, we will understand the Higgs mechanism as a failure of the electric charge to be a superselected quantum number. DHR showed that the presence of gauge symmetry leads to superselection sectors labelled by the quantised gauge charges and, conversely, if one is given only these superselection rules one can reconstruct the compact gauge group giving rise to them [110–112]. This is a category-theoretic generalisation of Tannaka-Krein duality [113]. For us, however, the details of the reconstruction are not so relevant: we just need to understand how the presence of gauge symmetry leads to a superselection structure.

The starting point is a net $\mathcal{O} \to \mathcal{U}(\mathcal{O})$ satisfying the axioms of section 5.1, such that the observables in a region $\mathcal{O}$ are the self-adjoint elements of $\mathcal{U}(\mathcal{O})$. The net $\mathcal{U}(\mathcal{O})$ is also called the *observable algebra*. Gauge fields are, however, not observable and do not belong to $\mathcal{U}(\mathcal{O})$. Thus, we need to consider a larger net $\mathcal{F}(\mathcal{O})$ called the *field algebra*, such that gauge transformations act on this larger net. More precisely, we have the following definition [112].

Definition 5.18. Let π_0 denote the vacuum representation of the net $\mathcal{U}$ of local observables. A *field system with gauge symmetry* for $\mathcal{U}(\mathcal{O})$ is a triple $(\pi, G, \mathcal{F})$ consisting of a representation π of $\mathcal{U}$ on a Hilbert space H containing π_0 as a subrepresentation on $H_0 \subset H$, a compact group G represented by a strongly continuous family of unitaries $g \in \mathcal{G} \mapsto U(g) \in B(H)$ leaving H_0 pointwise fixed, and a net $\mathcal{O} \to \mathcal{F}(\mathcal{O})$ of von Neumann algebras acting on H such that

(i) the $g \in G$ induce automorphisms α_g of $\mathcal{F}(\mathcal{O})$ with $\pi(\mathcal{U}(\mathcal{O})) = \mathcal{F}(\mathcal{O}) \cap U(G)'$, i.e. such that $\mathcal{U}$ is the algebra of gauge-invariant elements of $\mathcal{F}$;

(ii) The field algebra $\mathcal{F} = \overline{\bigcup_{\mathcal{O}} \mathcal{F}(\mathcal{O})}$ acts irreducibly on H;

(iii) H_0 is cyclic for $\mathcal{F}(\mathcal{O})$;

(iv) For any spacelike separated $\mathcal{O}_1, \mathcal{O}_2$ we have $[\mathcal{F}(\mathcal{O}_1, \mathcal{U}(\mathcal{O}_2))] = 0$.

The family of unitaries $U(g)$ representing the gauge group is assumed to commute with the Poincaré group. Moreover, the gauge automorphisms α_g act locally, i.e. $\alpha_g(\mathcal{F}(\mathcal{O})) = \mathcal{F}(\mathcal{O})$ for all $\mathcal{O}$. This reflects the fact that gauge symmetries are internal symmetries. The analysis that was carried out by DHR assumes a certain criterion for which representations of $\mathcal{U}$ are allowed [101]. It basically states that representations must look like the vacuum at large distance.

> **Selection criterion** A representation π of $\mathcal{U}$ is said to fulfill the DHR selection criterion if, for a sufficiently large causal diamond $\mathcal{O}$, it satisfies
>
> $$\pi|_{\mathcal{U}(\mathcal{O}')} \cong \pi_0|_{\mathcal{U}(\mathcal{O}')},$$
>
> where π_0 denotes the vacuum representation. In words: only those representations are considered which become unitarily equivalent to the vacuum representation at sufficiently large spatial distance.

The wonderful thing is that field systems with gauge symmetries yield representations satisfying the **selection criterion**. In fact, these representations decompose into superselection sectors labelled by the gauge group. This is formalised in the following result (Theorem 3.6 in [112]).

Theorem 5.19. Let $(\pi, G, \mathcal{F})$ be a field system with gauge symmetry for $\mathcal{U}$. Then

(a) $\pi(\mathcal{U})' \cap \mathcal{F} = \mathbb{C}I$;

(b) an automorphism γ of $\mathcal{F}$ is a gauge automorphism, i.e. $\gamma = \alpha_g$ for some $g \in G$, if and only if γ acts trivially on $\pi(\mathcal{U})$;

(c) $\pi(\mathcal{U})' = G''$ and $\pi = \bigoplus_\xi d(\xi)\pi_\xi$, where the π_ξ are inequivalent irreducible representations of $\mathcal{U}$ fulfilling the **selection criterion** and having parastatistics of finite order $d(\xi)$.

Here G'' denotes the double commutant of the algebra consisting of $U(g)$ with $g \in G$, and the ξ are characters of G, i.e. unitary equivalence classes of irreducible representations of G [114, p. 85]. It is not so important for us what precisely is meant by "parastatistics of finite order" and the statistical dimension $d(\xi)$. What matters is the fact that the representation π of $\mathcal{U}$ decomposes into superselection sectors π_ξ, which are mapped into each other by the action of gauge group. However, the field systems considered so far do not necessarily give all superselection sectors, so we define the following [112].

Definition 5.20. A field system with gauge symmetry $(\pi, G, \mathcal{F})$ is *complete* if each equivalence class of representations of $\mathcal{U}$ satisfying the **selection criterion** and having finite statistics is realised as a subrepresentation of π. In other words: π describes *all* the relevant superselection sectors.

What is clear, then, is that if the field system with gauge symmetry is complete, the decomposition in part (c) of Theorem 5.19 ranges over all equivalence classes of representations fulfilling the **selection criterion** (and having finite statistics). This actually obtains for the reconstructed field systems in the DHR analysis. For this reconstruction one needs another assumption: **duality**. It states that

$$\pi_0(\mathcal{U}(\mathcal{O}'))' = \pi_0(\mathcal{U}(\mathcal{O}))$$

for all causal diamonds $\mathcal{O}$. This is much stronger than the **isotony** axiom from section 5.1. One also needs what Doplicher and Roberts call "property B", which can be derived from some standard assumptions in QFT [112]. It states that if $E \in \mathcal{U}(\mathcal{O})$ is a non-zero projection, then for any $\mathcal{O}_1$ containing the closure of $\mathcal{O}$ there is an isometry $W \in \mathcal{U}(\mathcal{O}_1)$ such that $WW^* = E$ and $W^*W = I$. Theorem 3.5 in [112] then gives the desired reconstruction.

Theorem 5.21. Let $\mathcal{U}(\mathcal{O})$ be a net of local observables satysifying property B and **duality** in the faithful, irreducible vacuum representation π_0 acting on the separable Hilbert space H_0. Then there exists a complete, normal field system with gauge symmetry for $\mathcal{U}$ and this system is unique up to equivalence.

We will not go into the meaning of normality (Definition 3.2 in [112]). The notion of equivalence used here is the following.

Definition 5.22. Two field systems with gauge symmetry $(\pi_1, G_1, \mathcal{F}_1)$ and $(\pi_2, G_2, \mathcal{F}_2)$ are *equivalent* if there is a unitary operator $U\colon H_1 \to H_2$ such that for all $\mathcal{O}$ and $a \in \mathcal{U}$ we have

$$\begin{aligned} U\pi_1(a) &= \pi_2(a)U, \\ UG_1 &= G_2U, \\ U\mathcal{F}_1 &= \mathcal{F}_2U. \end{aligned}$$

In the particular case where G is Abelian, the superselection sectors are labelled by the Pontryagin dual of G, and the set of sectors has the structure of a discrete Abelian group [110]. For $G = U(1)$, this discrete Abelian group is $\mathbb{Z}$ and it signifies the quantised electric charge [106]. This means that it is impossible to take coherent superpositions of states carrying different charges, an idea that is intuitively logical, for if we were to take such a superposition

$$|\psi\rangle = \alpha|q_1\rangle + \beta|q_2\rangle$$

with $q_1 \neq q_2$, then the relative phase between α and β would be changed by a gauge transformation, since such a transformation would act differently on $|q_1\rangle$ and $|q_2\rangle$. States with different charges lie in disjoint folia, and the charge carrying fields map from one sector to another [101].

5.4. Spontaneous symmetry breaking

Now that we have considered the role of global gauge symmetries in AQFT, we come back to SSB. We already briefly introduced the algebraic definition of SSB in the preamle to this chapter and we now recall it.

Definition 5.23. Let $\mathcal{A}$ be a C^*-algebra and $\alpha\colon \mathcal{A} \to \mathcal{A}$ a $*$-automorphism. Then a state $\omega\colon \mathcal{A} \to \mathbb{C}$ is said to spontaneously break the symmetry α if it cannot be unitarily implemented in the GNS representation π_ω, i.e. if there is no unitary operator $U \in B(H)$ such that

$$\pi_\omega(\alpha(a)) = U\pi_\omega(a)U^*, \qquad a \in \mathcal{A}.$$

Alternatively, we may characterise SSB by unitary inequivalence.

Definition 5.24. Let $\mathcal{A}$ be a C^*-algebra and $\alpha\colon \mathcal{A} \to \mathcal{A}$ a $*$-automorphism. Then a state $\omega\colon \mathcal{A} \to \mathbb{C}$ is said to spontaneously break the symmetry α if the GNS presentations π_ω and $\pi_{\alpha^*\omega}$ are not unitarily equivalent, i.e. if there is no unitary operator $U\colon H_\omega \to H_{\alpha^*\omega}$ such that

$$\pi_{\alpha^*\omega}(a) = U\pi_\omega(a)U^*, \qquad a \in \mathcal{A}.$$

Here the state $\alpha^*\omega$ is defined by $\alpha^*\omega(a) = \omega(\alpha^{-1}(a))$.

The equivalence of these definitions follows from the following result [58, p. 345].

Theorem 5.25. An automorphism $\alpha\colon \mathcal{A} \to \mathcal{A}$ of a unital C^*-algebra $\mathcal{A}$ can be implemented in the GNS representation π_ω of a state $\omega \in S(\mathcal{A})$ if and only if $\pi_{\alpha^*\omega}$ and π_ω are unitarily equivalent.

Proof. Let H_ω and $H_{\alpha^*\omega}$ denote the Hilbert spaces of the GNS representations π_ω and $\pi_{\alpha^*\omega}$, and let $\Omega_\omega, \Omega_{\alpha^*\omega}$ denote the respective unit cyclic vectors. We define an operator $W\colon H_\omega \to H_{\alpha^*\omega}$ by

$$W\pi_\omega(a)\Omega_\omega = \pi_{\alpha^*\omega}(\alpha(a))\Omega_{\alpha^*\omega}$$

for any $a \in \mathcal{A}$. This operator can be extended to all of H_ω because $\pi_\omega(\mathcal{A})\Omega_\omega$ is dense in H_ω. To ensure that it is well-defined we need to consider what happens if $\pi_\omega(a)\Omega_\omega = \pi_\omega(b)\Omega_\omega$ for some $a, b \in \mathcal{A}$. By definition of the GNS construction this means that $[a] = [b] \in \mathcal{A}/N_\omega$, i.e. $\omega(c^*c) = 0$ for $c = a - b$. To guarantee that W is well-defined we must have $[\alpha(a)] = [\alpha(b)] \in \mathcal{A}/N_{\alpha^*\omega}$. That is: we need $\alpha^*\omega(\alpha(c)^*\alpha(c)) = 0$. But we have $\alpha^*\omega(\alpha(c)^*\alpha(c)) = \alpha^*\omega(\alpha(c^*c)) = \omega(\alpha^{-1}(\alpha(c^*c))) = \omega(c^*c) = 0$, so W is indeed well-defined. It satisfies

$$W\Omega_\omega = W\pi_\omega(1)\Omega_\omega = \pi_{\alpha^*\omega}(\alpha(1))\Omega_{\alpha^*\omega} = \pi_{\alpha^*\omega}(1)\Omega_{\alpha^*\omega} = \Omega_{\alpha^*\omega},$$

and its adjoint $W^*\colon H_{\alpha^*\omega} \to H_\omega$ is given by

$$W^*\pi_{\alpha^*\omega}(a)\Omega_{\alpha^*\omega} = \pi_\omega(\alpha^{-1}(a))\Omega_\omega,$$

as can be seen from:

$$\begin{aligned}
&\langle W\pi_\omega(a)\Omega_\omega, \pi_{\alpha^*\omega}(b)\Omega_{\alpha^*\omega}\rangle = \langle \pi_{\alpha^*\omega}(\alpha(a))\Omega_{\alpha^*\omega}, \pi_{\alpha^*\omega}(b)\Omega_{\alpha^*\omega}\rangle = \langle \pi_{\alpha^*\omega}(b^*)\pi_{\alpha^*\omega}(\alpha(a))\Omega_{\alpha^*\omega}, \Omega_{\alpha^*\omega}\rangle \\
&= \langle \pi_{\alpha^*\omega}(b^*\alpha(a))\Omega_{\alpha^*\omega}, \Omega_{\alpha^*\omega}\rangle = \omega(\alpha^{-1}(b^*\alpha(a))) = \omega(\alpha^{-1}(b^*)a) = \langle \pi_\omega(\alpha^{-1}(b^*)a)\Omega_\omega, \Omega_\omega\rangle \\
&= \langle \pi_\omega(\alpha^{-1}(b))^*\pi_\omega(a)\Omega_\omega, \Omega_\omega\rangle = \langle \pi_\omega(a)\Omega_\omega, \pi_\omega(\alpha^{-1}(b))\Omega_\omega\rangle = \langle \pi_\omega(a)\Omega_\omega, W^*\pi_{\alpha^*\omega}(b)\Omega_{\alpha^*\omega}\rangle.
\end{aligned}$$

Here we have used that by definition of the GNS representation $\omega(a) = \langle \pi_\omega(a)\Omega_\omega, \Omega_\omega\rangle$. But it now clearly follows that W is unitary:

$$WW^*\pi_{\alpha^*\omega}(a)\Omega_{\alpha^*\omega} = W\pi_\omega(\alpha^{-1}(a))\Omega_\omega = \pi_{\alpha^*\omega}(\alpha(\alpha^{-1}(a)))\Omega_{\alpha^*\omega} = \pi_{\alpha^*\omega}(a)\Omega_{\alpha^*\omega},$$

and similarly $W^*W = 1$. In addition, we have $W\pi_\omega(a)W^* = \pi_{\alpha^*\omega}(\alpha(a))$ for all $a \in \mathcal{A}$, as can be seen from

$$\begin{aligned}
&W\pi_\omega(a)W^*\pi_{\alpha^*\omega}(b)\Omega_{\alpha^*\omega} = W\pi_\omega(a)\pi_\omega(\alpha^{-1}(b))\Omega_\omega = W\pi_\omega(a\alpha^{-1}(b))\Omega_\omega \\
&= \pi_{\alpha^*\omega}(\alpha(a\alpha^{-1}(b)))\Omega_{\alpha^*\omega} = \pi_{\alpha^*\omega}(\alpha(a)b)\Omega_{\alpha^*\omega} = \pi_{\alpha^*\omega}(\alpha(a))\pi_{\alpha^*\omega}(b)\Omega_{\alpha^*\omega}.
\end{aligned}$$

All of these hold regardless of whether any of the two conditions in the theorem are satisfied.

Suppose now, that π_ω and $\pi_{\alpha^*\omega}$ are unitarily equivalent. By definition this means that there exists a unitary operator $V\colon H_\omega \to H_{\alpha^*\omega}$ such that for all $a \in \mathcal{A}$: $V\pi_\omega(a)V^* = \pi_{\alpha^*\omega}(a)$. Then $U = V^*W$ is unitary and for all $a \in \mathcal{A}$ we have

$$U\pi_\omega(a)U^* = V^*W\pi_\omega(a)W^*V = V^*\pi_{\alpha^*\omega}(\alpha(a))V = \pi_\omega(\alpha(a)),$$

so α can be implemented in π_ω through $U = V^*W$.

Conversely, suppose α can be implemented in π_ω and let $V \in B(H_\omega)$ denote the corresponding unitary operator. Define the unitary $U = VW^*$. Then for any $a \in \mathcal{A}$ we have

$$U\pi_{\alpha^*\omega}(a)U^* = VW^*\pi_{\alpha^*\omega}(a)WV^* = V\pi_\omega(\alpha^{-1}(a))V^*\pi_\omega(\alpha(\alpha^{-1}(a))) = \pi_\omega(a),$$

so U intertwines $\pi_{\alpha^*\omega}$ and π_ω, i.e. they are unitarily equivalent. □

The two definitions of SSB that we have just proven to be equivalent are simple and general, but not so easy to check [44, p. 120]. We should like to have an *order parameter*, i.e. an element of the algebra whose ground state expectation value is not invariant under the symmetry in question when that symmetry is broken. This is how we will treat the Higgs mechanism in chapter 6, and to that end we have the following result (cf. proposition II.8.2 in [44]).

Proposition 5.26. Let $\mathcal{A}$ be a C^*-algebra with vacuum state $\omega_0 \in S(\mathcal{A})$ in whose GNS representation $(\pi_{\omega_0}, H_{\omega_0}, \Omega_{\omega_0})$ spacetime translations α_x are implementable through a strongly continuous family of unitary operators $U(x)$, such that Ω_0 is the unique translationally invariant state in H_{ω_0}. Then an *internal* symmetry (one that commutes with spacetime translations) $\alpha \in \mathrm{Aut}(\mathcal{A})$ is unbroken in ω_0 if and only if all ground state correlation functions are invariant under α, i.e. for all $a \in \mathcal{A}$

$$\omega_0(\alpha(a)) = \omega_0(a).$$

Proof. Suppose α is unbroken, i.e. that it can be implemented in π_{ω_0} by a unitary $U_\alpha \in B(H_{\omega_0})$. Then $\alpha^*\omega_0$ corresponds to the vector $U_\alpha\Omega_{\omega_0}$ in H_{ω_0}:

$$\alpha^*\omega_0(a) = \omega_0(\alpha^{-1}(a)) = \langle U_\alpha^*\pi_{\omega_0}(a)U_\alpha\Omega_{\omega_0}, \Omega_{\omega_0}\rangle = \langle \pi_{\omega_0}(a)U_\alpha\Omega_{\omega_0}, U_\alpha\Omega_{\omega_0}\rangle.$$

But $\alpha^*\omega_0$ is also translationally invariant since α is an internal symmetry:

$$\begin{aligned}\alpha^*\omega_0(\alpha_x a) &= \langle \pi_{\omega_0}(\alpha_x a)U_\alpha\Omega_{\omega_0}, U_\alpha\Omega_{\omega_0}\rangle = \langle U_\alpha(x)\pi_{\omega_0}(a)U_\alpha^*(x)U_\alpha\Omega_{\omega_0}, U_\alpha\Omega_{\omega_0}\rangle \\ &= \langle \pi_{\omega_0}(a)U_\alpha^*(x)U_\alpha\Omega_{\omega_0}, U_\alpha^*(x)U_\alpha\Omega_{\omega_0}\rangle = \langle \pi_{\omega_0}(a)U_\alpha U_\alpha^*(x)\Omega_{\omega_0}, U_\alpha U_\alpha^*(x)\Omega_{\omega_0}\rangle \\ &= \langle \pi_{\omega_0}(a)U_\alpha\Omega_{\omega_0}, U_\alpha\Omega_{\omega_0}\rangle = \langle U_\alpha^*\pi_{\omega_0}(a)U_\alpha\Omega_{\omega_0}, \Omega_{\omega_0}\rangle = \omega_0(\alpha^{-1}(a)) = \alpha^*\omega_0(a).\end{aligned}$$

By uniqueness, we must have $U_\alpha\Omega_{\omega_0} = \Omega_{\omega_0}$, implying that indeed $\alpha^*\omega_0(a) = \omega_0(\alpha^{-1}(a)) = \omega_0(a)$ for all $a \in \mathcal{A}$.

Conversely, suppose that for all $a \in \mathcal{A}$ we have $\omega_0(\alpha(a)) = \omega_0(a)$. Then we can implement α in π_{ω_0} by defining $U_\alpha \in B(H_{\omega_0})$ as

$$U_\alpha\pi_{\omega_0}(a)\Omega_{\omega_0} = \pi_{\omega_0}(\alpha(a))\Omega_{\omega_0}.$$

This is well-defined, for if $\pi_{\omega_0}(a)\Omega_{\omega_0} = \pi_{\omega_0}(b)\Omega_{\omega_0}$, then, equivalently, by definition of the GNS representation $c = a - b$ satisfies $\omega_0(c^*c) = 0$. But since we have supposed $\omega_0 \circ \alpha = \omega_0$ we get $\omega_0(\alpha(c^*c)) = 0$, which implies $\pi_{\omega_0}(\alpha(a))\Omega_{\omega_0} = \pi_{\omega_0}(\alpha(b))\Omega_{\omega_0}$, i.e. $U_\alpha\pi_{\omega_0}(a)\Omega_{\omega_0} = U_\alpha\pi_{\omega_0}(b)\Omega_{\omega_0}$. We need to check that U_α is actually unitary and indeed implements α. We have, for any $a, b \in \mathcal{A}$:

$$\begin{aligned}&\langle U_\alpha\pi_{\omega_0}(a)\Omega_{\omega_0}, U_\alpha\pi_{\omega_0}(b)\Omega_{\omega_0}\rangle = \langle \pi_{\omega_0}(\alpha(a))\Omega_{\omega_0}, \pi_{\omega_0}(\alpha(b))\Omega_{\omega_0}\rangle = \langle \pi_{\omega_0}(\alpha(b^*a))\Omega_{\omega_0}, \Omega_{\omega_0}\rangle \\ &= \omega_0(\alpha(b^*a)) = \omega_0(b^*a) = \langle \pi_{\omega_0}(b^*a)\Omega_{\omega_0}, \Omega_{\omega_0}\rangle = \langle \pi_{\omega_0}(a)\Omega_{\omega_0}, \pi_{\omega_0}(b)\Omega_{\omega_0}\rangle,\end{aligned}$$

so U_α is unitary. Evidently $U_\alpha^* = U_\alpha^{-1}$ is given by $U_\alpha^*\pi_{\omega_0}(a)\Omega_{\omega_0} = \pi_{\omega_0}(\alpha^{-1}(a))\Omega_{\omega_0}$, which results in:

$$\begin{aligned}U_\alpha\pi_{\omega_0}(a)U_\alpha^*\pi_{\omega_0}(b)\Omega_{\omega_0} &= U_\alpha\pi_{\omega_0}(a)\pi_{\omega_0}(\alpha^{-1}(b))\Omega_{\omega_0} = U_\alpha\pi_{\omega_0}(a\alpha^{-1}(b))\Omega_{\omega_0} \\ &= \pi_{\omega_0}(\alpha(a\alpha^{-1}(b)))\Omega_{\omega_0} = \pi_{\omega_0}(\alpha(a))\pi_{\omega_0}(b)\Omega_{\omega_0}.\end{aligned}$$

So U_α implements α, which is unbroken. This part did not depend on the uniqueness of ω_0 as a translationally invariant state and was in the same spirit as the proof of Theorem 5.25. □

We call an element $a \in \mathcal{A}$ whose ground state expectation value is *not* invariant under a symmetry α a *symmetry breaking order parameter*, since by Proposition 5.26 the existence of such an element shows that the symmetry is broken. The existence of an order parameter is a concrete test of SSB, and we shall use it in chapter 6 to detect mass generation in the Higgs mechanism. The idea of an order parameter can also be made infinitesimal via the following notion (Definition 9.18 in [58]).

Definition 5.27. A *derivation* on a C*-algebra $\mathcal{A}$ is a linear map $\delta \colon \mathcal{A} \to \mathcal{A}$ satisfying the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. An *unbounded derivation* is a linear map $\delta \colon \mathrm{Dom}(\delta) \to \mathcal{A}$ whose domain is a dense subspace of $\mathcal{A}$ and which satisfies the Leibniz rule. An (unbounded) derivation δ is called *symmetric* if $\delta(a^*) = \delta(a)^*$.

The following result (Proposition 9.19 in [58]) then makes precise the idea that differentiating in a C*-algebra gives a derivation.

Proposition 5.28. A continuous homomorphism $\alpha \colon \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ on a C*-algebra $\mathcal{A}$ defines an unbounded symmetric derivation δ by the norm limit

$$\delta(a) = \left.\frac{d\alpha_\lambda(a)}{d\lambda}\right|_{\lambda=0} := \lim_{\lambda\to 0}\frac{\alpha_\lambda(a) - a}{\lambda},$$

where $\alpha_\lambda = \alpha(\lambda)$ and $\mathrm{Dom}(\delta)$ consists of all elements of $\mathcal{A}$ for which this limit exists.

Now, if we have a Lie group G acting on $\mathcal{A}$, the elements of its Lie algebra $\mathfrak{g}$ generate 1-parameter subgroups of automorphisms α_λ of $\mathcal{A}$, with $\lambda = 0$ corresponding to the identity [101, p. 138]. By Proposition 5.28 this gives a derivation δ. The infinitesimal version of the symmetry implementability condition is then

$$\omega_0(\delta a) = 0 \qquad \text{for all } a \in \mathrm{Dom}(\delta).$$

This condition can be used in the algebraic version of Goldstone's theorem [30, 115, 116]. A particularly simple variant of it is presented by Haag (Theorem III.3.27 in [101]).

Theorem 5.29. Suppose the vacuum state ω_0 of a net $\mathcal{U}(\mathcal{O})$ satisfying the axioms of section 5.1 is separated by an energy gap $m \neq 0$ from other states in its folium and there is a uniform bound

$$|\omega_0(\delta a)| \leq |\Phi(R)| \cdot (\|\pi_{\omega_0}(a)\Omega_{\omega_0}\| + \|\pi_{\omega_0}(a^*)\Omega_{\omega_0}\|) \qquad \text{for all } a \in \mathcal{D}(\mathcal{O}_R),$$

where $\mathcal{D}(\mathcal{O}_R)$ is the dense domain of δ for the causal diamond $\mathcal{O}_R$ of radius R centered at the origin, such that for some $n > 0$

$$\lim_{R\to\infty} R^{-n}\Phi(r)R^n \to 0.$$

Then one has $\omega_0(\delta a) = 0$, i.e. the symmetry giving the derivation δ is unbroken. Contrapositively, if the symmetry is broken we cannot have an energy gap $m \neq 0$.

We will not prove this purely algebraic Goldstone theorem. In chapter 6 an important aspect of the non-perturbative results on the Higgs is understanding how precisely the Goldstone theorem is avoided when the global gauge symmetry of the field algebra $\mathcal{F}$ from section 5.3 is broken. This then also begs the question of what happens to the superselection rules induced by that global gauge symmetry when it is spontaneously broken. We will see that for $G = U(1)$, the electric charge is superselected only when the $U(1)$ symmetry remains unbroken.

We now have an overview of the basic structures arising in the algebraic approach to quantum field theory, so we are ready to apply these ideas to the Higgs mechanism.

