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Degrees of Objectivity? *Mathemata* and Social Objects

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Abstract
A down-to-earth admission of abstract objects can be based on detailed explanation of where the objectivity of mathematics comes from, and how a 'thin' notion of object emerges from objective mathematical discourse or practices. We offer a sketch of arguments concerning both points, as a basis for critical scrutiny of the idea that mathematical and social objects are essentially of the same kind—which is criticized. Some authors have proposed that mathematical entities are indeed institutional objects, a product of our collective imposition of function onto reality (the phrase comes from Searle) and of surrogation or hypostasis. Yet there are significant disanalogies between the typical social objects and *mathemata*, on which basis I argue that one should make a clear distinction between both. The comparison of mathematical with social objects helps understanding how non-physical objects can figure prominently in our explanations of reality. Yet mathematical objects have a different kind of cognitive grounding, and the more elementary of them emerge under relatively very simple sociocultural conditions. The differences are also reflected in the wide scope of use of mathematical concepts, and the much higher degree of variation found among social objects. On the basis of all of these features, I defend the thesis that one can significantly distinguish degrees of objectivity, and I use the distinction to articulate a graded ontology where one can locate the different kinds of mathematical and social objects.

Keywords Abstract objects · Objectivity in mathematics · Thin ontology · Conceptual structuralism

A large proportion of publications on the philosophy of mathematics deal with the question of mathematical ontology. In recent years, this question has been reactivated by the comparison with social ontology. Could the mode-of-being of money and marriage, perhaps, help clarifying the status of numbers and fields? Here is a relevant quote, taken from Feferman's "ten theses" (2009), namely thesis no. 10:

> *The objectivity of Mathematics lies in its stability and coherence under repeated communication, critical scrutiny and expansion by many individuals often working independently of each other. Incoherent concepts, or ones which fail to withstand critical examination or lead to conflicting conclusions are eventually filtered out from mathematics. The objectivity of mathematics is a special case of intersubjective objectivity that is ubiquitous in social reality.*

Is the objectivity of mathematics a special case of intersubjective objectivity? I think so, a very special case indeed.[1] Is it of the same kind as the reality of social things? The comparison is enlightening, but here I have doubts which I shall try to articulate.

To begin with, Feferman seems to share the standpoint of Kreisel and Putnam. In the mid-twentieth century, these two authors proposed a Copernican turn, by claiming that objectivity comes first and is the deeper question – objects have, in math, only the status of surrogates. This idea has since been elaborated by several philosophers, and arguably that part of the main problem has been solved already.

To be precise, this Copernican shift will only be completed when we have (1) explained in sufficient detail where the objectivity of mathematics comes from, and (2) explained how a 'thin' notion of object emerges from objective mathematical discourse or practices. I believe that the second part of the problem has been solved, especially by authors like Parsons (2009) and Tait (2005)[2]. If so, we should be able to explain it in simple terms, to the first man

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[1] I discussed the problem in Ferreirós (2016), chap. 6, and in a forthcoming paper (2023).

[2] For the first part, see Sect. 4 below and Ferreirós (2016).

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or woman on the street. Let me make a try by way of introduction to the paper.[3]

Consider numbers: I live on no. 17 of my street, and I own four books authored by Descartes. These are perfectly objective facts, although the first depends on a social convention (about using numbers to identify buildings on streets). One may notice that 4 is an even number, and 17 a prime, hence 17 is not divisible by 4; but there are many numbers divisible by 4. In these statements, we have introduced singular terms, predication, relations, and quantifiers. The role of abstract objects (the numbers to which '4' and '17' refer) in our knowledge depends essentially on symbolism, on semiotic systems (or symbolic frameworks). Certainly the relation between the sign '4' and the number four is entirely conventional, yet given the role played by '4' in our semiotic practices (including how it is *used* in counting and in calculation), it can only refer to the fourth natural number.

The point is that we have certain things for which we seriously employ the resources of singular terms, predication, relations, and quantifiers. Where is the deep ontological problem? The problem comes from bewilderment caused by *hypostasis*, or reification: 'four' passes from attribute (four books) to entity (number 4). Ultimately the problem comes from believing that the only objects to which sound scientific (or philosophic) discourse can refer, are natural objects (physical things, *actual* things). Yet I surmise that hypostatization is a common logico-linguistic phenomenon. Consider a non-mathematical example. We talk about my family, we predicate of families (this one is big, that other is multilingual), we even quantify on them (there are many families in my building). The mode-of-being of numbers seems to cause us philosophical trouble, in a way that the existence of families does not.

In principle we would need far fewer objects, reforming our symbolic frameworks so that they involve complex relations of relations of relations…, yet hypostasis is almost inevitable.[4] We speak of *the* natural number four, the practice of taking objects as surrogates to aid us with representational activities is ubiquitous. In due course, we humans bewilder because we compare such objects with chairs, tables, coins, or atoms – we are like children in wonder when they see a kaleidoscope. Wait, a coin is somewhere (e.g. my pocket) while number 17 is nowhere, and timeless… What *is it*, then? Our cultural and semiotic practices make us believe in a dream world, an otherworld inside the kaleidoscope.

Well, 17 is an objective component of our knowledge, and our knowledge is supported by semiotic systems. Semiotic systems are not mental items, nor are they subjective in any reasonable sense of this word. Our semiotic systems, in particular our language and our logical symbolism, allow us to talk about abstract objects, predicate on them, specify relations, quantify on them. That's all it is.

The *natural* numbers are no human invention: we have just discovered numerical relations in the world around us—and later, based on that, the properties of numbers themselves—just like we have discovered that there are many kinds of trees. Ultimately, number is just a discovery of humankind, at least if we restrict to *natural* numbers.[5] But also, the ultimate ontological referents of number (if I may be allowed to use this vocabulary) are not *things*, but relations (relational situations in the world; I give some examples below); and what underlies our use of number and our analyses of numerical properties, is *semiotics*, not physics or chemistry.

