What are Mathematical Practices? The Web-of-Practices Approach

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In this chapter I shall revisit the proposal made in *Mathematical Knowledge and the Interplay of Practices* (2016) for the analysis of practices in terms of an intricate spider-web which extends from 'technical' (pre- or non-mathematical) practices to high-level mathematical ones, also including links to scientific practices of modeling, data control, etcetera. In order to offer a refreshed perspective on the topic, we shall (i) reconsider and refine my working definition of what a mathematical practice is, (ii) analyze the relations and differences between mathematical practices and other kinds of cultural practices, e.g. those studied in ethnomathematics, and (iii) consider a particular case, the web of practices and theory related with the central concept of function, which has articulated a large portion of the mathematics developed in the last 300 years.

The reader can profitably consider the chapter as made up of two parts, a general discussion of the notion of mathematical practice (hereafter MP) and the limits of its use, comprised by sections 1 to 3; and a particular case study that is presented schematically in order to exemplify the idea of the web of practices, which occupies sections 4 and 5 – this concerns the evolution of the concept of *function*, which is very complex, to be sure. The presentation of my approach to the notion of mathematical practice is synthetic but more theoretical than in the book (2016). The considerations in sect. 3 about mathematical cognition and ethnomathematics have to do with the 'limits' of MPs.

1. **Defining mathematical practices: cognitive aspects**.

Chapter 2 of (Ferreirós 2016, p. 33) proposed a 'working definition' of mathematical practice, restricted to the limited purposes of an epistemological analysis. This was meant to acknowledge that studies of mathematical practice may have goals different from the philosophical ones – not the case here, and that for those purposes it may be useful to characterize mathematical practice differently. (It also meant to emphasize that I preferred to keep the 'definition' simple, in order to employ it as a tool for analysis; it would have been possible, but probably of little use, to present things in a much more sophisticated way.) Here we aim to revise and update that definition, making it a bit more precise and self-contained. In any event, I should warn the reader that this first part of the chapter is meant as a kind of précis and clarification of aspects of the book, but some important elements cannot be elaborated upon in detail. Thus I'll be referring you to the book itself for elaboration and details.

By a 'practice' in general we mean a recognizable type of activity that is *done*, and can be *taught* and *learned*, by human agents. We always associate mastery of a practice with one or more particular ways of solving some problems, and we only acknowledge that somebody knows or masters a practice when some *success conditions* are met—they know *how to* do it. In many cases there will be explicit norms and rules associated with a practice, but in *all* cases there are implicit norms.[1] In relation to such implicit norms, and how a given agent may learn them, other agents typically act as regulators—the community is most relevant to an individual's attainment of proficiency. Any reasonable analysis of the factors that regulate mathematical practice, making mathematical knowledge possible, must include the resources and abilities of the single mathematician, and her or his interaction with others in the community. So,

> <u>First</u>, reference to the community of mathematicians is essential, and one can state briefly the main reason: there is no practice without practitioners.

But next,

> <u>Second</u>, each mathematical practice is linked with a recognizable type of symbolic framework—formulas or diagrams or signs of a certain kind—employed by the agents to solve problems in the pursuit of particular goals, and to share information.

[1] Think of the practice of riding a bike or sailing a boat: we have explicit rules in the form of traffic regulations, and also rules for competitions; but, even at the most basic level, there are success conditions and associated norms (the ability to maintain control and balance, to brake at or negotiate a curve, and so on).

The symbols we are considering here are external representations, and it's important to keep in mind that diagrams are counted among them.[2] Mathematics is public and shared, it is "common knowledge" (see the Liu Hui quotation below).

At this point, it's relevant to introduce an aside, a note of warning that will accompany our reflection in what follows. Notice that, in retrospect, it will always be possible to link such mathematical practices with particular mathematical theories—and thus to *reconceive* them from a theoretical standpoint, given that we, today, possess a large body of mathematics. But in fact, for our purposes, the adequate thing is to keep separate, at the level of analysis, the practice of mathematics and its products (see sections 4 and 6). This implies also to be careful distinguishing a given practice from any possible (and, in the chronological order, future) theoretical reconstruction. We come back to this below with remarks on retroprojection (section 4), but let's continue with the exposition.

