When Mathematics Became Useful to Science

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Mathematics is the "language of nature," a privileged mode of expression in science. We think it latches onto something essential about the physical universe, and we seek theories that reduce phenomena to mathematical laws. Yet, this attitude could not arise from the philosophies dominant before the early modern period. In orthodox Aristotelianism, mathematical categories are too impoverished to capture the causal structure of the world. In the revived Platonism of its opponents, the natural world is too corrupt to exemplify mathematical perfection. Modern mathematical science required a novel *tertium quid*, due to Pietro Catena.

Introduction

Modern science holds that mathematics is a privileged mode of reasoning and expression. Thus, even if not always successfully, science aims for the reduction of phenomena to mathematical laws. But what exactly are we saying when we assert, with Galileo, that the book of nature "is written in the language of mathematics"?[1] We are saying, perhaps *inter alia*, that mathematics is certain, applicable, and productive. First, mathematical demonstrations are *certain* insofar as the premises necessarily entail the conclusions. Second, mathematical constructions and derivations are *applicable* to natural phenomena—they express what happens. Finally, mathematics is *productive*. The mathematical relationship expressed by a "law of nature" is the reason why—the *propter quid*—the world behaves as it does.[2]

To give a simple example, consider Newton's second law of motion: "A change in motion is proportional to the motive force impressed."[3] That is, a change of a body's motion—an acceleration—will be in proportion to the force applied. So, if a body of mass 2 units is subjected to a force of magnitude 6, it will accelerate at 3 of the appropriate units. This is *certain*, since it is deduced from the law. This is *applicable* insofar as it is true of actual bodies and their actual motions (ignoring

[1] (Galilei, et al. 1960, 184)
[2] This does not entail that the explanation is causal. Also, this is compatible with instrumentalism. For an instrumentalist, the mathematical laws just are the best explanations available, and they apply to nature insofar as they generate accurate predictions.
[3] (Newton 1999, 416)

impediments like friction). And it is *productive*, since the proportion is the reason why the body behaves as it does.

This raises a question: How did authors come to think that mathematics is certain, applicable, and productive in the study of nature? The mathematization of science has been attributed to the early modern combination of two disciplines—the natural philosophy taught in the universities and the practical mathematics of astronomers, mechanicians, and other professionals. Modern commentators agree that this was due, in some mixture, to the resurgence of Platonism arising from the humanist recovery of ancient texts (e.g., Plato, Plotinus, Proclus) and the development of Aristotelian "mixed sciences," also invigorated by a recovered corpus (e.g., Archimedes, Pseudo-Aristotle).[4] However, neither Platonism nor Aristotelianism provided an adequate framework for this disciplinary combination, a difficulty revealed by a contemporary controversy, the *quaestio de certitudine mathematicarum*, carried out in a series of treatises from the mid-sixteenth to early seventeenth centuries. While all parties agreed that mathematics is certain, Platonists held that mathematics is too abstract to apply to the natural world, while Aristotelians argued that mathematics is too accidental to explain it.

This paper argues the combination of natural philosophy and mathematics was not motivated by Platonism or Aristotelianism, but by a novel approach that emerged during the *quaestio*. This was due to Pietro Catena, who developed a view of mathematics that was at once certain, applicable, and productive in natural philosophy. Thus, it was this novel Catena-ism that allowed the reconciliation of practical mathematics with natural philosophy and motivated the mathematization of science.

The *Status Quo Ante*

Prior to the early modern period, mathematicians and natural philosophers constituted distinct intellectual spheres. Natural philosophers aimed their enquiry at the explanation of natural phenomena, guided by Aristotelian method, especially as described in the *Posterior Analytics*. Here, the ideal explanation of some effect—a *demonstratio potissima*—is a syllogism that simultaneously shows the existence of the cause of the effect (the *quia*) and that the cause was the reason why the effect occurred (the *propter quid*).[5] In other words, the effect described by the syllogism's conclusion flows from the nature or essence described by the premises, such that the syllogism's logical necessity recapitulates physical necessity. The *demonstration potissima* thus explains natural effects.