Already Gauss said, around 1830, that mathematics is "the science of relations," that the mathematician abstracts entirely from the content of relations, and concentrates only on their forms, on comparing them; Poincaré and others would repeat the claim decades later. But it's easier (logically or psychologically) to pack complex networks of relations into new entities, *mathemata* (I will use this name as shorthand for 'mathematical objects').

Both Frege and Peirce taught that existence, as in $\exists x$, does not involve actual existence (of the physical or natural kind) but is, so to say, broader. In Peircean terminology, *reality* is broader than the limited realm of physical existence (Peirce 1902, 375). And this teaching can be repeated without making any concession to Plato (meant is the usual, perhaps simplistic interpretation of Plato's views about the World of Ideas, where the One and the Dyad live): all that is needed to sustain the timeless reality of 4 and 17 is stable semiotic practices.

Perhaps you might wish to say that abstract objects are objects of knowledge, though not objects of nature. In any event, there is no otherworld involved here. And yet, you may reply, there surely is a big difference between number 17 and the legend of Ulysses and the sirens. I cannot be claiming that mathematical objects are like literary fictions, or even comparable to institutional objects such as a Court

[3] The following seven paragraphs constitute an attempt to explain the basic idea concerning (2) in most basic terms, so that anyone might grasp it. Thus the style is not the usual one in a philosophy paper (in particular, I offer no careful argument). I therefore ask readers to be indulgent, or else to skip these paragraphs and go to the last one of the introduction.

[4] Whether the reasons for our tendency to hypostasize are logico-linguistic, or merely psychological, is an interesting subject, into which we cannot go here.

[5] Any human culture that surpasses a certain threshold of complexity is strongly prone to develop a number system (see Overmann 2013). I should warn readers that my claims above are valid only for elementary mathematical notions (natural numbers, fractions, basic geometric figures) and *cannot* be applied to advanced mathematical ideas without modification (see the end of Sect. 3).

of Justice. Again I do agree: there is a great difference in the first case, a significant difference in the second. But they are differences in degrees. This is the idea I shall try to explain.

The structure of the paper is as follows. First we discuss a thin conception of abstract objects as allied to structuralism and distilled into quantificational logic. Then (Sect. 2) we consider social ontology, and in particular a proposal of considering *mathemata* as a kind of institutional objects. Next (Sect. 3) I propose some of the most relevant disanalogies between *mathemata* and social objects, which I articulate under the headings of (i) amplitude or scope, (ii) grounding of the system, (iii) cultural variation, and (iv) 'thickness' of the sociocultural preconditions. In Sect. 4 we take up, very briefly, the question of the roots of mathematical objectivity, as preparation for Sect. 5, where we consider in some detail the idea of degrees of objectivity, applying it to social and mathematical objects in the form of a scale or linear ordering of them. Finally, some conclusions are offered.

1 Thin Abstracts Objects, Structuralism, and Logic

Linnebo (2013) defines "Mathematical platonism" as the conjunction of three theses: *existence*: There are mathematical objects; *abstractness*: Mathematical objects are abstract; and *independence*: Mathematical objects are independent of intelligent agents and their language, thought, and practices (2013, Sect. 1). I suggest that the third thesis is too strong, unnecessary, and confusing. I propose that a theory of thin objects can be based on a second conjunction of three theses:

1. *Existence* of mathematical objects (not to be assimilated naïvely with physical objects);
2. *Abstractness*; and
3. *Objectivity*: mathematical objects, though not independent of intelligent agents, are independent of mental processes –of anyone's mental processes–, and objective insofar as they are 1. strongly intersubjective, cognitively speaking, and 2. linked with the analysis of relational patterns in the natural world and the world of our actions.

I hope the formulation just given, regarding objectivity, is clarifying and suggestive. Of course it's not an analysis of the topic, which deserves much greater care.

Two aspects of the 'thin' conception of objects deserve to be mentioned: its links with structuralism, and with formal logic. Methodological structuralism (the 'modern' 20th-century practice in mathematics) and some versions of philosophical structuralism suggest a close connection with thin abstract objects.[6] To discuss this connection briefly and not in detail, let me begin by saying this: mathematics is, first and foremost, *conceptual work*. Mathematics is a science "that draws necessary conclusions" about "hypothetical states of affairs", as Peirce said (1902). One needn't be talking about actual states of affairs in the natural world, it's enough to entertain certain conceptions of structures; math analyzes these hypothetical conceptions (which of course have often been stimulated by actual phenomena), and draws conclusions, elaborates methods, finds solutions. This is conceptual work, I claim (Ferreirós 2023).

Structuralism is often opposed to the idea that natural numbers are *sui generis* objects, existing objectively in full independence of we humans. Mathematical objects "serve only as relata of key relations" (Hellman 2005). Mathematics is, speaking in a very general sense, the science of relations and structures, not primarily a science studying peculiar kinds of objects.

How do we arrive at abstract objects? Consider *relations*: this is to the left of that, A is smaller than B, F is conjugate to G. Common sense tells us that relations are not (natural) objects: this lamp and this table are objects; the lamp is on the table; being-on-the-table is not of the kind of a lamp. But the mathematician studies relations, and compares them, and composes them. When you study relations, you start saying things like this: To each relation R there is an inverse R^{-}; two relations R and S on the same domain can be composed, which gives a third relation S·R; and so on. We elaborate new language in which we refer to relations, predicate on them, discuss relations among relations, quantify on relations. Now we have a new domain whose (abstract) objects are relations.

Following Frege and Peirce, it's a primitive way of thinking to believe that all objects of knowledge are akin to physical objects or reducible to physical objects. That was also a Quinean theme, with a slightly different bent as Quine (1953) refused to even make the distinction between physical objects, and objects in general. Consider a randomly variable quantity of the usual kind encountered in statistical work, in the natural or social sciences; when the distribution is not known, we usually assume that it will be a Gaussian distribution, in accordance with the probability density function.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

here the signs σ and μ denote the mean and the variance, while the signs π, $\sqrt{2}$ and *e* denote the well-known real numbers. As far as Quine is concerned, they are all objects – ingredients of the well-founded, scientific "myth-making"

[6] A relevant recent source for this is Reck & Schiemer (2020).

by which we make sense of the world. Following Frege and Peirce, one might want to say that the mean σ is an average obtained from *actual* values of certain features that can be measured in reality, while the number π is non-actual but nevertheless perfectly objective (definable as the ratio of circumference and diameter, or by means of the Madhava-Leibniz series).

