Cantor's illusion simplified

Richard L. Hudson 1-6-2025

abstract

This analysis shows Cantor's diagonal definition in his 1891 paper was not compatible with his horizontal enumeration of the infinite set M. The diagonal sequence was a counterfeit which he used to produce an apparent exclusion of a single sequence to prove the cardinality of M is greater than the cardinality of the set of integers N.

keywords: Cantor, diagonal, infinite

1. the argument

Translation from Cantor's 1891 paper [1]:

Namely, let *m* and *n* be two different characters, and consider a set [*Inbegriff*] *M* of elements

$$E = (x_1, x_2, \dots , x_v, \dots)$$

which depend on infinitely many coordinates $x_1, x_2, \dots , x_v, \dots$, and where each of the coordinates is either *m* or *w*. Let *M* be the totality [*Gesamtheit*] of all elements *E*.
To the elements of *M* belong e.g. the following three:

$$E^{I} = (m, m, m, m, \dots),$$
$$E^{II} = (w, w, w, w, \dots),$$
$$E^{III} = (m, w, m, w, \dots).$$

I maintain now that such a manifold [*Mannigfaltigkeit*] *M* does not have the power of the series 1, 2, 3, …, *v*, ….

This follows from the following proposition:
"If $E_1, E_2, \dots, E_v, \dots$ is any simply infinite [*einfach unendliche*] series of elements of the manifold *M*, then there always exists an element E_0 of *M*, which cannot be connected with any element E_v."
For proof, let there be

$$E_1 = (a_{1.1}, a_{1.2}, \dots , a_{1,v}, \dots)$$
$$E_2 = (a_{2.1}, a_{2.2}, \dots , a_{2,v}, \dots)$$
$$E_u = (a_{u.1}, a_{u.2}, \dots , a_{u,v}, \dots)$$
$$\dots\dots\dots\dots\dots\dots$$

where the characters $a_{u,v}$ are either *m* or *w*. Then there is a series $b_1, b_2, \dots b_v, \dots$, defined so that b_v is also equal to *m* or *w* but is *different* from $a_{v,v}$.

Thus, if $a_{v,v}$ = m, then b_v = w.
Then consider the element

$$E_0 = (b_1, b_2, b_3, \ldots)$$

of *M*, then one sees straight away, that the equation

$$E_0 = E_u$$

cannot be satisfied by any positive integer u, otherwise for that u and for all values of v.

$$b_v = a_{u,v}$$

and so we would in particular have

$$b_u = a_{u,u}$$

which through the definition of b_v is impossible. From this proposition it follows immediately that the totality of all elements of *M* cannot be put into the sequence [*Reihenform*]: $E_1, E_2, \ldots, E_v, \ldots$ otherwise we would have the contradiction, that a thing [*Ding*] E0 would be both an element of M, but also not an element of M.
(end of translation)

2. Cantor's argument

The symbols {0, 1} will be substituted for {m, w} for visual clarity. The term 'list' will be substituted for 'enumeration'.
Cantor defines an infinite set M consisting of elements E_n. Each E_n is an infinite one dimensional horizontal sequence composed of two symbols 0 and 1. He does not specify a rule of formation for sequences, thus they are assumed to result from a random process such as a coin toss. There is one sequence per row, and all sequences are unique differing in one or more positions. He then assigns coordinates to the array of symbols using a two dimensional (u, v) grid.

u \ v	1	2	3	4	5	6	7	8	…
1	0	0	1	1	0	0	1	1	
2	1	1	1	1	1	1	1	1	
3	0	1	0	1	0	1	0	1	
4	1	0	1	1	1	0	1	1	
5	0	1	1	1	0	0	1	0	
6	0	1	0	1	0	1	0	1	
7	1	1	1	1	0	0	0	0	
8	1	0	1	0	0	1	1	1	
9	1	0	1	0	1	1	0	0	
10	1	0	0	1	0	1	0	0	
⋮									
D	0	1	0	1	0	1	0	1	
E_0	1	0	1	0	1	0	1	0	…

fig.1

2.1 orientation

Cantor then defines a diagonal sequence D (red) composed of symbols with coordinates (u, u). The negation of a sequence differs in all positions. Using D as a template, he interchanges all 0's and 1's to produce E_0 as the negation of D or (not D). He declares, E_0 as a horizontal sequence, cannot be anywhere in the list since it will differ from each D coordinate (u, u) .

3. issues

u \ v	1	2	3	4	5	6	7	8	…
1	0	0	1	1	0	0	1	1	
2	1	1	1	1	1	1	1	1	
3	0	1	0	1	0	1	0	1	
4	1	0	1	1	1	0	1	1	
5	0	1	1	1	0	0	1	0	
6	0	1	0	1	0	1	0	1	
⋮									
78	1	0	1	0	1	1	0	0	
79	0	0	0	0	0	0	0	0	
⋮									

fig.2

Before defining D, a sequence and its negation have a simultaneous existence in the list with no problems. One example being u2 and u79, as shown in fig.2. The sequences are independently formed and entered. Since all sequences are parallel, none should interact with any other, nor restrict the location of another within the list.

u \ v	1	2	3	4	5	6	7	8	…
1	0								
2		1							
3			0						
4				1					
5					0				
6	0	1	0	1	0	1	0	1	
7							0		
8								1	
9									
10									
⋮									

fig.3

In fig.3 the sequence of symbols in D can appear anywhere in the list in horizontal form without conflict with the same sequence in the diagonal form.

u \ v	1	2	3	4	5	6	7	8	...
1	0								
2		1							
3			0						
4				1					
5					0				
6	1	0	1	0	1	?	1	0	
7							0		
8								1	
9									
10									
⋮									

fig.4

In fig.4, the sequence of symbols in E_0 cannot appear anywhere in the list in horizontal form without conflict with D in the diagonal form. They cannot coexist in the same list since coordinates of the intersection (6,6) cannot be 0 and 1 simultaneously.

4. conclusion

1. The diagonal D was defined by Cantor using a specific rule of formation, as one element from each sequence with coordinates (u, u). Its orientation allows it to interact with all sequences in the list. The diagonal D must begin at u1 if its purpose is to exclude E_0 from the list.
2. There was no sequence-negation conflict before the diagonal D was defined, as shown in fig.2.
3. The diagonal D differs in orientation from all other sequences, is redundant as shown in fig.3, and if ignored, eliminates the exclusion of E_0.
4. For every sequence beginning with "0" there is its negation beginning with "1", thus they occur in pairs. The list cannot be missing just one sequence. Both must be members of the set M.
5. Cantor defined the argument in geometric terms, thus orientation/direction can be a factor in the analysis.
6. Cantor's argument uses misdirection in the form of the diagonal D. His interpretation of the conflict in fig.4 is the source of his contrived 'contradiction'.

reference

[1] THE LOGIC MUSEUM Copyright © E.D.Buckner 2005