[4] For a historiographical overview, see (Gorham, et al. 2016, 8-14).
[5] (Piccolomini 1565, 102v) See (Longeway 2009).

But the expectation that proper explanations be productive introduces a significant constraint. In the Aristotelian framework, the *propter quid* of a natural effect derives from an *intrinsic* nature—"what belongs to something in itself."[6] For instance, the nature of a heavy body—its heaviness—is the relevant cause of its fall. Consequently, proper explanations must refer to the natures of the objects under consideration. Conversely, one cannot apply demonstrations outside of their own proper domain, since the natures described would not be intrinsic in the other domain. This is the Aristotelian prohibition against *metabasis*.[7] Mathematical demonstrations are certain and productive when one is examining mathematical objects. But mathematical objects are not objects in the natural world. A bronze globe is not a mathematical sphere, so, strictly speaking, the properties of the sphere do not imply anything about the globe. In other words, mathematics is not *applicable* to the natural world.

Even Aristotle recognized this was too strict. There are disciplines where the nature of the objects permits the use of mathematics to account for phenomena. These are the mixed sciences of practical mathematics; the ancient archetypes being harmonics, optics, astronomy, and mechanics.[8] To allow these to count as legitimate enquiries, Aristotle introduced *subordination* to work around the prohibition against *metabasis*.[9] In a subordinate science, some aspect of pure mathematics is combined with a hypothesis drawn from natural philosophy to create a mixed mathematical discipline. For instance, astronomy is the geometry of circles and spheres to which is added the physical premise that planets move circularly. The mathematics then allows calculation of planetary positions.

By design, subordination makes mathematics *applicable*. But subordination also makes this applicability accidental, and thus not productive. Mathematical astronomy does not describe the physical natures of the planets—it concerns only the accidental properties of their motion. The mathematics predicts locations, but it says nothing about why they move—nothing about the *propter quid*. That is the concern of natural philosophers.

Such, in broad outline, was the mid–sixteenth-century state of play. Productivity was the concern of natural philosophers, who rejected the applicability of mathematics to their enquiries.

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[6] *Posterior Analytics* I.6 (Aristotle 1984, 1:122)
[7] *Posterior Analytics* I.7 (Aristotle 1984, 1:122) See (Capecchi 2018, 3).
[8] Other fields were later added; e.g., geography, architecture, ballistics, fortification, hydraulics, calendrics, gnomonics, perspective, accounting, and provisioning.
[9] *Posterior Analytics* I.13 (Aristotle 1984, 1:127-29) See (McKirahan 1978).

Mathematicians, meanwhile, held fast to the instrumental applicability permitted by subordination, but did not aim at productivity.

De Certitudine Mathematicarum

At the same time, though, the two disciplinary spheres came into increasingly close contact, primarily because of the rising importance of civil and military engineering, which resulted in the establishment of university chairs in mathematics.[10] This, in turn, caused some philosophy professors to defend their own preeminence by attacking the intellectual standing of mathematics. One example is *De certitudine mathematicarum disciplinarum* by the Paduan philosopher Alessandro Piccolomini, published in 1547. Extending a point made by Proclus, Piccolomini offers a radical critique of mathematics. Whereas it had been long agreed that mathematics is not productive in natural science because it is not applicable (because of subordination), Piccolomini asserts that mathematics is not applicable *because* it is not productive—*even* in its own domain. More precisely, Piccolomini argues that mathematical demonstrations are not *potissimae*—they do not show the reason why their conclusions are true.