The *thin* conception of objects finds expression in formal logic (Tait 2005; Parsons 2009). Objects are whatever we deal with using the apparatus of singular terms, identity, predication and quantification, i.e. quantificational logic. Typically the logic will be a form of first-order logic, or perhaps a many-sorted calculus (this includes second-order logic, to the extent that this system is a formal logic). Quite interestingly, logic thus conceived is at the core of ontology – but of course we are talking about 'thin' objects, which might also be called 'logical objects' or 'ideal objects'. There is no reason to expect that the objects of knowledge will be actual, natural or physical objects.[7]

According to the structuralist view, it is sufficient for the reality of numbers (or functions, or spaces) that our discourse about them be coherent; i.e. that it be *conceptually consistent*. Here I agree with Parsons, Shapiro, Hellman, etc. and we all seem to be following Dedekind and Hilbert. It's a pity that the consistency of a system like elementary number theory cannot be established mathematically by finitary means. But we may be convinced, by other reasons, that our conception of the basic structure of natural numbers is coherent. For that matter, we may be quite sure that the conception of a realm of sets, as established by ZFC, is conceptually consistent.[8]

Like Parsons (2009), we may adopt a non-eliminative form of structuralism, and yet avoid conflict with the anti-metaphysical stance of Feferman (2009). There is little need for eliminative positions, which in any event are unconvincing due to the idealizations at the basis of mathematics. Elementary arithmetic, or even primitive recursive arithmetic, build on the basis of idealizations, transcending what is feasible for human beings and (possibly at least) what is actually existent. There is no reason why the hypothetical states of affairs that we conceive may not transcend the feasible, what is explicitly constructible.[9]

2 Social Ontology: are *mathemata* institutional entities?

To understand how non-actual objects can figure prominently in our explanations of reality, it is certainly helpful to reflect on the fact that such objects are prominent in our day-to-day lives. I'm Spanish and I'm a university professor, although no physico-chemical or biomedical exploration of my body or brain could determine those facts. Social ontology is a fundamental aspect of our lives, where we find social objects of different kinds (e.g. social groups) and also institutional objects such as money, private property, professorships, State borders, or Courts of Justice—things that determine our lives. Julian Cole (2015) has proposed that one should understand the abstract objects of mathematics as institutional objects. This is a helpful proposal, and an interesting metaphor, although I'll argue that there is reason to question it (in calling it a metaphor, I'm accepting the idea that metaphors play a great role in human cognition).

Institutional objects are very real and objective, even in cases when it could be argued that they don't exist physically.[10] Recently we celebrated May 1st (also known as May Day), an international holiday to commemorate the historic struggles and gains made by workers and the labour movement.[11] Some of us in Europe went out to take part in demonstrations, which made a very physical difference to some parts of our towns. Some of us simply didn't go to work, perhaps taking advantage of the free day to enjoy the countryside or the beach. Certainly there is nothing 'natural' to mark the difference between a holiday like this and a working day—it's just socially instituted. Some authors, like Searle, say that these things exist only because we believe in them, but it seems to be a category mistake to employ here the word 'belief'. I just know that May 1st is a holiday where I live, and an international holiday, and this happens to be true (though at some point in the future the holiday might cease to be celebrated, the institution might be dissolved).[12]

[7] With these ideas in mind, the reader might want to read the introduction again. Incidentally, it is worth mentioning that there are structuralist approaches to social ontology, in fact inspired by mathematical structuralism (see Ritchie 2020).

[8] Perhaps by a combination of ideas and reflections about how real numbers and analysis can be derived from ZFC, the coherence of conceiving the real line as a set of points, and the iterative conception of sets (see Ferreirós 2016, chap. 9 & 10).

[9] And it seems to me possible that a structural conception may be fully coherent, in the sense of conceptually consistent, and yet escape

Footnote 9 (continued)

full conceptual control by any available means. On this delicate matter see Feferman,' forthcoming.

[10] Many institutional objects do have some material component. Some authors argue that institutional objects are concrete objects composed of a normative component and a physical component; this idea is proposed by hylomorphic accounts of social objects (Passinsky 2021) and by the constitution view (Epstein 2015).

[11] This is not followed in the USA or Canada, where they shifted to Labor Day, avoiding the (19th century) socialist connotations behind the choice of date.

[12] The attempt to reduce institutional things to states of mind is part of an outdated mentalistic epistemology, but we don't need to discuss this in more detail.

Much has been written in recent years about social objects, especially social groups of different kinds (see e.g. Epstein 2018; Ritchie 2020; Passinsky 2021). Not so much work has been devoted to institutional objects, although Searle's influential ideas (1995) have to do mainly with institutional social facts. It is relevant to take into account that, although social objects have emerged spontaneously and informally in the context of human practices, within our highly organized societies most such things have become institutional objects (due to the existence of legal institutions that underlie them).

Cole (2015) argues that mathematical entities are institutional objects, a product of our *collective imposition of function* onto reality (the phrase comes from Searle), and also a product of *surrogation*. Humans have "the capacity to impose functions on objects and people where the objects and the people cannot perform the functions solely in virtue of their physical structure" (Searle 1995, 7). One may think that the concept of function in this sense is roughly synonymous to teleological role—a role that promotes some end(s), goal(s), or purpose(s). Yet Searle's approach emphasizes the idea that the functions in question are status functions and therefore have a normative role. There are *institutions* as a result of standing declarations—an institution being a plurality of activities governed at least in part by a system of standing declarations (e.g. somebody is declared President of the State).

