Beyond Natural Geometry: on the nature of proto-geometry

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Abstract

Keywords

In recent years, big claims have been made about geometric knowledge, its universality, and its roots in core cognition. Particularly relevant in this respect are contributions by Spelke and collaborators, as well as the celebrated but disputed results of Dehaene and collaborators concerning knowledge of geometry in the Amazonian group of the Mundurukú. There is little doubt that Ancient geometry takes advantage of the cognitive abilities of human agents in a most noteworthy way, by relying on very robust perceptive, manipulative and conceptual abilities, which furthermore were carefully regimented so as to maximize agreement and eliminate sources of error (Ferreirós, 2016,141). But, unlike other aspects of mathematical cognition like the natural numbers, on which there is substantial agreement among experts, the highly interesting topic of geometry and its cognitive roots is not yet sufficiently clear.

This paper is devoted to the thesis of universality of geometric notions (Dehaene *et al.*, 2006) and to critical reflections about the concept of "natural geometry" employed by Spelke (Spelke *et al.*, 2010). We aim to promote interdisciplinary work in this field, by bringing in logical results and historical evidence, always in a spirit of constructive cooperation with the empirical cognitive sciences. Turning to cognitive history and cognitive archeology in search for clarification of what basic proto-geometry may be, we shall consider very especially the case of ancient China. Particular attention will be given to the analysis of right-angled triangles in China (gou – gu) and the Theorem that is known in the West as Pythagorean, in the East as the Gou-gu theorem; we shall call it simply 'the Theorem'.

A first part of this paper (sections 1 and 2) is devoted to a review of empirical results leading to the theses of core cognition, which are the basis for Spelke notion of a "natural geometry". In the second part, we shall offer preliminary considerations arising out of a comparative analysis of Chinese geometry, and its apparent origins, with other ancient forms of geometric or proto-geometric culture. We will touch on some questions that emerge naturally, as long as there is interesting archeological or historical evidence to bear on them. In particular, we address the early formation of geometric conceptions (notions of circle, square, triangle); links between geometric practice and other forms of practice or knowledge (e.g. astronomical); and the question how to characterize basic geometry and distinguish it from sophisticated elaborations such as Euclid's geometry in the *Elements*.

1. Core cognition and geometry.

The notions of core cognition and core knowledge systems (CKS) have been promoted in recent years by Susan Carey and Elisabeth Spelke (1994). In contrast with the idea of massive modularity of the mind, proponents of core cognition propose that a small number of modules are the foundation of cognition (Spelke & Kinzler, 2007): five modules for objects, actions, numbers, space or geometry, and social partners. Carey (2009) considers core systems for agents, numbers, and objects, the latter including also causal and spatial relations. Core cognition is assumed to be available innately, serving as building blocks on which new cognitive skills will be founded (Vallortigara, 2012). Comparative experiments on infants, rats, monkeys, and even fish and insects, offer evidence for this assumption and for the CKS having a long evolutionary history (Cheng & Newcombe, 2005).

The CKS function automatically, are domain and task specific, and are encapsulated (Spelke & Kinzler, 2007). Each CKS has its own set of 'principles' built in, responding to features of the entities, events or problems within its domain. According to these experts, human adult cognition is built on the basis of such innate core systems, but even though more developed and sophisticated cognitive apparatus will develop, the CKS remain active and encapsulated (Kinzler & Spelke, 2007).

Carey's Program (Carey, 2009; Spelke *et al*., 2010) suggests that we must consider three relevant aspects while investigating cognitive development: 1) specify the CKS which provide the primitive concepts on which the relevant knowledge system will build, 2) describe the conceptual changes that occur during development, establishing similarities and cognitive differences among younger and older children, 3) characterize the processes of conceptual change that cause the emergence of older child's cognitive systems, the maturational changes.

On the basis of empirical evidence, it is argued that our cognitive system gives particular prominence to spatial invariants extracted from the environment; this constitutes geometric or, perhaps better, proto-geometric cognition. The evolutionary rationale, following Gallistel's interpretation, is that "geometric properties are likely to remain stable in the face of changes in featural properties" (Cheng *et al*., 2013,1034): geometric properties of our environment, like the distance between two trees, do not change significantly over time, while featural properties do (they may drop leaves or be covered with snow). Thus, it is evolutionarily sound to give preference to spatial invariants of the environment, and thus we have a core system encoding this kind of information.

A number of studies have led to the assumption of two CKS responsible for the origin of geometric knowledge. One the one hand, a system for large-scale navigation in the environment, "representing the distances and directions of large-scale, extended surfaces", and on the other a system responsible for the recognition of forms and the categorization of objects, "representing landmark objects and surface markings" (Spelke & Lee, 2012). Proto-geometric cognition is, according to this, the maturational outcome of the development of these core systems: the *navigational module* and the *landmark module*.

Geometric knowledge has to do with the metric relations of angle, distance and sense (direction) in the environment, which jointly determine its geometry.[1] Now, each of the relevant CKS is specialized and neither in isolation grasps all of the relevant

[1] This provides a sound characterization of geometry, also from a purely mathematical point of view. Although other axiomatic systems (like Hilbert's) are more usual, vector axiomatics corresponds to the relations of angle, distance and sense or direction.

geometric relations. The navigational module encodes properties of sense and distance, while the landmark module grasps properties of angle and distance (Spelke *et al.*, 2010; Spelke & Lee, 2012). Only the combination of both into a more encompassing cognitive system is able to supersede their limitations and give form to mature geometric concepts and intuitions.

How is proto-geometric knowledge developed? What are the relevant cognitive mechanisms? There are two main tentative hypotheses trying to explain how the two CKS get interconnected and expanded into a system encoding the three fundamental properties of angle, distance and sense. The first (Spelke *et al.*, 2010) postulates that the use of some cognitive tools, like maps, scale models, or pictures, makes the two systems come together. The second (Shusterman & Spelke, 2005; Pyers *et al.*, 2010, Spelke & Lee 2012) considers language responsible for the connection, so that the two CKS combine "when children begin systematically to produce spatial expressions, including the terms left and right" (Spelke & Lee 2012, 2790). At any rate, these are just two hypotheses, the second being more in line with proponents of modularity and nativism; at the same time, it has been argued that there is "substantial evidence to doubt that language is crucially important for the integration of geometric and feature information in humans" (Twyman & Newcombe, 2010,1325; see sec. 3 of that paper)

From the standpoint of cognitive archaeology, there is a very clear preference for the first type of hypothesis (Malafouris, 2010; Overmann, 2013). The evolutionary increase in size of the human brain is associated with the making of tools, starting some 2.5 million years ago (*homo habilis*). With *homo erectus*, the manufacture of Acheulean tools -principally handaxes- required advanced spatial cognitive abilities (the active coordination of dorsal and ventral information from the primary visual cortex) and hierarchical organization of action, mechanisms of cognitive control not found in the tools of earlier hominids (Coolidge & Wynn, 2016,387). Overmann (2013,35) argues that material culture "scaffolds" the development of explicit concepts of number, which in turn help structure timekeeping. She makes the case that developed number systems are never found in societies of low material complexity (Overmann, 2013,25), and a similar case can be made for geometric concepts. Giardino (2016) has argued that explicit external representations created with tools have been a key element in the emergence of both numerical and geometric notions (see also de Cruz, 2012).