Piccolomini considers, for instance, the 16th proposition of the first book of Euclid's *Elements*, which shows that an external angle constructed by extending the side of a triangle is greater than either internal opposite angle. Piccolomini points out that mathematical objects are not physical agents, and there is nothing in the triangle itself that *produces* the external angle or determines this particular property of it; "And therefore there is no one who can say how in the nature [*ratione*] and form of a triangle there is [contained] that the external angle is greater than either interior opposite [angle] ...."[11] Similarly, Piccolomini notes that a single mathematical proposition might be proven in multiple ways.[12] For example, Euclid's proposition I.32, which shows that the angles of a triangle sum to two right angles, can be proven either by extending a side (as Euclid does) or by drawing a parallel to one side through the opposite vertex (as Proclus does).[13] But this shows that mathematical proofs do not offer the "proper, unique, and immediate cause" of their conclusions. And this, Piccolomini concludes, "will suffice to show that *demonstrationes potissimae* giving immediate causes cannot be found in mathematics."[14] But since *potissimae* are required by Aristotelian method, mathematics is incapable of establishing knowledge. Euclid's proofs are not productive—the triangle's properties

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[10] (Biagioli 1989; Garber 2010)
[11] (Piccolomini 1565, 102v; De Pace 1993, 33)
[12] (Piccolomini 1565, 103r)
[13] (Piccolomini 1565, 103v)
[14] (Piccolomini 1565, 105v)

are not *causal* consequences of the triangle's nature. They are only *mathematical* consequences, and that is something different. The logical necessity of mathematics cannot replicate physical necessity.

Piccolomini does not go so far as to say that mathematics is not certain. He just holds that the mathematics offers a different *kind* of certainty than natural philosophy. Yet if the certainty of mathematics is not that of natural science, what is the epistemic basis of mathematical knowledge? Piccolomini asks the question himself:

> We therefore concede to the mathematical disciplines the highest order of certainty, but we deny that the cause of this ordering is correctly identified by the Latin [scholars]. What, then is the true cause of this certainty?[15]

This expanded the terms of the ensuing debate, from the applicability of mathematics to its productivity and now to its certainty. Piccolomini here problematized something no one had thought to worry about—the certitude of mathematical proof. The subsequent *quaestio de certitudine* would hinge upon this issue. It debated the basis of mathematical certainty in order to explain how mathematical proof was or was not productive and applicable in the investigation of nature. In other words, by seeking the source of certainty in mathematics, the debate's participants also sought the grounds of its use in natural science.

Aristotle or Plato?

To defend the certainty of mathematics in the existing philosophical landscape, there were basically two ways one could go: Aristotelianism or Platonism. Participants in the *quaestio* pursued both. For the sake of brevity, I will take Piccolomini to represent the Aristotelian faction, Francesco Barozzi the Platonists.[16] Yet both options reinforce mathematical certainty at the expense of applicability and productivity, though in different ways.

The Aristotelian option is to argue that mathematics is a science of "quantity"—which is an attribute of intelligible matter. In the Aristotelian theory of perception, the sensible forms of concrete objects—*species*—enter into the passive intellect via the senses and inform the common sense—the *phantasia*—thereby creating sensory objects.[17] So the objects in sensory experience are composites of forms and the *intelligible matter* of the *phantasia*. Consequently, by abstracting away all the formal aspects peculiar to individual sensations, one can uncover the attributes of the intelligible matter

[15] (Piccolomini 1565, 106r)
[16] See (Mancosu 1996; De Pace 1993; Giacobbe 1981; Higashi 2018; Feldhay 1998).
[17] See, e.g., (Pasnau 1997).

itself, one of which is its quantity. Mathematics is the study of the properties this *quantum phantasiatum*.

On this view, mathematics is certain because quantity is the "most immediate and manifest of all properties."[18] That is, quantity and its properties are *obvious*.

> Mathematical objects, by abstraction, offer their deepest and innermost selves to our senses, and disclose themselves. Insofar as these [mathematical objects] are all quantities, they manifestly surrender not only their properties, but also their subject and even their very forms to our senses. Quantity is truly the most sensible of sensibles [*omnium sensatorum sensatissimum*].[19]

When we construct a mathematical proof, we are manipulating intelligible matter, and we know it is conclusive because we can *sense* the result.[20]

This view also yields applicability to nature. Concrete objects in nature are constituted by substantial forms in composition with prime matter. Sensory objects are constituted by the sensible aspects of the very same forms and intelligible matter. Intelligible matter in the *sensorium* and prime matter in *concreta* thus have the same potency—the capacity to be actualized by the forms. But mathematics is the science of this potentiality. So it is applicable to the prime matter, as well. The quantity in the intellect is like the quantity in the physical world. And since prime matter is constitutive of all concrete objects, mathematics is universally applicable.