The other core ingredient is surrogation, by which Cole means reification (or hypostasis) that can be explained as a logico-linguistic phenomenon. As he argues, we find it significantly easier to engage in various representational activities when their subject matter—what we are trying to inquire after, reason about, analyze—is treated as an object in our representations. We have encountered several examples of this above: 17 refers to an object, even if there is nothing in building no. seventeen of the street that may physically correspond to 17 as its reference; and 17 has the property of being prime, even if what this may represent in an actual, physical situation is not an actual object.

Imagine that I have 17 books. The fact that 17 is prime corresponds to the fact that I cannot distribute my books into groups of equal size, so that there are two or more books in each group. One might say that this is a *relational situation*, which is the actual counterpart of the arithmetic fact. Making explicit all of these aspects and details is cumbersome, and we find it significantly easier to just treat 17 as an object.

Reflection reveals, as Cole says, that the practice of taking objects as surrogates to aid us with representational activities is ubiquitous; e.g., we have positions in organizations, natural numbers, possible worlds, and abstract fictional characters (like Zeus). And certainly our numbers, not only 17 but also π, are as real at least as facts about the President of my country, or the legal borders between the USA and Mexico, or between Europe and Africa, that explain many hard facts about the lives of many people.

Yet, for some reason, the identification of mathematical objects with institutional entities (in the sense of social ontology) is not quite convincing. I will try to articulate why.

3 Disanalogies

Is a number system like money? Are the real numbers like the pound? Or comparable to marriage? One feels the urge to say that there's something much more basic about numbers, which makes them almost independent of sociocultural context.

Take again May Day, the day of workers: it would be irrelevant if not impossible to understand around 1500, for the labour movement and its social preconditions (industrialization, capitalism) didn't exist. Similarly, private ownership of the land was hard to understand for indigenous groups in the Americas, and it would have been impossible to understand for people in the late Paleolithic. Yet the meaning and function of number is almost independent of social or cultural context, in the sense that, for most cultural contexts, one can imagine situations that make the use of numbers natural. People in the year 1500 had no trouble understanding the number fifteen hundred, regardless of whether we are talking about Europeans, Chinese, or Incaic people who hadn't been in contact with the Europeans yet.

One would like to say, too, that number is much closer to our biophysical conditions, to our basic and actual nature. Indeed, numbers can very easily be introduced into a culture that lacked them, precisely because they are much less dependent on sociocultural context (than legal borders, presidencies, or religious figures).

Let's consider some of the key *differences* between a social institution and a mathematical object. I will argue that there are differences in the *grounding* of the system; in the *range of variation* among cultures and peoples; in the 'thickness' of their *sociocultural preconditions*; and in the *amplitude* of the system's relevance. I begin with the latter:

1. **Amplitude or Scope** Number and spatial figures are ubiquitous, and this is already a difference with social institutions, which tend to be specific to this or that sphere. There is an obvious counterexample, however: money is nowadays ubiquitous, perhaps as much as number; there are still societies that lack a currency, and this tends to coincide with lack of a system of numerals. If you go back in history, perhaps to the Middle Ages, you find that money was then less omnipotent, and number was a more widespread phenomenon.

2. **Grounding** By a difference in the grounding of the system, I mean cognitively and biologically. Even if one is very skeptical about claims in mathematical cognition concerning the ANS and the OTS,[13] the evidence suggests that we are endowed from early life with abilities that allow us to individuate objects in the environment and track them. And these abilities are recruited to allow us subitizing, and to enable the learning of the system of numerals we encounter in our oral culture (Carey 2009). Thus number has strong cognitive roots, arguably stronger and much more basic than most social institutions. I rush to add that family, or perhaps just the mother–child bond, is arguably a potential counterexample: we have the innate ability to identify faces, and babies immediately establish bonds with their mother.

In a sense this counterexample actually helps reinforce the key point: the relation between mother and child is basic to social life, but it's also biologically most basic. One can hardly argue that the mother–child bonds are 'merely' sociocultural, and in the same sense one cannot argue that numbers like 4, 17, ½ or $^2/_3$ are 'merely' a matter of history and social construction. This is unlike the King of Spain, the President of China, the Court of Justice of the EU, and even unlike private property (*pace* Locke).

3. **Variation** By a difference in variation among peoples, I mean to say that there is a certain degree of arbitrariness in most social institutions (marriage, legal norms, etc.), which is significantly higher than is the case for numbers or figures. To put it otherwise, social institutions are *conventional* in a sense that the most basic *mathemata* are not (meant here are natural and rational numbers, geometric figures). The *Mozi* Canon (book 10), which goes back to around the fifth century BCE, states: "*Yuan* (circular) is having the same lengths from one center", and the accompanying exposition comments that compasses draw it roughly.[14] The sense of this proposition is clear and can travel through many different cultures and societies, from the Far East to the Far West, from the North to the South.[15]

By contrast, there exist different forms of marriage and different kinds of family – not just the nuclear family that is typical among Westerners, but also matrifocal or patrifocal families, extended families, and more. Even at this most basic level of human life, there is significant variation that extends beyond the plurality found in number systems. Yes, we all know that there is diversity here too, as the number system may be base 10 or 5 or 20, and the notation may be Roman or Indo-arabic. Nevertheless there are *structural differences* among the different kinds of families, while there is structural homogeneity (essentially an isomorphism) between the different kinds of numeral systems. You may think for yourself about social institutions less basic and 'natural' than the families, and it's obvious that we find wide variations.

4. **Thickness** Perhaps most important difference is the '*thickness' or complexity of the sociocultural preconditions* involved in the different kinds of entities that we are discussing. In summary, most of the examples of social institutions that are usually given presuppose a complex social organization, of the kind of the ancient river civilizations (in Mesopotamia, or Harappa, around the Nile or the Yellow River). These were societies with large permanent settlements featuring urban development, social stratification, specialization of labor, centralized organization, and written means of communication. A famous and paradigmatic example is the Code of Hammurabi, promulgated by the Babylonian king who reigned from c. 1792 to 1750 BCE. Meanwhile, the preconditions for the emergence of number systems are considerable less. According to recent work by Overmann (2013), material complexity precedes the transition to numeration systems, but the relevant social and material complexity can be found among hunter-gatherers, thus in social organizations much less complex, stratified and specialized than those mentioned before.