6. Non-Perturbative Results

The usual perturbative approach to the Higgs mechanism, in which a gauge is fixed in which the Higgs field has a non-zero vacuum expectation value (VEV), such that perturbation theory can be performed by considering small fluctuations around this VEV, suffers from conceptual and technical problems. Chief among these problems is the gauge-dependence of the Higgs field VEV. More specifically, there are gauges in which the Higgs field VEV vanishes, such as the Landau gauge [39, p. 149]. Fröhlich, Morchio and Strocchi also deduce the vanishing of the Higgs field VEV in the temporal gauge from the exponential decay of the gauge-invariant two-point function of the Higgs field [38]. In such gauges, the gauge bosons remain massless to all orders in perturbation theory [39, p. 135]. This implies that the masses of the gauge bosons, which the Higgs mechanism was supposed to "generate", are gauge-dependent and therefore unphysical. How, then, is it possible that perturbation theory based on gauge-fixing has made such incredibly accurate predictions which have been tested in particle accelerators? This is what the FMS method from section 3.3 explains by relating the gauge-invariant formulation of the Higgs mechanism to the gauge-dependent perturbative treatment. However, another possibility for avoiding these problems is to work non-perturbatively from the very start, which is what this chapter is devoted to.

The vanishing of the Higgs field VEV relates to the question of whether a symmetry breaking local order parameter exists. It is clear that the Higgs field VEV cannot function as a suitable order parameter, but in fact this holds for any gauge-dependent quantity. This can be understood heuristically by considering the path integral argument 3.1, which is related to Elitzur's theorem, which in turn implies that "breaking of local symmetry such as the Higgs phenomenon, for example, is always explicit, not spontaneous. The local symmetry must be broken first explicitly by a gauge-fixing term leaving only global symmetry" [51, p. 3981]. The remaining global symmetry may then be *spontaneously* broken. This is precisely what the Coulomb gauge condition is so useful for: it forces us to dress our fields in order to make them invariant under local gauge transformations, but *not* under global gauge transformations.

Crucially, however, this does not mean that the results that we will derive in the Coulomb gauge in this chapter are gauge-dependent. The very opposite is in fact the case: we use the Coulomb gauge precisely because it removes the unphysical local gauge group, but leaves the global gauge group as its remnant gauge symmetry. The Coulomb gauge condition makes our fields gauge-invariant in the Redundant sense of gauge from section 4.3.2. The Coulomb gauge is thus physical, and we implement it through the dressing field $\exp(-ie\Delta^{-1}\partial^i A_i)$, making use of the radiative projection from equation 4.3, which was used at length in chapter 4. However, this field is only a dressing field for the local gauge group and not for the global gauge group, and thereby circumvents the problems of the DFM from chapter 3.

We begin this chapter by introducing the Wightman axioms in section 6.1. Subsequently, we consider the implications of the Gauss law in gauge theories in section 6.2. We then consider two ways of imposing the Gauss law in QED: the local gauge quantisation in section 6.3 and the Coulomb gauge quantisation in section 6.4. Using our results from the Coulomb gauge we prove the central theorem of this thesis in section 6.5, and finally we consider the non-Abelian generalisation in section 6.6.

6.1. Wightman axioms

The Wightman axioms are a less abstract set of axioms than the Haag-Kastler axioms from chapter 5. They define quantum fields as operator-valued distributions. The algebras generated by these operator-valued distributions are then supposed to yield the abstract nets of algebras from the Haag-Kastler axioms. In our presentation of the Wightman axioms, which were invented in the 1950s but published in 1964 [99], we follow [46, 117]. First, we need to introduce some preliminary concepts.

Definition 6.1. We denote by $C_c^\infty(\mathbb{R}^n)$ or $\mathcal{D}(\mathbb{R}^n)$ the locally convex topological vector space of smooth complex-valued functions on $\mathbb{R}^n$ with compact support. The topology is the one induced by the family of seminorms

$$p_K(f) = \sup_{x\in K} \left| D^k f(x) \right|,$$

where $K \subset \mathbb{R}^n$ is compact and

$$D^k = \frac{\partial^{|k|}}{(\partial x^1)^{k_1} \cdots (\partial x_n)^{k_n}}$$

is the differential operator for the multi-index $k = (k_1, ..., k_n)$, with $|k| = k_1 + ... + k_n$. An element of $C_c^\infty(\mathbb{R}^n)$ is called a *test function*.

Definition 6.2. A *distribution* δ on $\mathbb{R}^n$ is a continuous linear functional $\delta\colon C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$. The space of these distributions is the dual space $C_c^\infty(\mathbb{R}^n)' = \mathcal{D}(\mathbb{R}^n)'$ [117, p. 35].

According to the Wightman axioms, a quantum field is not just any distribution but an *operator*-valued and *tempered* distribution. This latter term requires us to consider a less restrictive space of test functions.

Definition 6.3. The *Schwarz space* $\mathcal{S}(\mathbb{R}^n)$ is the space of smooth functions on $\mathbb{R}^n$ whose derivatives decrease faster than any power of the Euclidean distance. More precisely, we define, for any smooth function f and non-negative integers $r, s \in \mathbb{N}$:

$$\|f\|_{r,s} = \sum_{|k|\leq r} \sum_{|l|\leq s} \sup_{x\in\mathbb{R}^n} |x^k D^l f(x)|.$$

Here $k, l \in \mathbb{N}^n$ are multi-indices and $x^k = x^{k_1} \cdots x^{k_n}$. It is not difficult to see that $\|\cdot\|_{r,s}$ has the properties of a norm [117, p. 33]. We thus define $\mathcal{S}(\mathbb{R}^n)$ as the space of all $f \in C^\infty(\mathbb{R}^n)$ such that $\|f\|_{r,s} < \infty$ for all $r, s \in \mathbb{N}$, equipped with the norm topology.

Definition 6.4. A *tempered distribution* T is a continuous linear functional $T\colon \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$. The space of tempered distributions is the dual $\mathcal{S}(\mathbb{R}^n)'$. Since $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ we have $\mathcal{S}(\mathbb{R}^n)' \subset \mathcal{D}(\mathbb{R}^n)'$, i.e. every tempered distribution is a distribution [117, p. 35].

Now, a tempered distribution T can always be written in the special form

$$T(f) = \sum_{0\leqslant|k|\leqslant s} \int_{\mathbb{R}^n} F_k\left(x_1, \dots x_n\right) D^k f\left(x_1, \dots x_n\right) dx_1 \dots dx_n, \qquad f \in \mathcal{S}(\mathbb{R}^n),$$

where k is still a multi-index and the F_k are continuous functions satisfying $|F_k(x)| \leq C_k(1 + |x|^j)$, with j dependent on k [117, pp. 34-35]. Thus, we can symbolically write

$$T(x) = \sum_{0\leqslant|k|\leqslant s} (-1)^{|k|} D^k F_k(x).$$

The derivative of a (tempered) distribution is then defined in analogy to partial integration [117, p. 37].

$$\frac{\partial T}{\partial x_j}(f) = -T\left(\frac{\partial f}{\partial x_j}\right), \qquad f \in \mathcal{S}(\mathbb{R}^n).$$

We refer the reader to chapter 2 of [117] for many more mathematical details. With these notions at hand we present the Wightman axioms [46, p. 69-70] [117, p. 97-100].

- **Relativistic quantum theory:** the states of the theory are described by unit rays in a separable Hilbert space H. Spacetime translations are described by a strongly continuous family of unitary operators $U(a) \in B(H)$ for $a \in \mathbb{R}^4$. The spectrum of their generators P_μ is contained in the closed forward lightcone and there is a vacuum state vector $\Psi_0 \in H$ which has the property of being the unique translationally invariant state in H.

- **Field operators:** the theory is formulated in terms of fields $\varphi_1(x), .., \varphi_N(x)$ which are operator-valued tempered distributions on $\mathbb{R}^4$. This means that for every test function $f \in \mathcal{S}(\mathbb{R}^4)$ we have operators $\varphi_1(f), ..., \varphi_N(f)$ which, together with their adjoints $\varphi_j(f)$, are defined on a common dense domain $D \subset H$ containing Ψ_0. For any $\Phi, \Psi \in D$ we require $\langle \Phi, \varphi_j(f)\Psi \rangle$, regarded as a functional of $f \in \mathcal{S}(\mathbb{R}^4)$, to be a tempered distribution. We assume Ψ_0 to be cyclic for the fields, i.e. that by applying polynomials of the smeared fields to Ψ we obtain a dense set $D_0 \subset H$. Thus we can take $D = D_0$.

- **Relativistic covariance:** the Lorentz transformations $\Lambda \in SO^+(1,3)$ are described by a strongly continuous family of unitary operators $U(\Lambda(A))$, where $A \in SL(2, \mathbb{C})$ is in the universal covering group of $SO^+(1,3)$ (recall that $SL(2,\mathbb{C})$ is isomorphic to $Spin^+(1,3)$ from Definition B.14). Under Poincaré transformations $U(a, \Lambda(A)) = U(a)U(\Lambda(A))$ the fields transform as

$$U(a, \Lambda(A))\varphi_j(x)U(a, \Lambda(A))^{-1} = \sum_k S_{jk}(A^{-1})\varphi_k(\Lambda x + a),$$

  where $S(A)$ is a finite-dimensional representation of $SL(2, \mathbb{C})$.

- **Locality:** at spacelike separated points the fields either commute or anticommute. That is: for $x, y \in \mathbb{R}^4$ with $(x-y)^2 > 0$ we have

$$[\varphi_j(x), \varphi_k(y)]_\mp = 0,$$

  where $[\cdot, \cdot]_\mp$ denotes the commutator or anticommutator respectively. This condition should be understood as stating that, on test functions f and g whose supports are spacelike separated, we have $[\varphi_j(f), \varphi_k(g)]_\mp = 0$.

Although the question of the equivalence of the Haag-Kastler and Wightman axioms is in general very complicated [101], the idea is that the algebraic approach provides a more abstract, economical framework which encompasses field theories in the Wightman sense. The algebra $\mathcal{A}(\mathcal{O})$ of observables in a region $\mathcal{O} \subset \mathbb{R}^4$ is generated by all polynomials of smeared fields $\varphi(f)$ with $\text{supp}(f) \subset \mathcal{O}$. States in the Wightman sense, i.e. unit vectors $\Psi \in H$, then yield algebraic states by setting $\omega_\Psi(a) = \langle \Psi, a\Psi \rangle$.

6.2. Local Gauss laws

Having presented the Wightman axioms, we now know what quantum fields are (supposed to be). However, it turns out that the Gauss law prevents a straightforward quantisation of gauge theories in terms of the Wightman axioms. To explain why, we first derive the Gauss law through Noether's second theorem classically in section 6.2.1, and then show that the Gauss law leads to a conflict with locality when we attempt to extend these results to quantum fields in section 6.2.2. We refer the reader to [118] for a detailed mathematical analysis of the role of the Gauss law in quantum gauge theories.

6.2.1. Noether's second theorem

In chapter 4 we showed the Gauss law to be the constraint in the Hamiltonian formulation of Yang-Mills theory. Formally, we understood the Gauss constraint as the momentum map for the action of the local gauge group. We will now examine these ideas in a way that is closer to the practice of physicists. As always, we let G denote a compact matrix Lie group with Lie algebra $\mathfrak{g}$. Let t^a denote the generators of $\mathfrak{g}$, for $a = 1, ..., n$. Then a gauge transformation with parameter $\epsilon^a \in C^\infty(\mathbb{R}^4)$ acts infinitesimally on classical gauge fields A^a_μ and matter fields φ_i as [46, p. 141]

$$\delta^\epsilon \varphi_i(x) = i\epsilon_a(x) t^a_{ij} \varphi_j(x),$$
$$\delta^\epsilon A^a_\mu(x) = i\epsilon_c(x) T^c_{ab} A^b_\mu(x) + \partial_\mu \epsilon^a(x).$$

Here T^c_{ab} are the generators for the adjoint representation, i.e. $T^c_{ab} = i f^a_{cb} = -i f^a_{bc}$ with f^a_{bc} the structure constants (see section 2.2.1), which are zero for G Abelian.

Since Noether's first theorem tells us that symmetries give rise to conservation laws, we would naively expect the above gauge transformations to give rise to an infinite number of conservation laws. Noether's second theorem, however, tells us something different: the presence of a gauge symmetry with n-dimensional structure group (or more generally the invariance under the action of a Lie group parametrised by n functions) gives n relations between the Euler-Lagrange equations, i.e. n constraints.

Indeed, if a classical Lagrangian $\mathcal{L}(\varphi, \partial_\mu\varphi, A_\nu, \partial_\mu A_\nu)$ is invariant under the infinitesimal gauge transformations given above, then we have [46, p. 142]

$$\begin{aligned} i\left(\frac{\delta\mathcal{L}}{\delta\varphi_i}(t^a\varphi)_i + \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_i}(t^a\partial_\mu\varphi)_i + \frac{\delta\mathcal{L}}{\delta A^b_\nu}(T^a A_\nu)^b + \frac{\delta\mathcal{L}}{\delta\partial_\mu A^b_\nu}(T^a\partial_\mu A_\nu)^b\right)\varepsilon^a \\ + \left(i\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_i}(t^a\varphi)_i + \frac{\delta\mathcal{L}}{\delta A^a_\mu} + i\frac{\delta\mathcal{L}}{\delta\partial_\mu A^b_\nu}(T^a A_\nu)^b\right)\partial_\mu\varepsilon^a + \frac{\delta\mathcal{L}}{\delta\partial_\mu A^a_\nu}\partial_\mu\partial_\nu\varepsilon^a = 0. \end{aligned} \tag{6.1}$$

Here $(t^a\varphi)_i = t^a_{ij}\varphi_j$ and $(T^a A_\mu)^b = T^a_{bc} A^c_\mu$ etc. The standard argument is then that, since the $\epsilon(x)$ are arbitrary smooth functions the expressions in front of ϵ^a and $\partial_\mu\epsilon^a$ must vanish. For the expression in front of $\partial_\mu\partial_\nu\epsilon^a$ only the $\mu \leftrightarrow \nu$ symmetrised part needs to vanish, for the antisymmetrised part will give zero when multiplied with the symmetric $\partial_\mu\partial_\nu\epsilon^a$. Thus, we define the antisymmetric tensor

$$F^{\mu\nu}_a = -\frac{\delta\mathcal{L}}{\delta\partial_\mu A^a_\nu} = -F^{\nu\mu}_a.$$

For $G = U(1)$, this antisymmetric tensor is the well-known electromagnetic field strength $F_{\mu\nu}$. As of yet, we do not know how it relates to the distribution of charges (both of matter and of non-Abelian gauge charges). However, we recognise, as part of the expression in front of $\partial_\mu\epsilon^a$ in equation 6.1 above, the Noether current coming from Noether's first theorem for the global symmetry under the action of G

$$J^{a\mu} \equiv i\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_i}(t^a\varphi)_i + i\frac{\delta\mathcal{L}}{\delta\partial_\mu A^b_\nu}(T^a A_\nu)^b \equiv j^{a\mu}(\varphi) + j^{a\mu}(A).$$

In the Abelian case the gauge current $j^{a\mu}(A)$ vanishes, for then we have $(T^a A_\nu)^b = T^a_{bc} A^c_\mu = 0$ (since $T^a_{bc} = 0$). Indeed, in Abelian gauge theories the gauge bosons are uncharged. We can now rewrite the requirement that the expression in front of $\partial_\mu\epsilon^a$ in equation 6.1 vanish as:

$$\frac{\delta\mathcal{L}}{\delta A^a_\mu} + J^{a\mu} = 0.$$

This gives

$$J^{a\mu} = -\frac{\delta\mathcal{L}}{\delta A^a_\mu} = -\frac{\delta\mathcal{L}}{\delta A^a_\mu} + \partial_\nu\frac{\delta\mathcal{L}}{\delta\partial_\nu A^a_\mu} - \partial_\nu\frac{\delta\mathcal{L}}{\delta\partial_\nu A^a_\mu} = -E[A]^a_\mu + \partial_\nu F^{\nu\mu}_a,$$

where $E[A]^a_\mu$ are the Euler-Lagrange terms for the gauge field. But $E[A]^a_\mu$ vanishes when the equations of motion are satisfied, so in that case we obtain the relation $J^{a\mu} = \partial_\nu F^{\nu\mu}_a$ between the antisymmetric field tensor and the Noether current. The antisymmetry of $F_a^{\mu\nu}$ gives the continuity equation $\partial_\mu J^{a\mu}(x) = 0$. The statement $J_\mu^a = \partial^\nu F_{\nu\mu}^a$, with $F_{\mu\nu}^a = -F_{\nu\mu}^a$, is called the *Gauss law*, and it defines the *Gauss charge*

$$Q^a = \int dx^3 J_0^a(\mathbf{x}, 0).$$

This charge generates gauge transformations through the Poisson bracket [46, p. 143], just like we saw in section 4.3.3:

$$\{Q^a, \varphi_i\} - \delta^a \varphi_i, \qquad \{Q^a, A_\mu^b\} = \delta^a A_\mu^b.$$

The statement of the second Noether theorem, then, is that the charges defined by the action of the *global* gauge group G through Noether's first theorem are actually Gauss charges [46, p. 143], i.e. can be expressed in terms of the antisymmetric field strength tensor. Strocchi takes the existence of Gauss laws to be the central physical consequence of gauge symmetries: "we shall prove that the validity of Gauss laws has very strong consequences for the quantisation of gauge theories and leads to crucial differences with respect to standard quantum field theories" [46, p. 146]. We now turn to quantum fields.

6.2.2. Locality

Many of the consequences of the Gauss law for quantisation relate to the fact that it is in conflict with locality. From now on we specialise to scalar quantum electrodynamics (QED) and we come back to non-Abelian gauge theory only in section 6.6. In QED, the local Gauss law is $j_0(x) = \nabla \cdot \mathbf{E}(x)$. We wish to extend the classical infinitesimal generation of gauge symmetries by the Gauss charge to the quantum case, replacing Poisson brackets by commutators for quantum fields $A_\mu(x)$ and $\varphi(x)$ assumed to satisfy the Wightman axioms. These tempered distributions act on test functions $f \in \mathcal{S}(\mathbb{R}^4)$ via integration, like the Dirac delta distribution. To extend the ideas from section 6.2.1 to such quantum fields, we need to regularise the Noether charge:

$$Q_R = \int d^3x dt f_R(\mathbf{x}) \alpha(t) j_0(\mathbf{x}, t) \equiv j_0\left(f_R \alpha\right),$$

where $f_R(\mathbf{x}) = f(\mathbf{x}/R) \in \mathcal{D}(\mathbb{R}^3)$ with $f(x) = 1$ for $|\mathbf{x}| < 1$ and $f(\mathbf{x}) = 0$ for $|\,\mathbf{x}| < 1 + \delta$ with $\delta > 0$, and where $\mathrm{supp}(\alpha) \subset [-\gamma, \gamma]$ with $\int \alpha(t) dt = 1$ [46, p. 147]. The limit Q_R for $R \to \infty$ does not actually exist, but this is no problem, since we only need the limits of the commutator of Q_R with local fields. These limits do exist and are actually independent of the choice of smearing function α [46, p. 147]. It it then not too hard to show that, for a theory with a scalar field $\varphi(\mathbf{x}, t)$ and current $j_\mu(\mathbf{x}, t)$ which are local with respect to each other, we have [46, p. 148]

$$i \lim_{R\to\infty} [Q_R, \varphi(g, t)] = i \lim_{R\to\infty} [j_0\left(f_R, t\right), \varphi(g, t)] = \delta^e \varphi(g, t), \qquad g \in \mathcal{D}(\mathbb{R}^3),$$

where δ^e denotes the derivation corresponding to the global $U(1)$ gauge symmetry. This is indeed the desired quantum version of the classical Noether relation between the charge (the integral over the current density) and the gauge symmetry transformation. This result, however, is spoiled in the presence of the Gauss law (Proposition 2.1 in [46]).

Proposition 6.5. If a field φ is local with respect to the electric field $\mathbf{E}$, then the Gauss law gives $\lim_{R\to\infty}[j_0(f_R\alpha), \varphi] = 0$, implying that Q_R cannot generate the global $U(1)$ transformations.

Proof. By the local Gauss law we have

$$\lim_{R\to\infty}[j_0(f_R\alpha), \varphi(y)] = \lim_{R\to\infty} \int d^3x dt f_R(\mathbf{x}) \alpha(t) [j_0(\mathbf{x}, t), \varphi]$$
$$= \lim_{R\to\infty} \int d^3x dt f_R(\mathbf{x}) \alpha(t) [\nabla \cdot \mathbf{E}(x), \varphi(y)] = -\lim_{R\to\infty} \int d^3x dt \partial_i f_R(\mathbf{x}) \alpha(t) [E^i(x), \varphi(y)],$$

where in the last step we used partial integration. But since $f_R(\mathbf{x})$ is constant outside $R \leq |\mathbf{x}| \leq R(1+\delta)$ we see that

$$\operatorname{supp}(\partial_i f_R \alpha) \subset \{R \leq |\mathbf{x}| \leq R(1+\delta), |t| \leq \gamma\}.$$

This support becomes spacelike with respect to any $y \in \mathbb{R}^4$ for large enough R, and the commutator $[E^i(x), \varphi(y)]$ vanishes when x is spacelike with respect to y since φ is local with respect to $\mathbf{E}$. Thus, the above integral becomes zero in the limit, showing that $\lim_{R\to\infty}[j_0(f_R\alpha), \varphi(y)] = 0$. This means that the regularised charge cannot generate the gauge symmetry. □

Conversely, this proposition tells us that if a field φ is charged in the sense that $\delta^e\varphi \neq 0$, then it cannot be local with respect to the electric field. Thus, if we denote by $\mathcal{F}$ the local field algebra, then charged states cannot be in the closure of $\mathcal{F}\Psi_0$ but must instead be generated from the vacuum by non-local charged fields [46, p. 149]. In algebraic language, if $\mathcal{O}$ is a bounded region of spacetime, then for sufficiently large R we have $Q_R \in \mathcal{A}(\mathcal{O}')$. For any algebraic state ω which is localised in $\mathcal{O}$ we then get $\omega(Q_R) = \langle\Psi_0, Q_R\Psi_0\rangle = 0$, showing that ω is uncharged.

6.3. Local gauge quantisation

The conflict between locality and the Gauss law forces us to choose between two approaches to the quantisation of gauge theories. Firstly, we could insist on the Gauss law as an operator equation and accept that the field algebra becomes non-local [46, p. 146]. This is what we will do in the next section by employing the Coulomb gauge. The second option is to weaken the Gauss law to a condition on the physical states, keeping the field algebra local. In that case, however, the inner product defined by the vacuum correlation functions of the local field algebra cannot be positive definite (cf. Proposition 7.3.3 in [46] for QED).

This is the approach that we present in this section. We do so because, even though we will prove our main result on the Abelian Higgs mechanism in the Coulomb gauge, we do need to understand the local gauge quantisation of QED: since the Coulomb fields are defined in terms of the local fields, we will in fact need some results about the local fields to prove statements about the Coulomb fields in sections 6.4 and 6.5.

Before we begin, a brief comment is in order. From now on, we work in specific gauges. This may come as a surprise, as it seems that this makes our results gauge-dependent, such that we run into the problems concerning the perturbative expansion in specific gauges described in the preamble to this chapter. We avoid these perturbative issues, however, by working non-perturbatively. Besides, as we already mentioned, the Coulomb gauge actually enables us to work gauge-invariantly in the sense that Coulomb fields are invariant under local gauge transformations.

Following Strocchi, we consider the Feynman-Gupta-Bleuler (FGB) quantisation of QED. The FGB gauge, more commonly known as the Feynman gauge, arises through the addition of the gauge fixing term $-\frac{1}{2}(\partial^\mu A_\mu)^2$ to the Lagrangian. This leads to a violation of the Gauss law, since the Euler-Lagrange equation now implies:

$$j_\mu = -\frac{\delta\mathcal{L}}{\delta A^\mu} = -\partial_\nu \frac{\delta\mathcal{L}}{\delta\partial_\nu A^\mu} = \partial^\nu F_{\nu\mu} - \partial_\nu \frac{\delta}{\delta\partial_\nu A^\mu}\left(-\frac{1}{2}(\partial^\alpha A_\alpha)^2\right) = \partial^\nu F_{\nu\mu} + \partial_\nu\partial_\mu A^\mu.$$

What we do instead is impose a *weak Gauss law* by defining physical states Ψ, Φ to be those that satisfy

$$\langle\Psi, (j_\mu - \partial^\nu F_{\nu\mu})\Phi\rangle = 0,$$

and considering all other states to by unphysical. In other words, we let go of the Gauss law as an operator equation on the fields but require it on the physical states. This leads to the following general definition [46, p. 154].

Definition 6.6. A *local gauge quantisation* of QED consists of:

(i) a local field algebra $\mathcal{F}$ generated by operator-valued tempered distributions $\varphi(x), A_\mu(x)$ satisfying the Wightman axioms and transforming covariantly under the Poincaré group;

(ii) a group of local gauge transformations of $\mathcal{F}$ defining its gauge-invariant subalgebra $\mathcal{F}_I$;

(iii) a Poincaré-invariant vacuum functional $\omega_0 \colon \mathcal{F} \to \mathbb{C}$, also written $\omega_0(a) = \langle a \rangle_0$, satisfying the spectral condition and, for all $a \in \mathcal{F}_I$, the positivity condition $\omega_0(a^* a) \geq 0$;

(iv) the weak Gauss law $\langle \Psi, (j_\mu - \partial^\nu F_{\nu\mu})\Phi \rangle = \langle \Psi, \mathcal{L}_\mu \Phi \rangle = 0$ defining physical states Ψ, Φ.

Here the quantum fields $j_\mu = ie(\varphi(D_0\varphi)^* - \varphi^* D_0 \varphi)$ and $F_{\mu\nu}$ are obtained from the generating fields by multiplication, conjugation and differentiation of distributions (operations which are well-defined since the operator-valued fields share a common dense domain $D \subset H$ as per the Wightman axioms). For the FGB gauge we have $\mathcal{L}_\mu = \partial_\mu \partial_\nu A^\nu$. As for the second condition: the requirement of $F \in \mathcal{F}$ being invariant under gauge transformations is actually equivalent to $[\partial^\mu A_\mu, F] = 0$ [46, p. 153].