But that very argument runs against the productivity of mathematics. An attribute of potentiality, like quantity, can never contribute the *reason why* something is actual.[21] So mathematics is not consideration of substance, but the *lack* of it:

> it is indeed the case that quantity is truly the most imperfect of all accidents [*imperfectissima omnium accidentum*]; since it alone among all the accidents, not having a formal account, … follows [only] from matter itself.[22]

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[18] (De Pace 1993, 43)
[19] (Piccolomini 1565, 106v)
[20] Note that this is an early version of the wax argument in Descartes's second Meditation, where he asserts that mathematical extension is the distinctive feature of all sensations and imaginations.
[21] (De Pace 1993, 51)
[22] (Piccolomini 1565, 104r)

In other words, mathematics is near useless in natural science—it is about an accident of passive matter, not about the active causes that bring the world into being.

This conclusion gets Piccolomini what he ultimately wants—the denigration of mathematics with respect to natural philosophy. As Piccolomini's ally Pereira bluntly concludes,

> It is my opinion that the mathematical disciplines are not proper sciences. ... [For] the mathematician neither considers the essence of quantity, nor treats of its affections as they flow from such essence, nor declares them by the proper causes on account of which they are in quantity, nor makes his demonstrations from proper and 'per se' but from common and accidental predicates.[23]

The obviousness of mathematics makes it first in the order of knowing (*ordo cognoscendi*), but it is last in the order of being (*ordo essendi*). It is "most imperfect"; just "common and accidental."[24] Useful only as practical rules of thumb, mathematics belongs outside the university, facilitating the manual arts.

Piccolomini's attack occasioned defenses of the dignity of mathematics. Francesco Barozzi, who translated Proclus and lectured on mathematics at Padua, published one such response in the *Opusculum in quo una Oratio et duae Questiones, altera de certitudine et altera de medietate mathematicarum continentur* (1560). Here, Barozzi takes the other obvious tack—he Platonizes. On this view, mathematics is about forms, not matter. Specifically, Barozzi holds that the act of abstraction in the intellect does not simply discover the quantitative properties of intelligible matter; rather it recognizes or, better, recollects the existence of real mathematical objects. So, for instance, if one abstracts away the material aspects in a perception of a bronze globe, one is not left with potential quantity, but with a *sphere*, a mathematical object subsisting in the intelligible matter.[25] For Barozzi, mathematics is the science of these abstracted realities.

This science is certain because, unlike natural substances, mathematical objects are invariant and immutable. They are more perfect than the concrete objects of sense, and this perfection is inherited by mathematical demonstrations. That is, the certainty of the demonstrations derives from the immutability of their subjects. Similarly, mathematical science is also productive. The demonstrations capture the causal natures of the mathematical entities. So Euclid's proofs are

[23] (Pereira 1576, 24; De Pace 1993, 91; Mancosu 1996, 13)
[24] (De Pace 1993, 97)
[25] (Barozzi 1560, 37v)

demonstrationes potissimae. For instance, the fact that its angles sum to two right angles flows from the real, abstract essence of a triangle. On this view, mathematics precedes natural philosophy both in the order of knowing and in the order of being. It is not a mere servant to the manual arts, and it deserves an exalted place in the university, contra Piccolomini.