In the hunter-gatherer groups studied by Overmann, using previous ethnographic studies, one finds early methods for counting like the use of beads or tally sticks. Examples of beads with high associated social value, probably employed in necklaces or the like, have been found as far back as 90,000 years before present (Blombos Cave). In connection with this, one has to emphasize the role of material artifacts and semiotic practices, be they with things (beads, calculi), body parts (e.g. fingers), or words (numerals).[16]

Thus number words are very frequent in human languages, to the point that almost all human cultures have been numerate in some very basic sense—yet many societies lacked numeral terms beyond 'five' (Hurford 1987). Number is of course ubiquitous in the industrialized world, but experts claim e.g. that 85% of nearly 200 languages from Aboriginal Australia surveyed do not have numerals beyond 'five' (Nuñez 2017).[17] In this connection, it's interesting to

[13] ANS is the Analog Number system, OTS the Object-Tracking system, well accepted cognitive items in the literature. See Geary, Berch, & Koepke (2015).

[14] These books are from the Mohist school in ancient China, see Boltz & Schemmel (2016). Translation from the digital Chinese Text Project, see https://ctext.org/mozi/canon-i (accessed 1 July 2021).

[15] Incidentally, the concepts of east, west, north and south are also a good example, not merely mathematical in our sense of this word (though it was "mathematics" in the *traditional* Middle Ages sense).

[16] On this topic see Overmann (2013), Everett (2017).

[17] Some cultures are alleged to lack any precise number words: for the case of the Pirahã, see Everett 2017, chap. 5. In most of these cases, however, there are precise terms at least for 'one', 'two', 'three' (maybe 'four') and then 'many', and if need arises the members of such cultures manage to establish one-to-one correspondences.

remark that linguists have established that number words and number symbols have a very significant endurance (without change) through the centuries; this indicates highly stable practices, making them a special case compared to other words and forms of writing (Hurford 1987). You may take it as an empirical symptom of crucial differences.

The case of the real numbers (or that of sophisticated methods for the measuring of areas, and similar geometric feats) is significantly different, requiring complex sociocultural conditions, just like many social objects.[18] Thus, the claims we are making at this point are valid for the more elementary areas of mathematics (e.g., basic arithmetic) but cannot be applied to advanced math without qualifications. With that proviso, one can say that mathematical objects have cognitive roots hardly comparable with usual social entities. They are more solidly and stably grounded. Also relevant may be the *direction of fit*: social objects fit collective intention, often in a one directional way; basic mathematical objects have two directions of fit, they fit (and represent) relational patterns in experience and action.

4 Grounds for Objectivity

As explained in the introduction, there are two aspects to the Kreisel-Putnam idea of objectivity as the central question: (1) to explain in detail the roots of mathematical objectivity, and (2) to explain how a 'thin' notion of object emerges from objective mathematical practices or discourse. The second was analyzed in Sect. 1, while the answer to the first question can only be a long one (see Ferreirós 2016). Still, here I present a brief summary of key ideas.

The ontology of thin objects discussed previously would, at a minimum, only require the consistency of the corresponding theory. Of course we are not asking for formal proofs of logical consistency; from an agent-based standpoint it's enough to grant conceptual coherence. Thus e.g. one may safely assume that the classical theory of fractions (rational numbers) and the theory of real numbers are conceptually coherent, and this underwrites our admission of such numbers as mathematical objects. But mere consistency are *insufficient* to explain the relevance of mathematical structures: our reasons to admit real numbers are much stronger.

If the 'logical' justification just suggested were the only one, the idea that mathematical objects are fictions would be reasonable, and their degree of objectivity would fall behind institutional objects like money. But the basic concepts of natural number and fraction are even indispensable for common life in moderately complex cultures; their objectivity is more strongly grounded than is the case with social or institutional objects.

The *cognitive* basis of mathematics is one of the keys to offering an explanation of its strong intersubjectivity. Regarding the roots of mathematical ideas, one must take into account not only core cognitive systems (e.g. the basic 'number sense' that psychologists study) but also oral language, the numerals, and some other 'technical' practices employed by human agents to count.[19] The grounding of mathematical concepts or objects involves at least the following components: basic cognitive abilities, linked with action in the biophysical environment, and with social interactions; practices, including what I call 'technical' practices such as counting, drawing designs, or measuring[20]; and symbolic or semiotic components, i.e. external representations including notations and diagrams. There are good reasons to consider writing and drawing techniques among the essential cognitive tools that make possible the emergence of mathematical practices. (Of course, writing is also of the essence for the establishment of social institutions that last long, generation after generation—think again of the Code of Hammurabi.)

If such a viewpoint is accepted, it means that there is no mathematics without some cultural preconditions – which however does not turn math into a 'mere' cultural product.

I also emphasize the idea of a *pragmatist* perspective on mathematical knowledge because of the importance that 'technical' practices (e.g. the techniques of counting and measuring) have, in establishing among humans shared methods and semiotic systems, and ultimately shared conceptions. It's on the basis of those practices, and aided by notational systems, that people share notions such as those of natural and rational number. It's on the basis of practices of drawing that people share notions such as those of square and circle (Ferreirós & García-Pérez 2020).

Obviously this is only the starting point for the development of mathematical ideas, results, and theories, but those cognitive and pragmatist roots make possible the *strong intersubjectivity* of mathematics. As we saw in the previous section, the outcome can only in part be compared with social institutions, even the most enduring and 'universal' of them.

Another key element for an explanation of its strong intersubjectivity is well known. Mathematical structures are of

[18] See Ferreirós (2016), chap. 8, Schemmel (2016), chap. 4.