Let us now consider the canonical quantisation of QED in the FGB gauge. We depart from the canonical equal time commutators $[\dot{A}_\mu(x), A_\nu(y)]_{x_0=y_0} = -i\eta_{\mu\nu}\delta(\mathbf{x}-\mathbf{y})$ and $[\dot{\varphi}(x), \varphi(y)]_{x_0=y_0} = i\delta(\mathbf{x}-\mathbf{y})$, with all other equal time commutators vanishing. The equation $j_\mu = \partial^\nu F_{\nu\mu} + \partial_\mu \partial^\nu A_\nu$ gives

$$\partial_0 \partial^\mu A_\mu = j_0 - \partial^\mu F_{\mu 0} = j_0 - \partial^i F_{i0} = j_0 - \partial^i(\partial_i A_0 - \partial_0 A_i) = j_0 - \Delta A_0 + \partial_0 \partial^i A_i,$$

and this can used to derive the other equal time commutators [46, p. 152]:

$$\begin{aligned}
&[\partial^\mu A_\mu(x), A_0(y)]_{x_0=y_0} = [-\dot{A}_0(x) + \partial^i A_i(x), A_0(y)]_{x_0=y_0} = [-\dot{A}_0(x), A_0(y)]_{x_0=y_0} = -i\delta(\mathbf{x}-\mathbf{y}),\\
&[\partial^\mu A_\mu(x), A_i(y)]_{x_0=y_0} = [-\dot{A}_0(x), A_i(y)]_{x_0=y_0} = i\eta_{0i}\delta(\mathbf{x}-\mathbf{y}) = 0,\\
&[\partial_0 \partial^\mu A_\mu(x), A_0(y)]_{x_0=y_0} = [j_0(x) - \Delta A_0(x) + \partial_0 \partial^i A_i(x), A_0(y)]_{x_0=y_0} = 0,\\
&[\partial_0 \partial^\mu A_\mu(x), A_i(y)]_{x_0=y_0} = [\partial^i \dot{A}_i(x), A_i(y)]_{x_0=y_0} = -i\partial_i \delta(\mathbf{x}-\mathbf{y}),\\
&[\partial^\mu A_\mu(x), \varphi(y)]_{x_0=y_0} = 0,\\
&[\partial_0 \partial^\mu A_\mu(x), \varphi(y)]_{x_0=y_0} = [j_0(x), \varphi(y)]_{x_0=y_0} = e\delta(\mathbf{x}-\mathbf{y})\varphi(y).
\end{aligned}$$

Canonical quantisation then leads to the unequal time commutation relations [46, p. 152]

$$\begin{aligned}
[\partial^\nu A_\nu(x), A_\mu(y)] &= -i\partial_\mu D(x-y),\\
[\partial^\nu A_\nu(x), \varphi(y)] &= eD(x-y)\varphi(y),
\end{aligned}$$

where $D(x) = \Delta(x; m^2 = 0)$ is the propagator. Now, since the FGB quantisation yields a local field algebra and therefore does not contain charged fields by the arguments from section 6.2.2, we expect the charge, i.e. the generator of the global $U(1)$ transformations, to commute with all observables - that is: we expect the charge to be superselected (cf. Theorem 7.5.1 [46]).

Theorem 6.7. In the FGB quantisation of QED, all observables commute with the charge Q defined on the local states by

$$QF\Psi_0 = \lim_{R\to\infty} [j_0(f_R\alpha), F]\Psi_0, \qquad F \in \mathcal{F}.$$

Proof. Let $F \in \mathcal{F}_I$ be an observable. By the locality of $\mathcal{F}$ we have

$$[Q, F] = \lim_{R\to\infty} [Q_R, F] = \lim_{R\to\infty} [Q_R - \partial^i F_{i0}(f_R\alpha), F].$$

By the "Gauss law" $j_0 = \partial^i F_{i0} + \partial_0 \partial^\nu A_\nu$ we moreover get $Q_R - \partial^i F_{i0}(f_R\alpha) = \partial_0 \partial^\nu A_\nu(f_R\alpha)$, so that

$$[Q, F] = \lim_{R\to\infty} [\partial_0 \partial^\nu A_\nu(f_R\alpha), F] = -\lim_{R\to\infty} [\partial^\nu A_\nu(f_R \partial_t \alpha), F].$$

But, as we remarked above, F being gauge-invariant means $[\partial^\nu A_\nu, F] = 0$, so we find $[Q, F] = 0$. □

This proof of the superselection of the charge can be extended to the non-Abelian case, as we will discuss in section 6.6. We will also prove a version in the Coulomb gauge (Theorem 6.10).

6.4. The Coulomb gauge

Having studied the most important tenets of the local gauge quantisation of QED, we now take the other of the two approaches to handling the Gauss law. We accept the existence of non-local charged fields, such that charged states can be obtained by applying these non-local fields to the vacuum. If A_μ and φ are the gauge and matter fields in the local FGB quantisation of QED, then the transformation to the non-local Coulomb gauge is the by now well-known radiative projection from section 4.1.2:

$$\varphi_C = e^{-ie\Delta^{-1}\partial^i A_i}\varphi,$$
$$A^C_\mu = A_\mu - \partial_\mu \Delta^{-1}\partial^i A_i.$$

The second equation gives

$$A^C_0 = A_0 - \partial_0\Delta^{-1}\partial^i A_i = A_0 - \Delta^{-1}\partial^i\partial_0 A_i = A_0 - \Delta^{-1}\partial^i(\partial_i A_0 - F_{i0}) = \Delta^{-1}\partial^i F_{i0} = \Delta^{-1} j^C_0.$$

Again using $\partial_0\partial^\nu A_\nu = j_0 - \Delta A_0 + \partial_0\partial^i A_i$, this gives the Coulomb current density

$$j^C_0 = \Delta A_0 - \partial_0\partial^i A_i = j_0 - \partial_0\partial^\nu A_\nu.$$

The field φ_C is charged since it transforms under the global charge group $U(1)$, but it is gauge-invariant in the Redundant sense of gauge transformations from section 4.3.2. We will now shed light on two issues. The first is the relation between the electric charge, i.e. the generator of the global $U(1)$ symmetry, and the Noether current. It is standard to assume that the electric charge is the integral over the current density, but in fact the Gauss law spoils this relation. This insight is at the heart of our main result on the Abelian Higgs mechanism in section 6.5. Secondly, we discuss the fact that the electric charge is superselected. In Theorem 6.13 we will see that the electric charge can only be superselected when the global $U(1)$ gauge symmetry is *unbroken*.

Let $\mathcal{F}_C$ denote the field algebra generated by the Coulomb fields φ_C and A^C_i. We assume its vacuum correlations to be well-defined (see footnote 49 in [46, p. 171] for references on this). The generator of the global $U(1)$ symmetry is the electric charge Q^e, defined by

$$[Q^e, \varphi_C] = e\varphi_C, \qquad Q^e\Psi_0 = 0.$$

The following result (Proposition 4.1 in [42], Proposition 5.3 in [46]) shows that the correspondence between the generator of the global $U(1)$ symmetry and the current density fails.

Proposition 6.8. In the Coulomb gauge of QED, with field algebra $\mathcal{F}_C$ and vacuum vector Ψ_0, we have that for all $\Psi, \Phi \in \mathcal{F}_C\Psi_0$ the limit

$$\lim_{R\to\infty} (\Psi, [j^C_0(f_R\alpha), \varphi_C]\Phi)$$

exists. However, the limit is generally dependent on α, meaning that the time-independent global $U(1)$ symmetry cannot be generated by such integrals of the charge density.

Proof. The commutators $[\partial^\nu A_\nu(x), A_\mu(y)] = -i\partial_\mu D(x-y)$ and $[\partial^\mu A_\mu(x), \varphi(y)] = eD(x-y)\varphi(y)$ in the FGB quantisation give, by the above epxression of φ_C in terms of φ:

$$\begin{aligned}[\partial^\nu A_\nu(x), \varphi_C(y)] &= [\partial^\nu A_\nu(x), e^{-ie\Delta^{-1}\partial^i A_i(y)}]\varphi(y) + e^{-ie\Delta^{-1}\partial^i A_i(y)}[\partial^\nu A_\nu(x), \varphi(y)]\\ &= -ie\Delta^{-1}\partial^i[\partial^\nu A_\nu(x), A_i(y)]e^{-ie\Delta^{-1}\partial^i A_i(y)}\varphi(y) + e^{-ie\Delta^{-1}\partial^i A_i(y)} eD(x-y)\varphi(y)\\ &= -e\Delta^{-1}\partial^i\partial_i D(x-y)\varphi_C(y) + eD(x-y)\varphi_C(y) = 0.\end{aligned}$$

This is just the statement that φ_C is gauge-invariant (except of course under global gauge transformations). Thus, we have $\partial^\nu A_\nu \mathcal{F}_C\Psi_0 = \mathcal{F}_C\partial^\nu A_\nu\Psi_0$, implying that $(\Psi, (j^C_0 - j_0)\Phi) = 0$ for any states

$\Psi, \Phi \in \mathcal{F}_C$. We then also get $[j_0(f_R, x_0), \varphi_C(y)] = [j_0^C(f_R, x_0), \varphi_C(y)]$, but this last commutator is just equal to $[\partial^i F_{i0}(f_R, x_0), \varphi_C(y)] = -[F_{i0}(\partial^i f_R, x_0), \varphi_C(y)]$. By locality of the FGB fields $\lim_{R\to\infty}[F_{i0}, \varphi] = 0$, so since

$$[F_{i0}(\partial^i f_R, x_0), \varphi_C(y)] = [F_{i0}(\partial^i f_R, x_0), e^{-ie\Delta^{-1}\partial^i A_i(y)}]\varphi(y) + e^{-ie\Delta^{-1}\partial^i A_i(y)}[F_{i0}(\partial^i f_R, x_0), \varphi(y)],$$

it is the long distance behaviour of the commutator $[F_{i0}(x), e^{-ie\Delta^{-1}\partial^i A_i(y)}]$ that we need to study to show that the limits from the statement of the proposition exist but are α-dependent. In order to estimate the behaviour of this commutator, we note that the commutator $[F_{\mu\nu}(x + \mathbf{a}), A_j(\mathbf{z}, y_0)]$ vanishes for points $(\mathbf{x} + \mathbf{a} - \mathbf{z}) > (x_0 - y_0)^2$ by the locality of the FGB fields. Thus, for fixed x_0, y_0 it has compact support in the variable $\mathbf{x} + \mathbf{a} - \mathbf{z}$, so that its convolution

$$C_{\mu\nu}(x + \mathbf{a}, y) = -\frac{ie}{4\pi}\int d^3z \partial_z^j |\mathbf{y} - \mathbf{z}|^{-1}[F_{\mu\nu}(x + \mathbf{a}), A_j(\mathbf{z}, y_0)]$$

with $\partial_z^j|\mathbf{y} - \mathbf{z}|^{-1}$ (which decreases as $|\mathbf{z}|^{-2}$) decreases as $|\mathbf{a}|^{-2}$ for $|\mathbf{a}| \to \infty$. By the same reasoning, the commutator of $C_{\mu\nu}(x + \mathbf{a}, y)$ with $\Delta^{-1}\partial^i A_i(y)$ in turn decreases as $|\mathbf{a}|^{-4}$. This means that, in expanding the exponential $\exp(-ie\Delta^{-1}\partial^i A_i(y))$, we find that $C_{\mu\nu}(x + \mathbf{a}, y)$ commutes with all other terms, up to terms decreasing as $|\mathbf{a}|^{-4}$, so that we obtain [47, p. 59]

$$\begin{aligned}
&[F_{i0}(x + \mathbf{a}), e^{-ie\Delta^{-1}\partial^j A_j(y)}]\varphi(y) \to e^{-ie\Delta^{-1}\partial^j A_j(y)}[F_{i0}(x + \mathbf{a}), -ie\Delta^{-1}\partial^j A_j(y)]\varphi(y) + \mathcal{O}(|\mathbf{a}|^{-4})\\
=&[F_{i0}(x + \mathbf{a}), \frac{ie}{4\pi}\int d^3|\mathbf{y} - \mathbf{z}|^{-1}\partial_z^j A_j(\mathbf{z}, y_0)]\varphi_C(y) + \mathcal{O}(|\mathbf{a}|^{-4}) = C_{i0}(x + \mathbf{a}, y)\varphi_C(y) + \mathcal{O}(|\mathbf{a}|^{-4})
\end{aligned}$$

for $|\mathbf{a}| \to \infty$. Now, it follows from the commutator $[\partial^\nu A_\nu(x), A_\mu(y)] = -i\partial_\mu D(x-y)$ from section 6.3 that $C_{\mu\nu}(x+\mathbf{a}, y)$ commutes with $\partial^\nu A_\nu$, i.e. that it is a gauge-invariant field. Thus, the *cluster property* applies to it. This property can be derived from the uniqueness of the translationally invariant state [46, p. 73] and states that for any $A, B \in \mathcal{F}_I : \langle A\Psi_0, U(\mathbf{a})B\Psi_0\rangle \to \langle B\rangle_0\Psi_0$ for $|\mathbf{a}| \to \infty$. Here $U(\mathbf{a})$ is the space translation operator. Thus, we find that for $|\mathbf{x}| \to \infty$:

$$[F_{i0}(x), \varphi_C(y)] \to -\frac{ie}{4\pi}\int d^3z\partial_z^j|\mathbf{y} - \mathbf{z}|^{-1}\langle[F_{i0}(x), A_j(\mathbf{z}, y_0)]\rangle_0\varphi_C(y) + \mathcal{O}(|\mathbf{x}|^{-4}),$$

understood to hold in matrix elements between two Coulomb states $\Phi, \Psi \in \mathcal{F}_C\Psi_0$. Let us now introduce the Källén-Lehmann spectral representation

$$\langle F_{\mu\nu}(x)F_{\lambda\sigma}(y)\rangle_0 = i d_{\mu\nu\lambda\sigma}\int d\rho(m^2)\Delta^+(x - y; m^2)$$

of the two-point function of the electromagnetic field, where $\rho(m^2)$ is the spectral measure and where $d_{\mu\nu\lambda\sigma} = \eta_{\nu\sigma}\partial_\mu\partial_\lambda + \eta_{\mu\lambda}\partial_\nu\partial_\sigma - \eta_{\nu\lambda}\partial_\mu\partial_\sigma - \eta_{\mu\sigma}\partial_\nu\partial_\lambda$ [47, p. 58]. Then we can define the commutator function $F(x) = F^+(x) - F^+(-x)$, where $F^+(x) = i\int d\rho(m^2)\Delta^+(x; m^2)$. This allows us to write [46, p. 172]

$$\langle[F_{i0}(x), A_j(z)]\rangle_0 = (\eta_{ij}\partial_0 - \eta_{0j}\partial_i)F(x - z) = \delta_{ij}\partial_0 F(x - z).$$

Thus, we obtain the following expression (again understood as a matrix element between Coulomb states) for $R \to \infty$:

$$\begin{aligned}
[j_0(f_R, x_0), \varphi_C(y)] \to&\frac{ie}{4\pi}\int d^3z\partial_z^j|\mathbf{y} - \mathbf{z}|^{-1}\int d^3x\partial_x^i f_R(\mathbf{x})\delta_{ij}\partial_0 F(x - z)\varphi_C(y)\\
=&\frac{ie}{4\pi}\int d^3z\partial_z^j|\mathbf{y} - \mathbf{z}|^{-1}\int d^3x\partial_j^x f_R(\mathbf{x})\partial_0 F(x - z)\varphi_C(y)\\
=&ie\frac{-1}{4\pi}\int d^3z\partial_z^j\partial_j^z|\mathbf{y} - \mathbf{z}|^{-1}\int d^3x f_R(\mathbf{x})\partial_0 F(x - z)\varphi_C(y)\\
=&ie\Delta^{-1}\Delta\int d^3x f_R(\mathbf{x})\partial_0 F(x - y)\varphi_C(y) = ie\partial_0\int d^3x f_R(\mathbf{x})F(x - y)\varphi_c(y).
\end{aligned}$$

By locality, $F(x)$ vanishes for large enough $|\mathbf{x}|$, so that the $R \to \infty$ limit exists. Thus, we have shown that the limits from the statement of the proposition exist. In fact, by the well-known expression for the propagator [42, p. 3179]

$$F(x-y) = \int d\rho(m^2)\Delta(x-y;m^2) = \int d\rho(m^2)\int \frac{d^4k}{2E(\mathbf{k})}\delta(k^2+m^2)e^{-ik(x-y)},$$

with $E(\mathbf{k}) = \sqrt{\mathbf{k}^2+m^2}$, we obtain

$$\begin{aligned}
\lim_{\mathbb{R}\to\infty}[j_0(f_R,x_0),\varphi_C(y)] &= \lim_{\mathbb{R}\to\infty} ie\partial_0\int d^3x f_R(\mathbf{x})\int d\rho(m^2)\int\frac{d^4k}{2E(\mathbf{k})}\delta(k^2+m^2)e^{-ik(x-y)}\varphi_c(y)\\
&= ie\partial_0\int d^3x\int d\rho(m^2)\int\frac{d^4k}{2E(\mathbf{k})}\delta(k^2+m^2)e^{-ikx}e^{iky}\varphi_c(y)\\
&= -ie\partial_0\int d\rho(m^2)\int\frac{d^4k}{2E(\mathbf{k})}\delta(k^2+m^2)\delta(\mathbf{k})e^{ik_0x_0}e^{iky}\varphi_c(y)\\
&= -ie\partial_0\int d\rho(m^2)\int\frac{dk_0}{2m}\delta(k_0^2-m^2)e^{ik_0(x_0-y_0)}\varphi_c(y)\\
&= e\partial_0\int d\rho(m^2)\frac{1}{2mi}\left(e^{im(x_0-y_0)}-e^{-im(x_0-y_0)}\right)\varphi_c(y)\\
&= e\partial_0\int d\rho(m^2)\frac{1}{m}\sin(m(x_0-y_0))\varphi_c(y)\\
&= e\int d\rho(m^2)\cos(m(x_0-y_0))\varphi_C(y).
\end{aligned}$$

This is clearly dependent on time, unless $d\rho(m^2) = \delta(m^2)$, i.e. unless $F_{\mu\nu}$ is a free field. In the interacting case, however, it gives the α-dependence of the charge-density commutator [46, p. 173]. □

Now, the failure of the integral of the charge density to generate the time-independent global $U(1)$ transformations can be remedied by time-averaging the integral of j_0 with an improved smearing. That is, we define $Q_{\delta R} = j_0(f_R\alpha_{\delta R})$ where $\alpha_{\delta R} = \alpha(x_0/\delta R)/\delta R$, with $\alpha \in \mathcal{D}(\mathbb{R})$, $0<\delta<1$ and $\operatorname{supp}(\alpha) \subset [-\epsilon,\epsilon]$ with $\epsilon \ll 1$ [42, p. 3180]. However, this only works in the case of *unbroken* $U(1)$ symmetry and can in fact be related to the existence of massless photons, as shown by the following very important result (cf. Proposition 5.4 in [46], Proposition 5.1 in [42]).

Proposition 6.9. In the Coulomb gauge the global $U(1)$ symmetry ($\delta^e\varphi_C = ie\varphi_C, \delta^e A_i^C = 0$) is generated by the integral of the charge density, i.e. for all $F \in \mathcal{F}_C$

$$\delta F = i\lim_{\delta\to 0}\lim_{R\to\infty}[Q_{\delta R},F], \tag{6.2}$$

if and only if the spectral measure $d\rho(m^2)$ of the electromagnetic field contains a $\delta(m^2)$ contribution, i.e. if there are massless photons. Moreover, one generally has

$$\lim_{R\to\infty} j_0(f_R\alpha_{\delta R})\Psi_0 = 0, \tag{6.3}$$

so that if there are massless photons the $U(1)$ symmetry is unbroken. In this case, one can express the electric charge Q^e as an integral of the charge density j_0 not only in the commutators with charged fields, but also in the matrix elements of Coulomb charged states $\Psi, \Phi \in \mathcal{F}_C$:

$$\langle\Psi, Q^e\Phi\rangle = \lim_{\delta\to 0}\lim_{R\to\infty}\langle\Psi, j_0(f_R\alpha_{\delta R})\Phi\rangle. \tag{6.4}$$

Proof. To prove equation 6.2, it suffices to consider $F = \varphi_C, A_i^C$, since these fields generate $\mathcal{F}_C$ by definition. Starting with $F = \varphi_C$, the expressions in the final part of the proof of Proposition 6.8 give

$$\begin{aligned}
\lim_{R\to\infty}[j_0(f_R\alpha_{\delta R}), \varphi_C(y)] &= \lim_{R\to\infty} ie\partial_0 \int d^4x f_R(\mathbf{x})\alpha_{\delta R}(x_0)\int d\rho(m^2)\int \frac{d^4k}{2E(\mathbf{k})}\delta(k^2+m^2)e^{-ikx}e^{iky}\varphi_c(y) \\
&= \lim_{R\to\infty} ie \int d^4x f_R(\mathbf{x})\alpha_{\delta R}(x_0)\int d\rho(m^2)\int d^4k\frac{ik_0\delta(k^2+m^2)}{2E(\mathbf{k})}e^{-ikx}e^{iky}\varphi_c(y) \\
&= \lim_{R\to\infty} e \int d\rho(m^2)\int d^4k\tilde{f}_R(\mathbf{k})\tilde{\alpha}_{\delta R}(k_0)\frac{k_0\delta(k^2+m^2)}{2E(\mathbf{k})}e^{iky}\varphi_c(y) \\
&= \lim_{R\to\infty} e \int d\rho(m^2)\int d^3k\tilde{f}_R(\mathbf{k})\mathrm{Re}\left(\tilde{\alpha}_{\delta R}(E(\mathbf{k}))e^{-iE(\mathbf{k})y_0}\right)e^{i\mathbf{k}\cdot\mathbf{y}}\varphi_c(y),
\end{aligned}$$

where $\tilde{f}_R$ and $\tilde{\alpha}_{\delta R}$ denote the Fourier transforms of f_R and $\alpha_{\delta R}$. By the scaling properties of the Fourier transform this expression becomes

$$\begin{aligned}
&\lim_{R\to\infty} e \int d\rho(m^2)\int d^3kR\tilde{f}(R\mathbf{k})\mathrm{Re}\left(\tilde{\alpha}(\delta RE(\mathbf{k}))e^{-iE(\mathbf{k})y_0}\right)e^{i\mathbf{k}\cdot\mathbf{y}}\varphi_c(y) \\
=&\lim_{R\to\infty} e \int d\rho(m^2)\int d^3k\tilde{f}(\mathbf{k})\mathrm{Re}\left(\tilde{\alpha}(\delta R\omega_R(\mathbf{k}))e^{-i\omega_R(\mathbf{k})y_0}\right)e^{i\mathbf{k}\cdot\mathbf{y}/R}\varphi_c(y),
\end{aligned}$$

where $\omega_R(\mathbf{k}) = \sqrt{\mathbf{k}^2R^{-2}+m^2}$. For $m \neq 0$ we have $\delta R\omega_R(\mathbf{k}) \to \delta Rm \to \infty$ as $R \to \infty$. Since α is of fast decrease this implies $\tilde{\alpha}(\delta R\omega_R(\mathbf{k})) \to 0$, so by the dominated convergence theorem the above expression for the commutator vanishes in the $R \to \infty$ limit if the measure of the point $m^2 = 0$ is zero, i.e. if there is no $\delta(m^2)$ contribution [46, p. 175]. For $m = 0$ we have $\omega_R(\mathbf{k}) \to |\mathbf{k}|R^{-1}$, so that $\tilde{\alpha}(\delta R\omega_R(\mathbf{k})) \to \tilde{\alpha}(\delta|\mathbf{k}|)$ and $e^{-i\omega_R(\mathbf{k})y_0} \to 1$. If we then also take the $\delta \to 0$ limit we get $\tilde{\alpha}(\delta|\mathbf{k}|) \to \tilde{\alpha}(0) = 1$, and since also $\int d^3k\tilde{f}(\mathbf{k}) = f(0) = 1$ we find that if there is a $\delta(m^2)$ contribution to the spectral measure we indeed get

$$\lim_{\delta\to 0}\lim_{R\to\infty}[Q_{\delta R}, \varphi_C(y)] = e\varphi_C(y).$$

As for $F = A_i^C = A_i - \partial_i\Delta^{-1}\partial^j A_j$, we know from the locality of the FGB fields that the limit of $[\partial^j F_{j0}(f_R, x_0), A_i]$ vanishes, so that the only contribution to the commutator comes from the $\partial_i\Delta^{-1}\partial^j A_j$ part of A_i^C. But in the proof of Proposition 6.2 we have seen that $[F_{i0}(x), \Delta^{-1}\partial^j A_j]$ is proportional to the commutator C_{i0}, which decreases as $|\mathbf{x}|^{-2}$. But the commutator $[F_{i0}(x), \partial_k\Delta^{-1}\partial^j A_j]$ contains yet another spatial derivative, so it decreases as $|\mathbf{x}|^{-3}$. Thus, it gives a vanishing contribution when smeared with f_R in the $R \to \infty$ limit [46, p. 175].