In the literature, there is also extensive discussion (Izard & Spelke, 2009) of exactly when these two core systems emerged, each apparently with a different evolutionary function, and how each one of them evolves in the cognitive development of the subject. Equally, there is a long tradition of research on how widespread these core systems are along the animal kingdom, as one can see in the general survey (Spelke & Lee 2012), or in studies about fishes (Lee *et al.*, 2013), rats (Cheng, 1986), or even insects (Cartwright & Collett, 1982)

Three main types of experiments have been performed in support of the above-mentioned picture. First, experiments based on a disorientation paradigm, where subjects (fish, rats, children aged 4 or 5) are placed in a room with a certain geometric shape and presented a certain goal situated in a corner (food, an exit, a toy). After training and subsequent disorientation, the subject is allowed to search again in the room (Cheng 1986, see the review in Spelke & Lee 2012). Subjects make use of geometric information while searching for the goal, ignoring visual tracks such as painting on the wall, drawn figures, etc. The geometric information about the shape of the room is provided by the navigational CKS, as proven by the fact that only information about length and direction of the walls is employed; angular information is ignored, which gives rise to characteristic errors.

In a second type of experiment, human children are presented with a simple map of a room with objects disposed in a particular geometric shape (Fig 1). The subjects are asked to place an object, say a doll, in the corresponding place as shown in the map (Spelke *et al.*, 2010; Winkler-Rhoades *et al.*, 2013). The maps employed are as abstract as possible, they represent simple figures in 2D, no kind of iconic element is presented, and thus a rather abstract correspondence with the 3D environment has to be established. In this case, characteristic mistakes are made by children below 6, which can be explained by their inability to employ angular information – if the map represents objects forming an isosceles triangle, the children will distinguish clearly the top from the base, but may exchange the two corners at the base. Thus, only around the age of 6 are children able to extract angular information with an accuracy comparable to their grasp of distance and directional relations.

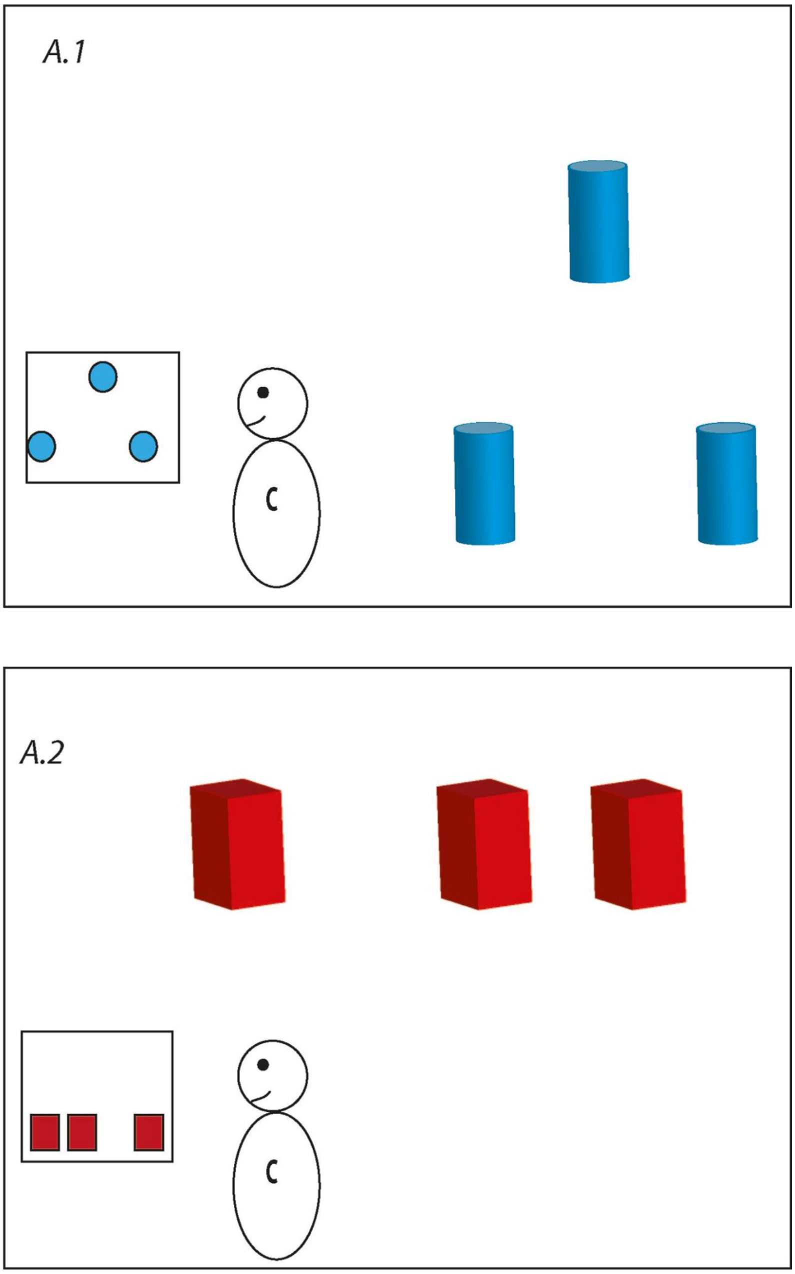

Fig. 1 We have here two different situations: In A.1, we see a child with a map representing an isosceles triangular disposition; in A.2, a child with a map representing a linear disposition; in this second task, we can see that the orientation of the map relative to the array has been changed. In such experiments the subject has to put a doll in the array in the same geometric disposition that the map represents.

This second type of experiment is much more complex than the first. Experimenters have found more failures when the child is confronted with the map while staying in the middle of the represented environment – they can better handle the overall picture of what the map represents when they are outside the depicted environment. Even so, "children represented at least some of the spatially invariant properties that link a map to the world to which it refers despite differences between the map and array in size, mobility, orientation, dimensionality and perspective" (Winkler-Rhodes *et al.*, 2013, 373).

A third type of experiment was performed for the first time by Dehaene and collaborators (2006) to test whether a population uneducated in mathematics, with a poor mathematical vocabulary, and not used to maps (the Mundurukú in Amazonia) possessed some kind of spontaneous geometric knowledge (Fig 2). The point was to evidence the universality of geometric cognition, at least in the sense of showing that the possession of certain geometric intuitions is universal. Subjects were presented with groups of six drawn figures and asked to identify the "weird" one, the deviant shape. Here are some examples:

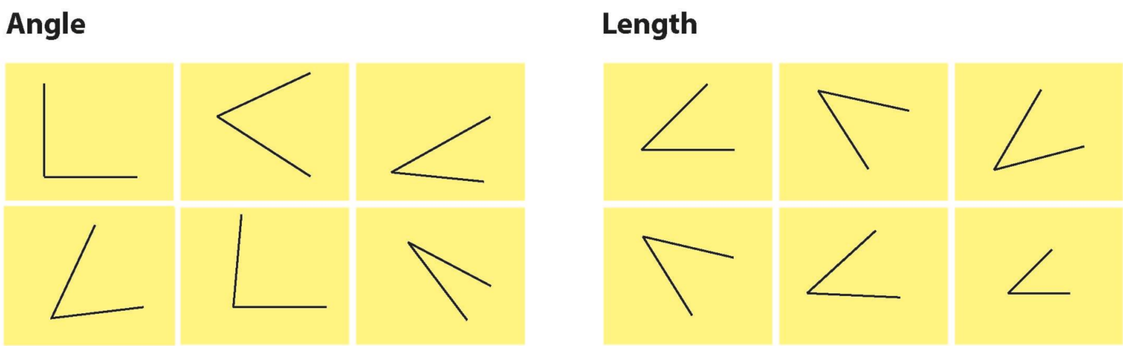

Fig 2. This type of experiment aims to test sensitivity to visual forms. Experimental subjects are asked to find the "weird" picture among six that are shown. On the left, sensitivity to angular information is tested – the first picture is different, a right angle; on the right, the different picture is the last and smallest one.