[19] It is well known that Dehaene (2011) speaks of the "number sense", but it would be better to avoid the word 'number' here, reserving it for the precise concept that allows one to distinguish 7 from 8, or to count up to twenty. This crisp conception can be found in human cultures all over the world, in all continents, going far back in history and prehistory. The proposal of speaking about a 'quantic sense' was made by Núñez (2017).

[20] Ferreirós (2016), Sects. 2.5, 3.4, 5.1, and 7.2.

great importance for scientific practice, for modelling and explanations; examples can be multiplied, we gave one in Sect. 2. This goes a long way to explaining the relevance of mathematical structures, the strong grounding enjoyed by mathematical ideas going beyond the elementary. Notice that relational patterns may or may not correspond exactly to the given in physical phenomena, they are often determined – in part – by constitutive hypotheses. The idea that scientific representation can be a pure mirroring of world phenomena is too naïve.

Interestingly, very different cultures have given rise to apparently convergent conceptual developments: let me just give the examples of number π as the measure of the circle (with unit diameter), and the so-called Pythagorean theorem (gou-gu procedure); both can be found in ancient Greece and ancient China, in the context of mathematical developments within widely diverging cultural contexts. This could be connected with the topic of the range of variation of sociocultural products, discussed in the previous section.

Notice, once again, the order of the investigation: first to expose the deeper reasons for the objectivity of mathematical results, and then to understand the sense and limits of the claim that numbers are real in the sense of Peirce.[21] Existence or reality is here meant in the sense of objectivity and thin ontology (Sect. 1), *mathemata* being surrogates of objectively studied relational patterns.

I have tried here to summarize very briefly what is in fact a long argument about mathematics as a unique human product, indeed—restricting to elementary math – a product that one is justified to call more than human, in the sense we discussed in simple terms in the introduction. To say that mathematics is a social construction is to emphasize the collective sociocultural aspect in a way that, I surmise, is biased and misleading.[22] If you remember that human beings are part of natural reality, if you remind our biophysical nature, then to say that mathematics is a human product should not ring any bell of 'psychologism' or 'subjectivity' or the like. There's nothing psychological or subjective to the concept of natural number (Ferreirós 2016, chap. 7): in fact, I would go as far as to say that the basic concept of number is largely a *discovery* we make about the constitution of reality, and the constitution of our own patterns of thinking and action.

For reasons we are going to discuss, this claim cannot be extended to the more sophisticated conception of real numbers, basic as it is for us.

[21] So-called *real* numbers being less real (in Peirce's sense) than other kinds of numbers, e.g. the rationals; while, on the other hand, the *complex* numbers are just as real as the *real* numbers.

[22] I prefer the label 'humanism' employed by R. Hersh, to the label 'social construction', and I hope a sensible reader can reconstruct from the previous pages my main reasons.

5 Degrees of Objectivity

Let us now consider the idea of degrees of objectivity. Although I find it difficult to elaborate on the topic, there are two key ideas that I'd like to argue for. Namely, that there are different degrees of objectivity among mathemata; and that some mathematical objects are more strongly objective than social objects.

One can try to apply the idea of degrees to institutional objects vs. mathematical objects, but also to different types of objects within each kind. Let me first say a word on social objects. It seems reasonable to establish differences of degree among them: money is more solid stuff (if I may speak this way) than e.g. state borders. Private property and monetization are so strongly entrenched in our social lives, globally on Earth, that it seems impossible to imagine ways to avoid them. And yet we know, from history and anthropology, that one can live without them. Meanwhile, state borders are more fluid stuff, they are negotiable and can even tend to disappear (as is currently the case within the EU). But I'm no expert on these matters, so I leave further discussion to the connoisseurs.

Coming back to math, there's a difference between the natural numbers and the reals in how basic and deeply grounded they are. True that $\mathbf{N}$ and $\mathbf{R}$ are the most important and basic structures in all of mathematics, according to modern and contemporary conceptions of the subject, but it makes perfect sense to make a distinction at the level of epistemology. The idea of natural numbers belongs in elementary mathematics, while the idea of real numbers is prototypical of *advanced* mathematics. The key distinguishing trait is that advanced math is characterized by the intrusion of hypothetical assumptions—in this case, the *completeness* or continuity of the real number system.[23] But you may ask, what is a *hypothesis* in this context? How is it different from an idealization?

The natural numbers (and the fractions) as we use them correspond to given relational patterns in our environment; to give an example, there are around 100 billion stars in our galaxy (10^{11} in scientific notation). But when we consider the series of natural numbers, we take the successor function to be applicable in full generality, without considerations of feasibility. Someone might object that numbers beyond some order (call it, zillions) make no practical sense and cannot be given concrete application by human beings – yet we discount such limitations of feasibility. Developing a fully general theory has some advantages; the situation is perhaps like that of the physicist, who discards friction in a

[23] See Ferreirós (2016), chap. 7 and 8. I refer to the axiom of completeness, which can be stated in terms of the notion of 'least upper bound', or by means of Dedekind cuts, or by nested closed intervals.

first approximation to how a mechanical system works. So much for idealization.

Hypothetical assumptions are a different matter. Consider the system of rational numbers (fractions), which is *dense* in the mathematical sense: between any two fractions there is another one (hence infinitely many). This is already the product of idealization and it may well be the case that there's nothing in the relational patterns in our environment, corresponding to *that*. Now, when we pass from rational numbers to the structure of real numbers, we introduce the extra requirement that the system be complete; *completeness* in the strict mathematical sense. This is not like disregarding some aspects or limitations in the situation, it's rather *adding* crucial new information into how the situation is determined. Notice also that such a move necessitates the transition from potential infinity to the actual infinite (e.g. in the presentation via Dedekind cuts, upper and lower cut *must* be taken to be actually infinite sets).[24]

I characterize advanced mathematics as those theoretical developments which rely on such constitutive hypotheses. But I also argue that reliance on hypotheses does not impede the obtaining of *objective* results. An example I have given in detail is the following (Ferreirós 2016, chap. 9): many properties of the natural and real numbers are established *before* even considering that they may (or not) form a set; once the hypothesis that **N** and **R** are actually infinite sets is admitted, those previous properties enforce the conclusion that one cannot establish a bijection between real and natural numbers. The argument is clear, even for those who may want to argue that the hypothesis should be rejected, or that an infinite totality such as **N** is different in kind from **R** (thus rejecting the basic tenet behind classical set theory).