Now, to prove equation 6.3, we use the Gauss law (which holds as an operator equation in the Coulomb gauge) to write

$$\begin{aligned}
\langle j_0(x)j_0(y)\rangle_0 &= \langle\partial^i F_{i0}(x)\partial^j F_{j0}(y)\rangle_0 = \partial_x^i\partial_y^j\langle F_{i0}(x)F_{j0}(y)\rangle_0 = \partial_x^i\partial_y^j d_{i0j0}F^+(x-y) \\
&= \partial^i\partial^j(\eta_{00}\partial_i\partial_j + \eta_{ij}\partial_0\partial_0 - \eta_{0j}\partial_i\partial_0 - \eta_{i0}\partial_0\partial_j)F^+(x-y) \\
&= \partial^i\partial^j(-\partial_i\partial_j + \delta_{ij}\partial_0\partial^0)F^+(x-y) = \partial^i\partial_i(-\partial^j\partial_j + \partial^0\partial_0)F^+(x-y) \\
&= -\Delta\Box F^+(x-y) = -\Delta\Box\int d\rho(m^2)\int d^4k\frac{\theta(k_0)\delta(k^2+m^2)}{2E(\mathbf{k})}e^{-ik(x-y)} \\
&= -\int d\rho(m^2)\int d^4k\frac{\theta(k_0)\delta(k^2+m^2)}{2E(\mathbf{k})}k^2|\mathbf{k}|^2e^{-ik(x-y)} \\
&= \int d\rho(m^2)m^2\int d^3k\frac{|\mathbf{k}|^2}{2E(\mathbf{k})}e^{iE(\mathbf{k})(x_0-y_0)}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.
\end{aligned}$$

Thus, we obtain

$$
\begin{aligned}
\|Q_{\delta R}\Psi_0\|^2 &= \int d^4x \int d^4y f_R(\mathbf{x}) f_R(\mathbf{y}) \alpha_{\delta R}(x_0)\alpha_{\delta R}(y_0) |\langle j_0(x) j_0(y)\rangle_0| \\
&= \int d^4x \int d^4y \int d\rho(m^2) m^2 \int d^3k f_R(\mathbf{x}) f_R(\mathbf{y}) \alpha_{\delta R}(x_0)\alpha_{\delta R}(y_0) \frac{|\mathbf{k}|^2}{2E(\mathbf{k})} e^{iE(\mathbf{k})(x_0-y_0)} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\
&= \int d\rho(m^2) m^2 \int d^3k \tilde{f}_R(\mathbf{k}) \tilde{f}_R(-\mathbf{k}) \tilde{\alpha}_{\delta R}(-E(\mathbf{k})) \tilde{\alpha}_{\delta R}(E(\mathbf{k}))(y_0) \frac{|\mathbf{k}|^2}{2E(\mathbf{k})} \\
&= \int d\rho(m^2) m^2 R \int \frac{d^3k}{2\omega_R(\mathbf{k})} \left|\tilde{f}(\mathbf{k}) \tilde{\alpha}(\delta R \omega_R(\mathbf{k}))\mathbf{k}\right|^2 .
\end{aligned}
$$

Since α is of fast decrease, we have $|\tilde{\alpha}(\delta R\omega_R(\mathbf{k}))|^2 \le C_N(1+(\delta R)^2 m^2)^{-N}$ for any $N \in \mathbb{N}$ [46, p. 175]. Additionally, for $R \to \infty$ we have $\int d^3k |\tilde{f}(\mathbf{k})\mathbf{k}|^2/2\omega_R(\mathbf{k}) \le C/m$, and since the tempered $d\rho(m^2)$ is polynomially bounded there exist a finite measure $d\rho'(m^2)$ and a large $M \in \mathbb{N}$ such that [47, p. 61]

$$
\|Q_{\delta R}\Psi_0\|^2 \le C_N C \int d\rho'(m^2) \frac{Rm}{(1+(\delta R)^2 m^2)^{N'}}.
$$

Clearly this integral goes to zero as $R \to \infty$, so we conclude that $\lim_{R\to\infty} Q_{\delta R}\Psi_0 = 0$. Equation 6.4 then immediately follows from 6.2 and 6.3, since $Q^e\Psi_0 = 0$. □

We have thus arrived at a very important result: the Noether relation between the electric charge and the current density holds only if there are massless photons, in which case the global $U(1)$ symmetry is unbroken. The only thing that remains to be done for a full characterisation of the Abelian Higgs mechanism, then, is to show the converse, i.e. that if the global $U(1)$ group is broken, then there are massive bosons. Before we prove this in section 6.5, where we finally obtain what is probably the central statement of this entire thesis, namely Theorem 6.13, let us briefly focus on the importance of superselection, as in section 5.3, in the story we are telling. Strocchi succintly summarises the role of the global gauge group in relation to superselection:

> Thus, the important (if not the exclusive) role of the gauge group is that of providing, through the invariants of its representations, the superselected quantum numbers or charges, which classify the irreducible representations of the observable algebra $\mathcal{A}$. The unobservable charged fields play the role of intertwiners (or charge raising/lowering operators) between the vacuum representation (vacuum sector) and the charged sectors. This justifies the introduction of the enlarged algebra $\mathcal{F}$ generated by the observables and the charged fields (field algebra), since $\mathcal{F}$ is the carrier of the irreducible representations of the gauge group and provides the corresponding invariants. [46, p. 169].

In the unbroken case, the electric charge is indeed such a superselected quantum number, as shown by the following result.

Proposition 6.10. Any observable $A \in \mathcal{F}_I$ commutes with the electric charge Q^e on the Coulomb states:

$$
\langle \Psi, [Q^e, A]\Phi\rangle = \lim_{\delta\to 0}\lim_{R\to\infty} \langle \Psi, [j_0(f_R\alpha_{\delta R}), A]\Phi\rangle = 0.
$$

Proof. By the Gauss law we get

$$
\lim_{\delta\to 0}\lim_{R\to\infty} \langle \Psi, [j_0(f_R\alpha_{\delta R}), A]\Phi\rangle = \lim_{\delta\to 0}\lim_{R\to\infty} \langle \Psi, [\partial^i F_{i0}(f_R\alpha_{\delta R}), A]\Phi\rangle = -\lim_{\delta\to 0}\lim_{R\to\infty} \langle \Psi, [F_{i0}(\partial^i f_R\alpha_{\delta R}), A]\Phi\rangle.
$$

But $\partial^i F_R(\mathbf{x})$ is nonzero only for $\mathbf{x} > R$, whereas $\alpha_{\delta R}(x_0)$ is nonzero only for $|x_0| < \epsilon\delta R$, where we have $\mathrm{supp}(\alpha) \subset [-\epsilon, \epsilon]$, $\delta < 1$ and $\epsilon << 1$. Thus, given any compact reigon $\mathcal{O}$ in spacetime, for R large enough the points x at which $\partial^i f_R(\mathbf{x})\alpha_{\delta R}(x_0)$ does not vanish become spacelike to $\mathcal{O}$. By the locality of the observable A with respect to the observable field $F_{\mu\nu}$, we then find that the RHS of the above equation vanishes. □

6.5. Abelian Higgs mechanism in Coulomb gauge

We are now finally in a position to rigorously study the Higgs mechanism in QFT by putting together everything we have worked out in the last few chapters. Before we state our main result in the Coulomb gauge, however, we briefly consider the Goldstone theorem in local gauges, because we need its characterisation of Goldstone modes. To this end we have Theorem 17.1 from [44].

Theorem 6.11. Let $\mathcal{F}$ be a local field algebra generated by Wightman fields with α^λ a 1-parameter group of $*$-automorphisms of $\mathcal{F}$ which commutes with spacetime translations and is assumed to define a derivation δ_α on $\mathcal{F}$. Assume furthermore that α^λ is generated by the charge of a local covariant conserved current j_μ, i.e. that for all $F \in \mathcal{F}$ we have

$$\delta_\alpha F = i \lim_{R\to\infty} [Q_R, F],$$

where $Q_R = j_0(f_R, \alpha)$. If the symmetry α^λ is spontaneously broken in the sense that there exists some $F \in \mathcal{F}$ such that $\langle \delta_\alpha F \rangle_0 \neq 0$, then the Fourier transform of $\langle j_0(x) F$ contains a $\delta(k^2)$ singularity, which is the massless *Goldstone mode*.

We will not present the proof, for we do not need it for our main Theorem 6.13. Before we present that result, however, we remark that the above Goldstone theorem can be avoided in local gauge quantisations of QED, as shown by Proposition 7.6.1 in [46].

Proposition 6.12. In local gauge quantisations of QED with local field algebra $\mathcal{F}$, the spontaneous breaking of the global $U(1)$ group by an order parameter $\langle \delta^e F \rangle_0 \neq 0$ implies that the Fourier transform of the two-point function $\langle j_0(x) F \rangle_0$ contains a $\delta(k^2)$. However, this singularity does not come from the energy-momentum spectrum of the physical vectors Ψ that satisfy the weak Gauss law $\langle \Psi, (j_\mu - \partial^\nu F_{\nu\mu})\Psi \rangle = 0$. In other words: the Goldstone modes are unphysical.

In section 6.6 we will consider a generalisation of this result to the non-Abelian case. Now that we know that the Goldstone modes in the Wightman approach must be understood as $\delta(k^2)$ contributions to the Fourier transform of $\langle j_0(x) F \rangle_0$, where $F \in \mathcal{F}$ is a symmetry breaking order parameter in the sense of Proposition 5.26, we have everything we need to state the central result of this chapter and probably of this thesis (cf. Theorem 7.6.2 in [46], Theorem 2.8.3 in [47]).

Theorem 6.13. Let $\mathcal{F}_C$ denote the Coulomb field algebra with vacuum Ψ_0, generated by the gauge field A_i^C and the complex scalar Higgs field φ_C. Let α^λ denote the 1-parameter family of $*$-automorphisms of $\mathcal{F}_C$ corresponding to the continuous global $U(1)$ symmetry with generator Q^e (the electric charge), and let j_0 be the associated conserved Noether current. Then the following results hold.

A. If the spectral measure $d\rho(m^2)$ of the electromagnetic field $F_{\mu\nu}$ contains a $\delta(m^2)$ contribution, i.e. if there are massless vector bosons, then

(i) the global $U(1)$ gauge transformations are generated by the improved smeared current density, i.e. for any $F \in \mathcal{F}_C$

$$\delta^e F = \left.\frac{d\alpha^\lambda(F)}{d\lambda}\right|_{\lambda=0} = i \lim_{\delta\to 0} \lim_{R\to\infty} [j_0(f_R \alpha_{\delta R}), F];$$

(ii) we have $\lim_{R\to\infty} j_0(f_R \alpha_{\delta R})\Psi_0 = 0$, so that $\langle \delta^e F \rangle_0 = 0$ for any $F \in \mathcal{F}_C$, meaning that the global $U(1)$ symmetry is unbroken (by Proposition 5.26);

(iii) the electric charge Q^e can be expressed in terms of j_0 not only in the commutators with fields $F \in \mathcal{F}_C$, but also in the matrix elements of the Coulomb charged states $\Phi, \Psi \in \mathcal{F}_C \Psi_0$:

$$\langle \Phi, Q^e \Psi \rangle = \lim_{\delta\to 0} \lim_{R\to\infty} \langle \Phi, j_0(f_R \alpha_{\delta R})\Psi \rangle,$$

implying that Q^e is a superselected charge.

B. If the global $U(1)$ symmetry is broken by some $F_{SSB} \in \mathcal{F}_C$ such that $\langle \delta^e F_{SSB} \rangle_0 \neq 0$, then

(i) the spectral measure $d\rho(m^2)$ of $F_{\mu\nu}$ cannot contain a $\delta(m^2)$ contribution, i.e. there are no massless vector bosons;

(ii) the global $U(1)$ gauge transformations are not generated by the current density, in fact

$$\lim_{\delta \to 0} \lim_{R \to \infty} [j_0(f_R \alpha_{\delta R}), F] = 0 \tag{6.5}$$

for any $F \in \mathcal{F}_C$, and $j_0(f_R \alpha_{\delta R})$ annihilates the vacuum, so that for any $\Psi \in \mathcal{F}_C \Psi_0$:

$$\lim_{\delta \to 0} \lim_{R \to \infty} j_0(f_R \alpha_{\delta R}) \Psi = 0, \tag{6.6}$$

i.e. we have *current charge screening*;

(iii) the two-point function $\langle j_\mu(x) F_{SSB} \rangle_0$ does not vanish, and its Fourier spectrum coincides with the energy-momentum spectrum of $F_{\mu\nu}$, so that the absence of massless vector bosons coincides with the absence of massless Goldstone modes.

Proof. The **A** part of the proof is just a neat overview of the contents of Propositions 6.9 and 6.10. As for the first part of **B**, that is just the contrapositive of **A**(ii). Furthermore, in the proof of Proposition 6.9 we saw that $\lim_{R \to \infty} [j_0(f_R \alpha_{\delta R}), \varphi_C(y)]$ vanishes if there is no $\delta(m^2)$ contribution to the spectral measure, and that $\lim_{R \to \infty} [j_0(f_R \alpha_{\delta R}), A_i^C(y)]$ vanishes anyway. Since $\mathcal{F}_C$ is generated by these fields equation 6.5 follows. Now, we actually proved equation 6.3, i.e. $\lim_{R \to \infty} j_0(f_R \alpha_{\delta R}) \Psi_0 = 0$, in general, using only the Gauss law. Thus equation 6.6 also follows.

Lastly, **B**(iii) follows from the proof of the Goldstone Theorem 6.11, for which we refer the reader to Chapter 17 of [44]. Indeed, in that proof, the fact that $\langle \delta^e F_{SSB} \rangle_0 \neq 0$ is used to exlude the possibility that $\langle j_\mu(x) F_{SSB} \rangle_0$ vanishes [46, p. 184]. In the present case, the relation from the proof of Proposition 6.8 stating that $[j_0(f_R, x_0), \varphi_C(y)] \to ie\partial_0 \int d^3x f_R(\mathbf{x}) F(x-y) \varphi_C(y)$ when $R \to \infty$ shows that, if we have $\langle \delta^e \varphi_C \rangle_0 = ie \langle \varphi_C \rangle_0 \neq 0$, then the two-point function $\langle j_\mu(x) \varphi_C \rangle_0$ cannot vanish since it is proportional to the vector boson two-point function. Since φ_C generates all elements of $\mathcal{F}_C$ that can break the $U(1)$ symmetry (after all A_μ is invariant), the same applies to all $F \in \mathcal{F}_C$. Thus, the Goldstone spectrum is proportional to the spectrum of $F_{\mu\nu}$, meaning in particular that there can be no massless Goldstone modes if there are no massless vector bosons. □

6.6. Non-Abelian generalisation

Although Theorem 6.13 is a beautiful result proving that the Abelian Higgs mechanism can be understood as an instance of SSB (in the algebraic sense) of the global $U(1)$ gauge symmetry, there is one evident flaw: it does not tell us anything about the non-Abelian case and in particular the electroweak theory. Back in section 1.3, however, we explicitly identified RQ1.3 as the question of how results on the Abelian Higgs mechanism generalise to the non-Abelian setting. Now, it must be admitted right away that, to the author's knowledge, there is nothing like a non-Abelian version of Theorem 6.13. If such a version would be attempted, one would expect to need a non-Abelian generalisation of the Coulomb gauge in order to construct a field algebra of non-Abelian charged fields that are invariant under the gauge transformations generated by the Gauss constraint, i.e. under the action of $\mathcal{G}_0^\infty$. It is hard to say, however, how realistic this would be, since non-Abelian gauge theories are generally quite poorly understood non-perturbatively.

Still, not all hope is lost, for we could also attempt to generalise the other approach outlined in this chapter, i.e. that of local gauge quantisation, to the non-Abelian case. This would mean that we would generalise Theorem 6.12, which shows that the Goldstone bosons associated to the breaking of the global $U(1)$ symmetry in QED are unphysical. Of course, this result is much weaker than Theorem 6.13, for it does not say anything about the existence of massive vector bosons - it only tells us that there are no Goldstone modes. On the other hand, we have seen in Theorem 6.13 that the absence of massless vector bosons coincides with the absence of massless Goldstone modes, so there might be a way forward there.

Now, the envisioned generalisation of Theorem 6.12 has actually been performed by De Palma and Strocchi [43]. They work in the BRST quantisation of Yang-Mills theory, defined by the gauge fixing

$$\mathcal{L}_{GF} = -\partial^\mu B^a A_{a\mu} + \frac{1}{2}\xi B^a B_a - i\partial^\mu \bar{c}^a (D_\mu c)_a,$$

where ξ is a parameter, A^a_μ is the gauge potential with a running over the Lie algebra generators, B^a is the Nakanishi-Lautrup field and $c^a, \bar{c}^a$ are the Faddeev-Popov ghosts [43, p. 5]. In the BRST quantisation, the inner product between states in $\mathcal{F}\Psi_0$, where $\mathcal{F}$ is the field algebra, is not semidefinite, similar to the FGB gauge [43]. Physical vectors $\Psi \in \mathcal{F}\Psi_0$ are then identified by the *BRST subsidiary condition* $Q_B\Psi = 0$, where Q_B is the nilpotent BRST charge. This gives us a condition of physicality similar to imposing the weak Gauss law in the local gauge quantisation of QED.

The result by De Palma and Strocchi is then as follows. We will not prove it, for this would require us to go into BRST quantisation, which is beyond the scope of this thesis. For more details, we refer the reader first and foremost to [43], but also to Proposition 7.5.2 in [46] and Theorem 2.7.1 in [47].

Theorem 6.14. In the BRST gauge of Yang-Mills theory with structure group G, if the global gauge group G is broken by the vacuum expectation value of an element F of the field algebra $\mathcal{F}$, i.e. $\langle \delta F\rangle_0 \neq 0$, then the Fourier transform of the two-point function $\langle J^a_\mu(x)F\rangle_0$, where J^a_μ are the conserved Noether currents defined by

$$J^a_\mu = \partial^\nu F_{\nu\mu} + \partial_\mu B^a + f^a_{bc} A^b_\mu B^c - i f^a_{bc} \bar{c}^b (D_\mu c)^c,$$

contains a $\delta(k^2)$, i.e. there are massless Goldstone modes. However, these modes do not belong to the physical spectrum.

With this result we end this chapter and thereby the main body of this thesis. In chapter 7 we will summarise all the main ideas we have developed and attempt to extract a coherent picture of the Higgs mechanism.

7. Conclusion

In chapters 2, 3, 4, 5 and 6 we studied global gauge symmetries and the Higgs mechanism from various angles, with the aim of answering the research questions we distilled from the philosophical literature in sections 1.2 and 1.3.

Chapter 2 provided a gauge-theoretical overview of the full Higgs mechanism in the Standard Model from the perspective of fibre bundles. It defined global gauge transformations (Definition 2.6) and explained how vacuum configurations of the Yang-Mills-Higgs Lagrangian on a trivial bundle over a connected and simply connected spacetime correspond to a choice of vacuum vector in a global vacuum gauge (Proposition 2.28). This is how the Higgs mechanism is usually presented: one uses a vacuum vector $(0, v/\sqrt{2})$ for the Higgs field, but such a choice can only be described once a vacuum gauge has been chosen. We derived the consequences of such a choice of vacuum vector in a vacuum gauge: the generation of masses for broken gauge bosons and leptons through the Yukawa coupling. We stressed that the Higgs mechanism solves not one but three issues this way: masses for gauge bosons, different masses for fermions in the same gauge multiplet and masses for twisted chiral fermions.

However, the mathematical machinery of chapter 2 did not allow us to address the main conceptual problem of the Higgs mechanism: the alleged fact that gauge symmetries are mere descriptive conventions, such that gauge symmetry breaking cannot represent anything *real*. We therefore considered a proposed solution to this problem in chapter 3: getting rid of gauge symmetry altogether by means of the dressing field method (DFM). In the DFM one turns the original fields into gauge-invariant composite objects by means of a dressing field. It is similar to the notion of bundle reduction in differential geometry. Berghofer et al. then write that the DFM shows that there is no SSB in the electroweak theory [41], since the $SU(2)$ gauge symmetry can be completely removed by a dressing field, such that there is no symmetry left to break at all. We discussed several problems with this approach, related to the assumption that the Higgs field is nowhere vanishing, which is needed to define a dressing field through the polar decomposition.

In chapter 4 we then explained why the very approach of completely removing gauge symmetry by using gauge-invariant composite fields is misinformed. The crucial point is that not all gauge transformations are unphysical. This can be made mathematically precise in the language of constrained Hamiltonian analysis, where 'unphysical' is formalised as *generated by the first-class constraints*. In Yang-Mills theory the relevant constraint is the Gauss law, which generates only local, small gauge transformations. Thus, if we wish to reparametrise our space of fields in terms of gauge-invariant objects along the lines of the DFM, we should consider fields which are invariant only under the gauge transformations generated by the Gauss constraint. In electromagnetism with a scalar field this can be achieved via the Coulomb gauge, since the remnant symmetry group of the Coulomb gauge is precisely the group of global gauge transformations. In section 4.2 we showed that, when the Abelian Higgs model is reformulated in terms of the Coulomb fields, the gauge boson and Higgs field gain mass only when the global $U(1)$ remnant symmetry is broken.

These results are, however, only classical. They can be used as a basis for perturbation theory, but this leads to problems with the apparent gauge-dependence of perturbative calculated quantities such as gauge boson masses, as explained in the preamble to chapter 6. We therefore devoted chapters 5 and 6 to developing a non-perturbative understanding of the Higgs mechanism in QFT. In chapter 5 we did not study the Higgs mechanism proper, but instead we presented some of the general structure of axiomatic QFT and specifically the appearance of the global gauge group. We briefly discussed DHR superselection theory, which explains how the presence of a global gauge symmetry leads to a superselection structure in the unitary equivalence classes of irreducible representations of the net of algebras. These equivalence classes are labelled by the quantised gauge charges, such that charged fields intertwine between the vacuum representation and charged representations. Thus, the global gauge symmetry, which gives rise to the conserved gauge charges through Noether's first theorem, is embedded into the very structure of gauge quantum field theories. Indeed, DHR have proved that the compact global gauge group can be recovered from the observable superselection structure of a QFT. We ended chapter 5 by studying SSB in algebraic quantum theory, proving that SSB in the algebraic sense can be detected by means of a symmetry breaking order parameter (Proposition 5.26) and by formulating the Goldstone theorem algebraically (Theorem 5.29).

With the structures from chapter 5 in mind, we then set out in chapter 6 to characterise the Higgs mechanism as an instance of SSB of global gauge symmetry in QFT by means of a symmetry breaking order parameter. To do this, we used work by Morchio and Strocchi based on the Wightman axioms, which give a more concrete implementation of the nets of local algebras from AQFT. We explained how one faces a dilemma when quantising gauge theories: one can choose to require the Gauss law as an operator equation on the Wightman fields, but then the field algebra contains non-local charged fields. The other option is to require a weak Gauss law only on the physical states, leading to a local field algebra. With the latter approach, one can prove that the Goldstone modes associated with breaking the global gauge group are unphysical, both in the Abelian and the non-Abelian case. The former approach can be implemented for electromagnetism by means of the Coulomb gauge, using the same field transformation as in chapter 4. One can then prove several powerful results (Propositions 6.8 and 6.9, Theorem 6.13) which show that the Noether relation between the electric charge as the generator of the global $U(1)$ symmetry and the smeared current density holds only if there is no symmetry breaking order parameter, in which case the electric charge is superselected in the sense of chapter 5. If there is such an order parameter, then this relation fails and there must be massive gauge bosons.

7.1. Answering the research questions

Having summarised the main ideas and results from this thesis, let us now revisit our original research questions, explain how we have addressed them and discuss what issues remain enigmatic. We begin with the subquestions RQ1.1-RQ1.3, such that we can combine the answers to these subquestions into a response to our main RQ1.

RQ1.1: *How can the apparent contradiction be resolved between the implication of the DFM that there is no SSB in the Higgs mechanism, and results presenting the Higgs mechanism as SSB of global gauge symmetry?*

In section 1.2 we pointed out that there is a tension between the DFM, which is used by Berghofer et al. to argue that there is no SSB in the electroweak theory [41], and the results by Struyve and by Morchio, Strocchi and De Palma [42, 43], which characterise the Higgs mechanism as a breaking of global gauge symmetry. We dedicated chapter 4 to dissolving this tension. Indeed, we found that the basic idea of the DFM is not wrong. In fact, we implemented the DFM at length in chapters 4 and 6 by using the dressing field $\exp(-ie\Delta^{-1}\partial^i A_i)$ for the Abelian Higgs model in the Coulomb gauge. However, this is a dressing field only for the local gauge group of transformations generated by the Gauss constraint and not for the global gauge group. Thus, we understand now that Berghofer et al. were looking in the wrong place for their application of the DFM. They attempted to use it to reduce the structure group $SU(2)$ in the electroweak theory, but this was based on the problematic assumption that the Higgs field is everywhere non-vanishing. This assumption in effect alters the structure of field space, permitting fewer configurations than in the full theory. What we have done instead, following Gomes, is to apply the DFM on infinite-dimensional field space like in section 4.3.4 (especially Proposition 4.7). This automatically leads to the Coulomb gauge as the physical gauge in which the fields are invariant under the local gauge group but not under the global gauge group. Using the Coulomb dressed fields allows us to remove precisely the Redundant part of the gauge group, as is the aim of the DFM.

RQ1.2: *Why should the global gauge group not be considered merely as a subgroup of the local gauge group, but rather as having a different physical significance?*

We studied the global gauge group from two main perspectives: in classical field theory in chapter 4 and in AQFT in chapter 5. In chapter 4 we explained how the notion of *gauge redundancy* can be formalised via constrained Hamiltonian analysis. The symplectic form on (infinite-dimensional) phase space is non-degenerate by definition, but when it is pulled back to the constraint surface this need no longer be the case. The symplectic form on the constraint surface may have null directions, which are identified as gauge orbits in Teh's Redundant sense. These gauge orbits are in fact generated by the first class constraints. For Yang-Mills theory on a Cauchy surface, the only first class constraint is the Gauss law. However, the Gauss law only generates gauge transformations which asymptotically approach the identity. Thus, it does not generate global gauge transformations, which act non-trivially even at infinity. This fact is at the heart of the argument that global gauge symmetries should not be considered as a mere subgroup of the total gauge group, which also includes local gauge symmetries (meaning those that become the identity at infinity). It can also be understood more formally through the notion of symplectic reduction, which requires a free group action. The global gauge group does not act freely on the space of Yang-Mills fields, for it leaves the connection 1-forms with values in the center of the Lie algebra of the structure group invariant. For an Abelian structure group this is just the entire space of connection 1-forms. Thus, one cannot perform a symplectic reduction of the whole gauge group, but only of the group of local gauge transformations, which do act freely. If one instead considers the bigger field space of Yang-Mills connection 1-forms together with a scalar matter field, then the global gauge group still does not act freely, for the scalar field may vanish. This possibility was of course precisely precluded in the DFM analysis of Berghofer et al.

However, the global gauge group does not necessarily exhaust all physical gauge transformations. Indeed, we worked out very precisely in section 4.3.3 what gauge transformations are physical and unphysical. This depends on the boundary conditions imposed on the fields. We required all fields to fall off asymptotically at a rate of order $\mathcal{O}(r^{-2})$. The group of gauge transformations preserving these boundary conditions was then shown to consist of those smooth maps $g\colon M \to G$ which satisfy $g \to \text{const} + \mathcal{O}(r^{-1})$ as $r \to \infty$. We denoted this group by $\mathcal{G}^I$ and wrote $\mathcal{G}_0^\infty$ for the group of small transformations $\{e^{\xi(x)} : \xi \to 0 + \mathcal{O}(r^{-1})\}$ generated by the Gauss constraint and going to the identity asymptotically. The quotient $\mathcal{G}^I/\mathcal{G}_0^\infty$ is then the group of physical gauge transformations. This quotient includes the global transformations, but is not limited to them in the non-Abelian case, for then there are topological contributions through so-called winding numbers. Such a topologically non-trivial symmetry breaking pattern in the universe is known as a *cosmic string* [119]. It is important to note that, in order to even be able to speak of the identity at infinity, i.e. to be able to define $\mathcal{G}^I$, we must have a section/trivialisation/frame at infinity. Of course, in defining the gauge group $\mathcal{G}$ as the set of maps $M \to G$, we already assume we have a global section, such that the section at infinity is automatically taken care of. If this were not the case, however, and we were working on an untrivialised bundle, then we would need to choose a trivialisation at infinity. This is in fact similar to the situation in gravity, where the diffeomorphisms that are Redundant fall off sufficiently rapidly towards infinity, while diffeomorphisms that correspond to boundary symmetries, and therefore potentially are non-Redundant, induce non-trivial diffeomorphisms at infinity. In that case, a choice of coordinates is used to define the metric at infinity as well as the diffeomorphisms preserving it [120, 121].

In AQFT the role of the global gauge group relates to superselection sectors, i.e. unitary equivalence classes of primary or irreducible representations of the net of operator algebras, as formalised by DHR. However, DHR theory makes no reference to the local gauge group and therefore cannot help us very much in answering RQ1.1. It only provides yet more support for the idea that global gauge symmetries should be considered to have direct empirical significance, not only because they can figure in Galileo's ship scenarios in classical field theory, but also because they are intimately connected to the superselection structure of quantum field theories and provide the quantised gauge charges that label such superselection sectors. Superselection rules are observable in nature from the fact that no coherent superpositions of states carrying different superselected quantum numbers are possible. This seems to be a more direct form of empirical significance than the existence of conserved charges through Noether's theorem, which philosophers usually view as indirect empirical significance.