The Mundurukú answers were controlled with groups of infants and adults from the United States, and they turned out comparable in accuracy to the children responses, even though children in the USA receive mathematical education. The conclusion is that both groups, culturally very different, answer in similar ways due to sharing innate CKS which provide them with geometric intuitions. Thus, geometric intuitions would be universal, even if their further elaboration within cultures may give rise to variations. Furthermore, Dehaene and collaborators (2006) conclude that human geometric intuitions are essentially Euclidean, insofar as very different people are able to extract angle, length and sense information from a given scene.[2]

In what follows we shall accept the conclusion that two basic CKS are available for dealing with geometric features of the environment, a navigational module and a landmark module that recognizes forms and objects. Nevertheless, as we shall see, there are reasons to criticize the conclusions of Spelke, Dehaene and collaborators, or at least the concepts in which those conclusions are framed.

2. On Spelke's "natural geometry".

[2] Prevalence of difficulties to solve certain problems, e.g. about mirror symmetries which seem to be one of the most difficult tasks. Combine here text and footnotes 9, 10.

On the basis of the results and ideas that we have surveyed, Spelke & Lee (2012) identify their target as "natural geometry":

> we develop this hypothesis [about core cognition] by focusing on the domain of Euclidean geometry and its fundamental relations of distance, angle and direction. Euclidean concepts have three striking properties. First, they are extremely simple: just five postulates, together with some axioms of logic, suffice to specify all the properties of points, lines and forms. Second, they are exceedingly useful: almost all human cultural accomplishments depend on these concepts, from the measurement of space and time to the pursuit of science, technology and the arts. Third, the objects of Euclidean geometry go beyond the limits of perception and action: points are so small they have no size and so cannot be detected by any physical device; lines are so long they cannot be fully seen or traversed. (Spelke & Lee, 2012,2785)

While one can grant that geometric knowledge is exceedingly useful, underlying such diverse cultural accomplishments as the measurement of time, the creation of buildings, the design of artistic work, the study of the visible universe, and many others, there are a number of critical comments that must be made concerning the other two points.

Spelke & Lee equate their "natural geometry" with "Euclidean geometry and its fundamental relations of distance, angle and direction." But here lie many complexities and logical difficulties. Their belief that Euclidean concepts are extremely simple, requiring "just five postulates, together with some axioms of logic," reveals a surprising lack of awareness of the general conclusions of detailed mathematical and foundational studies. Since the time of Pasch and Hilbert (end of the 19th century) it is well known that, *precisely if* one wants to base geometry on simple principles of logic, what is required is a system of axioms that goes far beyond Euclid's famous five postulates. On the other hand, as Ferreirós (2016, chap. 5) has suggested, a highly reasonable interpretation of Euclid's five postulates is that they are not primarily axioms in the modern sense of Hilbert, but rather carefully regimented *stipulations that rule the construction of diagrams* -see also Manders (2008). Rules for construction, that is, betraying the central role of practical work with ruler and compass in the practices that accompany the reading of Euclid's text.

Much better is the identification of Euclidean plane geometry with a system based on fundamental relations of distance, angle, and direction – relations which have the advantage of being directly tied to the needs of navigation in the environment. Notice however that now we are talking about the basic notions of a modern theoretical system (Euclidean geometry) which a careful scientist should distinguish from Euclid's geometry as presented in the *Elements*. Furthermore, we want to argue that cognitive grasp and practical handling of some "fundamental relations of distance, angle, and direction" in a natural environment is not to be construed as possessing knowledge of Euclidean geometry. The crucial reason is suggested precisely in the third point underscored by Spelke & Lee, namely the idealizations involved in both Euclid's *Elements* and modern Euclidean geometry.

We could elaborate on this for a long time, but let us go immediately to the main issue: *neither Euclidean geometry (in any of its modern versions) nor Euclid's geometry in the* Elements *can be a good candidate for representing basic, practical geometry*. Euclid's theoretical development builds on the basis of idealizations that far outstep the limits of a purportedly "natural" understanding of geometric features in any environment. The difference between Euclid's ideal figures (perfect circles and triangles, invisible points, etc.) and drawn figures is an old topic since the times of Plato and the Neoplatonists.[3] It is not our intention to accuse Spelke or Dehaene of ignoring this simple, traditional point; but they have downplayed it in a way that is not defensible, falling back on rhetorical abuse of this identification of the "natural" with the "Euclidean".

The fact is that the label "natural geometry" clearly presupposes too much. In the current context of cognitive science, it suggests that human beings will develop geometric knowledge in the wild, i.e., regardless of whether they live in the Amazonian jungles or in the urban jungle of New York. A scientist should avoid highly charged and rhetorical terminology, to prefer more objective descriptions. Given that Spelke and collaborators, like many others, accept the idea that full geometric knowledge needs cultural ingredients and cognitive tools to develop, we find that it would be more adequate to employ denominations like "proto-geometry."

The accusation of rhetorical abuse is strong, so let us explain it. Consider the famous paper (Dehaene *et al.*, 2006) on "core knowledge of geometry" among the Mundurukú. Fig. 1 in that paper (Fig. 3) presents some of the test slides that were used in the study with this group of Amazonian indigenes.

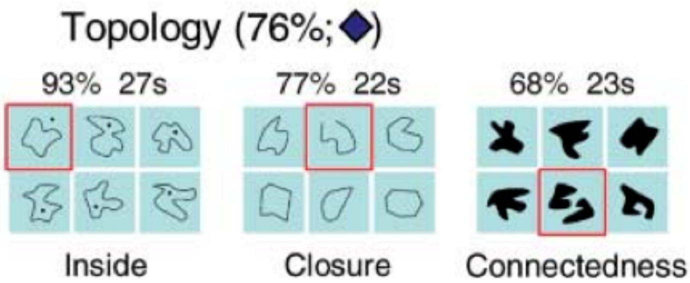

Fig 3: Three slides from (Dehaene *et al.*, 2006,382)

While it is certainly true that the notions of inner point, closed path and connected domain are important in the theoretical field of topology, it is certainly an abuse of interpretation to report the empirical findings obtained by Pica saying, e.g., that 93% of the Mundurukú subjects displayed knowledge of the topological distinction between inside and outside. It would be worthless to devote too much time to discussion of this issue. The reader will easily convince herself that the intuitive thinking behind recognition that the top-left of the first six pictures is "weird" or different, may be based on many possible aspects. Many European children, for instance, will look at those slides as pictures of rabbits or wolves or other animals (including some weird ones), except for the first (top-left) which has the eye out of place.[4]

What we just described as an *eye*, is interpreted by Dehaene *et al.* as a *point* in most of the slides. Of course, a drawn point can be many things – and in a way Dehaene's description is to be preferred because it is more neutral, more objective. But in the interest of objectivity, one should then stress that we mean a drawn point and *not a Euclidean point*. The student of Euclid's *Elements* must learn to see a point as having no dimensions at all, absolutely "no parts" as his Def. 1 makes clear. (We shall come back to the notion of point below, in connection with Chinese geometry and with the

[3] See Plato, *Republic* 510d-511a, and Proclus in particular (Morrow, 1992). Interestingly, our current cognitive-science nativists prefer to sidestep this traditional topic of innatist conceptions of geometry. For an analysis in terms of conceptual hypotheses, see Ferreirós (2016, chap. 5).