Since our notion of practice aims to help understanding mathematical knowledge, i.e., developing an epistemology of mathematics, it's important to underscore that mathematics, as we understand it here, is not just practical knowledge but has at its core theoretical knowledge. Thus,

> Third, we understand by mathematical practice one that is oriented to the production of knowledge, with a bias toward *know-that* rather than know-how at the level of goals.

Mathematical practices incorporate a theoretical orientation, which is characteristically linked with attention to some epistemic *values*, such as accuracy or precision, consistency, simplicity, and with the *goals* of the practice, such as generality, understanding, certainty. If you consider the gou-gu rule (a.k.a. the Pythagorean theorem), for practical purposes of building and the like, its *particular* examples are most useful (e.g. the fact that triangles of sides 3-4-5 or 8-15-17, which comply with the rule, are right angled). From a theoretical standpoint, however, the theorem as proved by Euclid or by Liu Hui is most valuable due to the great *generality* of the result. This is akin to the difference between approximative procedures, aimed at practical solutions, and perfectly exact or precise results.[3]

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[2] Diagrams are not just figures, there ought to be rules of construction and/or rules of operation on the elements of the diagram (see De Toffoli & Giardino 2014). On the symbolic component of mathematical practices, see (Ferreirós 2016, 46-50).

[3] More on this topic in (Ferreirós 2016, 115-117, 119-122). On the question of generality, see (Chemla, Chorlay & Rabouin 2016).

Attention to *processes*, and not merely to more or less perfect results, is characteristic of studies of mathematical practice. From the standpoint of practice, theories are one and only *one* of the products created by the mathematical community (besides, any given theory is in a dynamic process of reconception and changes in presentation). Other highly relevant products put forward by mathematicians are problems, conjectures, and methods; and so:

> Fourth, we shall consider *problems* and *proofs* (including proof-*methods*) and *theories* as central elements of mathematics, or if you wish as the central products of the processes that conform mathematical practices.

Problem solving, theorem proving, and theory shaping are some of the most important aspects of mathematical activity. Conjectures and methods can be made sense of as revolving around those key goals; e.g., in the process of solving a problem, some conjecture may be proposed, and subsequently refined (see Lakatos 1976; on conjectures, Mazur 1997). Or take what we just called *theory shaping*, a very important activity: it aims to select those methods that are judged most appropriate and relevant for a given subject matter, and to rework systematically a theoretical corpus so as to develop it in accordance with that approach. The theory will be reconceived and the proofs of theorems will be fine-tuned accordingly; often the process will lead to decisions about the principles and methods that are to be employed preferentially, about the assumptions that are decisive for establishing what one regards as the central results in the field, or about the axiomatic basis on which the theory is to be erected. Value judgments are of great importance in such cases—judgments about the most adequate conceptual framework; about the "purity" of the methods employed with respect to the subject matter; about which methods or ways of proving theorems are more fruitful, more explanatory; and so on.[4]

Summing up: We have argued that it's adequate for our purposes to concentrate on theoretical approaches to methods and results, clearly setting the mark above so-called 'practical mathematics'; which allows us to make a difference between mathematical practices and their prerequisites, among which we prominently find (i) basic cognitive skills and (ii) technical practices.[5] We have suggested constraints associated with the role of agents, with the cultural level at which we shall locate practices (in connection with

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[4] There's a lot of literature on these issues, see e.g. (Mancosu 2008) and other chapters in this Handbook.

[5] This notion of a 'technical' practice, applied to measuring and to drawing (geometric) designs, is discussed in (Ferreirós 2016, 38-42, 113-114). See also section 6.

symbolic frameworks, problems, results, methods, and their justification), with our interest in know-that (not just know-how), and with the main goals of practice that will be our concern. For our limited epistemological purposes, then:

> (I) *Mathematical practice is what a community of agents do when, on the basis of their cognitive abilities and cultural techniques, they employ the resources of symbolic frameworks and methods* (perhaps together with other instruments) *to solve problems, justify results and procedures, shape theories, and eventually to elaborate new frameworks*.

Let me insist that problem solving, theorem proving (or perhaps better, to avoid the Western notion of *theorem*: the justification of results and procedures),[6] and theory shaping are some of the most important goals of mathematical activity.