While there's a kind of inevitability about the natural numbers and the positive rationals, the situation is different with the real numbers. Both adoption and rejection of a complete system of real numbers, are perfectly reasonable and rational alternatives – which mark the difference between so-called 'classical' mathematics, established from around 1850, and forms of constructivism or predicativism.

Let us finally consider the idea of *ordering* different kinds of objects, with special emphasis on mathematical and social ones. What follows is just a provisional suggestion, a preliminary attempt that takes into account what we have previously discussed. The list, ordered from most objective to less, might begin as follows:

1. Numbers (natural or rational) and geometric figures
2. Families or basic social units
3. Institutions of justice (peace-makers, courts)
4. …
5. Money//Real numbers, & many other mathl. objects

Further down the list one might find such things as State borders, Universities and professorships, and so on. And yet further down, we might want to include fictional entities like Don Quixote, Zeus, or Ulysses and the sirens.

The implication is that basic numbers are more strongly objective than such basic social institutions as family (in its diverse forms) or basic social units. The reason cannot be just a matter of spread or scope, since many societies lack a sophisticated system of numbers, but of course they don't lack basic social units, nor (I assume) some institutions of justice. At this level I have taken into account what was said in Sect. 4 about the grounding (cognitive, pragmatic, biophysical) and the role in describing basic features of all kinds of real situations. Number is not just strongly intersubjective, it also reflects basic objective features of experience; it has a key role in depicting relational patterns. That's why number is ubiquitous, it plays many relevant roles in relation to 'technical' practices and other practices (social, religious, scientific, etc.).

Also reflected in that tentative and very provisional list, are the four features discussed in Sect. 3: amplitude or scope, grounding, variation, and 'thickness'. It seems reasonable to regard the real numbers (non-elementary, hypothetical as they are) as comparable to an institution like money: while the real numbers often enjoy the same amplitude of use as other numbers,[25] they are not so strongly grounded, and they presuppose complex sociocultural systems, i.e., material complexity and social complexity of a kind much higher than, say, the natural numbers.

In fact, to study in a fine-grained way the possible ordering of different kinds of entities, according to degree of objectivity, one might employ the four features of Sect. 3. Consider the example I've given of institutions of justice, e.g., peacemakers (to employ a category that may be used in very different cultural contexts). The reader may have wondered why, in the previous list, such institutions were deemed more objective than money. The idea behind this proposal is that conflicts and situations of injustice emerge in social groups from very early stages, and the organization of some way of handling such issues is a feature commonly found in all kinds of human groups. Think of a 'primitive' culture, perhaps a group of hunter-gatherers: they will not have a form of private property of land, nor economic

[24] Another example might be the notion of a *set of all subsets* of a given set. I have discussed this, in terms of the contrast between definable sets and *arbitrary* sets, in Ferreirós (2016), Sects. 8.5, 9.3, 9.4, 10.4.

[25] Though their use in such cases is not essential, meaning that they typically can be replaced by rational numbers (for practical purposes, we don't need to consider the exact value of π, we can assume e.g. that $\pi = 3.14159265$, which is just a rational number).

exchanges of sufficient complexity for money to appear; but they will probably have the institution of a peacemaker – perhaps some kind of 'chief' who has this among his or her status functions.

Consider now the four features applied to this kind of justice institutions. They are grounded in cultural practices, but do not connect intimately with core cognitive capacities (as is arguably the case with numbers and with basic social units, at least thinking of the mother–child bond); their scope is certainly narrower, since they are relevant only in situations of social conflict; there is a great variation in the forms adopted by such institutions in different cultures. Yet, their sociocultural preconditions seem to be rather thin, making a case that is perhaps comparable to natural numbers, or presupposing even less in terms of material complexity of the social practices.[26]

In what precedes, I have considered mainly institutional objects, although the ideas are applicable also to social objects in general. This is for two reasons. First, it has been explicitly proposed to interpret mathemata as institutional objects; second, under conditions of social complexity such as we are used to, many social objects become institutionalized. Thus most of the examples we have employed, considered in the context of so-called 'modern' societies, are cases of institutional object.

6 Conclusions

We have seen that there is an analogy between mathematical and social objectivity, which helps understand the way in which abstract objects like numbers or fields can play central roles in our dealings with reality (e.g. in explanations). Attention to social objects, which are fundamental in our lives and can even determine our fates, helps clear the way to a freer understanding of the role of epistemic objects which are not simply a reflection of natural or physical entities.

But we have also questioned how far the analogy goes. I have argued that the analogy with social objects breaks down, considering a number of features such as the grounding of mathematical knowledge (cognitively, biophysically, semiotically) or the relative simplicity of the sociocultural preconditions for its emergence. Such differences are also reflected in the scope of application of mathematical ideas, which tend to be ubiquitous, and the very low degree of inter-cultural variation of (at least) elementary mathematical knowledge.

[26] The minimum of 'thickness' seems to be found in the case of families (or basic social units), which require less than number. Yet this may be compensated by the scope (narrower) and degree of variation (much greater) of such social units.

The four features of amplitude or scope, grounding of the system, cultural variation, and 'thickness' of the sociocultural preconditions (Sect. 3) can be applied to both social and mathematical objects, in the attempt to grasp differences in their objectivity. Thus we have spoken of degrees of objectivity, measured qualitatively and arranged in a linear scale, provisionally at least (see Sect. 5). This may be a useful proposal for attempts to consider in depth the epistemic and metaphysical import of different kinds of entities.