[4] An ethnographic study among the Mundurukú could be done, to see how they would describe the situation.

history of Greek geometry.) Similar remarks and perhaps even stronger reasons for suspicion should apply to the slides that Dehaene *et al.* interpret as displaying the notions of "parallel lines" and "secant lines" (see below).

To summarize, there appear to be sound experimental reasons to accept the existence of CKS grounding some basic geometric intuitions, and providing intuitive access to some fundamental relations of distance, angle, and direction. But one must be careful to try to differentiate carefully between this basic cognitive access and the possession of knowledge of Euclidean geometry.

The point is related to a general problem facing this type of studies. This is the problem of conflation of advanced mathematical concepts with basic cognitive 'givens' – Euclidean points or lines in the case of geometry. There is often a process of *reification* of a subtle and complex theoretical framework (in this case, Euclidean geometry), which is *projected back* from the realm of explicit theory into supposed cognitive systems inside our minds and brains.[5] This happens not only while studying geometric cognition, but also in the more basic domain of number cognition. Thus, the system that Dehaene described soberly as an "accumulator" (which captures imprecisely the 'size' of displays of objects or sensorial stimuli bigger than 4 or 5) is characterized by Spelke *et al.* (2010, 874-75) as a system for "comparing sets," i.e., for representing sets and their approximate cardinal values. Here, the mathematical *concept of a set*, which has very recent origins (about 150 years ago) and which arguably has no intuitive counterpart at the level of commonsense intuitions,[6] is again reified and projected back into our brains. The case would merit careful discussion, but we cannot enter into it here.

3. Cognitive and cultural origins of geometry.

Experimental results like the ones obtained by Pica among the Mundurukú, and studied in Dehaene *et al.* (2006), show at most the availability of certain intuitions – and the disposition to form associated concepts – at the level of spatial cognition in the above sense. (Incidentally, even the Mundurukú have a word for 'circle' – *iroyruy'at* meaning curved figure, circle –, probably because their camping sites are arranged in a more or less circular form; but they do not have words for square, rectangle or triangle.) One should probably avoid broader claims about 'geometry' or 'topology' reserving these terms for explicit conceptual and symbolic developments. There is empirical evidence about geometry, indeed, but one must look for it in experiments with subjects trained in mathematics, or else in the historical and archeological record. In fact, we shall explore this venue of inquiry, paying attention to ancient geometric forms of knowledge in diverse cultures, with the aim of making comparative analyses.

Let me offer another example. Among the tests classified by Dehaene *et al.* (2006) as having to do with Euclidean geometry, we find the following three:

[5] Spelke is known to come from an original nativist viewpoint, in the tradition of Cartesianism and Kantianism, and of course the label "natural geometry" fits nicely with her preferred view; in Spelke et al. (2010, 863), she introduces the label with direct reference to a passage of Descartes's *Geometrie*.

[6] See Ferreirós (2007) and especially the Appendix. The key point is that a set is a mathematical object, submitted to operations analogous to the operations on numbers, while the collections, groups or classes we encounter in everyday life (and about which we have intuitions) are completely unlike that.

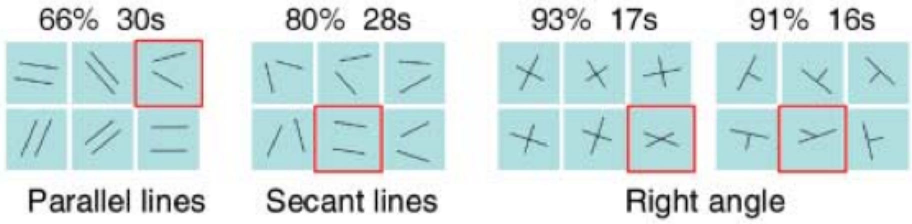

Fig 4: Three more slides from (Dehaene *et al.*, 2006,382)

We learn here that 66% of the Mundurukú subjects tested were able to distinguish as "weird" or different the convergent lines on the first test (compared to the others, which we categorize as parallel), and 80% of them regarded as different the parallel lines in the second test (while the others are what we call convergent). One may construe this as showing the availability of certain intuitions about line directions, but to present the evidence saying that the Mundurukú would develop – under certain unspecified circumstances – a form of Euclidean geometry, would be clumsy and unconvincing. If we aim to find evidence for the universality of some kind of geometric *concepts*, close to what became Euclid's geometry, we will need stronger results.

In connection with the critical remarks of the previous section, one ought to make distinctions between *visual* cognition (including concrete features and cues), *spatial* cognition (more abstract, dealing with relations of angle, length, etc.) and *geometry* strictly speaking (which is explicit knowledge in symbolic form) – not to mention even more abstract theories like topology. Contrary to what the label "natural geometry" seems to suggest, geometric ideas are not developed under all circumstances of human life and culture. As far as one can judge, sophisticated forms of handicraft and explicit measurement practices seem to have been in place in all human societies that developed explicit forms of geometry – the ancient Chinese texts quoted below support this point. (This should be taken into account to nuance Spelke's remark, quoted above, that Euclidean concepts made possible "the measurement of space and time;" the actual situation seems to be more complex, with ideas and practices from both areas, geometry and measurement, influencing each other.) It is exactly the same with a precise concept of number – to be contrasted with intuitions of numerosity – and with arithmetic: they are not found under all circumstances, and their emergence is linked with concrete practices such as counting.

One can *investigate the practices and the material culture* underlying the emergence of geometric concepts, but this is mostly *a task for cognitive archeology and cognitive history*. As regards archaeology, pioneering work such as (Keller, 2004a) argues for the importance of a proto-geometric period that would cover the whole Paleolithic era, when tool manufacture, decoration of objects, and symbolic representations of figures attest to the emergence of sophisticated processes of spatial cognition. According to Keller, the peoples of the Upper Paleolithic "invented the surface of representation, the figure in general together with its elements (line and point), and certain basic figures, such as the rectangle and circle" (Keller, 2004b,94). Discussing Paleolithic work in general, he argues:

> It was geometry in the sense that it involved *thoughtful* work applied to a [local] *space* (a pebble, the wall of a cave, the surface of an object carved from bone or ivory) in order to obtain a *preconceived figure* (a sharp-edged flake, carved pebble or rock, handaxe, Levallois flake, retouched blade, microlith, painting, engraving, drawing or

stencil), through a process of *structuring* space (two perpendicular symmetrical planes for the handaxe, the system of Levalloisian debitage, the interior/exterior of a representation, friezes, etc.)." (Keller, 2014,1)

In our terminology, such concrete practices have to do with proto-geometry and with the emergence of geometric conceptions. We shall find further examples in what follows, having to do with much more recent periods (less than 100,000 years) which led to the Neolithic revolution. Starting around 100,000 years ago, we find particularly significant cognitive development (Coolidge y Wynn 2016), with personal ornamentation, cave art, ritualized burial, technology of bow and arrows, enigmatic figures, and so on.