One possible objection is that my definition does not single out *mathematical* practices. There are other kinds of symbolic practice—especially writing, but also the employment of musical notation—that comply with my definition. By writing we solve certain problems (e.g. communication between people, or the storage of relevant information), sometimes we justify certain results and ideas (e.g., by logical argument and critique), and we may even articulate and develop theories (say, philosophical theories). I accept this objection. In fact, I don't think it possible to characterize mathematics without reference to the *contents* involved. Mathematical practices have to do with number and form (spatial figures), or at least with results and patterns that can be linked with numbers and forms, or instantiated in numerical and figural instances (e.g., geometric configurations).

Notice that concepts of number and concepts of geometric form are articulated prior to the introduction of mathematical practices in the proper sense: they belong to proto-mathematics.[7] So we add to the previous clause (I) the following:

[6] This ought to make room for including ancient Chinese maths, and Indian maths, as proper practices; Liu Hui (see Chemla & Suchun 2005) or Bhaskara (see Plofker 2009) were clearly interested in know-what, far beyond the realm of know-how.

[7] For this idea of proto-mathematics elaborated in the case of geometry, see (Ferreirós & García-Pérez 2019); interestingly, we agree on many points –including the use of this expression, proto-geometry– with (Schemmel 2016), a work that only came to my attention recently.

(II*) those practices are recognized as* mathematical *due to their links with the notions of number and form* (i.e., spatial figures)*, which may be explicit or implicit; generally speaking, mathematics has to do with the study of* patterns, *numerical and geometric patterns being prominent*.

The *Nine Chapters of mathematical procedures* (*Jiŭzhāng Suànshù*, 九章算术) may serve as a paradigm example of a mathematical practice defined as per (I) and (II). We find there the symbolic ingredients; procedures for the solution of problems, which are formulaic and explicitly recorded; instruments like the trysquare and compass;[8] the search for patterns and interconnections between those procedures, the systematic organization of knowledge into nine 'branches' of the tree of mathematical knowledge.

This is how the great Chinese mathematician Liu Hui (in the third century) writes about mathematical training, as it existed around 200 BCE, in his preface to the *Nine Chapters*:

> Mathematics (*suan*) is part of the six arts: the ancients employed them to select people of talent, to instruct the children of high dignitaries. … As regards the transmission of methods, one can certainly make common knowledge, as with the trysquare (*ju*), the compass (*gui*), the numbers, and measurement; there is nothing there particularly difficult. (My translation, following Chemla & Shuchun 2005,127)[9]

In the Chinese tradition, as in the Greek one of Euclid, problems of drawing figures and of measurement were considered in one sweep; and they were approached using traditional methods based on compass and ruler, a most noteworthy coincidence.[10] This underscores commonalities linked with the above condition (II). And if anyone thinks that the theoretical orientation is not sufficiently explicit in the *Nine Chapters*, the work of Liu Hui has an undeniable theoretical bias—an explicit interest in justifications, in knowing

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[8] The trysquare is an L-shaped instrument, a *gnomon* in Greek terminology, used to draw straight segments and right angles, and to find distances.

[9] The above translation does not coincide with (Shen *et al.* 1999, 53), where the last sentence reads: "The course of [the arithmetical arts] is not particularly difficult using the methods handed down from generation to generation, just like the compass and trysquare in measurement, with which we draw figures."

[10] This is not so different from early Modern views: D'Alembert in Vol. 7 (1757), p. 629 of the *Encyclopèdie* says: "Geometry is the study of the properties of area, inasmuch as one considers this simply as extent and form." And Legendre started his classic *Éléments de Géométrie* saying that "La géométrie est une science qui a pour objet la mesure de l'étendue," the measure of extension.

what is *precise* and what is not, a theoretical orientation.

Notice too that we have not required a focus on mathematical innovation or research, the elaboration of new problems, or new results. In (Ferreirós 2016) I intentionally did *not* adopt the position that only research mathematicians do mathematics, properly speaking, since we should be interested in how individuals attain knowledge of mathematics. Doing research and developing new mathematics is certainly of enormous interest, but the question is also how aspiring members of the community learn to do mathematics and mathematical research, how they obtain their professional know-how.

2. **Defining mathematical practices: sociocultural aspects**.

We emphasized that mathematics is the collective work of a community of agents. It should be a truism that mathematical knowledge is a collective product: no single individual could have developed the cumulative knowledge that was mastered by Liu Hui (or by Archimedes), which can only be amassed due to the existence of social institutions making it possible. To emphasize this point is by no means to argue for any strong form of social constructivism—that would imply to forget the role of biological conditions, cognitive abilities, semiotic components, logical aspects, and so on. However, it has to be made clear that mathematics has cultural preconditions and requires social institutions.