Yet, if mathematics is about perfect mathematical entities, then it does not apply to less perfect natural entities. Investigating the mathematical properties of a sphere tells us little—mere accidents at most—about the bronze globe. Its behavior cannot be reduced to mathematical proportions. Here is Barozzi's ally Biancani on the point:

> However, we should know that even if these mathematical entities do not exist in that perfection, this is merely accidental, for it is well known that both nature and art intend to imitate primarily those mathematical figures, although because of the grossness [*ruditatem*] and imperfection of sensible matter, which is incapable of receiving perfect figures, they do not achieve their end…. For this reason we should hold that these geometrical entities which are perfect in all respects are *per se* and true beings; whereas natural as well as artificial figures, which exist in the nature of things, as they are not intended [*per se*] by any efficient cause, are beings *per accidens*, and are imperfect and false.[26]

So the Platonists, in the end, agree with Piccolomini: mathematics is of little use for natural science. For Piccolomini, mathematical categories are too impoverished to capture the physical structure of the world. For Barozzi, the world is too imperfect to live up to the perfection of mathematical truth.[27] Either way, we do not get the modern synthesis. Mathematics is certain, of course, but either not productive in or not applicable to the natural world. And mathematics and natural philosophy must remain separate.

The *Tertium Quid*

Those that maintained that mathematics *is* useful in the study of nature had to chart a new course. The Paduan professor of mathematics Pietro Catena recognized that both Platonism and Aristotelianism suffered from the same fundamental fault. Both held that the certainty of mathematics derived from its objects—either mere intelligible matter or genuine intelligible entities. But insofar as the mathematical objects are distinct from natural things, the certainty of the one could

[26] (Mancosu 1996, 180)
[27] (De Pace 1993, 183)

never be applied to the other. The epistemic power of mathematical demonstration is confined, in different ways, to the domain of mathematical objects, whatever they are (quantities or forms).

Catena reversed the order of dependence. He held that certainty does not depend on the existence of any entity. Rather the intellect, immediately recognizes certainty by the nature of reason itself, without sensing or recollecting. That is, there are necessary truths that reveal themselves as certain as soon as they are thought.[28] Among these are propositions about magnitude—namely, the Euclidean axioms and common notions. Catena writes of the "common principles," such as "if equals are taken from equals, the remainders are equal," that "their truth is known almost by nature—and I say 'almost' because as soon as the terms of their certainty are known, they are known with certainty."[29] Such principles are "incontrovertible and indubitable to the lights of nature."[30]

Catena then applies a version of the ontological argument: since these truths are necessary, it follows that their objects necessarily exist. In other words, there are entities that instantiate the axioms and common notions. These are the mathematical universals. For instance, there *is* a universal triangle that is described by the proof that the angles of a triangle sum to two right angles. The necessity of the *proof* implies the existence of the *angles*:[31]

> in the 32nd proposition of the first book of the *Elements*, where from a given triangle it is concluded that it has three angles equal to two right angles. Not only would it show what it is necessary to preconceive [*praeaccipere*], but also their [the angles] being. For from what is given, not only would it signify what is preconceived, which is also what is sought, but both what it signifies and what exists ....[32]

That is, the proof does not just entail belief ("preconception") of the property of the triangle, but the very existence of the triangle with that property. The *logical* necessity produces the *physical* necessity.

The mathematical universals, moreover, are realized by particulars insofar as the particulars add constraining conditions to the universal. For instance, a scalene is a triangle with the added condition that its sides are unequal. Consequently, the universal properties are inherited by the particulars. Everything that is true of triangles in general remains true of the scalene. But this is also

[28] (De Pace 1993, 190)
[29] (Catena 1556, 71)
[30] (Catena 1563, 5r)
[31] (De Pace 1993, 192)
[32] (Catena 1556, 66)

true, says Catena, of concrete particulars in the natural world. These are also mathematical universals under particular physical conditions. The bronze globe is a sphere under the conditioned of being bronze.[33] So the properties of the globe are identical to the properties of the abstract mathematical sphere, and the logic of mathematics extends from demonstrations in the mind to the mathematical entities and thence down through the more and more particular instantiations of them. The mathematics is applicable to nature insofar as it studies the mathematical properties that concrete objects *actually possess*—there is no *metabasis*. It is a "*univoca ratio*" pervading *all* the sciences.[34] The study of the rainbow is just a particularization of the study of refraction, which is just a particularization of pure geometry.[35]

By the same token, Catena's view yields the productivity of mathematics. As noted above, the existence of mathematical objects flows from the certainty of mathematical demonstration. So the certainty of mathematical demonstration exerts a kind of causal power. The premises produce their conclusions, both in the demonstrations themselves and in the mathematical entities. The logic of mathematics, that is, involves rational causes—"*causae illativae*"—that are prior to and more fundamental than the four natural causes, and that generate the *propter quid* of a demonstration.