A great opportunity for inquiry is afforded by the richness of historical evidence concerning geometric knowledge in Ancient cultures, e.g. in the Far East. We shall explore the case of China, which is quite well understood historically. Even better, this comes with a rich archeological record in which geometrically shaped objects do play a relevant role. We shall see some examples from the proto-Chinese Liangzhu culture, dating back to the 3rd millennium BCE, some 4500 years ago.

Spelke *et al.* (2010, 865) acknowledge the role of "uniquely human, culturally variable artifacts" such as pictures, models, and maps, in making possible the combination of core cognitive ingredients into the more complex and robust cognitive system that enables us to grasp spatial concepts (like square and round). It is well known that Euclid's work builds using only straight lines and circles, conforming a geometry of *ruler and compass* (which is idealized in the form of very simple assumptions made explicit in the postulates).[7] Taking this into account, it is highly interesting to find similar ingredients in the ancient Chinese tradition. In both cases, the emergence of explicit geometric knowledge is linked with the ruler (more precisely, the trysquare or *gnomon*) and the compass, but also with arithmetic and measurement practices.

The famous Chinese mathematician Liu Hui (3rd century), in his preface to the *Nine Chapters*, writes about mathematical training as it existed BCE:

> Mathematics (*suan*) is part of the six arts: the ancients employed them to select people of talent, to instruct the children of high dignitaries. Though it is called the «nine arithmetical (*shu*) arts», they enable us to exhaust the subtle and to reach the tiny [infinitesimal], to explore without limits. As regards the transmission of methods (*fa*), one can certainly make common knowledge, as with the trysquare (*ju*), the compass (*gui*), the numbers and measurement; there is nothing there particularly difficult. (translation following Chemla & Shuchun (2005,127))[8]

Thus, also in the Chinese tradition problems of drawing figures and of measurement were approached using traditional methods based on what basically are a compass and

[7] It may be relevant to mention here that Plato criticized geometers (including the great Eudoxus) for relying on mechanical devices (see Plutarch (1917), *Life of Marcellus*, cap. 14) and for looking for practical goals such as military applications (*Republic*). This is relevant as it shows that there was a strong Greek tradition emphasizing the link between geometry and mechanics – so not all Greek mathematicians agreed with Plato's highly idealized conception. Some experts believe that even Archimedes belongs in this mechanical tradition.

[8] Compare with Shen *et al.* (1999,53), the last sentence here reads: "The course [of the arithmetical arts] is not particularly difficult using the methods (*fa*) which have been handed down from generation to generation, just like the compass (*gui*) and trysquare (*ju*) in measurement, with which we draw figures."

ruler – the trysquare is an L-shaped instrument, in Greek terminology a *gnomon*, used to draw straight segments and right angles, and to find distances.

A famous Chinese text that is even earlier, and according to some may date from several centuries BCE, contains a brief conversation between the Duke of Zhou and an expert skilled in mathematics and astronomy, Shang Gao (probably a state officer). They discuss the origins of geometric and arithmetic knowledge, which in their view is intimately linked with the knowledge of Heaven and Earth – the main topic of the *Zhou bi*, which is the classic astronomical text of the Chinese tradition. And, among other things, they underscore the role of some instruments:

> The Duke of Zhou exclaimed 'How grandly you have spoken of the numbers! May I ask how the trysquare is used?'[9]
> Shang Gao said 'The level trysquare is used to set lines true. The supine trysquare is used to sight on heights. The inverted trysquare is used to plumb depths. The recumbent trysquare is used to find distances. The rotated trysquare is used to make circles, and joined trysquares are used to make squares.' (Cullen, 1996,174)

The trysquare was useful for carpenters and architects, but also prominent in astronomy and geometry. The Greeks, too, assigned an important place to the gnomon (see Cullen (1996) for its history). Anaximander (610–546 BCE) is credited with having introduced this Babylonian instrument; Oenopides of Chios, around 450 BCE, used the phrase *drawn gnomon-wise* to describe a line drawn perpendicular to another.[10]

If we zoom out and include archeological evidence, some quite interesting notions suggest themselves. These have to do with the emergence of basic geometric *concepts* in relation to cultural and material practices; also, linked with it, the emergence of some geometric *problems*, including famous ones; and, in particular, the interrelations between geometry and astronomy, which can be found in many different cultures (of course also in ancient Greece). Notice that explicit geometric figures, and the corresponding linguistic terms, express for the first time not merely intuitions of form, but *intuition + concept*, a precise conceptual idea that opens the question, for the first time, to the possibility of precise investigation and rigorous results. Thus, in the concepts of square and circle, and the corresponding (carefully crafted) drawings or material pieces, one finds at the same time geometric form and exact measure.

From a standpoint that might be called “archaeogeometrical,”[11] one should begin with the question how the most elementary geometric concepts are formed. This is a topic discussed by Giaquinto (2007, chap. 2 & 3), who argues for the crucial role of visual thinking, based on perceptual contents, in the formation of the basic concept of ‘square’ and the elaboration of a belief that “A square is symmetric in terms of its diagonal.” This is of course reminiscent of the celebrated passage of Plato (in the dialogue *Meno*) where an uneducated slave is able to form geometric ideas and come to true conclusions about them. Symmetry considerations seems to have been decisive

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[9] Shang Gao had previously replied: “The patterns for these numbers come from the circle and the square. The circle comes from the square, the square comes from the trysquare, and the trysquare comes from [the fact that] nine nines are eighty-one.” (This last expression is a reference to the multiplicative table.) As one can see, geometry and arithmetic are curiously mingled, and linked with drawing and measuring instruments. All of this will become clearer in what follows.