Mathematicians are a group of learned practitioners, they are erudites, *savants*, philo-sophers in the ancient sense of the word, and one can always see that their collective work is possible thanks to the existence of some institutions. I.e., their practices are made possible by the stratification of societies and the emergence of division of labor: Babylonian mathematics was made possible by the institution of the scribes, just like Greek mathematics was made possible by the institution of schools like the Academy in Athens and the Museion in Alexandria. The reinvention of mathematics in the nineteenth century was intimately connected with the emergence of a strictly professional class of mathematicians within industrialized societies. Fermat was an amateur, and Gauss was still (professionally) an astronomer, but these kinds of situations increasingly disappeared as the nineteenth century advanced, leaving the scene to professionals.[11]

[11] See e.g. C. Goldstein's chapter in *Eléments d'histoire des sciences*, ed. Michel Serres (Paris: Bordas, 1989).

As regards the cultural preconditions, we have seen that mathematical practices are, at core, semiotic practices. Thus they depend *inherently* on forms of *external representation*, such as writing, numerical notation, and figural designs such as those usually called geometric.[12] Of course, the cultural existence, development, and preservation of such symbol systems has further preconditions in material culture, such as the development of writing media: cuneiform tablets made of clay and worked with a stylus (c. 3200 BCE at Uruk in Sumer); the Egyptian papyri and the scrolls preserved in ancient libraries like the famous one of Alexandria; modern printed books, together with the book commerce and the organized libraries that accompanied them; or systems of digital archives in our computerized internet age.

Semiotic components, what I've called *symbolic frameworks*, belong among the cognitive preconditions of mathematical thinking, which by no means is merely a 'natural' product of the brain. But there is more. The semiotic component is central to the emergence of communication, the sharing of information and the interconnection of cognitive processes among agents, and thus to the intersubjective character of mathematics which turns it into true knowledge. For those reasons, mathematical cognition depends on sociocultural niches like the 'paidological' niches that I talked about in (Ferreirós 2016, 19-22; see Menary 2015).

Babylonian mathematics was made possible by the institution of the scribes, which in turn depended on the invention of writing and numbers. Notice also that the "appearance of texts clearly representative of theoretical thinking is a markedly uncommon phenomenon in the ancient world" (Boltz & Schemmel, in Schemmel ed. 2016, 142) and the presence of external representations of knowledge does not lead by itself to theoretical knowledge. For instance, we find "no sources from ancient Egypt or Mesopotamia that document a theoretical reflection on spatial language" (*op. cit.*, 123), which however we find in ancient Greece and ancient China, in the Warring States period.

It would be of little use to enter more deeply here into such components of cultural and mathematical history, which are well studied in a number of works.[13] Let us now turn to another question, about the lower reaches of MPs.

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[12] On this topic, see (Schemmel ed. 2016).

[13] See among others (Netz 2000), (Hoyrup 2002), (Chemla & Suchun 2005), (Robson & Stedall 2009), (Schemmel 2016), (Overmann 2019).

3. The limits of mathematical practices.

The previous working definition of MP may be perceived by some as too restrictive. Wouldn't it be advisable to work with a more inclusive approach that opens the way to acknowledge the mathematical aspects present in *all* human cultures? Many contributions of the last decades, to the history of non-Western mathematics, the study of mathematical cognition, and not least ethnomathematics, have adopted such a stance. Let us briefly review their perspectives.

A. *Cognitive science of mathematics* (*mathematical cognition*): according to some authors, the number concept has been wired into our brains by evolution (and into the brains of rats, pidgeons, dogs too); the same is true of the space concept, Euclidean space. This means that the basic notions that we have underscored in (II) as definitory of the contents of mathematics are, one may say, 'natural' and 'universal' (see e.g. Izard, Pica *et al*. 2011). At most, then, there could be cultural novelties having to do with the methods for dealing with such notions and developing them. But mathematics is in essence, according to these authors, pan-human and innate. (Cultural elements are often acknowledged as contributing to the emergence of elaborate concepts, e.g. by (Carey 2009), although the role of the cultural in cognition is mostly ignored; see (Beller, Bender *et al*. 2018).) Such claims are in conflict with traditional philosophy of mathematics.