RQ1.3: *To what extent can results on the Abelian Higgs mechanism be used to interpret the complete non-Abelian Higgs mechanism in the Standard Model?*

We have presented a wide variety of results on the Higgs mechanism in this thesis - some very generally for any compact matrix Lie group G (much of section 2.3, as well as section 5.3), many only for $G = U(1)$ (the very important section 4.2 and almost all of chapter 6), and some for the electroweak structure group $G = SU(2) \times U(1)$ specifically (mostly some examples from chapter 2 and the DFM in section 3.2.2). It is important, therefore, to ask whether the picture we are outlining of global gauge symmetry breaking works generally or only for the Abelian Higgs model. For this purpose we described the full Higgs mechanism in the Standard Model at length in chapter 2, by way of comparison class for any formulation other than the orthodox one.

The first important point to make in this regard is that our core argument on the direct empirical significance of gauge symmetries, developed in chapter 4, holds in the Abelian as well as the non-Abelian case. In section 4.3.3, we studied the Gauss constraint as the momentum map for the action of the local gauge group generally. The result that the quotient $\mathcal{G}^I/\mathcal{G}_0^\infty$ is the group of physical gauge symmetries derived there holds in both the Abelian and non-Abelian cases. The difference, however, lies in the question of what this quotient concretely consists of. For $G = U(1)$, it is isomorphic to the group of global gauge transformations, as proved in Proposition 4.4. For $G = SU(2)$ it is isomorphic to $SU(2) \times \mathbb{Z}$, i.e. the global gauge transformations times all the possible winding numbers of maps $S^3 \to S^3$. Thus, in the non-Abelian case, one must consider the possibility that the vacuum configuration of the Higgs field breaks the physical gauge symmetry $\mathcal{G}^I/\mathcal{G}_0^\infty$ in a topologically non-trivial way. Still, this does not change anything about the basic ideas developed in chapter 4.

The difficulty with the non-Abelian case, then, comes from the fact that it is not straightforward to implement a non-Abelian generalisation of the radiative projection and the Coulomb gauge, in order to obtain fields which are invariant under the unphysical gauge group $\mathcal{G}_0^\infty$ but not under the physical gauge group $\mathcal{G}^I/\mathcal{G}_0^\infty$. Indeed, as remarked in section 4.3.4, in the non-Abelian case the action of the gauge group on the space of Yang-Mills fields is *field-dependent*, in the sense that, besides the uniform term $g^{-1}dg$, a transformation $g \in \mathcal{G}^I$ also changes a connection 1-form A by a term $g^{-1}[A, g]$, which evidently depends on A. However, the Coulomb gauge can in fact be generalised [122], so an important and interesting direction for future research would be to use this to give the non-Abelian version of Struyve's gauge-invariant account of the Higgs mechanism as in section 4.2, building on the work that Lusanna and Valtancoli have already done [53–55].

In quantum field theory, the generality of our results depends on the axiomatic approach pursued. In AQFT, the theorems from section 5.3 are completely general and obtain for any compact global gauge group. Thus, also non-Abelian gauge charges label superselection sectors. Much of our work based on the Wightman axioms in chapter 6, however, does not readily generalise to the non-Abelian case. Indeed, we are not aware of any non-Abelian generalisation of our main result, Theorem 6.13. It may be very difficult to obtain something similar in the non-Abelian case, since the rigourous non-perturbative quantisation of gauge theory, and especially non-Abelian gauge theory, is poorly understood. On the other hand, the approach of local gauge quantisation by allowing for a Hilbert space inner product which is not positive-definite did generalise to the non-Abelian setting, as explained in section 6.6. The theorem by De Palma and Strocchi presented there is a non-Abelian version in the BRST gauge of Proposition 6.12, showing that the Goldstone modes associated to the breaking of global gauge symmetries are unphysical. Although this result is nothing like a generalisation of Theorem 6.13, it does strongly suggest that global gauge symmetry breaking in the electroweak theory is completely analogous to the Abelian Higgs model.

With these answers to our three subquestions in mind, we return to the main research question of this thesis.

RQ1: *What role does global gauge symmetry breaking play in the Higgs mechanism?*

Global gauge symmetry breaking is the physical means by which the Higgs mechanism can be understood to generate the masses of gauge bosons and leptons. Global gauge symmetry breaking does not suffer from the conceptual problems of local gauge symmetry breaking since global gauge symmetries are physical, as they are not generated by the Gauss constraint. Global gauge symmetry breaking also avoids Elitzur's theorem. Explicit derivations of mass generation by means of global gauge symmetry breaking have only been given for the Abelian case, both in classical and quantum field theory. In QED, the existence of a photon mass in case of SSB of the global $U(1)$ symmetry follows from the failure of the relation between the electric charge and the smeared Noether current density. When the global $U(1)$ symmetry is unbroken, the electric charge is superselected and can be expressed in terms of the smeared current density. In the case of broken global $U(1)$ symmetry, however, the current charge is screened (see also [123]), making it impossible for this charge to generate global gauge transformations. We can think of this intuitively as follows: in the case of current charge screening, the current charge distribution (which is defined in terms of the electromagnetic field) falls of quickly, meaning that it cannot be detected asymptotically, so that it cannot satisfy the Gauss law (which allows one to calculate the charge distribution from the electric field arbitrarily far away). This asymptotic falling-off implies that the electromagnetic force is short-ranged, i.e. that the photon is massive.

7.2. Superconductivity and dynamical symmetry breaking

Yet, even if global gauge symmetry breaking is physical, a major question that remains is:

How can global gauge symmetry breaking in the Higgs mechanism be understood as a dynamical process?

Indeed, throughout this thesis we have not said much about the Higgs mechanism as a temporal process that allegedly happened shortly after the Big Bang, when the temperature of the universe had droppped sufficiently. This also leads us back to the analogy with superconductivity with which we started chapter 1. Superconductivity is observed experimentally as a dynamical process whenever a superconductor is cooled below its critical temperature. Let us first examine this analogy and then discuss proposals for dynamical symmetry breaking and their potential for the Higgs mechanism.

Fraser has argued that the analogy between superconductivity and the Higgs mechanism, of which we sketched the historical development in section 1.1, is purely *formal*, i.e. uses the same equations, but does not point to a common underlying causal or physical mechanism [9]. According to Fraser, the BCS model gives the microscopic, causal explanation for the phase transition modelled phenomenologically by the GL theory: below the critical temperature electrons condense into Cooper pairs. Since, as of yet, there is no such causal explanation of the Higgs mechanism, the analogy between the GL theory of superconductivity and the Higgs mechanism can only be formal. Indeed, Fraser points out that in the usual presentations of the Higgs mechanism, the parameters of the model do not depend on time, either explicitly or implicitly via temperature-dependence. She summarises the situation as follows: "SSB in superconductors is a temporal process. Both the BCS and GL models offer descriptions of temporal processes that include phase transitions during which symmetry is spontaneously broken. The BCS model moreover offers a description of a causal process during which symmetry is spontaneously broken. A causal process is a temporal process satisfying some additional requirements. Clearly, since SSB in the Higgs model is not a temporal process, it is not a causal process either" [9, p. 83]. Thus, by

Fraser's account, superconductivity cannot be used to find a physical interpretation of the notion that particles "gain mass" in the Higgs mechanism.

What is peculiar about Fraser's study, however, is that she considers thermal QFT to be a theory "beyond the Standard Model" and compares superconductivity only to the Abelian Higgs model as we have studied it in this thesis, i.e. without any consideration for temperature-dependence. If one does this, it should not come as a surprise that the causal structure of the GL theory does not map onto the Higgs mechanism, since one studies the Higgs mechanism at zero temperature from the beginning. It would be a better analogy to compare the GL theory to the Higgs mechanism in QFT at finite temperature.

But what do the results from this thesis imply for the analogy between superconductivity and the Higgs mechanism? Do they show that the analogy is more than formal? Do they provide the causal mechanism that Fraser is missing? I submit that our results do indeed suggest that the analogy is physical and not merely formal. After all, superconductivity is itself an instance of global gauge symmetry breaking *and not of local gauge symmetry breaking* [124]. If a superconductor were to break local gauge symmetry it would also contradict Elitzur's theorem. The global gauge symmetry that is broken in a superconductor is the global $U(1)$ phase symmetry of the GL wave function for a superconducting island, i.e. the phase of the perfectly ordered BCS state [124, p. 77]. However, van Wezel stresses that this global phase is unmeasureable and gauge-dependent and can only be measured relatively. This leads to the famous Josephson effect that we mentioned in section 1.1.1, in which a current flows between two superconductors with a relative global GL phase difference. But how should we understand van Wezel's statement in the light of our results from chapter 4, which show that global gauge symmetries are physical? Here it is important to recall Galileo's ship from section 4.3.1. Even if a symmetry transformation is physical, we cannot observe its effect from within the system it is applied to. If global gauge transformations are applied to a subsystem like a superconductor, then we can only observe the difference from outside that superconductor. If we apply a global gauge transformation to the entire universe, then observing its effect is impossible, but that does not preclude the symmetry being physical in the sense of Galileo's ship (where the boost is also only observable when applied to a subsystem, namely the ship).

These considerations strengthen the analogy between superconductivity and the Abelian Higgs mechanism, since both should be understood as SSB of global $U(1)$ symmetry, which is physical in the sense of Galileo's ship, thereby making the analogy physical - though not yet causal. Indeed, even if we understand the Higgs mechanism as breaking of global gauge symmetry like in the superconductor, we still have not explained how it should be viewed as a dynamical process unfolding in time. It is important to mention, however, that this problem exists generally for all instances of SSB in quantum systems. We may call this the *general problem of spontaneous symmetry breaking*. It amounts to the question: is SSB really *spontaneous?* Does it occur by itself, or must symmetry breaking always be explicit? How can a quantum system with symmetry ever evolve from a symmetric state to an asymmetric state?

These questions remain unanswered for all instances of SSB in quantum systems; crystals, the superconductor, the Higgs mechanism etc. Thus, what we have really achieved in this thesis is to reduce the discussion of the Higgs mechanism to the discussion of SSB *in general*, by showing that global gauge symmetries are physical just like e.g. rotations in a ferromagnet. It seems likely therefore that an understanding of the Higgs mechanism as a causal process would go hand in hand with a understanding of dynamical symmetry breaking in general. We stress that this problem of dynamical symmetry breaking has not been solved for superconductors either, even though we know a superconducting phase transition to unfold in time because we observe it in the laboratory. The profound question of how precisely a quantum system exhibiting a symmetry could possible move from a symmetric to an asymmetric state remains open.

Indeed, the unitary, symmetric dynamics of a quantum system simply do not allow for such a process. Thus, there seem to be two main ways in which we could conceive of symmetry breaking in quantum systems: by arguing that it must always be explicit, i.e. occur by means of an external perturbation, or by allowing for non-symmetric evolution of the system itself. The latter option immediately reminds us of the notion of *measurement*, in which a wave function supposedly collapses non-unitarily.[1] In such a collapse, a system could very well move from a symmetric to an asymmetric state. One might even conjecture, then, that SSB and the measurement problem really are two sides of the same coin. This conjecture is pursed by van Wezel and others [125–128]. In fact, van Wezel submits that dynamical symmetry breaking cannot occur by means of unitary Schrödinger evolution alone, even in the presence of infinitesimal symmetry breaking perturbations [127]. Thus, the very presence of phase transitions in the world around us would imply that unitary quantum dynamics must itself break spontaneously:

> Infinitesimal preferences for any particular ordered state cannot induce a spontaneous dynamical breakdown of symmetry, because even in the thermodynamic limit the unitarity of time evolution requires all states with distinct order parameter orientations to appear as components with at most infinitesimal differences in weight in the final state wave function. The experimental observation that individual phase transitions are realised in our everyday world, are traversed as a function of time, and do result in a single symmetry-broken state each time they are encountered, directly suggests the existence of non-unitary time evolution. [127, p. 11]

Landsman, van de Ven and others have also pursued the similarity between measurement and SSB, but by focusing on symmetry breaking perturbations and not by assuming unitarity violation [129–131]. They call this idea the *flea on Schrödinger's cat* [129]. Landsman stresses the importance of recognising that SSB can only occur in an infinite quantum system, whereas the quantum systems we see around us are finite (e.g. finite crystals). Thus, SSB must be understood as an instance of *asymptotic emergence* [132]. SSB may formally only exist in the limit, but one must show that the effects of this asymptotically emergent behaviour can already be observed *before* the limit. For the Higgs mechanism this would be the limit of going from finite to infinite volume [129, p. 20].

Now, the point of mentioning these general approaches to dynamical symmetry breaking is to show that, by understanding the Higgs mechanism as global gauge symmetry breaking, it loses the puzzling edge of *gauge* symmetry breaking. Since global gauge symmetry is physical and the Higgs mechanism can be formulated that way, the problem of interpreting the mechanism as a dynamical process amounts to the same problem as for the vastly more general class of all symmetry breaking quantum systems. By a very long detour, we thus return to Wüthrich's statement from section 1.2.3:

> None of Lyre's worries, therefore, gives us reason to doubt that the Higgs mechanism can have the same ontological status as any other mechanism of spontaneous symmetry breaking, which we observe, for example, in ferromagnets or superconductors. [29, p. 10]

The only conceptual difference between, say, a crystal and the Higgs mechanism, is the fact that we usually think of the Higgs mechanism as applying to the entire universe. Thus, it is not as clear where a symmetry breaking perturbation could come from, since there seems to be no real environment. One might speculate that such a perturbation could come from a ubiquitous stochastic field, other quantum fields, or even gravity. It is a highly involved issue, but at least we know now that there is no conceptual difference in principle between the Higgs mechanism and other symmetry breaking quantum phenomena.

7.3. Suggestions for further research

From what has been said in this chapter it is clear that there are two broad directions for future research on conceptual and philosophical aspects of the Higgs mechanism: its formulation as global gauge symmetry breaking in the non-Abelian case, and the Higgs mechanism as a dynamical process.

[1] Measurement, however, is usually taken to be induced by something external to the quantum system that is being measured. Thus, there is an external stimulus to the measurement process, meaning that the effected collapse it not truly spontaneous.

In the first direction, the two obvious things to be done are to generalise the gauge-invariant account of the Abelian Higgs mechanism from section 4.2 as well as Theorem 6.13 to the electroweak theory. The work by Lusanna and Valtancoli [53–55] could be used for the former, and the non-Abelian generalisation of the Coulomb gauge [122] may be useful for the latter. It might be simplest to first focus on global gauge symmetries only and then take into account the topological contributions arising from the fact that the physical gauge group $\mathcal{G}^I/\mathcal{G}_0^\infty$ for $G = SU(2)$ has a countably infinite number of connected components.

In the second direction, it seems the first logical step to pursue would be to formulate the Higgs mechanism at finite temperature at all, by considering global gauge symmetry breaking KMS states at a temperature of around the Higgs field VEV, rather than ground states, and compare to results from the lattice [133]. One could then consider adding an infinitesimal global gauge symmetry breaking perturbation to try to rederive something like the Higgs field "rolling down the side of the scalar potential", but understood globally and not locally.

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Popular Summary

What is mass? At first sight, this looks like a fairly trivial question. The mass of an object signifies how heavy it is, right? Yet, it turns out that mass is an incredibly complicated concept it the physical theories that describe the microscopic world. In fact, one cannot even straightforwardly define masses of fundamental particles. One need a special tool to do so, called the *Higgs mechanism*.

This mechanism was proposed by six physicists in 1964, among whom Peter Higgs himself. It predicted the existence of a new particle: the *Higgs boson*. This particle was finally detected in 2012 at CERN, after almost half a century of intensive searching and building bigger and bigger particle accelerators. It was the Large Hadron Collider (LHC) that was able to do the job.

With the Higgs mechanism and the accompanying particle in place, the so-called Standard Model of particle physics was finally complete. The Standard Model describes the smallest particles that physicists know of: electrons, photons, quarks, etc. It contains two kinds of particles: leptons and bosons, and describes the interactions between these particles by means of three fundamental forces: the electromagnetic, weak and strong interactions. Each of these forces is carried by a (number of) boson(s). Such force-carrying bosons are known as *gauge bosons*. We will shortly explain why. The electromagnetic interaction is carried by the photon, i.e. the light particle. The weak interaction is carried by the three so-called W^+, W^- and Z bosons. Finally, the strong force is carried by *gluons*. We note that gravity is not described by the Standard Model. Indeed, finding a quantum theory of gravity is the greatest open problem in theoretical physics.

The weak interaction is responsible for radioactive decay. It is very weak, hence the name. However, it is not only weak, but also *short-ranged*, meaning that it acts only on very short distances. In this respect it differs from electromagnetism: we have all experienced the electromagnetic force in daily life when playing with magnets or touching electrostatically charged objects. The weak force cannot be easily observed in this way, because it is noticeable only on the atomic scale. The explanation for this difference is that the weak gauge bosons (i.e. the $W^\pm$ and Z) are *massive*, whereas the photon is *massless*. Before 1964 it was already known that the weak gauge bosons must be massive. The problem, however, was the fact that the theory decribing gauge bosons, called *Yang-Mills theory*, could only describe massless bosons like the photon. Thus, physicists needed to find a way to describe massive gauge bosons. This is what the Higgs mechanism allowed them to do.

To understand how the Higgs mechanism accomplishes the desired description of massive gauge bosons, we need to know why it was impossible to do this directly in Yang-Mills theory in the first place. Here we come back to that word *gauge* mentioned before. The force-carrying bosons are called gauge bosons because the fields describing them are *gauge fields*. A gauge field, in turn, is called that because it exhibits a very special kind of symmetry: a *gauge symmetry*. A gauge symmetry is a type of symmetry that is a bit like having a set of rules for changing one's perspective on something without actually changing the thing itself. Imagine one has a map of a city. One can look at the map from different angles, zoom in or out, or even use different colors to highlight various features. No matter how one looks at the map, the streets and buildings remain the same. The rules for changing one's perspective on the map while keeping the map itself unchanged are similar to what we mean by gauge symmetry in physics. In physics, gauge symmetry is about the idea that certain fundamental forces, like electromagnetism or the forces inside atomic nuclei, do not change even if one changes one's point of view on them in a specific mathematical way. These changes in perspective are called "gauge transformations." They are like the different ways one can look at a map without altering the actual layout of the city. The electromagnetic, weak and strong forces are described by gauge fields, and the corresponding gauge symmetries ensure that the physical properties of these fields stay consistent, no matter how we apply gauge transformations.

We can now understand the fundamental idea of the Higgs mechanism: *gauge symmetry breaking*. In the Higgs mechanism, the original gauge symmetry of the Standard Model is *broken*, meaning that it is reduced, such that the fields in the Standard Model no longer exhibit the gauge symmetry they used to have. The field that performs this gauge symmetry breaking is the Higgs field. Because the Higgs field is *coupled* to the force fields, the symmetry breaking by the Higgs field gives rise to mass terms for the gauge bosons whose gauge symmetry is broken. Now, a gauge symmetry is mathematically described by an object known as a *Lie group*. Such a Lie group has a certain dimension, and the Lie group describing the electroweak sector of the Standard Model, i.e. the electromagnetic and weak interactions, is four-dimensional. In the Higgs mechanism, this four-dimensional symmetry group is reduced to a one-dimensional symmetry group. The three broken dimensions then give rise to the three massive weak bosons, whereas the one *unbroken* dimension that is left corresponds to the massless photon.

However, philosophers of physics have taken issue with this standard account of the Higgs mechanism. They have pointed out that gauge symmetries are mere "mathematical redundancies" or even "descriptive fluff". It is our choice how to look at the map of the city. A gauge transformation is like a transformation of *coordinates*: it changes how we describe something, but not what something really is. Thus, the Higgs mechanism cannot be a real, physical process that generates mass for fundamental particles. Something physical like mass cannot be gained by breaking or removing a mere mathematical, descriptive convention. At least, this is what philosophers have come to believe.

The answer to this philosophical challenge of the Higgs mechanism that we work out in this thesis is that most, but not all gauge symmetries represent mathematical redundancies. There is a subset of gauge symmetries called *global* gauge symmetries that are physical, and the Higgs mechanism can be understood as a breaking of this global gauge symmetry only. Global gauge symmetries are contrasted with *local* gauge symmetries, which are indeed mere "descriptive fluff."

To understand why global gauge symmetries differ from local gauge symmetries, it is important to note that, in physics, we assume fields like the electromagnetic field to "fall off" as we move very far away. This is because these fields carry energy, and if we do not assume the fields to vanish as we move further and further away, then their total energy becomes *infinite*, which leads to great problems. Thus, we assume fields to become weaker and weaker asymptotically, until finally they become zero "at infinity." Now, roughly speaking, local gauge symmetries respect this condition on the fields, because they only act on a small region of space. Global gauge symmetries, however, act *everywhere*, including at infinity. For this reason, they are different from local gauge symmetries and have an empirical significance that local gauge symmetries lack.

The point of this thesis, then, is to reformulate the Higgs mechanism as global gauge symmetry breaking instead of local gauge symmetry breaking, and to explain how precisely global gauge symmetries differ from local gauge symmetries. The intended reformulation is not so easy, because we have to first remove the local gauge symmetry, while keeping the global gauge symmetry. Luckily, this can indeed be done, at least in simple cases. The question of how to do it in more complicated cases, however, remains open. Maybe a future scientist reading this can solve it?

A. Functional Analysis and Operator Algebras

In this appendix we provide a brief introduction to the functional-analytic and operator-algebraic concepts used in the algebraic approach to quantum theory. We start with the basics of the theory of operators on Hilbert spaces and then we generalise these notions to abstract algebras of operators. We end with the GNS construction, which is central to the algebraic definition of spontaneous symmetry breaking. We mostly follow [58, 102, 109, 134] and do not provide detailed proofs, though we will refer the reader to the literature.

A.1. Operators on Hilbert spaces

The basic functional-analytic objects in quantum theory are Hilbert spaces. Quantum states are usually modelled as unit vectors in a Hilbert space and observables are the self-adjoint operators on that Hilbert space. In our treatment, Hilbert spaces are always defined over the field of the complex numbers $\mathbb{C}$. We recall that a metric space M is called *complete* if every Cauchy sequence in M converges to a point in M, and that an inner product $\langle \cdot, \cdot \rangle$ on a vector space gives a norm $\|x\| = \sqrt{|\langle x, x \rangle|}$. Using this we define:

Definition A.1. A *Hilbert space* is a complex vector space H together with an inner product $\langle \cdot, \cdot \rangle$ such that, relative to the metric $d(x, y) = \|x - y\|$ induced by the norm, H is a complete metric space.

Hilbert spaces are special cases of the following:

Definition A.2. A *Banach* space is a normed vector space which is complete which respect to the topology induced by this norm.

A great advantage of Hilbert spaces is the possibility of defining orthogonality [134, p. 7].

Definition A.3. If H is a Hilbert space and $f, g \in H$, then f and g are said to be *orthogonal* if $\langle f, g \rangle = 0$. This is also written $f \perp g$. Two subsets $A, B \subset H$ are said to be orthogonal if for all $f \in A, g \in B : f \perp g$.

For any $A \subset H$ we denote the *orthogonal complement* $A^\perp = \{f \in H | \text{for all } g \in A : f \perp g\}$, and $A^\perp$ is always a closed linear subspace of H [134, p. 10]. Clearly $H^\perp = 0$ and $0^\perp = H$. We now wish to define orthogonal projections, for which we use the following theorem [134, p. 10].

Theorem A.4. If M is a closed linear subspace of the Hilbert space H and $h \in H$, let Ph be the unique point in M such that $h - Ph \perp M$. Then
(a) P is a linear transformation on H,
(b) $\|Ph\| \leq \|h\|$ for all $h \in H$,
(c) $P^2 = P \circ P = P$,
(d) $\ker P - M^\perp$ and $\operatorname{ran} P = M$.

This allows us to define:

Definition A.5. If M is a closed linear subspace of the Hilbert space H and P_M is the linear map defined by theorem A.4, then P_M is called the *orthogonal projection* of H onto M.

We note the following corollary [134, p. 10].

Corollary A.6. If $A \subset H$ where H is a Hilbert space, then $(A^\perp)^\perp$ is the closed linear span of A. Thus, if A is a closed linear subspace, then $(A^\perp)^\perp = A$.

Next we will consider linear functionals on Hilbert spaces, i.e. linear maps from a Hilbert space to the complex numbers. It turns out that these are continuous if and only if they are bounded, as the following theorem shows [134, p. 11].

Theorem A.7. Let H be a Hilbert space and $L\colon H \to \mathbb{C}$ a linear functional. Then the following are equivalent:
(a) L is continuous,
(b) L is continuous at 0,
(c) L is continuous at some point,
(d) There is a constant $c > 0$ such that for all $h \in H : |L(h)| \leq c\,\|h\|$.

A similar result holds for operators, i.e. linear maps from one Hilbert space to another [134, p. 26].

Theorem A.8. Let H_1 and H_2 be Hilbert spaces and $A\colon H_1 \to H_2$ a linear transformation. Then the following are equivalent:
(a) A is continuous,
(b) A is continuous at 0,
(c) A is continuous at some point,
(d) There is a constant $c > 0$ such that $\|Ah\| \leq c\,\|h\|$ for all $h \in H_1$.

If one of these conditions holds, we call an operator $A\colon H_1 \to H_2$ *bounded* and define its norm to be

$$\|A\| = \sup\{\|Ah\| : h \in H_1, \|h\| \leq 1\}.$$

The set of all bounded operators $A\colon H \to H$ on a Hilbert space H is denoted by $B(H)$. It can be made into a topological space, but there are different ways to do so. The easiest way is to equip $B(H)$ with the *norm topology* induced by the norm $\|\cdot\|$. Two other very important topologies are:

Definition A.9 (WOT and SOT)**.** If H is a Hilbert space, the *weak operator topology* (WOT) on $B(H)$ is the topology defined by the seminorms $\{p_{h,k} : h, k \in H\}$ where $p_{h,k}(A) = |\langle Ah, k\rangle|$. The *strong operator topology* (SOT) is the topology defined by the family of seminorms $\{p_h : h \in H\}$ where $p_h(A) = \|Ah\|$.

Convergence in these topologies then amounts to the following.

Proposition A.10. Let H be a Hilbert space and let $\{A_i\}$ be a net in $B(H)$.
(a) $A_i \to A$ (WOT) if and only if $\langle A_i h, k\rangle \to \langle Ah, k\rangle$ for all $h, k \in H$.
(b) $A_i \to A$ (SOT) if and only if $\|A_i h - Ah\| \to 0$ for all $h \in H$.

A.2. Operator algebras

Now that we have introduced Hilbert spaces and bounded operators on them, we consider more general algebras of operators. The quintessential such operator algebra is $B(H)$, where H is a Hilbert space. In fact, we will see that $B(H)$ is a so-called C^*-algebra and that every C^*-algebra can be faithfully represented as a subalgebra of $B(H)$ for some Hilbert space H. We will always take algebras to be defined over the field of complex numbers. We start with the notions of involution in an algebra and of morphisms of such involutive algebras.