[10] Heath (1981,78-79). Diogenes Laertius, "Life of Anaximander", in Book II of: *The Lives and Opinions of Eminent Philosophers*, trans. C. D. Yonge. (1853, Bohn). [see http://www.classicpersuasion.org/pw/diogenes/dlanaximander.htm]

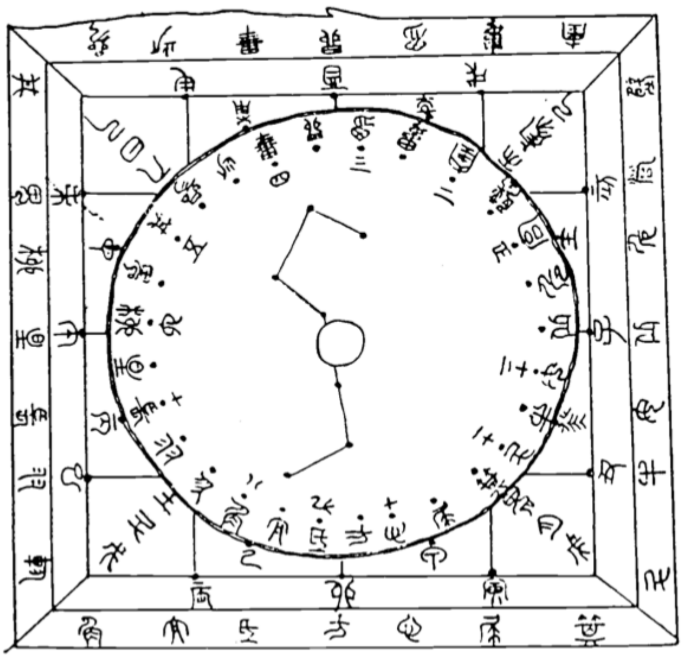
[11] Arqueoastronomy is by now a well-established and highly interesting field of studies, but the corresponding analysis of proto-geometric notions is not well developed.

indeed in the case of many basic concepts, such 'square' and 'circle,' and archaeological material from ancient China is rich in this respect.[12]

First, there is evidence for the very early formation of some basic geometric ideas, in particular *square* and *circle*. Rich Jade pieces found in tombs of the Liangzhu culture include very prominently the so-called *bi* disks and *cong* tubes (some 4,500 years ago); the meticulous care and perfection with which the circular shapes (*bi*) were crafted is worthy of attention. Second, in all likelihood that process was promoted by ritual and mythological connotations, so that one encounters the importance of cultural ingredients as driving forces in the attention to geometric shapes. This explains why the *bi* disks and *cong* tubes were prominent in rituals, as cultural elements of high prestige. At some point before the common era, perhaps 300 BCE or a bit later, the following sentence was written:

> The square pertains to Earth, and the circle pertains to Heaven. Heaven is a circle, and Earth is a square. (preface to the *Zhou bi,* Cullen 1996, 174)

Indeed, in *gai tian* cosmology, the drawing of a square and circle became a symbolic representation of the world (Fig. 5); as the poet said: *The square earth is a chariot; / The round heaven is its canopy*.[13] Thus the symbolically very prominent question of Earth and Heaven led to the study of square and circle, with their perfectly equal sides, and equal radii. It is relevant to add here that the part of the *Zhou bi* dealing with circle and square is deemed probably its oldest (see figure below).[14]



[12] The evidence of a relevant presence of symmetry configurations in many cultures, *independently* of each other, is strong. Consider Babylonian bronze friezes, like the famous Frieze of Archers from Suse (around 500 BCE) or the Assyrian tree of life (throneroom of Ashurnasirpal II, around 850 BCE, in the British Museum); Chinese pieces like the Jade head from Liangzhu culture ((late Neolithic period, ca. 3300-2250 BCE), or some beautiful Aztec work like the Double-headed serpent (around 1500; turquoise pieces applied to a wood baser).

[13] The chariot served as a scale model for the whole World, old chariots having a square form and bearing a circular canopy.

[14] On all of these topics, see Cullen (1996).

Fig. 5. *Shi* cosmic model, early second century BCE, from Cullen (1996), 45.

Thirdly, ancient Chinese scholars were thus led to consider the relations between a circle and the square circumscribed around it (Fig. 6). This led to "the methods of square and circle," about which it is said:

> The square and circle are of universal application in all activities of the myriad things. The compasses and the trysquare are deployed in the work of the Great Artificer. A square may be trimmed to make a circle, or a circle may be cut down to make a square. (Cullen 1996, 182)

Investigating the circle inside the square led to a rough estimate of their lengths, with 4 as the (exact) dimension of the perimeter of the square and 3 as the (very rough) dimension of the circumference. This question opens the celebrated problem of the measurement of the circle in comparison to its diameter = the side of that square; in effect, relations between the circumference of a circle and the perimeter of a square built around it, constitute a prominent element in chapter 1 of the *Nine Chapters* (see problem 31).[15]

Figure 14. Circle and square.

Fig. 6: Diagram that accompanies text about "the methods of square and circle." From Cullen 1996, 182.

Fourth, as in other cultures, one finds the association of explicit geometry with astronomical considerations: in China, the *gnomon* or trysquare was from early on central to observations (and measurement) of the shadow cast by the sun, with a view to the management of time and the calendar. Ancient regulations known as the 'Rites of Zhou' included stipulations about the need to regulate the seasons (the calendar) on the basis of observations done with a gnomon (*bi* or *biao*). This simplest and most ancient astronomical instrument, basically just a vertical post erected so that one can measure the shadow thrown by the midday sun, seems to have been used in ancient China for centuries before the common era.[16] Now, the gnomon and its shadow form a *gou* and *gu*; the configuration as a whole presents us with a right-angled triangle.

[15] Some of the procedures proposed here – though only some – are perfectly precise: "Another procedure: multiplying diameter and circumference one by another, divide by 4."

[16] One can hardly rule out the possibility that the ancient Chinese, too, got to know the use of the gnomon from the Mesopotamians. But here one must remember Hoyrup's views: "There are some similarities between what we find in the last of the *Nine Chapters* and the things that develop in the Near East at a

Fifth, it is these activities with the gnomon ("the supine trysquare is used to sight on heights" and to measure the position of the sun) that in all likelihood have promoted the study of gou – gu in ancient China, i.e., the theory of right-angled triangles. In fact, these topics are found in the *Zhou bi*, the astronomical treatise, but not in the *Suan shu shu* (around 200 BCE); they will later be developed into the material in the *Nine Chapters* (see below).

4. Geometric knowledge in ancient China: gou – gu and the Theorem.

The Theorem states that, in a right-angled triangle, the square on the hypothenuse is equal to the sum of the squares on the other two sides. Its great relevance is due to several factors. First, concrete applications like a triangle of sides 3, 4 & 5 offer a practical method for constructing right angles (the Babylonians knew this and many other examples of "pythagorean" triples, e.g., $5^2 + 12^2 = 13^2$). Second, the result establishes a deep connection between arithmetic and geometry, and it became the basis (much later, after Descartes) for the definition of distance. Third, the Theorem also leads to seemingly paradoxical results, as when applied to the simple case of a triangle with the two sides equal (the hypothenuse is then $\sqrt{2}$, a 'number' that was only accepted by Westerners in modern times). Fourth, the result can be generalized to all triangles with a right angle, and it admits many different proofs – the Chinese offered detailed justifications in the 3rd century, using cut-and-paste methods (and thus different from Euclid's proof which culminates book I of the *Elements*).[17]

But there is a very interesting, fifth reason to pay particular attention to the Theorem. This result, when considered in an axiomatic context incorporating the characteristic idealizations of Euclid's geometry, is equivalent to the Axiom of Parallels. Famously, the Parallel Axiom distinguishes Euclidean geometry from its non-Euclidean counterparts, and is actually needed to prove the Theorem.[18] (Something similar happens with other geometric results as well, in particular with the famous theorem that the angles in a triangle add up to 180° (two right angles) – but in fact this theorem, prominent among the Greek, was never mentioned by the ancient Chinese.) To put it differently, the geometric knowledge of the ancient Chinese (as it existed BCE) *implies acceptance* of the Parallel Axiom, at least if we are to analyze its logical basis within an axiomatic framework of Euclidean kind. Thus, the historical evidence of ancient Chinese mathematics and its comparison with Greek geometry, both being independent, does indeed offer some basis for a claim concerning the *universality* of Euclidean geometry. But this is a delicate matter; we come back to it in section 5.