The crucial point for many experts in mathematical cognition is that there exist specific numerical (and spatial) capabilities in our basic cognitive setup, and that they are independent from other cognitive capabilities, especially linguistic ones.[14] Yet we should make clear that speaking of an innate "number sense" is ungranted by the available data, or at the very least misleading: the number concept, in a minimally precise rendering of this notion, goes beyond the basic cognitive abilities of subitizing and of judging size (what is sometimes called the AMS or 'analogue magnitude system'). So, what should the philosophy of mathematics answer to the claims of cognitive scientists?

B. *Ethnomathematics*: experts in this field have found "mathematics" in basically all cultures: the famous *quipu* of the Incas, string games of Papua New Guinea, the patterns of Arabic geometric decoration, polyrhythmic music from the pygmies, and so on (see Ascher 1991, Eglash 2000, Gerdes 2004). Often these experts suspect racism or ethnocentrism whenever more restrictive conceptions of "mathematics" are defended:

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[14] See, among others, (Dehaene, Izard *et al*. 2006), (Izard, Pica *et al*. 2011). For discussion and criticism of such views, see (Núñez 2009), (Overmann 2011 and 2019), (Ferreirós & García-Pérez 2019).

> Ethnomathematics holds that mathematical ideas are pan-human and are developed within cultures. ... The imposition of order on space, for example, is universal, but its particular expression varies with and within culture, and may change over time. (Ascher & Ascher 1994)

Notice also that an important part of ethnomathematics deals in effect with applications of modern mathematics (formal languages, graph theory, knot theory, combinatorics) to the study of non-Western cultural products. The mathematical nature of string games (Vandendriessche 2015) or polyrythmic music (Arom 1985, Chemillier 2002) is gauged by applying modern mathematical tools. There is here an intriguing combination of the opposition to ethnocentrism and the imposition of math-centrism.

C. *History of non-Western mathematics*: one finds very significant bodies of mathematical information in cultures that did not develop the famous "Greek" or Euclidean notion of proof and systematic deductive organisation. Cultures and civilisations that have carefully elaborated mathematical notions and methods, to solve problems in ways that can be compared with so-called "Western" mathematical practices: Babylonian or Egyptian mathematics many centuries before the common era, Chinese mathematics from before C.E., Indian mathematics long before 1300, and so on. In fact, this body of work can be analyzed with the approach to PM that I outlined in the previous two sections – for it does not presuppose a given way of proving, neither the Euclidean nor what one might call the "Hilbertian" one.

In all of these cases we find results or ideas *comparable* to results in the mathematics we know, but not always commensurate with them. In all of them, the old wisdom of hermeneutics applies: we can only interpret other languages, other cultures by analogy with our own; even historians, in the midst of all their sophisticated attempts to break the hermeneutic circle, cannot avoid to relate the ideas and techniques of other times or spaces with our own. But also, hermeneutics has always emphasized the need to make a conscious effort to question one's own cultural assumptions while developing an acquaintance with the foreign. If you study a non-urban, unstratified society with oral culture, it would not be wise to assume that they will display marked differences between, say, the areas of religion and sports, like the ones we may be accustomed to. I assume it wouldn't be polemical to say that some cultures do not have "sports",[15] and so one may

[15] *Sport* describes a category of activities that only coalesced in the West in the nineteenth century and was then carried around the globe by Western colonialism and imperialism and later globalisation (Guttmann 1994). One may find it wise to broaden the notion of sport so as to include Greek Olympic games, Roman

ask why would it be polemical to say that some cultures do not have "mathematics". A matter of cultural pride, there's little doubt.

In the historical and ethnographical cases, we find ideas or practices that can be profitably compared to mathematical results, or studied with mathematical methods, but of course the possibility to interpret something with mathematics does not imply that such a thing is mathematics. The classical example of this is music. Django Reinhardt was not like Bach. Is the musician who plays by rote learning, with no erudite knowledge of musical notation, counterpoint, and the like, still unknowingly a mathematician?

If we wanted to embrace the approach of ethnomathematics, we'd have to revise the previous notion of mathematical practice. Notice, by the way, how the ethnomathematical perspective may also be seen as a natural by-product of the nativist ideas of some cognitive scientists. Both trends are inclined to emphasize that there's something innate or pan-human in mathematics. But to make room for this, one would have to modify the basic definition of mathematical practice. Not only should the emphasis on knowledge—on know that, the theoretical—be avoided, but also we'd have to cease requiring that there exist symbolic components—formulas or diagrams, or some kind of external means of representation—as a basic element of the practice.