Definition A.11. An involutive algebra or $*$-algebra $\mathcal{A}$ is an algebra (over $\mathbb{C}$) together with a map $*\colon \mathcal{A} \to \mathcal{A}$ that satisfies, for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$:
1. $(a + b)^* = a^* + b^*, (ab)^* = b^* a^*$,
2. $(\lambda a)^* = \bar{\lambda} a^*$,
3. $(a^*)^* = a$.

Definition A.12. A $*$-morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ between $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is an algebra morphism compatible with involution, i.e. for all $a, b \in \mathcal{A}, \lambda \in \mathbb{C}$:
1. $\varphi(ab) = \varphi(a)\varphi(b)$,
2. $\varphi(\lambda a + b) = \lambda\varphi(a) + \varphi(b)$,
3. $\varphi(a^*) = \varphi(a)^*$.

We now wish to equip algebras with a norm that works well with the algebra multiplication.

Definition A.13. A *normed algebra* $\mathcal{A}$ is an algebra that is also a normed vector space such that

$$\|ab\| \leq \|a\| \|b\|, \qquad a, b \in \mathcal{A}.$$

If $\mathcal{A}$ is unital, then it is a *unital normed algebra* if in addition $\|1_{\mathcal{A}}\| = 1$.

We can require normed algebras to be complete with respect to their norm topology, giving:

Definition A.14. A (unital) *Banach algebra* $\mathcal{A}$ is a (unital) normed algebra that is also a Banach space (Definition A.2) with respect to the same norm.

Definition A.15. An *involutive Banach algebra* is a Banach algebra $\mathcal{A}$ with involution $*\colon \mathcal{A} \to \mathcal{A}$ such that for all $a \in \mathcal{A} : \|a^*\| = \|a\|$.

Finally we are in a position to define the algebras that are used to describe quantum systems.

Definition A.16. A C^**-algebra* $\mathcal{A}$ is an involutive Banach algebra satisfying the C^*-property

$$\|a^*a\| = \|a^*\| \|a\| = \|a\|^2, \qquad a \in \mathcal{A}.$$

Example A.17. The archetypal example of a C^*-algebra is $B(H)$, the algebra of bounded operators on a Hilbert space H with the operator norm, with involution given by the adjoint and the identity operator as its unit. In fact, we will shortly see that any C^*-algebra can be represented as a subalgebra of $B(H)$ for some Hilbert space H.

Now, rather than the norm topology, we could also consider the more permissive SOT or WOT on a C^*-algebra. This leads to:

Definition A.18. A *von Neumann algebra* is a $*$-subalgebra of $B(H)$ that is closed in the SOT.

Remark A.19. Since the SOT is weaker than the norm topology, a von Neumann algebra is also closed in the norm topology and hence a C^*-algebra.

Von Neumann algebras can actually also be characterised algebraically using the notion of a *commutant* S' of a set $S \subset B(H)$, i.e. all operators $S' = \{A \in B(H) : AB = BA \text{ for all } B \in S\}$ that commute with all of S. This gives a beautiful link between topology and algebra.

Theorem A.20 (Double commutant)**.** Let $\mathcal{A}$ be a $*$-algebra on a Hilbert space H with $1_H \in \mathcal{A}$. Then $\mathcal{A}$ is a von Neumann algebra on H if and only if $\mathcal{A} = \mathcal{A}''$.

The following theorem from [109] shows that it actually does not matter whether we consider the strong or weak closure for von Neumann algebras.

Theorem A.21. Suppose $\mathcal{A}$ is a $*$-algebra on a Hilbert space H containging 1_H. Then
(a) The weak closure of $\mathcal{A}$ is $\mathcal{A}''$,
(b) $\mathcal{A}$ is a von Neumann algebra if and only if it is weakly closed.

Having defined the algebras that model observables, let us introduce the notion of a state.

Definition A.22. A state $\omega\colon \mathcal{A} \to \mathbb{C}$ on a unital $*$-algebra $\mathcal{A}$ is a positive linear functional of unit norm, i.e. such that $\omega(1_{\mathcal{A}}) = 1$ and $\omega(a^* a) \geq 0$ for all $a \in \mathcal{A}$.

Remark A.23. If $\mathcal{A}$ is a unital $*$-algebra then the set of states on $\mathcal{A}$ is naturally a convex set. That is, if $\omega_1, \omega_2\colon \mathcal{A} \to \mathbb{C}$ are states, then for any $\lambda \in [0,1]$ the linear combination $\lambda\omega_1 + (1-\lambda)\omega_2$ is also a state. We denote the state space by $S(\mathcal{A})$, equipped with the weak* topology inherited from $S(A) \subset A^*$.

Definition A.24. A state $\omega\colon \mathcal{A} \to \mathbb{C}$ on a $*$-algebra $\mathcal{A}$ is called *pure* if $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, where ω_1 and ω_2 are states and with $\lambda \in (0,1)$, implies $\omega_1 = \omega_2$. In other words, pure states The pure states of $\mathcal{A}$ comprise the pure state space $P(\mathcal{A})$, which is the extreme boundary of $S(\mathcal{A})$ [58, p. 334]. A state that is not pure is called *mixed*.

A.3. GNS construction

We will now outline the GNS construction, which links representations of C$*$-algebras with states. For technical details we refer the reader to [109, 134]. We begin by defining representations of C$*$-algebras and then present the GNS construction itself.

Definition A.25. A representation of a C*-algebra $\mathcal{A}$ is a faithful $*$-homomorphism $\pi\colon \mathcal{A} \to B(H)$, where H is some Hilbert space.

Definition A.26 (Gelfand-Naimark-Segal construction)**.** To each positive linear functional $\tau\colon \mathcal{A} \to \mathbb{C}$ on a C*-algebra $\mathcal{A}$ we can associate a representation by first setting

$$N_\tau = \{a \in \mathcal{A} \mid \tau(a^* a) = 0\},$$

which is a closed left ideal of $\mathcal{A}$. The map $(\mathcal{A}/N_\tau)^2 \to \mathbb{C}$ given by

$$([a],[b]) \mapsto \tau(b^* a)$$

is then a well-defined inner product on $\mathcal{A}/N_\tau$. Let H_τ denote the Hilbert space completion of $\mathcal{A}/N_\tau$. Then for any $a \in \mathcal{A}$ we define an operator $\pi(a) \in B(\mathcal{A}/N_\tau)$ by

$$\pi(a)([b]) = [ab].$$

It can be checked that $\|\pi(a)\| \leq \|a\|$, such that $\pi(a)$ has a unique extension to a bounded operator $\pi_\tau(a)$ on H_τ. The map $\pi_\tau\colon \mathcal{A} \to B(H_\tau)$ given by $a \mapsto \pi_\tau(a)$ is a $*$-morphism. We thus obtain the *GNS representation* (H_τ, π_τ) associated to the positive linear functional τ. If $\mathcal{A}$ is non-zero, we define its *universal representation* to be the direct sum of all the GNS representations (H_τ, π_τ), where τ ranges over the state space $S(\mathcal{A})$ defined in Remark A.23.

The GNS construction and the ensuing universal representation allow us to prove the essential result that we can faithfully represent every C*-algebra as some operator algebra on a Hilbert space. To do so, we need the following theorem [109, p.90].

Theorem A.27. If a is a normal element of a non-zero C*-algebra $\mathcal{A}$, then there is a state $\omega \in S(\mathcal{A})$ such that $\|a\| = |\omega(a)|$.

Using this we can prove the desired result, following [109, p. 94].

Theorem A.28 (Gelfand-Naimark)**.** If $\mathcal{A}$ is a C*-algebra, then it has a faithful representation. Specifically, its universal representation is faithful.

Proof. Let (H, π) be the universal representation of $\mathcal{A}$ and suppose that $a \in \mathcal{A}$ such that $\pi(a) = 0$. By the theorem above there is a state $\omega \in S(\mathcal{A})$ such that $\omega(a^*a) = \|a^*a\|$ (since a^*a is normal). Denoting $b = (a^*a)^{\frac{1}{4}}$ we then have

$$\|a\|^2 = \|a^*a\| = \omega(a^*a) = \omega(b^4) = \|\pi_\omega(b)([b])\|^2 = 0,$$

by definition of the GNS construction and since $\pi_\omega(b^4) = \pi_\omega(a^*a) = 0$, which implies $\pi_\omega(b) = 0$ since π_ω is a $*$-morphism. But $\|a\|^2 = 0$ implies $a = 0$, so π is injective. □

Now, if we apply the GNS construction to a state, we know even more about the representation. We call a representation $\pi\colon \mathcal{A} \to B(H)$ *cyclic* if there is a vector $\Omega \in H$ such that $\overline{\pi(\mathcal{A})\Omega} = H$, where the bar denotes the closure. In words this means that each element in H is a limit of a sequence $\pi(a_n)\Omega$ with the $a_n \in \mathcal{A}$. In this case we call Ω a *cyclic vector* for π. It turns out that the GNS construction of a state is cyclic [109, p. 141].

Theorem A.29. Let A be a C^*-algebra and $\omega \in S(A)$. Then there is a unique vector $\Omega_\omega \in H_\tau$ such that

$$\omega(a) = \langle [a], \Omega_\omega \rangle, \qquad a \in \mathcal{A}.$$

Moreover, Ω_ω is a unit cyclic vector for (H_ω, π_ω) and for all $a \in \mathcal{A}$

$$\pi_\omega(a)\Omega_\omega = [a],$$

which thus gives $\omega(a) = \langle \pi_\omega(a)\Omega_\omega, \Omega_\omega \rangle$. We call Ω_ω the *canonical cyclic vector* for (H_ω, π_ω).

The GNS representation is the unique representation characterised this way. That is, if for a state $\omega\colon \mathcal{A} \to \mathbb{C}$ we have two representations π_1, π_2 on Hilbert spaces H_1, H_2 with unit cyclic vectors Ω_1, Ω_2 such that $\omega(a) = \langle \pi_1(a)\Omega_1, \Omega_1 \rangle = \langle \pi_2(a)\Omega_2, \Omega_2 \rangle$ for all $a \in \mathcal{A}$, then there is a unitary operator $U\colon H_1 \to H_2$ intertwining the representations and sending $\pi_1(a)\Omega_1 \mapsto \pi_2(a)\Omega_2$. Like in group theory, we have a notion of (ir)reducibility for representations of C^*-algebras.

Definition A.30. A representation $\pi\colon \mathcal{A} \to B(H)$ of a C^*-algebra $\mathcal{A}$ is called *irreducible* when H has no nontrivial closed subspaces stable under $\pi(\mathcal{A})$. That is, if $K \subset H$ is a closed subspace and $\pi(\mathcal{A})K \subset K$, then $K = 0$ or $K = H$.

There is an elegant characterisation of irreducibility of GNS representations of states [109, p. 144].

Theorem A.31. Let $\omega \in S(\mathcal{A})$ be a state on a C^*-algebra $\mathcal{A}$. Then (H_ω, π_ω) is irreducible if and only if ω is pure.

B. Spinors

To fully understand the Higgs mechanism in the Standard Model it does not suffice to consider only gauge fields and scalar fields, which are all bosonic. We must also be able to describe fermions, and mathematically we use *spinors* for this. Defining spinors requires some work though: they do not transform simply under the Lorentz group, but instead under a double covering of the Lorentz group called the spin group. We therefore first have to define the spin group and its spinor representation, then we need the notion of a *spin structure*, and only then can we consider spinor bundles. In addition, we need to understand how to couple spinor fields to gauge fields via the *twisted* spinor bundle and we must take into consideration *chirality*. In this appendix we introduce all these notions, building on the definitions and results from chapter 2. Understanding spinors is necessary for appreciating the second and third points about the Higgs mechanism listed in the preamble to that chapter.

B.1. Clifford algebras

Defining the spin group is most readily done by consdering *Clifford algebras*, which are unital associative algebras generated by a vector space over a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ carrying a symmetric bilinear form $Q\colon V \times V \to \mathbb{K}$. A Clifford algebra is the “freest” such algebra generated by V subject to the condition $v^2 = -Q(v) \cdot 1$ for all $v \in V$. This idea of being the “freest” is formalised through a *universal property*.

Definition B.1. Let V be a $\mathbb{K}$-vector space with symmetric bilinear form Q. A *Clifford algebra* over (V, Q) is a pair $(\mathrm{Cl}(V, Q), \psi)$ where $\mathrm{Cl}(V, Q)$ is an associative $\mathbb{K}$-algebra with unit 1 and $\psi\colon V \to \mathrm{Cl}(V, Q)$ is a linear map satisfying

$$\{\psi(v), \psi(w)\} = \psi(v)\psi(w) + \psi(w)\psi(v) = -2Q(v, w) \cdot 1, \qquad v, w \in V,$$

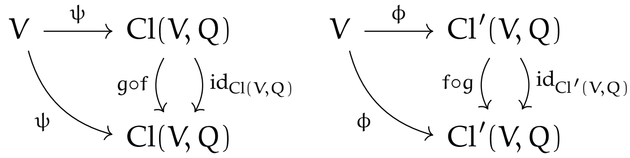
such that the following universal property is satisfied: if A is some other associative $\mathbb{K}$-algebra with unit 1 and $\phi\colon V \to A$ a $\mathbb{K}$-linear map such that for all $v, w \in V : \{\phi(v), \phi(w)\} = -2Q(v, w) \cdot 1$, then there exists a unique algebra homomorphism $f\colon \mathrm{Cl}(V, Q) \to A$ making the following diagram commute

$$\begin{array}{ccc} V & \xrightarrow{\ \psi\ } & \mathrm{Cl}(V, Q) \\ & \underset{\phi}{\searrow} & \downarrow f \\ & & A \end{array}$$

A Clifford algebra always exists. Indeed, it can be constructed by taking the tensor algebra $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ and quotienting by the two-sided ideal $I(Q)$ generated by the set $\{v \otimes v + Q(v, v) \cdot 1 \mid v \in V\}$. The product on $\mathrm{Cl}(V, Q) = T(V)/I(Q)$ is then defined as $[a] \cdot [b] = [a \otimes b]$ for any $a, b \in T(V)$. In addition, Clifford algebras are unique - so we can speak of *the* Clifford algebra - in the following sense.

Proposition B.2. Suppose $(\mathrm{Cl}(V, Q), \psi)$, $(\mathrm{Cl}'(V, Q), \psi')$ are both Clifford algebras over (V, Q). Then there exists an algebra isomorphism $f\colon \mathrm{Cl}(V, Q) \to \mathrm{Cl}'(V, Q)$ such that $f \circ \psi = \psi'$.

Proof. By the universal property of both Clifford algebras there exist two homomorphisms of algebras $f\colon \mathrm{Cl}(V, Q) \to \mathrm{Cl}'(V, Q)$ and $g\colon \mathrm{Cl}'(V, Q) \to \mathrm{Cl}(V, Q)$ making the following diagrams commute:



But by the universal property of Clifford algebras this implies $g \circ f = \mathrm{id}_{\mathrm{Cl}(V,Q)}$ and $f \circ g = \mathrm{id}_{\mathrm{Cl}'(V,Q)}$, so f is indeed an algebra isomorphism. □

We need the following to define the spin group.

Definition B.3. Let V be a vector space with symmetric bilinear form Q and let $T^0(V)$ and $T^1(V)$ denote the subspaces of $T(V)$ containing the elements of even and odd degree respectively. We set

$$\mathrm{Cl}^0(V,Q) = T^0(V)/(T^0(V) \cap I(Q)),$$
$$\mathrm{Cl}^1(V,Q) = T^1(V)/(T^1(V) \cap I(Q)).$$

We call $\mathrm{Cl}^0(V,Q)$ and $\mathrm{Cl}^1(V,Q)$ the *even* and *odd* parts of the Clifford algebra. In particular, $\mathrm{Cl}^0(V,Q)$ is spanned by products of an even number of vectors. It is not hard to see that

$$\mathrm{Cl}(V,Q) = \mathrm{Cl}^0(V,Q) \oplus \mathrm{Cl}^1(V,Q).$$

This gives $\mathrm{Cl}(V,Q)$ a $\mathbb{Z}_2$-grading.

Example B.4. The two most importantant examples of Clifford algebras are those of the standard real and complex vector spaces $(R^{s,t},\eta)$ and $(\mathbb{C}^d, q)$. Here $\mathbb{R}^{s,t}$ is the vector space $\mathbb{R}^{s+t}$ with standard basis $e_1, ..., e_{s+t}$ and η defined by

$$\begin{aligned} \eta(e_i,e_i) &= 1, && 1 \le i \le s, \\ \eta(e_i,e_i) &= -1, && s+1 \le i \le s+t, \\ \eta(e_i,e_j) &= 0, && i \neq j. \end{aligned}$$

Clearly $\mathbb{R}^{1,t}$ and $\mathbb{R}^{s,1}$ are Minkowski spacetime. We denote the Clifford algebra over $(R^{s,t},\eta)$ by $\mathrm{Cl}(s,t)$. The standard non-degenerate symmetric bilinear form q on $\mathbb{C}^d$ on the standard basis $e_1, ..., e_d$ is

$$\begin{aligned} q(e_i,e_i) &= 1 && 1 \le i \le d, \\ q(e_i,e_j) &= 0 && i \neq j. \end{aligned}$$

We denote the corresponding Clifford algebra by $\mathbb{C}\mathrm{l}(d)$. These two Clifford algebras are related by (cf. Lemma 6.3.2 in [59])

$$\mathbb{C}\mathrm{l}(s+t) \cong \mathrm{Cl}(s,t) \otimes_{\mathbb{R}} \mathbb{C}, \tag{B.1}$$

and the complex representations of $\mathrm{Cl}(s,t)$ are equivalent to complex representations of $\mathbb{C}\mathrm{l}(s+t)$. Moreover, for $n \geq 1$ the complex Clifford algebra satisfies $\mathbb{C}\mathrm{l}(n-1) \cong \mathbb{C}\mathrm{l}^0(n)$, as we can send $\psi(e_i) \mapsto \psi(e_i)\psi(e_n)$ for any $1 \le i \le n-1$ (cf. Lemma 6.3.3 in [59]).

Now, a great deal can be said about the structure of both the real and complex Clifford algebras. For the complex case there is the following particularly simple result (cf. Theorem 6.2.23 in [59]).

Theorem B.5. The complex Clifford algebra and its even part are isomorphic to certain complex algebras as follows. For n even we have

$$\mathbb{C}\mathrm{l}(n) \cong \mathrm{End}(\mathbb{C}^N),$$
$$\mathbb{C}\mathrm{l}(n)^0 \cong \mathrm{End}(\mathbb{C}^{N/2}) \oplus \mathrm{End}(\mathbb{C}^{N/2}),$$

where $N = 2^{n/2}$. For n odd we have

$$\mathbb{C}l(n) \cong \mathrm{End}(\mathbb{C}^N) \oplus \mathrm{End}(\mathbb{C}^N),$$
$$\mathbb{C}l(n)^0 \cong \mathrm{End}(\mathbb{C}^N),$$

where $N = n^{(n-1)/2}$.

We want to highlight that the splitting of $\mathbb{C}l(n)^0$ into two copies of $\mathrm{End}(\mathbb{C}^{N/2})$ for n even is at the root of the notion of *chirality* in physics. This splitting ultimately yields the difference between *left-handed* and *right-handed* spinors, and the Higgs mechanism is needed to define masses for such *chiral* spinors, which cannot be done otherwise. Theorem B.5 allows us to define a particularly important representation of Clifford algebras.

Definition B.6. The *spinor representation* $\rho\colon \mathbb{C}l(n) \xrightarrow{\cong} \mathrm{End}(\Delta_n)$ of the complex Clifford algebra on the space of Dirac spinors $\Delta_n = \mathbb{C}^N$ is defined through Theorem B.5. For n even it is just the standard representation and for n odd it is obtain by projection onto the first copy of $\mathrm{End}(\mathbb{C}^N)$. There is also an induced complex spinor representation on $\mathrm{Cl}(s,t)$ by equation B.1.

If we restrict to the even part of the complex Clifford algebra - as we will do when defining the spin group - we get the following results on (ir)reducibility.

Corollary B.7. If n is odd we identify $\mathbb{C}l^0(n)$ with the first summand in $\mathbb{C}l(n) \cong \mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n)$. The induced representation $\mathbb{C}l \xrightarrow{\cong} \mathrm{End}(\Delta_n)$ is then irreducible, with $\Delta_n \cong \mathbb{C}^N$ where $N = 2^{(n-1)/2}$. If n is even the induced representation splits into *left-handed* and *right-handed Weyl-spinors*:

$$\mathbb{C}l^0(n) \xrightarrow{\cong} \mathrm{End}(\Delta_n^+) \oplus \mathrm{End}(\Delta_n^-),$$

where $\Delta_n^+ \cong \Delta_n^- \cong \mathbb{C}^{N/2}$ with $N = 2^{n/2}$.

Definition B.8. Having defined the spinor representation, we call the bilinear map $\mathbb{R}^{s,t} \times \Delta_n \to \Delta_n$ given by $(\nu, \chi) \mapsto \nu \cdot \chi = \rho(\psi(\nu))\chi$ the *mathematical Clifford multiplication* of a vector and a spinor. The *physical Clifford multiplication* is the same but with a factor $-i$.

Clifford algebras arose out of Dirac's study of spinors using gamma matrices. How, then, does our abstract approach relate to the usual physical representation?

Definition B.9. Let $\rho\colon \mathrm{Cl}(s,t) \to \mathrm{End}(\mathbb{C}^N)$ be an algebra representation. Then we call $\gamma_a = \rho \circ \psi(e_a)$ the *mathematical gamma matrices* and $\Gamma_a = -i\gamma_a$ the *physical gamma matrices*. Clifford multiplication of a basis vector $e_a \in \mathbb{R}^{s,t}$ with a spinor $\chi \in \mathbb{C}^N$ is then just equal to $e_a \cdot \chi = \Gamma_a$. The anticommutators are $\{\gamma_a, \gamma_b\} = -2\eta_{ab} I_N$ and $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} I_N$. We set $\gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$ and $\Gamma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b]$.

B.2. The spin group

The spin group is a subgroup of the Clifford algebra $(\mathbb{R}^{s,t}, \eta)$. Before we define this, however, we consider the Lorentz group, of which the spin group is a double covering. The Lorentz group, in turn, is a subgroup of the pseudo-orthogonal group.

Definition B.10. Let η denote the standard symmetric bilinear form on $\mathbb{R}^{s+t}$ with signature (s,t). Then the *pseudo-orthogonal group* of signature (s,t) is defined as

$$\mathrm{O}(s,t) = \{A \in \mathrm{GL}(s+t, \mathbb{R}) \mid \eta(A\nu, Aw) = \eta(\nu, w) \text{ for all } \nu, w \in \mathbb{R}^{s+t}\}.$$

It is a well-known fact that $\mathrm{O}(s,t)$ is a closed subgroup of $\mathrm{GL}(s+t, \mathbb{R})$ and therefore a Lie group by the closed subgroup theorem.

To define the Lorentz group we need the following notion (cf. Definition 6.1.11 in [59]).

Definition B.11. Let $V^+ = \mathrm{span}\{e_1, ..., e_s\} \subset \mathbb{R}^{s,t}$ and let $\pi\colon V \to V^+$ denote the projection. Clearly V^+ is maximally η-positive definite. Note that if any other subspace $W \subset V$ is maximally η-positive definite then $\pi|_W \to V^+$ must be an isomorphism and W can be given a unique orientation such that $\pi|_W$ is orientation preserving if V^+ has been given an orientation. We fix an orientation on V^+. Then we define the *time-orientability* of $A \in O(s,t)$ to be +1 or -1 depending on whether $A|_{V^+} \to A(V^+)$ preserves or does not preserve orientation, where $A(V^+)$ is maximally η-positive definite (since A preserves η) and thus has the unique orientation defined by requiring $\pi|_{V(A^+)}$ to be orientation-preserving.

Definition B.12. We call

$$\begin{aligned}
O^+(s,t) &= \{A \in O(s,t) \mid A \text{ has time-orientability +1}\},\\
SO(s,t) &= \{A \in O(s,t) \mid \det A = 1\},\\
SO^+(s,t) &= SO(s,t) \cap O^+(s,t),
\end{aligned}$$

the *orthochronous*, the *proper* or *special* and the *proper orthochronous* pseudo-orthogonal groups. In the case of $s = 1$ or $t = 1$ they are called the orthochronous, proper and proper orthochronous *Lorentz groups*.

Remark B.13. $SO^+(s,t)$ is the connected component of the identity in $O(s,t)$ (Proposition 6.1.17 in [59]).

We will now briefly put aside the various pseudo-orthogonal groups and define the (s)pin groups. Then we will show that the latter are double-coverings of the former. This formalises the idea that "an electron needs to be rotated twice before coming back to its original state." To define the (s)pin groups, we remark that an element $x \in Cl(s,t)$ is invertible if there exists some $y \in Cl(s,t)$ such that $xy = yx = 1$, and similarly for $\mathbb{C}l(n)$. The open subset $Cl^*(s,t) \subset Cl(s,t)$ of all invertible elements forms a Lie group [59, p. 349], and the (s)pin groups are subgroups of this Lie group.

Definition B.14. We denote

$$\begin{aligned}
S^{s,t}_+ &= \{v \in \mathbb{R}^{s,t} \mid \eta(v,v) = 1\},\\
S^{s,t}_- &= \{v \in \mathbb{R}^{s,t} \mid \eta(v,v) = -1\},\\
S^{s,t}_\pm &= S^{s,t}_+ \cup S^{s,t}_-.
\end{aligned}$$

Using this we define the following groups:

$$\begin{aligned}
\mathrm{Pin}(s,t) &= \{v_1 v_2 \cdots v_r \mid v_i \in S^{s,t}_\pm\},\\
\mathrm{Spin}(s,t) &= \mathrm{Pin}(s,t) \cap Cl^0(s,t) = \{v_1 v_2 \cdots v_{2r} \mid v_i \in S^{s,t}_\pm\},\\
\mathrm{Spin}^+(s,t) &= \{v_1 v_2 \cdots v_{2p} w_1 w_2 \cdots w_{2q} \mid v_i \in S^{s,t}_+, w_i \in S^{s,t}_-\},
\end{aligned}$$

endowed with the topology from $Cl(s,t)$. These groups are called the *pin group*, *spin group* and *orthochronous spin group* respectively.

Our aim now is to prove that the groups in Definition B.14 are double coverings of the pseudo-orthogonal groups. For this we write $\deg u = 0$ if $u \in Cl^0(s,t)$ and $\deg u = 1$ if $u \in Cl^1(s,t)$ for any $u \in \mathrm{Pin}(s,t)$. Canonically identifying $\mathbb{R}^{s,t}$ as a vector subspace of $Cl(s,t)$ we then have the following.