It may be useful to summarize here our brief discussion of the nature of mathematical objectivity (essentially in line with ideas previously presented in Ferreirós 2016). Mathematics is, first and foremost, *conceptual work* – a science that draws necessary conclusions about "hypothetical states of affairs" (Peirce) or "ideal-world pictures" (Feferman) which are typically presented as axiomatically characterized structures. Such ideal-world pictures are conceptions of structures. Mathematics, speaking in a very general sense, is a science of relations and structures, not a science studying peculiar kinds of objects. But this is compatible with the reification or hypostatization of thin objects, which we regard as a basic logico-linguistic phenomenon that affects all kinds of theoretical developments. Mathemata could thus be labelled 'logical objects', for the reasons discussed in Sect. 1; we have argued for a form of *conceptual* structuralism which admits thin objects (see Ferreirós 2023).

The *objectivity* of mathematical objects can be characterized as follows. On the one hand, it is a form of intersubjectivity:

- though not independent of intelligent agents, mathemata are independent of my (of anyone's) mental processes; they are communally shared like institutional objects;

but, furthermore, they're *strongly* intersubjective (one might simply say, objective) in the following sense:

- they have strong cognitive roots (consider again Sect. 4 and the features discussed in Sect. 3), and they are key to the analysis of relational patterns in the world of our experience and actions.

Hence their role in all kinds of practices, from simple techniques and more or less complex activities of common life, up to scientific modelling practices; mathemata play a central role in representing basic and recurrent relational patterns. Due to their cognitive background, links with human actions and techniques, and role in modelling natural patterns, they form a very special class of logical objects. This should clarify why the (surrogate) objectivity of mathemata goes beyond typical instances of social objects.

But the conceptualist approach includes the thesis that there are differences among mathematical objects, which may be discussed in terms of degrees of objectivity; they are essentially due to the differences between elementary and advanced mathematics. Objectivity need not be conceived as an all-or-nothing affair, if we admit that it is largely a matter of *different* forms of intersubjectivity. For this reason, it makes sense to consider a gradation of objectivity: the grounds on which we share a conception of natural numbers are significantly different, deeper than the grounds on which we share the idea of private property of land, or money. To recap, the comparison with social objects is enlightening, but also limited.

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References

Boltz WG, Schemmel M (2016) Theoretical reflections on elementary actions and instrumental practices: The example of the *Mohist Canon*. In M. Schemmel, ed. (2016) *Spatial thinking and external representation*. Max Planck Res Library, Stud 8:121–144

Carey S (2009) Where our number concepts come from. *J Philosophy* 106(4):220–254

Cole JC (2015) Social construction, mathematics, and the collective imposition of function onto reality.*Erkenntnis* 80(6):1101–1124

Dehaene S (2011) *The number sense*. Oxford University Press, UK

Epstein B (2015) *The Ant Trap: Rebuilding the Foundations of the Social Sciences*. Oxford University Press, UK

B. Epstein 2018. Social Ontology, *The Stanford Encyclopedia of Philosophy* (Winter 2021 Edn), ed. E.N. Zalta. URL = <https://plato.stanford.edu/archives/win2021/entries/social-ontology/>.

Everett C (2017) *Numbers and the Making of Us: counting and the course of human cultures*. Harvard Univ Press, Cambridge

Feferman S 2009. Conceptions of the continuum. *Intellectica* 51:1 (2009), available for download in http://math.stanford.edu/~feferman/papers.html No. 85.

Feferman S (forthcoming). The Continuum Hypothesis is neither a definite mathematical problem nor a definite logical problem. Harvard Philosophy Dept. EFI lecture. Available for download in https://math.stanford.edu/~feferman/papers.html No. 115.

Ferreirós J (2016) *Mathematical knowledge and the interplay of practices*. Princeton University Press, New Jersey

Ferreirós J (2023) Conceptual structuralism. *J Gener Philos Sci*, in print.

Ferreirós J, García-Pérez M (2020) On the nature of proto-geometry. *Philos Psychol* 33(2):181–205

Geary DC, Berch DB, Koepke KM (2015) *Mathematical cognition and learning evolutionary origins and early development of number processing*. Elsevier Academic Press, Cambridge

Hellman G (2005) Structuralism. In: Shapiro S (ed) *Oxford handbook of philosophy of mathematics and logic*. Oxford University Press, UK, pp 536–562

Hurford JR (1987) *Language and Number: The emergence of a cognitive system*. Basil Blackwell, Oxford

Linnebo Ø (2013) Platonism in the Philosophy of Mathematics. *The Stanford Encyclopedia of Philosophy*, Winter 2013, E. N. Zalta (ed.), https://plato.stanford.edu/.

Núñez R (2017) Is there really an evolved capacity for number?. *Trends Cogn Sci* 21(6):403–490

Overmann K (2013) Material scaffolds in numbers and time. *Camb Archaeol J* 23(February):19–39

Parsons C (2009) *Mathematical thought and its objects*. Cambridge University Press, Cambridge

Passinsky A (2021) Norm and object: a normative hylomorphic theory of social objects. *Philosopher's Imprint* 21(25):1–21

Peirce C S (1902) The nature of mathematics. In: *Philosophical Writings* (New York, Dover, 1955), p. 135–149

Poincaré H (1902) *Science and Hypothesis*. Bloomsbury, London, 2017

Quine WV (1953) *From a Logical Point of View*. Harvard University Press, Cambridge

Reck E, Schiemer G (eds) (2020) *The prehistory of mathematical structuralism*. Oxford University Press, UK

Ritchie K (2020) Social structures and the ontology of social groups. *Philos Phenomenol Res* 100(2):402–424

Schemmel M (2016) Historical epistemology of space: from primate cognition to spacetime physics. Springer Verlag

Searle J (1995) *The Construction of Social Reality*. Allen Lane, London

Shapiro S 2008. Mathematical Structuralism. *The Internet Encyclopedia of Philosophy*, http://www.iep.utm.edu/m-struct/ .

Tait W (2005) *The Provenance of Pure Reason*. Oxford University Press, UK