However, one can offer rather robust arguments for the view that the above-mentioned experts in mathematical cognition are wrong: there are no concepts of number, nor concepts of spatial configurations, without cultural practices involving external representations. The case has been made by experts such as Overmann and Núñez as regards number, or Schemmel for space (see also Ferreirós & García-Pérez 2019).

Adopting the views of these experts, coming from fields as diverse as cognitive psychology, archeology, and historical epistemology—so that their views are based on careful consideration of cognitive science, anthropology, and history—it's natural to argue that one should differentiate three levels of cognition:[16]

- **basic cognition**, of the kind that is developed from innate cognitive abilities, 'naturally', by exposal to typical phenomenal experiences;[17]

gladiator games and chariot races, and Mesoamerican ball court games, even though this must be done rather carefully. But it would be uninformative to adopt such a broad notion of sport that one would *ipso facto* find them in all cultures.

[16] See also (Giardino 2016).

[17] Here we mean experiences characteristic of human life on earth, such as walking or running, regardless of cultural environment.

- **proto-mathematics** based on basic cognition amplified by external representations, which requires some socio-cultural conditions, including a certain degree of organizational and material complexity;[18]
- and **mathematics** properly speaking, which is the product of theoretical reflection on certain forms of proto-mathematics, made possible by the creation of specialized external representations.

Let me give an example. The "number sense" (to use Dehaene's phrase) is pan-human, available at the level of basic cognition, and it *must be* distinguished from the possession of a concept of number; the latter emerges at the second level,[19] not in all cultures but in many of them, often in relation to measuring practices (measuring with concrete tools requires counting, and thus presupposes number). Mathematics properly speaking begins when there is not only methods for the resolution of concrete problems, practical problems like counting or creating a building with right-angled corners, but also a next-higher interest in reflection about such methods and the patterns that emerge in them—questions such as whether a given method can be extended further, the exact limits of their validity, their degree of precision, etc.[20] These seem to have always required systems of writing, the elaboration of technical vocabulary, and in many cases special symbol systems for the practice of mathematics.

I claim that the distinction of those three levels is highly informative and even necessary. The labels may be judged more or less felicitous, and perhaps the ethno-mathematician will prefer to label 2. as mathematics, and 3. as theoretical mathematics. Such a terminological option seems good to me, provided the relevant pre-conditions and presuppositions are made clear. Of course, a change in the definition implies a change in the content and the conditions of applicability.

4. The web of mathematical practices: the case of functions.

In (Ferreirós 2016), I presented and discussed the thesis that mathematical knowledge

[18] On those preconditions, see e.g. (Overmann 2011 and 2019).

[19] The so-called 'number' sense is precise only at the range of subitizing, but is fuzzy in general; external representations (counting with sticks, words, or body parts) make it possible to obtain a crisp general concept of number. Instead of the 'number sense', one should perhaps speak of a *quantic sense*, making a contrast of crisp concept (number) with imprecise intuition (of quantity).

[20] For instance, the *gou-gu* rule is valid for certain triangles, like the 3-4-5 or the 5-12-13 triangle, but is it valid more generally for any right-angled triangle? The ratio of circumference to radius is close to 157/50, but is this result fully precise?

typically requires a web of practices, interconnected by systematic links.[21] The meaning and function of mathematical statements can be linked with a plurality of practices, starting from what I called 'technical practices' (see above) and going all the way to sophisticated scientific practices. This can be easily seen for any of the core concepts of mathematical knowledge, such as the notions of natural number, of real number, of geometric form (in the plane), or of function. Intentionally left aside in (Ferreirós 2016) were more complex notions such as function, but it may be useful to elucidate the thesis of the web of practices with this case in view.