Proposition B.15. Consider the map $R\colon \mathrm{Pin}(s,t) \times \mathbb{R}^{s,t} \to \mathbb{R}^{s,t}$ defined by

$$(u,x) \mapsto (-1)^{\deg u} u x u^{-1}.$$

This map is well-defined, for any $v \in S^{s,t}_\pm$ the map $R_v = R(v,\cdot)\colon \mathbb{R}^{s,t} \to \mathbb{R}^{s,t}$ is a reflection in the hyperplane $v^\perp \subset \mathbb{R}^{s,t}$, and we get a continuous homomorphism $\lambda\colon \mathrm{Pin}(s,t) \to O(s,t)$ that sends $u \mapsto R_u$.

Proof. To check that R is well-defined, let $v \in \mathbb{R}^{s,t}$ such that $\eta(v,v) = \pm 1$. Then $v^{-1} = \mp 1$ and $\deg v = -1$. For any $x \in \mathbb{R}^{s,t}$ this gives $R_v(x) = -vxv^{-1} = \pm vxv$. This shows that $R_v(x) = -x$ if $x \| v$ and $R_v(x) = x$ if $x \perp v$. Indeed, if $x = \alpha v$ then $R_v(x) = \pm \alpha v^3 = \pm \alpha(\mp v) = -\alpha v = -x$, and if $x \perp v$ then $R_v(x) = \pm vxv = \pm xv^2 = \pm x(\pm 1) = x$. We thus see that $R_v(x) \in \mathbb{R}^{s,t}$ and that $R_v(x)$ is a reflection in $v^\perp$. Since for $u = v_1 \cdots v_r \in \mathrm{Pin}(s,t)$ we can write

$$R_u = R_{v_1 \cdots v_r} = R_{v_1} \circ R_{v_2} \circ \cdots \circ R_{v_r}, \tag{B.2}$$

we conclude that $R_u(x) \in \mathbb{R}^{s,t}$. Moreover, as reflections are elements of $O(s,t)$, it follows from equation B.2 that λ is a (continuous) homomorphism. □

To prove our main result we need the following lemma, which is Theorem 6.5.12 in [59], where the proof can also be found.

Lemma B.16. Let $\mathbb{R} \in O(s,t)$ be a composition of reflections in hyperplanes $v_i^\perp$ with $v_i \in S^{s,t}_\pm$. Then $R \in SO(s,t)$ iff the number of vectors v_i is even, and $R \in SO^+(s,t)$ iff the numbers of vectors $v_i \in S^{s,t}_+$ and $v_i \in S^{s,t}_-$ are both even.

Theorem B.17. The homomorphism $\lambda\colon \mathrm{Pin}(s,t) \to O(s,t)$ defined in Proposition B.2 is open and surjective with $\ker \lambda = \{\pm 1\}$. Moreover, the preimages of $SO(s,t)$ and $SO^+(s,t)$ are $\mathrm{Spin}(s,t)$ and $\mathrm{Spin}^+(s,t)$, which are thus open subgroups of $\mathrm{Pin}(s,t)$. Lastly λ restricts to surjective homomorphisms

$$\lambda\colon \mathrm{Spin}(s,t) \to SO(s,t),$$
$$\lambda\colon \mathrm{Spin}^+(s,t) \to SO^+(s,t).$$

with kernels $\{\pm 1\}$.

Proof. The fact that λ is open and surjective follows from the Cartan-Dieudonné theorem and its proof (see Theorems 6.5.11 and 6.5.13 in [59]). Our main interest lies in showing that $\ker \lambda = \{\pm 1\}$. Suppose $\lambda(u) = R_u = I \in O(s,t)$. Then $\deg u = 0$ since R_u has to be composed of an even number of reflections (cf. Proposition B.15). We have $R_u(e_i) = ue_iu^{-1} = Ie_i = e_i$ for any $1 \le i \le s+t$. This gives

$$e_i u e_i = e_i u e_i u^{-1} u = e_i^2 u = -\eta(e_i, e_i) u.$$

Writing u in the standard basis as $u = \alpha e_{i_1} e_{i_2} \cdots e_{i_{2k}}$ with $\alpha \in \mathbb{R}$, $k \ge 1$ and taking $i = i_{2k}$ we then get

$$\begin{aligned} e_{i_{2k}} u e_{i_{2k}} &= \alpha e_{i_{2k}} e_{i_1} e_{i_2} \cdots e_{i_{2k}} e_{i_{2k}} = -\alpha \eta(e_{i_{2k}}, e_{i_{2k}}) e_{i_{2k}} e_{i_1} e_{i_2} \cdots e_{i_{2(k-1)}} \\ &= \alpha \eta(e_{i_{2k}}, e_{i_{2k}}) e_{i_1} e_{i_2} \cdots e_{i_{2k}} = -\eta(e_{i_{2k}}, e_{i_{2k}}) u = -\alpha \eta(e_{i_{2k}}, e_{i_{2k}}) e_{i_1} e_{i_2} \cdots e_{i_{2k}}. \end{aligned}$$

This implies that $\alpha = 0$, so $u \in \mathbb{R} \cdot 1$. But $u \in \mathrm{Pin}(s,t)$ so $u = \pm 1$, which means $\ker \lambda = \{\pm 1\}$. The other statements follow from Lemma B.16 and the definitions of the spin groups in Definition B.14. □

Corollary B.18. Since λ is a continuous, open and surjective homomorphism with $\ker \lambda = \{\pm 1\}$ we can define a unique Lie group structure on $\mathrm{Pin}(s,t)$, $\mathrm{Spin}(s,t)$ and $\mathrm{Spin}^+(s,t)$ such that λ becomes a smooth double covering of Lie groups.

We end this section by restricting the irreducible spinor representation of the even Clifford algebra from Proposition B.7 to the orthochronous spin group. We denote it by $\kappa\colon \mathrm{Spin}^+(s,t) \to GL(\Delta_n)$ and we prove its compatibility with Clifford multiplication.

Proposition B.19. The spinor representation of the orthochronous spin group is compatible with Clifford multiplication in the sense that for any $u \in \mathrm{Spin}^+(s,t), v \in \mathbb{R}^{s,t}, \chi \in \Delta_n$ we have

$$\kappa(u)(v \cdot \chi) = (\lambda(u)v) \cdot (\kappa(u)\chi).$$

Proof. Let $\rho\colon Cl(s,t) \to \mathrm{End}(\Delta_n)$ denote the spinor representation of the Clifford algebra. Then we have

$$\kappa(u)(v \cdot \chi) = \rho(u)\rho(v)(\chi) = \rho(uvu^{-1}u)(\chi) = \rho(\lambda(u)v)\rho(u)(\chi) = (\lambda(u)v) \cdot (\kappa(u)\chi).$$

□

B.3. Spin structures and spinor bundles

Now that we have defined the orthochronous spin group as well as the homomorphism λ that makes it a double covering of the orthochronous Lorentz group, we can finally turn to so-called *spin structures* and *spinor bundles*. It is on the latter that spinor fields are defined. A preliminary definition is in order.

Definition B.20. A pseudo-Riemannian manifold (M, g) of signature (s, t) is called *orientable* if its frame bundle can be reduced to a principal $SO(s,t)$-bundle under the embedding $SO(s,t) \subset O(s,t)$, *time-orientable* if its frame bundle can be reduced to a principal $O^+(s,t)$-bundle under $O^+(s,t) \subset O(s,t)$ and *orientable and time-orientable* if its frame bundle can be reduced to a principal $SO^+(s,t)$-bundle under $SO^+(s,t) \subset O(s,t)$.

Definition B.21. Let (M, g) be an oriented and time-oriented pseudo-Riemannian manifold and denote the reduced frame bundle by $\pi_{SO^+}\colon SO^+(M) \to M$. A *spin structure* on M is a $\mathrm{Spin}^+(s,t)$-principal bundle $\pi_{\mathrm{Spin}}\colon \mathrm{Spin}^+(M) \to M$ with double covering $\Lambda\colon \mathrm{Spin}^+(M) \to SO^+(M)$ such that the following diagram commutes

$$\begin{array}{ccccc} \mathrm{Spin}^+(M) \times \mathrm{Spin}^+(s,t) & \longrightarrow & \mathrm{Spin}^+(M) & & \\ {\scriptstyle \Lambda\times\lambda}\big\downarrow & & {\scriptstyle \Lambda}\big\downarrow & \searrow^{\pi_{\mathrm{Spin}}} & \\ SO^+(M) \times SO^+(s,t) & \longrightarrow & SO^+(M) & \xrightarrow[\pi_{SO^+}]{} & M \end{array}$$

Here the unlabeled horizontal arrows denote the right group actions of $\mathrm{Spin}^+(s,t)$ and $SO^+(s,t)$.

We can thus think of spin structures as double coverings of the reduced frame bundle, fibre-wise looking like λ, such that the actions of $\mathrm{Spin}^+(s,t)$ and $SO^+(s,t)$ are compatible with respect to the covering. In other words, a spin structure is a λ-equivariant bundle morphism $\Lambda\colon \mathrm{Spin}^+(s,t) \to SO^+(s,t)$, that is, a λ-reduction of $SO^+(s,t)$ [59, p. 378]. A well-known result shows that a spin structure on $SO^+(M)$ exists if and only if the second Stiefel-Whitney class $w_2(M)$ vanishes, and that if there is a spin structure, there is a bijection of isomorphism classes of spin structures to the homology group $H^1(M;\mathbb{Z}_2)$ [59]. Proving this is not so relevant for the Higgs mechanism, though it is good to know that $\mathbb{R}^{s,t}$ admits a spin structure for any $s, t \geq 0$. We now turn to sections of the reduced frame bundle.

Definition B.22. A local section $e = (e_1, .., e_n)\colon U \to SO^+(M)$ is called an *n-bein* or *vielbein*, and a *tetrad* specifically for $n = 4$.

Proposition B.23. Suppose we have a spin structure $\Lambda\colon \mathrm{Spin}^+(M) \to SO^+(M)$. Then for every vielbein $e\colon U \to SO^+(M)$ on a contractible open subset $U \subset M$ there exist precisely two local sections $\epsilon_\pm\colon U \to \mathrm{Spin}^+(M)$ such that $\Lambda \circ \epsilon_\pm = e$.

Proof. The image $U' = e(U)$ of the vielbein e is also a contractible open subset diffeomorphic to U and therefore $\Lambda|_{\Lambda^{-1}(U')}$ is a trivial two-sheeted covering [59, p. 380]. This two-sheeted covering admits precisely two sections $s_\pm\colon U' \to \Lambda^{-1}(U')$. Defining $\epsilon_\pm = s_\pm \circ e$ then gives the result. □

Having defined spin structures we continue with spinor bundles, which are just vector bundles associated to spin structures through the spinor representation.

Definition B.24. Let $\mathrm{Spin}^+(M) \to M$ be a spin structure on M and let $\kappa\colon \mathrm{Spin}^+(s,t) \to GL(\Delta)$ denote the spinor representation (we have dropped the index n on Δ_n). Then the *spinor bundle* is the associated complex vector bundle $S = \mathrm{Spin}^+(M) \times_\kappa \Delta$. Sections of S are called *spinor fields*.

Proposition B.25. There exists a well-defined Clifford multiplication on bundles $TM \times S \to S$, written as $(X, \Psi) \mapsto X \cdot \Psi$, which restricts to a map $T_xM \times S_x \to S_x$ at every point $x \in M$ and which induces a multiplication of vector fields with spinor fields.

Proof. Denoting by ρ_{SO^+} the standard representation of $SO^+(s,t)$ on $\mathbb{R}^{s,t}$ we have $TM \cong SO^+(M) \times_{\rho_{SO^+}} \mathbb{R}^{s,t}$. This is a basic result from the correspondence between principal bundles and vector bundles by taking associated bundles and frame bundles. Since $S = \mathrm{Spin}^+(M) \times_\rho \Delta$ we can write the multiplication $TM \times S \to S$ as

$$(SO^+(M) \times_{\rho_{SO^+}} \mathbb{R}^{s,t}) \times (\mathrm{Spin}^+(M) \times_\rho \Delta) \to \mathrm{Spin}^+(M) \times_\rho \Delta,$$
$$([\Lambda(\epsilon), v], [\epsilon, \chi]) \mapsto [\epsilon, v \cdot \chi].$$

Here $v \cdot \chi$ denotes the standard Clifford multiplication from Definition B.8 and we have used the spin structure $\Lambda\colon \mathrm{Spin}^+(M) \to SO^+(M)$ on an element $\epsilon \in \mathrm{Spin}^+(M)$. We need to check that this multiplication is well-defined, for which we use Proposition B.19 on the compatibility of the spinor representation κ with Clifford multiplication. Let $A = \lambda(u) \in SO^+(s,t), u \in \mathrm{Spin}^+(s,t)$, where $\lambda\colon \mathrm{Spin}^+(s,t) \to SO^+(s,t)$ denotes the double covering. We have

$$\begin{aligned}
&([\Lambda(\epsilon)A, \rho_{SO^+}(A^{-1})v], [\epsilon u, \kappa(u^{-1})\chi]) = ([\Lambda(\epsilon)\lambda(u), \rho_{SO^+}(A^{-1})v], [\epsilon u, \kappa(u^{-1})\chi]) \\
=&([\Lambda(\epsilon u), \rho_{SO^+}(A^{-1})v], [\epsilon u, \kappa(u^{-1})\chi]) \mapsto [\epsilon u, \rho_{SO^+}(A^{-1})v \cdot \kappa(u^{-1})\chi] \\
=&[\epsilon u, \rho_{SO^+}(\lambda(u^{-1}))v \cdot \kappa(u^{-1})\chi] = [\epsilon u, \kappa(u^{-1})(v \cdot \chi)] = [\epsilon, v \cdot \chi],
\end{aligned}$$

since by Definition B.21 of a spin structure we know $\Lambda(\epsilon)A = \Lambda(\epsilon)\lambda(u) = \Lambda(\epsilon u)$. □

The following result is conceptually important, as it is at the root of the problem of defining mass terms for twisted chiral spinors that the Higgs mechanism solves. It can be compared to Corollary B.7 and it follows basically from Proposition B.25 (see Proposition 6.9.13.2 in [59]).

Proposition B.26. If $\dim M$ is even, then $S = S_+ \oplus S_-$ splits as a direct sum of complex Weyl spinor bundles defined by $S_\pm = \mathrm{Spin}^+(M) \times_\kappa \Delta^\pm$. Clifford multiplication with a vector then maps $S_\pm$ to $S_\mp$.

Before moving on to spin covariant derivatives we note that if we have a local vielbein $e\colon U \to SO^+(M)$ on a contractible open subset $U \subset M$ with $\epsilon_\pm\colon U \to \mathrm{Spin}^+(M)$ the associated sections from Proposition B.23, then if $\Psi\colon U \to S$ is a local section we can write $\Psi = [\epsilon_\pm, \psi_\pm]$ with $\psi_\pm = -\psi_\mp$ (this follows from the double cover structure with kernel $\{\pm 1\}$). Choosing one we can write $\Psi = [\epsilon, \psi]$, but we do have to check that expressions involving these objects are independent of this choice. Physical Clifford multiplication is given by $e_a \cdot \Psi = [\epsilon, \Gamma_a \psi]$, which is indeed independent of the choice of $\epsilon_\pm$, since $\Gamma_a \psi$ is linear in ψ [59, p. 382].

B.4. The twisted chiral spin covariant derivative

We need a way to differentiate spinors on an oriented and time-oriented pseudo-Riemannian manifold (M, g) equipped with spin structure $\Lambda\colon \mathrm{Spin}^+(M) \to SO^+(M)$ and Levi-Civita connection ∇_{LC}. Recall that the tangent bundle can be viewed as an associated bundle $TM = SO^+(M) \times_{\rho_{SO^+}} \mathbb{R}^{s,t}$. We denote by $A_{SO^+} \in \Omega^1(SO^+(M), \mathfrak{so}^+(s,t))$ the connection 1-form on the frame bundle $SO^+(M)$ induced by the Levi-Civita connection. In other words, we have $\nabla_{LC} = \nabla^{A_{SO^+}}$, i.e. the LC connection is the induced covariant derivative of A_{SO^+}. Choosing a local vielbein $e = (e_1, ...e_n)\colon U \to SO^+(M)$ and writing out a vector field $Y = Y^a e_a$ we then have, by the local expression 2.3 for the covariant derivative:

$$\nabla_X Y = (dY^b(X) + A^e_{SO^+}(X)^b_a Y^a) e_b,$$

where $A^e_{SO^+} = e^* A_{SO^+} \in \Omega^1(U, \mathfrak{so}^+(s,t))$ is the local connection 1-form. Defining local 1-forms $\omega_{ab} \in \Omega^1(U, \mathbb{R})$ by $\nabla_{e_a} = \omega_{ab} \eta^{bc} \otimes e_c$ (η of signature (s,t)) we get $A^e_{SO^+}(X)^b_a = \omega_{ac}(X)\eta^{cb}$ [59, p. 384]. Using the connection 1-form on the frame bundle we define a connection on the principal bundle $\mathrm{Spin}^+(M)$.

Proposition B.27. We can define a connection on the principal bundle $\mathrm{Spin}^+(M) \to M$ by

$$A_{\mathrm{Spin}^+} = (\lambda_*)^{-1} \circ (\Lambda^* A_{\mathrm{SO}^+}) \in \Omega^1(\mathrm{Spin}^+(M), \mathfrak{spin}^+(s,t)),$$

where $\lambda_* \colon \mathfrak{spin}^+(s,t) \to \mathfrak{so}^+(s,t)$ is the isomorphism of Lie algebras obtained from the double covering $\lambda \colon \mathrm{Spin}^+(s,t) \to \mathrm{SO}^+(s,t)$. This connection is called the *spin connection*.

Proof. In order to verify that the 1-form A_{Spin^+} satisfies the properties of a connection 1-form we use the compatibility of the actions of $\mathrm{Spin}^+(s,t)$ and $\mathrm{SO}^+(s,t)$ in Definition B.21 of a spin structure. In particular, for any $\epsilon \in \mathrm{Spin}^+(M)$, $g \in \mathrm{Spin}^+(s,t)$ we have $\Lambda(\epsilon \cdot g) = \Lambda(\epsilon) \cdot \lambda(g)$, so $\Lambda \circ r_g = r_{\lambda(g)} \circ \Lambda$. Using this, we find that for any $Y \in \mathfrak{X}\left(\mathrm{Spin}^+(M)\right)$:

$$\begin{aligned} r_g^* A_{\mathrm{Spin}^+}(Y) &= (\lambda_*)^{-1} \left(\Lambda^* A_{\mathrm{SO}^+}((r_g)_* Y)\right) = (\lambda_*)^{-1} \left(A_{\mathrm{SO}^+}(\Lambda_*(r_g)_* Y)\right) = (\lambda_*)^{-1} \left(A_{\mathrm{SO}^+}((r_{\lambda(g)})_* \Lambda_* Y)\right) \\ &= (\lambda_*)^{-1} \left(r_{\lambda(g)}^* A_{\mathrm{SO}^+}(\Lambda_* Y)\right) = (\lambda_*)^{-1} \left(\mathrm{Ad}_{\lambda(g)^{-1}} \circ A_{\mathrm{SO}^+}(\Lambda_* Y)\right) \\ &= (\lambda_*)^{-1} \left(\mathrm{Ad}_{\lambda(g)^{-1}} \circ \lambda_* \circ (\lambda_*)^{-1} \circ A_{\mathrm{SO}^+}(\Lambda_* Y)\right) = \mathrm{Ad}_{g^{-1}} \circ A_{\mathrm{Spin}^+}(Y). \end{aligned}$$

To check the action on fundamental vector fields, let $X \in \mathfrak{spin}^+(s,t)$ and $\epsilon \in \mathrm{Spin}^+(M)$. Then we have

$$\begin{aligned} &\left(A_{\mathrm{Spin}^+}\right)_\epsilon (\tilde{X}_\epsilon) = (\lambda_*)^{-1} \circ (\Lambda^* A_{\mathrm{SO}^+})_\epsilon \left(\frac{d}{dt}\Big|_{t=0} (\epsilon \cdot \exp(tX)) \right) \\ &= (\lambda_*)^{-1} \circ (A_{\mathrm{SO}^+})_{\Lambda(\epsilon)} \left(\frac{d}{dt}\Big|_{t=0} \Lambda(\epsilon \cdot \exp(tX)) \right) = (\lambda_*)^{-1} \circ (A_{\mathrm{SO}^+})_{\Lambda(\epsilon)} \left(\frac{d}{dt}\Big|_{t=0} \Lambda(\epsilon) \cdot \lambda(\exp(tX)) \right) \\ &= (\lambda_*)^{-1} \circ (A_{\mathrm{SO}^+})_{\Lambda(\epsilon)} \left(\left(\widetilde{\lambda_* X}\right)_{\Lambda(\epsilon)} \right) = (\lambda_*)^{-1} (\lambda_* X) = X. \end{aligned}$$

Here we have used the connection 1-form properties of A_{SO^+}. □

Definition B.28. We define the *spin covariant derivative* $\nabla \colon \Gamma(S) \to \Omega^1(M, S)$ to be the covariant derivative on the spinor bundle $S = \mathrm{Spin}^+(M) \times_\kappa \Delta$ induced from the spin connection A_{Spin^+}.

In a local trivialisation $\epsilon \colon U \to \mathrm{Spin}^+(M)$ on a contractible open subset $U \subset M$ as in Proposition B.23 we get a local connection 1-form

$$A^\epsilon_{\mathrm{Spin}^+} = \epsilon^* A_{\mathrm{Spin}^+} \in \Omega^1(U, \mathfrak{spin}^+(s,t)).$$

Writing a section $\Psi \in \Gamma(S)$ as $\Psi = [\epsilon, \psi]$ with $\psi \colon U \to \Delta$ we can write $\nabla_X \Psi = [\epsilon, \nabla_X \psi]$, where

$$\nabla_X \psi = d\psi(X) + \kappa_* \left(A^\epsilon_{\mathrm{Spin}^+}(X)\right) \psi.$$

Proposition 6.10.9 in [59] then gives us an explicit formula:

$$\nabla_X \psi = d\psi(X) + \frac{1}{4} \omega_{ab}(X) \gamma^{ab} \psi = d\psi(X) - \frac{1}{4} \omega_{ab}(X) \Gamma^{ab} \psi,$$

where γ^{ab}, Γ^{ab} are the commutators of gamma matrices from Definition B.9 with raised indices. We are now in a position to consider the Dirac operator, which is essential for the Standard Model.

Definition B.29. The *Dirac operator* $D \colon \Gamma(S) \to \Gamma(S)$ (also written $\not{D}$) is defined by

$$D\Psi = \eta^{ab} e_a \cdot \nabla_{e_b} \Psi, \qquad \Psi \in \Gamma(S),$$

where $\cdot$ denotes Clifford multiplication. In other words, it is the composition of the maps

$$\Gamma(S) \xrightarrow{\nabla} \Gamma\left(T^*M \otimes S\right) \xrightarrow{\eta} \Gamma(TM \otimes S) \xrightarrow{\cdot} \Gamma(S),$$

where $\eta \colon T^*M \to TM$ is the isomorphism induced by the pseudo-Riemannian metric.

Writing $\Psi = [\epsilon, \psi]$ and $D\Psi = [\epsilon, D\psi]$ we have

$$D\psi = \gamma^a \nabla_{e_a} \psi = i\Gamma^a \nabla_{e_a} \psi = i\Gamma^a \left(d\psi(e_a) - \frac{1}{4}\omega_{abc}\Gamma^{bc}\psi \right),$$

where $\omega_{abc} = \omega_{bc}(e_a)$. For $\dim M = n$ even the spin covariant derivative preserves the splitting $S = S_+ \oplus S_-$, whereas Clifford multiplication interchanges this splitting, so the Dirac operator $D\colon \Gamma(S_\pm) \to \Gamma(S_\mp)$ also exchanges right and left spinors for even dimension. So far we have not coupled spinors to gauge fields. In the Standard Model however, and particularly in the Higgs mechanism, this is necessary. Thus, we consider a principal G-bundle $P \to M$ with complex representation $\rho\colon G \to GL(V)$ and associated vector bundle $E = P \times_\rho V$, and a spinor bundle $S \to M$ associated to a spin structure on M.

Definition B.30. The vector bundle $S \otimes E$ is called the *twisted spinor bundle* or *gauge multiplet spinor bundle*.

Let us consider how we can describe the twisted spinor bundle in a local gauge $s\colon U \to P$. A section $\tau \in \Gamma(E)$ might then be written locally as $\tau = [s, \nu]$ with $\nu\colon U \to V$. Choosing a basis $\nu_1, ..., \nu_r$ of V we get a local frame $\tau_i = [s, \nu_i]$. This basis also allows us to identify $V \cong \mathbb{C}^r$. With a local trivialisation $\epsilon\colon U \to \mathrm{Spin}^+(M)$ as in Proposition B.23, a section $\Psi \in \Gamma(S \otimes E)$ can then be written locally as $\Psi = [\epsilon \times s, \psi]$ with $\psi\colon U \to \Delta \otimes \mathbb{C}^r$. More explicitly, we can write out a gauge multiplet

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_r \end{pmatrix} : U \longrightarrow \Delta \otimes \mathbb{C}^r, \tag{B.3}$$

where $\psi_i\colon U \to \Delta$. In section 2.3 we explain that the Higgs mechanism is needed to give different masses to spinors in the same gauge multiplet. We can use this local description to define a covariant derivative on the twisted spinor bundle, i.e. a covariant derivative of spinors that also "feels" the gauge field.

Definition B.31. Let A be a connection on the principal G-bundle P. We define the *twisted spin covariant derivative* $\nabla^A_X\colon \Gamma(S \otimes E) \to \Gamma(S \otimes E)$ by writing $\nabla^A_X \Psi = [\epsilon \times s, \nabla^A_X \psi]$ with $\psi\colon U \to \Delta \otimes V$ and where

$$\nabla^A_X \psi = d\psi(X) - \frac{1}{4}\omega_{ab}(X)\Gamma^{ab}\psi + (\rho_* A_s(X))\,\psi.$$

It is important to note that the matrices Γ^{ab} act on the Δ-part of ψ in the sense that they act separately on each spinor component ψ_i, whereas $\rho_* A_s(X)$ acts on the V-part of ψ, thus mixing the components of the gauge multiplet [59, p. 390]. This mixing blocks the possibility of directly defining different masses for multiplet components without using the Higgs field.