> Geometric induction ... always proceeds from [premises that] are true, primary, the illative causes of the conclusions [*causis illativis conclusionis*], and more known—not always from [premises that] are immediate, nor from causes that generate being, but from those that generate the *propter quid* of the inference.[36]

So, for instance, when the perspectivist shows that an object observed from further away looks smaller than when observed from closer, she shows this to be true *because* the angle on the same base diminishes as the vertex moves further away, as demonstrated in *Elements* I.21.[37] The demonstration is the reason why the phenomena are as they are.[38]

Continuity and Novelty

Catena's arguments echo what came before. Most importantly, he still seeks to satisfy the natural philosophical demand that sciences provide the *propter quid*. Yet Catena offers something genuinely novel, insofar as he rejects both the Aristotelian and the Platonic positions. Notice that, on Catena's

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[33] (De Pace 1993, 209-11; Catena 1556, 55)
[34] (De Pace 1993, 230)
[35] (Catena 1556, 83; De Pace 1993, 236)
[36] (Catena 1556, 28; De Pace 1993, 218)
[37] (Catena 1556, 84; De Pace 1993, 233-36)
[38] See also (Catena 1563, 7v).

view, physical necessity in nature recapitulates the logical necessity of mathematical demonstrations, not the other way around, as in Aristotelian method. Piccolomini had held that mathematics does not provide *demonstrationes potissimae*, and this inability to satisfy the canons of Aristotelian method entailed the denigration of mathematics. For Catena, the *causae illativae* precede the Aristotelian causes. And the failure to fit them into the *Posterior Analytics* is an indictment of Aristotle, not of mathematics. So mathematics, not Aristotelian method, is the proper means for studying the natural world. Mathematical demonstrations are "not only most certain in their own genera, but will accustom all other parts of philosophy to their light and offered certainty."[39] For instance, in the subordination of the mixed sciences, it is mathematics, not physics, that "render[s] manifest and with exactness all the properties of the objects."[40]

There is also an important reversal in the method of abstraction, according to Catena. On the Peripatetic view, when we observe a physical object, we must abstract away its essential nature to uncover accidental mathematical properties. According to Catena, however, physical properties are only accidental impediments to the mathematical understanding of the thing.

> And if the line, which is drawn in ink or produced by pen or pencil, is not straight, one should not say from this that Geometry errs because Geometry does not direct thought to what is placed before the eyes, but rather that it directs thought to what the soul inwardly grasps.... Indeed, Geometry concludes nothing from this colored line, as drawn by the pen; but the demonstration follows from the inward concept of the line. ... Therefore, the geometrical disciplines are most certain, and not because they are sensed, as some falsely say—because they are inwardly grasped.[41]

The real truth is not what is observed, and one has to ignore the vagaries introduced by the physical stuff. This is significant, because it is a precursor to the error theory by which Galileo and his successors set aside "impediments," such as friction and air resistance, in order to assert mathematical laws of nature, which are "inwardly grasped," not precisely instantiated by what is before the eyes.

Catena's position might be seen as a kind of Platonism—Catena is advocating the real existence of mathematical entities. But Catena is inverting Plato, as well. The certainty of

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[39] (Catena 1563, 2v)
[40] (Catena 1563, 7r-7v)
[41] (Catena 1556, 72; De Pace 1993, 197)

mathematics does not derive from the mathematical entities. Rather, the existence of the entities flows from the certainty of mathematics. In other words, certainty is derived from the epistemic power of reason itself, not from the ontic perfection of entities. This is as much as rejection of Barozzi as of Piccolomini.