To introduce the topic, let us begin with an example that offers a taste of ancient Chinese work. The famous Problem 13 in the last of the *Nine Chapters* is as follows: Suppose there is a bamboo of 1 unit height but which is broken, so that its extreme touches the ground at a distance of 3 deciunits from the base – at which height was it broken?[19] The answer is: 4 units 11/20 deciunits. And, as usual in the text, next comes the specification of the procedure to be used.

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slightly earlier moment … – but also so many differences that we must be confronted either with a case of independent development or with one of creative transformation."

[17] On all of this, see among others Stillwell (2010, chap. 1).

[18] For non-Euclidean geometry, see Stillwell (2010, chap. 18).

[19] Obviously we have 'modernised' the units, in the original it's 1 *zhang* of height and 3 *chi* from the base (10 chi = 1 zhang).

The procedure in question can be represented in modern mathematical language with this equation (*b* being the sought height, *a* is the distance from the base, *h* the total height = *b* + *c*):

$$\tfrac{1}{2}\,[\, h - a^2/h \,] = b.$$

In the *Nine Chapters* this is described verbally as a sequence of steps: multiply the gou (*a*) by itself, divide by the height *h*; subtract what you obtain from the height of the bamboo, and take half of it; the result is the gu (*b*) that you wanted. The deep question is, how can one obtain this surprising algorithm?

From the work of the commentators of the *Nine Chapters*, and especially of the famous Liu Hui (the Chinese Euclid, who lived in the 3rd century), it seems that the answer may be as follows. The writer of *Nine Chapters* explored the problem using the Gou-gu Theorem (what Westerners call Pythagoras theorem): we know that the square of $c = h - b$ equals the sum of the squares of gou (the base *a*) and gu (*b*). If we draw a big square with side *c*, and inside it a square of side *b*, as follows in fig. 7:

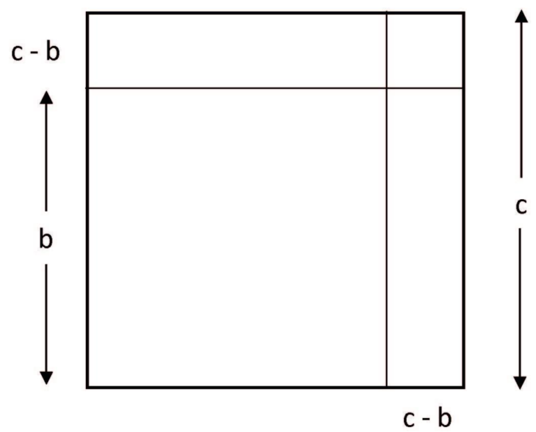

Figure 7. Geometric representation of the problem.

We see, on the basis of the Theorem, that the square of side *a* must equal the two rectangular areas. And from this one can conclude that $(c - b)(c + b) = a^2$ [a noteworthy, purely geometric way of obtaining a result related to a famous equality that we learn algebraically: 'sum times difference equals difference of squares', $(c - b)(c + b) = c^2 - b^2$]. Transforming that basic equality, with the aim of finding an explicit expression for *b*, one can find the procedure of the *Nine Chapters*.

How did Chinese knowledge of triangles emerge? How did they come to study gou – gu and to discover the Theorem? The most important known text before the *Nine chapters*, the *Suan shu shu* written around 200 BCE, is noteworthy because it does *not* contain material on right-angled triangles. Meanwhile, the astronomical classic *Zhou bi* (hard to date, probably around 300 BCE, but with material from previous centuries) does employ the Theorem as a basis for the solution of some simple problems, like finding the length of a half-chord in a circle (Cullen 1996, 81). On the basis of this and related evidence from the classic texts, it is now widely accepted that the topic of *gou-gu* emerged from within the context of astronomy, to become a key chapter of mathematical knowledge. This in effect represents a general pattern to be found in many different cultures: interest in geometry has historically been closely associated to astronomy, and indeed sustained by the interest in astronomical studies.[20]

Ancient regulations known as the ‘Rites of Zhou’ included stipulations about the need to regulate the seasons (the calendar) on the basis of observations done with a gnomon (*bi* or *biao*). The gnomon and its shadow form a *gou* and *gu*; the configuration as a whole presents us with a right-angled triangle. By tradition we follow the Greek in talking about “geometry” here, but one could also talk about topo-metry or even astro-metry, and it is necessary to keep in mind that the initial problems that gave rise to such traditions included the measuring of *time* and not only spatial configurations. In general, one must insist on the importance of the astronomical context, even before drawing came to play a central role in the study of geometric shapes.

Needless to say, the study of gou-gu eventually surpassed the limit of problems of interest to astronomical issues, becoming a subject of its own. This is what we find in the last of the *Nine Chapters*.

It is time now to go back to our most general question:

5. What might basic geometry be?

Let us try to offer a constructive alternative to the excessively simplistic notion of a “natural” geometry. *Basic, practical geometry* is likely to be *a rough body of knowledge*, lacking the idealized perfection that is postulated for Euclidean figures. A ‘point’ here would be a drawn, visible point – not the Euclidean point; a line would be a concrete, finite segment – not an infinite line (which in the history of geometry appears around the 17th century). This corpus would have to do with conceptions of geometric forms (circle, square, triangle, etc.) and their precise relations – linked with the concrete drawing of figures or crafting of material pieces (think of *bi* disks) by means of instruments. This basis of geometric notions on concrete material practices, on the use of instruments, and its relations with *measurement* (indicated already above), have consequences that ought to be carefully taken into account.

Assuming as we have done that the theory of two cognitive modules, the navigation module and the landmark module, is well grounded in experiment, one should still differentiate what this implies from any form of proto-geometric knowledge. An example of this might be, say, the ancient Chinese considerations about a circle inscribed on a square, and their relative sizes – considerations which led to the idea that, the perimeter of the square being 4, the circle will measure 3 more or less. (Incidentally, the same estimate is found in the *Bible*, book of Solomon.) The problem emerged in relation with cosmological and mythological considerations, as we have seen, and in so far as it was solved by methods of approximate measuring it may be regarded as proto-geometric. (The step to geometry in the proper sense would come with elaborate recipes or algorithms for its solution, see problem I.32 of the *Nine Chapters* – and even more clearly with attempts to justify and criticize such algorithms like we find in Liu Hui, around 263 CE.) What is the relation between the intuitions provided by the CKS, and proto-geometry?

Our proposal is that the CKS provide our cognitive systems with a bias toward extracting certain kinds of information from the environment: information abut shapes, lines, angles, is privileged over and above other kinds of information (like colors,

[20] It is even quite plausible that the text of Euclid’s *Elements* would have been lost, were it not for the interest in astronomical studies and models, whose careful analysis was impossible without geometry.

textures, etc.). Although a more sophisticated terminology might be advisable, let us call that kind of information "geometric cues." Thus, the navigational and landmark CKS promote the *extraction of information about geometric cues* in our perceptual dealings with the environment. But this should be construed at the level of intuitions, vague and merely approximate notions. Nothing comparable to the precise concept of a circle, and thus nothing that could be usefully compared with Euclid's geometry.