Definition B.32. We also have a Dirac operator D_A (also written $\not{D}_A$) of twisted spinors defined by

$$D_A = \eta^{ab} e_a \cdot \nabla^A_{e_b} \Psi,$$

which is the composition of the maps

$$\Gamma(S \otimes E) \xrightarrow{\nabla^A} \Gamma\left(T^*M \otimes S \otimes E\right) \xrightarrow{\eta} \Gamma(TM \otimes S \otimes E) \xrightarrow{\cdot} \Gamma(S \otimes E),$$

and which can locally be written as $D_A\Psi = [\epsilon \times s, D_A\psi]$ where

$$D_A\psi = i\Gamma^a \left(d\psi\,(e_a) - \frac{1}{4}\omega_{abc}\Gamma^{bc}\psi + (\rho_* A_a)\,\psi \right).$$

Although we now have a covariant derivative in the presence of gauge fields, we do not yet know how to deal with chirality. If $\dim M = n$ is even then $S = S_+ \oplus S_-$ splits by Proposition B.26, and this certainly happens on Minkowski spacetime $\mathbb{R}^4$. In the Standard Model this is important, for the structure group is actually represented on left-handed and right-handed vector spaces of different dimension, making the definition of mass terms for twisted chiral spinors problematic, as we show in section 2.3.

Definition B.33. Suppose $\dim M = n$ is even such that $S = S_+ \oplus S_-$ and let $\rho_\pm \colon G \to GL(V_\pm)$ be two representations with associated bundles $E_\pm = P \times_{\rho_\pm} V_\pm$. Then we call

$$(S \otimes E)_+ = (S_+ \otimes E_+) \oplus (S_- \otimes E_-)$$

the *twisted chiral spinor bundle*. We also write

$$(S \otimes E)_- = (S_- \otimes E_+) \oplus (S_+ \otimes E_-).$$

Definition B.34. Suppose $\dim M = n$ is even and let A be a connection 1-form on P. We define the *twisted chiral spin covariant derivative* ∇_A on the twisted chiral spinor bundle $(S \otimes E)_+$ locally by

$$\nabla^A_X \psi = [\epsilon \times s, \nabla^A_X \psi],$$

where, writing $\psi = \psi_+ + \psi_-$, we have

$$\nabla^A_X \psi = d\psi(X) - \frac{1}{4}\omega_{ab}(X)\Gamma^{ab}\psi + (\rho_{+*}A_s(X))\,\psi_+ + (\rho_{-*}A_s(X))\,\psi_-.$$

Definition B.35. We again define a Dirac operator $D_A \colon \Gamma((S \otimes E)_+) \to \Gamma((S \otimes E)_-)$ as the composition of maps in the same way as in Definitions B.29 and B.32. It decomposes into

$$D_{A_\pm} \colon \Gamma(S_\pm \otimes E_+) \to \Gamma(S_\mp \otimes E_-)$$

and locally it is given by

$$D_A \psi = i\Gamma^a \left(d\psi\,(e_a) - \frac{1}{4}\omega_{abc}\Gamma^{bc}\psi + (\rho_{+*}A_a)\,\psi_+ + (\rho_{-*}A_a)\,\psi_- \right).$$

Bibliography

[1] R. Ariew, ed., *Leibniz and Clarke: Correspondence*. Hackett Publishing Company, 2000.

[2] J. Earman, "Curie's Principle and spontaneous symmetry breaking," *International Studies in the Philosophy of Science*, vol. 18, no. 2-3, pp. 173–198, 2004.

[3] A. J. Beekman, L. Rademaker, and J. van Wezel, "An Introduction to Spontaneous Symmetry Breaking," 2019. Available at SciPost: `https://scipost.org/submissions/1909.01820v1/`.

[4] K. Brading, E. Castellani, and N. Teh, "Symmetry and Symmetry Breaking," in *The Stanford Encyclopedia of Philosophy* (E. N. Zalta and U. Nodelman, eds.), Metaphysics Research Lab, Stanford University, fall 2023 ed., 2023.

[5] J. Earman, "Laws, Symmetry, and Symmetry Breaking: Invariance, Conservation Principles, and Objectivity," *Philosophy of Science*, vol. 71, no. 5, pp. 1227–1241, 2004.

[6] v. Dirk and P. Kes, "The discovery of superconductivity," *Europhysics News*, vol. 63, 2011.

[7] F. London, H. London, and F. A. Lindemann, "The electromagnetic equations of the supraconductor," *Proceedings of the Royal Society of London. Series A - Mathematical and Physical Sciences*, vol. 149, no. 866, pp. 71–88, 1997.

[8] V. L. Ginzburg and L. D. Landau, "On the Theory of Superconductivity," in *On Superconductivity and Superfluidity: A Scientific Autobiography* (V. L. Ginzburg, ed.), pp. 113–137, Springer, 2009.

[9] D. Fraser and A. Koberinski, "The Higgs mechanism and superconductivity: A case study of formal analogies," *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, vol. 55, pp. 72–91, 2016.

[10] L. P. Gor'kov, "Microscopic derivation of the Ginzburg–Landau equations in the theory of superconductivity," *Sov. Phys. - JETP (Engl. Transl.); (United States)*, vol. 9:6, 1959.

[11] B. D. Josephson, "Possible new effects in superconductive tunnelling," *Physics Letters*, vol. 1, no. 7, pp. 251–253, 1962.

[12] Y. Nambu, "Quasi-Particles and Gauge Invariance in the Theory of Superconductivity," *Physical Review*, vol. 117, no. 3, pp. 648–663, 1960.

[13] T. W. B. Kibble, "Englert-Brout-Higgs-Guralnik-Hagen-Kibble mechanism (history)," *Scholarpedia*, vol. 4, no. 1, p. 8741, 2009.

[14] Y. Nambu and G. Jona-Lasinio, "Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. I," *Physical Review*, vol. 122, no. 1, pp. 345–358, 1961.

[15] Y. Nambu and G. Jona-Lasinio, "Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. II," *Physical Review*, vol. 124, no. 1, pp. 246–254, 1961.

[16] J. Goldstone, "Field theories with $\ll$ Superconductor $\gg$ solutions," *Il Nuovo Cimento (1955-1965)*, vol. 19, no. 1, pp. 154–164, 1961.

[17] J. Goldstone, A. Salam, and S. Weinberg, “Broken Symmetries,” *Physical Review*, vol. 127, no. 3, pp. 965–970, 1962.

[18] F. Englert and R. Brout, “Broken Symmetry and the Mass of Gauge Vector Mesons,” *Physical Review Letters*, vol. 13, no. 9, pp. 321–323, 1964.

[19] P. W. Higgs, “Broken Symmetries and the Masses of Gauge Bosons,” *Physical Review Letters*, vol. 13, no. 16, pp. 508–509, 1964.

[20] P. W. Higgs, “Broken Symmetries and the Masses of Gauge Bosons,” *Physical Review Letters*, vol. 13, no. 16, pp. 508–509, 1964.

[21] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, “Global Conservation Laws and Massless Particles,” *Physical Review Letters*, vol. 13, no. 20, pp. 585–587, 1964.

[22] W. Struyve, “Gauge invariant accounts of the Higgs mechanism,” *Studies in History and Philosophy of Modern Physics*, vol. 42, no. 4, pp. 226–236, 2011.

[23] S. Weinberg, “A Model of Leptons,” *Physical Review Letters*, vol. 19, no. 21, pp. 1264–1266, 1967.

[24] “Facts and figures about the LHC — CERN.” Available at: `https://home.cern/resources/faqs/facts-and-figures-about-lhc`, accessed on 2023-06-03.

[25] P. W. Higgs, “Spontaneous Symmetry Breakdown without Massless Bosons,” *Physical Review*, vol. 145, no. 4, pp. 1156–1163, 1966.

[26] J. Earman, “Rough guide to spontaneous symmetry breaking,” in *Symmetries in Physics: Philosophical Reflections* (E. Castellani and K. Brading, eds.), pp. 335–346, Cambridge University Press, 2003.

[27] C. Smeenk, “The Elusive Higgs Mechanism,” *Philosophy of Science*, vol. 73, no. 5, pp. 487–499, 2006.

[28] H. Lyre, “Does the Higgs Mechanism Exist?,” *International Studies in the Philosophy of Science*, vol. 22, no. 2, pp. 119–133, 2008.

[29] A. Wüthrich, “Eating Goldstone bosons in a phase transition: A critical review of Lyre’s analysis of the Higgs mechanism,” 2012. Available at: `https://philsci-archive.pitt.edu/9384/`.

[30] M. Stöltzner, “Constraining the Higgs Mechanism: Ontological Worries and the Prospects for an Algebraic Cure,” *Philosophy of Science*, vol. 79, no. 5, pp. 930–941, 2012.

[31] S. Friederich, “Gauge symmetry breaking in gauge theories—in search of clarification,” *European Journal for Philosophy of Science*, vol. 3, no. 2, pp. 157–182, 2013.

[32] S. Rivat, “On the Heuristics of the Higgs Mechanism,” *Journal for General Philosophy of Science / Zeitschrift für allgemeine Wissenschaftstheorie*, vol. 45, no. 2, pp. 351–367, 2014.

[33] D. Wallace, “In Defence of Naiveté: The Conceptual Status of Lagrangian Quantum Field Theory,” *Synthese*, vol. 151, no. 1, pp. 33–80, 2006.

[34] D. Fraser, “Quantum Field Theory: Underdetermination, Inconsistency, and Idealization*,” *Philosophy of Science*, vol. 76, no. 4, pp. 536–567, 2009.

[35] D. Wallace, “Taking particle physics seriously: A critique of the algebraic approach to quantum field theory,” *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, vol. 42, no. 2, pp. 116–125, 2011.

[36] D. Fraser, "How to take particle physics seriously: A further defence of axiomatic quantum field theory," *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, vol. 42, no. 2, pp. 126–135, 2011.

[37] J. Fröhlich, G. Morchio, and F. Strocchi, "Higgs phenomenon without a symmetry breaking order parameter," *Physics Letters B*, vol. 97, no. 2, pp. 249–252, 1980.

[38] J. Fröhlich, G. Morchio, and F. Strocchi, "Higgs phenomenon without symmetry breaking order parameter," *Nuclear Physics B*, vol. 190, no. 3, pp. 553–582, 1981.

[39] A. Maas, "Brout–Englert–Higgs physics: From foundations to phenomenology," *Progress in Particle and Nuclear Physics*, vol. 106, pp. 132–209, 2019.

[40] A. Maas, "The Fröhlich-Morchio-Strocchi mechanism: A underestimated legacy," 2023. Preprint available at arXiv:2305.01960, `http://arxiv.org/abs/2305.01960`.

[41] P. Berghofer, J. François, S. Friederich, H. Gomes, G. Hetzroni, A. Maas, and R. Sondenheimer, "Gauge Symmetries, Symmetry Breaking, and Gauge-Invariant Approaches," *Elements in the Foundations of Contemporary Physics*, 2023.

[42] G. Morchio and F. Strocchi, "Localization and symmetries," *Journal of Physics A: Mathematical and Theoretical*, vol. 40, no. 12, p. 3173, 2007.

[43] G. De Palma and F. Strocchi, "A non-perturbative argument for the non-abelian Higgs mechanism," *Annals of Physics*, vol. 336, pp. 112–117, 2013.

[44] F. Strocchi, *Symmetry Breaking*, vol. 732 of *Lecture Notes in Physics*. Springer, 2008.

[45] F. Strocchi, "Symmetries, Symmetry Breaking, Gauge Symmetries," 2015. Preprint available at arXiv:1502.06540, `http://arxiv.org/abs/1502.06540`.

[46] F. Strocchi, *An Introduction to Non-Perturbative Foundations of Quantum Field Theory*. International Series of Monographs on Physics, Oxford University Press, 2016.

[47] F. Strocchi, *Symmetry Breaking in the Standard Model: A Non-Perturbative Outlook*. Scuola Normale Superiore, 2019.

[48] F. Strocchi, "Some rigorous results on symmetry breakings in gauge QFT," *SciPost Physics Proceedings*, no. 6, p. 021, 2022.

[49] R. Nozick, "Invariance and Objectivity," *Proceedings and Addresses of the American Philosophical Association*, vol. 72, no. 2, pp. 21–48, 1998.

[50] N. K. Nielsen, "On the gauge dependence of spontaneous symmetry breaking in gauge theories," *Nuclear Physics B*, vol. 101, no. 1, pp. 173–188, 1975.

[51] S. Elitzur, "Impossibility of spontaneously breaking local symmetries," *Physical Review D*, vol. 12, no. 12, pp. 3978–3982, 1975.

[52] H. Lyre, "The Just-so Higgs Story: A Response to Adrian Wüthrich," *Journal for General Philosophy of Science / Zeitschrift für Allgemeine Wissenschaftstheorie*, vol. 43, no. 2, pp. 289–294, 2012.

[53] L. Lusanna and P. Valtancoli, "Dirac's Observables for the Higgs Model: I) the Abelian Case," *International Journal of Modern Physics A*, vol. 12, no. 26, 1997.

[54] L. Lusanna and P. Valtancoli, "Dirac's Observables for the Higgs Model: II) the non-Abelian SU(2) Case," *International Journal of Modern Physics A*, vol. 12, no. 26, 1997.

[55] L. Lusanna and P. Valtancoli, "Dirac's observables for the su(3)×su(2)×u(1) standard model," *International Journal of Modern Physics A*, vol. 13, no. 27, pp. 4605–4690, 1998.

[56] S. Friederich, "A Philosophical Look at the Higgs Mechanism," *Journal for General Philosophy of Science / Zeitschrift für Allgemeine Wissenschaftstheorie*, vol. 45, no. 2, pp. 335–350, 2014.

[57] W. Caudy and J. Greensite, "Ambiguity of spontaneously broken gauge symmetry," *Physical Review D*, vol. 78, no. 2, p. 025018, 2008.

[58] K. Landsman, *Foundations of Quantum Theory*, vol. 188 of *Fundamental Theories of Physics*. Springer International Publishing, 2017.

[59] M. J. Hamilton, *Mathematical Gauge Theory*. Universitext, Springer International Publishing, 2017.

[60] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*. Interscience Tracts in Pure and Applied Mathematics Nr. 15, Interscience Publishers, 1963.

[61] P. A. M. Dirac, "Gauge-invariant formulation of quantum electrodynamics," *Canadian Journal of Physics*, vol. 33, no. 11, pp. 650–660, 1955.

[62] J. Attard, J. François, S. Lazzarini, and T. Masson, "The Dressing Field Method of Gauge Symmetry Reduction, a Review with Examples," in *Foundations of Mathematics and Physics One Century After Hilbert: New Perspectives* (J. Kouneiher, ed.), pp. 377–415, Springer International Publishing, 2018.

[63] J. François, "Artificial versus Substantial Gauge Symmetries: A Criterion and an Application to the Electroweak Model," *Philosophy of Science*, vol. 86, no. 3, pp. 472–496, 2019.

[64] M. Zajac, "The dressing field method in gauge theories - geometric approach," *Journal of Geometric Mechanics*, vol. 15, no. jgm-15-01-007, pp. 128–146, 2023.

[65] C. Fournel, J. François, S. Lazzarini, and T. Masson, "Gauge invariant composite fields out of connections, with examples," *International Journal of Geometric Methods in Modern Physics*, vol. 11, no. 03, p. 1450016, 2014.

[66] T. Masson and J.-C. Wallet, "A Remark on the Spontaneous Symmetry Breaking Mechanism in the Standard Model," 2010. Preprint available at arXiv:1001.1176, `http://arxiv.org/abs/1001.1176`.

[67] P. Kosso, "The Empirical Status of Symmetries in Physics," *The British Journal for the Philosophy of Science*, vol. 51, no. 1, pp. 81–98, 2000.

[68] K. Brading and H. R. Brown, "Are Gauge Symmetry Transformations Observable?," *The British Journal for the Philosophy of Science*, vol. 55, no. 4, pp. 645–665, 2004.

[69] R. Healey, "Perfect Symmetries," *The British Journal for the Philosophy of Science*, vol. 60, no. 4, pp. 697–720, 2009.

[70] H. Greaves and D. Wallace, "Empirical Consequences of Symmetries," *The British Journal for the Philosophy of Science*, vol. 65, no. 1, pp. 59–89, 2014.

[71] S. Friederich, "Symmetry, Empirical Equivalence, and Identity," *British Journal for the Philosophy of Science*, vol. 66, no. 3, pp. 537–559, 2015.

[72] N. J. Teh, "Galileo's Gauge: Understanding the Empirical Significance of Gauge Symmetry," *Philosophy of Science*, vol. 83, no. 1, pp. 93–118, 2016.

[73] S. Murgueitio Ramírez, "A Puzzle Concerning Local Symmetries and Their Empirical Significance," *The British Journal for the Philosophy of Science*, vol. 73, no. 4, pp. 1021–1044, 2022.

[74] N. J. Y. W. Teh, "The Philosophy of Symmetry," *Elements in the Philosophy of Physics*, 2024.

[75] H. Gomes and A. Riello, "Unified geometric framework for boundary charges and particle dressings," *Physical Review D*, vol. 98, no. 2, p. 025013, 2018.

[76] H. Gomes, F. Hopfmüller, and A. Riello, "A unified geometric framework for boundary charges and dressings: Non-Abelian theory and matter," *Nuclear Physics B*, vol. 941, pp. 249–315, 2019.

[77] H. Gomes, "Gauging the boundary in field-space," *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, vol. 67, pp. 89–110, 2019.

[78] H. Gomes, "Holism as the empirical significance of symmetries," *European Journal for Philosophy of Science*, vol. 11, no. 3, p. 87, 2021.

[79] H. Gomes and A. Riello, "The quasilocal degrees of freedom of Yang-Mills theory," *SciPost Physics*, vol. 10, no. 6, p. 130, 2021.

[80] H. Gomes, *Why Gauge? Conceptual Aspects of Gauge Theory*. PhD thesis, Apollo - University of Cambridge Repository, 2022.

[81] H. Gomes and J. Butterfield, "How to Choose a Gauge? The Case of Hamiltonian Electromagnetism," *Erkenntnis*, pp. 1–35, 2022.

[82] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems*, vol. 17 of *Texts in Applied Mathematics*. Springer, 1999.

[83] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*. Princeton University Press, 1992.

[84] L. Lusanna, "Dirac-Bergmann Constraints in Physics: Singular Lagrangians, Hamiltonian Constraints and the Second Noether Theorem," *International Journal of Geometric Methods in Modern Physics*, vol. 15, no. 10, p. 1830004, 2018.

[85] R. Healey and R. Healey, *Gauging What's Real: The Conceptual Foundations of Contemporary Gauge Theories*. Oxford University Press, 2007.

[86] J. Earman, "Tracking Down Gauge: An Ode to the Constrained Hamiltonian Formalism," in *Symmetries in Physics: Philosophical Reflections* (K. Brading and E. Castellani, eds.), pp. 140–62, Cambridge University Press, 2003.

[87] J. B. Pitts, "A First Class Constraint Generates Not a Gauge Transformation, But a Bad Physical Change: The Case of Electromagnetism," *Annals of Physics*, vol. 351, pp. 382–406, 2014.

[88] O. Pooley and D. Wallace, "First-class constraints generate gauge transformations in electromagnetism (reply to Pitts)," 2022. Preprint available at arXiv:2210.09063, `http://arxiv.org/abs/2210.09063`.

[89] A. P. Balachandran, "Gauge Symmetries,Topology and Quantisation," in *AIP Conference Proceedings*, vol. 317, pp. 1–81, 1994. Preprint available at arXiv:hep-th/9210111, `http://arxiv.org/abs/hep-th/9210111`.

[90] J. M. Arms, "Linearization stability of gravitational and gauge fields," *Journal of Mathematical Physics*, vol. 20, no. 3, pp. 443–453, 1979.

[91] J. M. Arms, "The structure of the solution set for the Yang-Mills equations," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 90, no. 2, pp. 361–372, 1981.

[92] G. Belot, "Symmetry and gauge freedom," *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, vol. 34, no. 2, pp. 189–225, 2003.

[93] I. M. Singer, "The Geometry of the Orbit Space for Non-Abelian Gauge Theories," *Physica Scripta*, vol. 24, no. 5, p. 817, 1981.

[94] S. Friederich, "Symmetries and the identity of physicsl states," *EPSA15 Selected Papers*, pp. 153–165, 2017.

[95] S. M. Ramírez and N. J. Teh, "Abandoning Galileo's Ship: The Quest for Non-relational Empirical Significance," *The British Journal for the Philosophy of Science*, 2021.

[96] D. Wallace, "Isolated systems and their symmetries, part I: General framework and particle-mechanics examples," *Studies in History and Philosophy of Science*, vol. 92, pp. 239–248, 2022.

[97] D. Wallace, "Isolated Systems and Their Symmetries, Part II: Local and Global Symmetries of Field Theories," *Studies in History and Philosophy of Science Part A*, vol. 92, pp. 249–259, 2022.

[98] A. Riello, "Soft charges from the geometry of field space," *Journal of High Energy Physics*, vol. 2020, no. 5, p. 125, 2020.

[99] A. S. Wightman and L. Garding, "Fields as operator-valued distributions in relativistic quantum theory," *Arkiv Fys.*, vol. Vol: 28, 1965.

[100] R. Haag and D. Kastler, "An Algebraic Approach to Quantum Field Theory," *Journal of Mathematical Physics*, vol. 5, no. 7, pp. 848–861, 1964.

[101] R. Haag, *Local Quantum Physics*. Springer, 1996.

[102] K. Rejzner, *Perturbative Algebraic Quantum Field Theory*. Mathematical Physics Studies, Springer International Publishing, 2016.

[103] L. Ruetsche, *Interpreting Quantum Theories: The Art of the Possible*. Oxford University Press UK, 2011.

[104] J. Earman and G. Valente, "Relativistic Causality in Algebraic Quantum Field Theory," *International Studies in the Philosophy of Science*, vol. 28, no. 1, pp. 1–48, 2014.

[105] "Quasi-equivalent representations - Encyclopedia of Mathematics." Available at: `https://encyclopediaofmath.org/wiki/Quasi-equivalent_representations`, accessed on 2024-03-12.

[106] N. Landsman, "Algebraic theory of superselection sectors and the measurement problem in quantum mechanics," *International Journal of Modern Physics A*, vol. 06, no. 30, pp. 5349–5371, 1991.

[107] N. P. Landsman, "Between classical and quantum," 2005. Preprint available at arXiv:quant-ph/0506082, `http://arxiv.org/abs/quant-ph/0506082`.

[108] J. Earman, "Superselection Rules for Philosophers," *Erkenntnis (1975-)*, vol. 69, no. 3, pp. 377–414, 2008.

[109] G. Murphy, *C*-Algebras and Operator Theory*. Elsevier Science, 2014.

[110] S. Doplicher, R. Haag, and J. E. Roberts, "Fields, observables and gauge transformations I," *Communications in Mathematical Physics*, vol. 13, no. 1, pp. 1–23, 1969.

[111] S. Doplicher, R. Haag, and J. E. Roberts, "Fields, observables and gauge transformations II," *Communications in Mathematical Physics*, vol. 15, no. 3, pp. 173–200, 1969.

[112] S. Doplicher and J. E. Roberts, "Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics," *Communications in Mathematical Physics*, vol. 131, no. 1, pp. 51–107, 1990.

[113] S. Doplicher and J. E. Roberts, "A new duality theory for compact groups," *Inventiones mathematicae*, vol. 98, no. 1, pp. 157–218, 1989.

[114] H. Halvorson and M. Mueger, "Algebraic Quantum Field Theory," in *Handbook of the Philosophy of Physics* (J. Butterfield and J. Earman, eds.), Kluwer Academic Publishers, 2006.

[115] D. Kastler, D. W. Robinson, and A. Swieca, "Conserved currents and associated symmetries; Goldstone's theorem," *Communications in mathematical physics*, vol. 2, pp. 108–120, 1966.

[116] D. Buchholz, S. Doplicher, R. Longo, and J. E. Roberts, "A new look at goldstone's theorem," *Reviews in Mathematical Physics*, vol. 04, pp. 49–83, 1992.

[117] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That*. Princeton Landmarks in Mathematics and Physics, Princeton University Press,, first paperback print., with revised pref. and corrections. ed., 2016.

[118] D. Buchholz, F. Ciolli, G. Ruzzi, and E. Vasselli, "Gauss's law, the manifestations of gauge fields, and their impact on local observables," in *Trails in Modern Theoretical and Mathematical Physics*, Springer International Publishing, 2022.

[119] T. Vachaspati, L. Pogosian, and D. A. Steer, "Cosmic strings," *Scholarpedia*, vol. 10, no. 2, p. 31682, 2015. revision #192547.

[120] S. D. Haro, N. Teh, and J. N. Butterfield, "Comparing Dualities and Gauge Symmetries," *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, vol. 59, pp. 68–80, 2017.

[121] S. D. Haro, "The Invisibility of Diffeomorphisms," *Foundations of Physics*, vol. 47, no. 11, pp. 1464–1497, 2017.

[122] C. Cronstrom, "The Generalisation of the Coulomb Gauge to Yang-Mills Theory," 1998. Preprint available at arXiv:hep-th/9810002, `http://arxiv.org/abs/hep-th/9810002`.

[123] P. Forgács and A. Lukács, "Charge screening in the Abelian Higgs model," *The European Physical Journal C*, vol. 81, no. 3, p. 243, 2021.

[124] J. van Wezel and J. van den Brink, "Spontaneous symmetry breaking and decoherence in superconductors," *Physical Review B*, vol. 77, no. 6, p. 064523, 2008.

[125] J. van Wezel, "Quantum dynamics in the thermodynamic limit," *Physical Review B*, vol. 78, no. 5, p. 054301, 2008.

[126] J. van Wezel, "Broken Time Translation Symmetry as a Model for Quantum State Reduction," *Symmetry*, vol. 2, no. 2, pp. 582–608, 2010.

[127] J. van Wezel, "Phase transitions as a manifestation of spontaneous unitarity violation," *Journal of Physics A: Mathematical and Theoretical*, vol. 55, no. 40, p. 401001, 2022.

[128] A. Mukherjee, S. Gotur, J. Aalberts, R. van den Ende, L. Mertens, and J. van Wezel, “Quantum State Reduction of General Initial States through Spontaneous Unitarity Violation,” *Entropy*, vol. 26, no. 2, p. 131, 2024.

[129] N. K. Landsman and R. Reuvers, “A Flea on Schrödinger?s Cat,” *Foundations of Physics*, vol. 43, no. 3, pp. 373–407, 2013.

[130] C. J. F. van de Ven, “The classical limit and spontaneous symmetry breaking in algebraic quantum theory,” *Expositiones Mathematicae*, vol. 40, no. 3, pp. 543–571, 2022.

[131] C. J. F. van de Ven, G. C. Groenenboom, R. Reuvers, and K. Landsman, “Quantum spin systems versus Schroedinger operators: A case study in spontaneous symmetry breaking,” *SciPost Physics*, vol. 8, no. 2, p. 022, 2020.

[132] N. P. Landsman, “Spontaneous Symmetry Breaking in Quantum Systems: Emergence or Reduction?,” *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, vol. 44, no. 4, pp. 379–394, 2013.

[133] M. D’Onofrio and K. Rummukainen, “The Standard Model cross-over on the lattice,” *Physical Review D*, vol. 93, no. 2, p. 025003, 2016.

[134] J. B. Conway, *A Course in Functional Analysis*, vol. 96 of *Graduate Texts in Mathematics*. Springer, 2007.