The *quaestio de certitudine* prepared the ground for the integration of mathematics and natural philosophy, where mathematics could function as the language of expression and instrument of knowing of natural phenomena. In Catena we first get the modern, scientific attitude toward mathematics. He holds that mathematics is at once certain, applicable, and productive. Most importantly, since the certainty is *sui generis*, mathematics is not about any particular domain. It is more than just the study of abstract quantity; it is the tool for studying all of nature. The later, better known integrations of mathematics and natural philosophy, such as Galileo's and Newton's, were fruits of a tree planted by Catena.

References

Aristotle (1984), *The Complete Works of Aristotle*. Edited by Jonathan Barnes. 2 vols. Princeton: Princeton University Press.

Barozzi, Francesco (1560), *Opusculum, in quo una Oratio, et duae Quaestiones: altera de certitudine, et altera de mediatate Mathematicarum continentur*. Padua: Grazioso Percacino.

Biagioli, Mario (1989), "The Social Status of Italian Mathematicians, 1450-1600," *History of Science* 27: 41-94.

Capecchi, Danilo (2018), *The Path to Post-Galilean Epistemology*. Cham: Springer International.

Catena, Pietro (1556), *Universa loca in logicam Aristotelis in mathematicas disciplinas*. Venice: Francesco Marcolini.

--- (1563), *Oratio pro idea methodi*. Padua: Gratiosum Perchacinum.

De Pace, Anna (1993), *Le Matematiche e il Mondo: Ricerche su un dibattito in Italia nella seconda metà del Cinquecento*. Milan: Francoangeli.

Feldhay, Rivka (1998), "The Use and Abuse of Mathematical Entities," in Peter Machamer (ed.), *The Cambridge Companion to Galileo*, Cambridge: Cambridge University Press, 80-145.

Galilei, Galileo, Horatio Grassi, Mario Guiducci and Johannes Kepler (1960), *The Controversy on the Comets of 1618*. Translated by Stillman Drake and Charles Donald O'Malley. Philadelphia: University of Pennsylvania Press.

Garber, Daniel (2010), "Philosophia, Historia, Mathematica: Shifting Sands in the Disciplinary Geography of the Seventeenth Century," in Tom Sorrell, G. A. Rogers and Jill Kraye (eds.), *Scientia in Early Modern Philosophy*, Dordrecht: Springer, 1-17.

Giacobbe, Giulio Cesare (1981), *Alle Radici della Rivoluzione Scientifica Rinascimentale: Le Opere di Pietro Catena sui Rapporti tra Matematica e Logica*. Pisa: Domus Galilaeana.

Gorham, Geoffrey, Benjamin Hill and Edward Slowik (2016), "Introduction," in Geoffrey Gorham, Benjamin Hill, Edward Slowik and C. Kenneth Waters (eds.), *The Language of Nature: Reassessing the Mathematization of Natural Philosophy in the Seventeenth Century*, Minneapolis: University of Minnesota Press, 1-28.

Higashi, Shin (2018), *Penser les mathématiques au XVIe siècle*. Paris: Classiques Garnier.

Longeway, John (2009), *Medieval Theories of Demonstration*, in Edward N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*, Stanford: Stanford University.

Mancosu, Paolo (1996), *Philosophy of Mathematics & Mathematical Practice in the Seventeenth Century*. New York: Oxford University Press.

McKirahan, Richard D., Jr. (1978), "Aristotle's Subordinate Sciences," *The British Journal for the History of Science* 11 (3): 197-220.

Newton, Isaac (1999), *The Principia: Mathematical Principles of Natural Philosophy*. Translated by I. Bernard Cohen and Anne Whitman. Berkeley: University of California Press.

Pasnau, Robert (1997), *Theories of Cognition in the Later Middle Ages*. Cambridge: Cambridge University Press.

Pereira, Benedetto (1576), *De Communibus omnium rerum naturalium principiis et affectionibus Libri quindecim*. Rome: Venturini Tramezzini, apud Franciscum Zanettum.

Piccolomini, Allesandro (1565), *De Certitudine Mathematicarum Disciplinarum*. Venice: Traianum Curtium.