Proto-geometric concepts emerge at another level, in processes that – for all we know – involve in an *essential* way *concrete material practices*, doing stuff with *instruments*. The formation of a precise concept of circle is not independent from concrete practices of *drawing* circles (with a trysquare, or with a rope, or say with a compass); or else practices of constructing circular shapes like the beautiful *bi* disks mentioned above. Also meant here are concrete practices like those of *measurement*: a quantity of liquid is measured employing a fixed, conventionally established unit of measurement (the *sheng* in ancient China = about 210 *ml*); a given length is measured using a metrical unit (from the Qin period,[21] 1 *chi* was about 23.1 cm), which is applied in yuxtaposition, counting how many times it was applied, etc.

The crucial point is that precise concepts (i.e., the level of proto-geometry) cannot be attained without relying on concrete material practices, most often involving explicit, external *representations* – see Giardino (2016).[22] Notice in particular that there is no evidence of precise concepts of a proto-geometric kind in human cultures that do not already possess metrological systems. Hence, the hypothesis that geometric concepts could be formed merely on the basis of CKS plus language (Shusterman & Spelke, 2005) is too speculative to be seriously taken into account. For all we know, material instruments, concrete practices, and furthermore explicit representations, are a *sine qua non* for the emergence of geometric concepts. Geometric thinking – as opposed to vague intuitions of shapes – cannot be obtained without symbolic means and representations.

This is relevant for the argument, given above, that basic geometry may be 'Euclidean' in an interesting sense, i.e., of leading 'naturally' to the Parallel Axiom. The argument was based on the independent development of the Theorem in cultures as different as ancient Greece and ancient China. Nevertheless, in light of the current discussion, one has to conclude that such knowledge is not an immediate product of core systems of cognition, either isolated or in combination, but is based on extensive experience of drawing, measuring, and other material practices. The basic, practical geometry that led to the Pythagorean or Gou-gu Theorem is extensively informed by experience and action.

Coming back to the difference between basic geometry and more sophisticated and idealized theories, one should keep in mind that Euclid's *Elements* were the culmination of a long process of development of explicit geometric knowledge. This lasted for about 300 years at least,[23] and we know that Euclid provided sophisticated answers to controversial and long-debated questions. For example, his definition of a point is different from those offered by either Plato or the Pythagoreans; and the differences (see below) support our thesis that basic geometry is coarse and rough.

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[21] See (Shen *et al.*, 1999). 6 ff. 1 *bu* = 6 *chi* in the Qin period. It is noteworthy that a uniform standard of length was introduced officially in China during the Qin period, about 200 BCE.

[22] Examples are the figures included in this paper. We leave aside the whole issue of mental representations, assuming that, in any event, mental representations (say of a square) are formed only after obtaining experience with explicit, external representations.

[23] Assuming as usual that Euclid composed the Elements around 300 BCE; but notice that this date is based on dubious information – it is not absurd to consider that Euclid may have lived after Archimedes.

Something similar to the idea we are defending was suggested by Felix Klein long time ago, with his distinction between *exact* and *approximative* mathematical knowledge.[24]

When exactly did the idea of breadthless lines and dimensionless point emerge? Probably only with Euclid himself. According to Aristotle, *all* geometers define the prior by means of the posterior, a point being defined as the extremity of a line (Heath, 1908,155); such is Plato's definition of a point as the "beginning of a line" (extremely unclear, compared to Euclid's, regarding the issue). Before them, the Pythagoreans engaged in a combined study of geometric configurations and arithmetic (with their "figured numbers" of which the tetractys is an example); according to Proclus (Morrow 1992, 78; see also Heath 1908, 155), they defined a point as a "monad having position," which leaves quite in the dark the question whether their points were meant to be dimensionless. Another important mathematician, before Euclid, was Democritus; and his atomism was an explicit option for a discrete structure of all reality, so that his points would have dimension.

Coming to ancient Chinese documents, we find relevant material in the Mohist canon, in the *Mo Jing* (believed to have been compiled between the late 4th and mid 3rd century BCE). One can find here the statement that "the line is separated into parts, and that part which has no remaining parts (that is, cannot be divided into smaller parts) and forms the extreme end is a point." (Needham, 1959,91). One also reads that the point (understood as starting-point) is "the unit without parts which precedes all others."[25] This seems to suggest an atomistic, two- or three-dimensional conception of the point; and one must admit that all notions of basic practical geometry are compatible with such ideas.

There is also in the *Mo Jing* a notion that Needham (Needham 1959, 93) interpreted as an analogue of the idea of parallelism. However this interpretation is dubious, and in any even the text underscores the crude and rude nature of basic geometric ideas, involving little idealization. The notion is *ping* (level/flat) defined as follows: "is of the same height." Needham pairs this with another text saying, "like two persons carrying [e.g., a beam], who should be of the same height like brothers." If we insisted to find here traces of the extremely subtle issue whether parallel lines in the Euclidean plane, when prolonged indefinitely, may or may not meet, we would be indulging in a typical example of anachronistic reading. Yet this is what the identification of basic geometry with Euclidean geometry in effect does.

Spelke et al. (2010) wrote, quite correctly, that the objects of Euclidean geometry "go beyond the limits of perception and action: points are so small they have no size and so cannot be detected by any physical device; lines are so long they cannot be fully seen or traversed." They should have added that this is unlike anything that might be reasonably called "natural" geometry, for human basic geometric conceptions are of visible figures in a limited space. In fact, it is well known that Euclid in the *Elements* avoided with utmost care any formulation that might involve a reference to infinity, so that e.g. his "lines" are always finite segments – never infinitely long lines.

It may be natural to ask whether primitive forms of geometry assume a behavior of parallel lines in accord with Euclid's Parallel Axiom, but this can only be judged

[24] Concerning the question whether we can form some kind of intuitive or visualizable notion that corresponds to the exact modern idea of a function, his answer was: "No. The representation of a curve only has approximative precision; the analytical counterpart of the [intuitive] curve is not a function, but a strip." (Klein, 1883,218)

[25] The text of the Canon I, with translation, can be found in the Chinese Text Project, see http://ctext.org/mozi/book-10.

indirectly. We cannot ask about the behavior of straight lines as they go to infinity, but we can enquire about logically equivalent questions, such as the validity of the theorem about angle-sum in a triangle, or the validity of the Theorem.

Nonetheless, this basic or primal geometry may be better described (from an idealized standpoint) by Euclidean geometry than by other geometric systems such as non-Euclidean geometries. Euclidean geometry cannot be identified with basic geometric knowledge, but cognitive scientists (already since Helmholtz in the 19th century) find arguments to the effect that basic geometry is *compatible* with Euclidean geometry. We have confirmed this insight with relevant historical material, showing that the Theorem (which, in an axiomatic setting, implies the Parallel Axiom) was developed and adopted independently by people who worked in two very different cultural contexts.

*** Suggestions for empirical work. ***

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