4.1. The web-of-practices approach. Let me first present a few of the basic tenets of this approach. There is a multiplicity of co-existing practices, i.e., different levels of practice (and associated knowledge) that coexist and are systematically linked – in an intersubjective way, as shown by the fact that all this is taught and learned, collectively shared. In my book I presented elementary examples of three levels of practice interconnected, in cases such as the natural numbers, geometry, and proportion theory (see Ferreirós 2016, 38-39). But of course there can be more levels involved, as happens with core mathematical ideas such as the real numbers. The concept of real number, even in its formative stages, links back to measuring practices (this is what I call a 'technical' practice), to the calculus of fractions and plane geometry (properly MPs), to the algebraic theory of equations, and so on. The web-of-practices approach underscores the idea that an adequate epistemological analysis of our knowledge of the real numbers must consider those practices separately,[22] but must also analyze their systematic interconnections.

With a view to the configuration of mathematical knowledge prior to the seventeenth century, in (Ferreirós 2016, 39) I mentioned schematically the following three practices:

measuring practices / fraction arithmetic / proportion theory.[23]

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[21] I will not enter into the idea of systematic links here, but it's central to my analysis. See (Ferreirós 2016, 83-88, 34-43), and *passim*.

[22] Notice that this goes against some traditional tendencies in the philosophy of mathematics: from foundational studies we have inherited a tendency to adopt a single mathematical system (e.g. axiomatic set theory) as a reductive basis for all of mathematics. I claim that such a reductionist perspective makes it *impossible* to properly analyze the cognitive bases of mathematical knowledge, and its historical development.

[23] The reader can probably reconstruct on her own the systematic links involved here: to give only an example, measuring practices lead to a system of fractions as soon as we consider the idea of using

This interconnected web became a basis for introducing a new layer of practice and knowledge, the real number system. Some mathematicians, notably Stevin and Newton, proposed to *define* number as "that by which the quantity of any thing is explained" (as Stevin said in 1585), which implicitly linked the idea of number with proportion theory; with Newton (in 1720) this becomes explicit, since number is for him "the abstracted ratio [proportion] of any quantity, to another quantity of the same kind" (see Ferreirós 2016, chap. 8). The arithmetic of fractions is subsumed here, the system has the classical algebraic properties, and the underlying assumption that some domains of quantities – to which the theory of ratios is applied – are perfectly continuous implies the continuity of the real number system.

The real numbers **R** are a veritable knot linked with all kinds of MPs. When we examine the real numbers, a complex spider-web should have to be considered, linking number and geometry, the discrete and the continuous, algebra and analysis, and then topology and more. But this is not all: one can (and should?) always consider the 'applications' of those ideas in scientific or engineering contexts, with astronomical models usually leading the way (in the historical order of development). Another important thesis of the web-of-practices approach is that there are always connections with one or more 'technical' practices, but usually also with so-called 'applications', that is to say, with scientific practices (see Ferreirós 2016, 39-43). When dealing with advanced mathematical concepts, the emerging picture can be truly complicated and difficult to explore. The situation with functions is correspondingly complex, but let's try to offer a survey.

4.2. The function concept. It should be obvious that the notion of function is central to modern mathematics as developed from 1750 onwards, until the beginning of the twentieth century. A very large amount of advanced mathematics can be explained and justified in terms of this central notion, which has also been at the core of innovative ways of understanding physical reality. If a great many mathematical theories can be considered as "extensions of the general theory of sets" (Bourbaki 1949, 7), exactly the same can be said about the theory of functions.[24] However, complex cases like that of the

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submultiples of the basic measuring unit, in order to refine measurements (an example being aliquot parts of the unit). For proportion theory and the reals, see e.g. (Stein 1990).

[24] Set and function are interdefinable concepts: to see the point fully clear at a technical level, consider von Neumann's original version of the NBG axiomatic system, or else the system ECTS (see Leinster 2014).

emergence and development of the function concept are truly defying for historical-philosophical analysis.

In a highly commendable paper, offering a general overview, Israel Kleiner wrote:

> Mathematical concepts usually develop gradually, in response to mathematical needs. While the function concept dates back about 300 years, the 'instinct for functionality' may be said to be about 4,000 years old. (Kleiner 1993)

I wholeheartedly agree that mathematical concepts develop historically, and the function concept is an excellent example. But the last part of Kleiner's statement is far from being clear, indeed it's far from a formulation of 'fact'—at most it should be regarded as the formulation of a problem. How far back can we follow the idea of function? Maybe the 'instinct for correlation' that appears to be extremely ancient could be regarded already as an 'instinct for functionality'? Or perhaps not. How do you know? how *can* you know?

Figure 1: Tablet #92698 in the British Museum (Old Babylonian period), featuring squares of the numbers up to 59. Here we find a record of calculations, but no functional

After all, axiomatic set theory is, to a large extent, just a basis for developing the number systems and the theory of functions.

> thinking…[25]

Notice that in such cases there is a great danger involved in what I have called retroprojection. It’s easy to see “functional thinking” when we look with modern eyes into documents which come from practices of which (very likely) that thinking was *not* part. Say, a Mesopotamian table of squares (Figure 1; we come back to this example in the last section). The phenomenon of retroprojection is familiar from studies of mathematical cognition, where some authors go so far as to speak of the ‘number sense’ as a system for “set detection”.[26] The phenomenon is also familiar from studies of ethnomathematics, many of which adopt the methodology of analyzing cultural practices with modern mathematical tools (here retroprojection is an explicit part of the methodology, which makes it admissible; but this also invalidates some possible conclusions or extrapolations from the case studies). In our case, since we aim at analyzing the historical evolution of mathematical concepts, we must be especially aware of retroprojection and regard it as damaging to our aims.

In sharp contrast with the sentiment expressed by Kleiner, math educator W. L. Schaaf wrote in 1930:

> The keynote of Western culture is the function concept, a notion not even remotely hinted at by any earlier culture. And the function concept is anything but an extension or elaboration of previous number concepts—it is rather a complete emancipation from such notions. (Schaaf 1930, 500)

The idea expressed here may also be exaggerated, but in my opinion there is no evidence that the concept of function can be found early, and in particular not in the so-called origins of Western culture—ancient Greece or the Hellenistic period.

In the next two sections, I’ll develop some ideas about the web of practices that is knotted around the function concept. We start with early Modern developments that led to the 18th century idea of function (as ‘analytic expression’) and the 19th century ‘logical’ idea of function. Then, in the last section, I’ll enter into the difficult question of the most basic practices and cognitive experiences that form the basis of the idea of function. In the end I propose that two old conceptions—a dynamic and an algebraic one—coalesced to

[25] Neugebauer, *Mathematische Keilschriftttexte* I, pp. 68-73 (especially p. 71 n. 2). Budge, *Guide to the Babylonian and Assyrian antiquities* (London, 1922), p. 161.

[26] (Spelke *et al*. 2010, 874-75) Unlike the notion of class, which in its early stages was vague and imprecise, the idea of set is very modern; it didn’t exist before 1860, and of course there are no sets in the environment with which we might interact.

give rise to the early Modern idea of function – but this is only the start of a longer process.

5. 1600 onwards: a mixture of mathematical and scientific practices

As concerns the historical evolution of the function concept, from my perspective one should differentiate four phases: 1. the *prehistory* of the function concept, up to about 1650 (I leave open the question when does it begin, but perhaps in the Middle Age); 2. the *early history* in analytical geometry and infinitesimal analysis, leading up to the explicit formulations of the concept in the eighteenth century.[27] 3. the *classical period* which can be located in the nineteenth century, marked by the debate around Dirichlet's proposal of "arbitrary functions" and characterized by progressive expansion of the field investigated by mathematicians (e.g., discontinuous functions); 4. the "*modern*" or *contemporary period*, including full extensionalization of the function concept but also featuring a series of innovative structural conceptions (e.g., function spaces). However, it's not my purpose here to write a history of functions, but to investigate the web of practices which surrounds this crucial mathematical idea.

To speak about prehistory, I surmise, there has to be a more or less explicit conceptual understanding of notions that can be recognized as leading to the idea of function. But let us for the moment ignore the difficult question of the most elementary practices that are linked with our conceptions of function (I come back to this in section 6). The prehistory of the function concept, I argue, saw the emergence of a dynamic conception of the interrelation between variable magnitudes,[28] impulsed by the study of models in astronomy and mechanics.

Such models had to do with the understanding of physical phenomena in terms of functional dependence, where *time* figures prominently and rises to the status of independent variable par excellence. If this is right, the emergence of an idea of function didn't just occur in connection with any calculation or correspondence of values, but was crucially linked to a conceptual understanding of *dynamic interrelations*. And thus the

[27] By Bernoulli (1718), quoted below. Cf. also (Euler 1748; Klügel 1803; Cauchy 1821). It's well known that, at this point, functions were mostly conceived to be defined as analytical expressions; for details, see the expositions by (Luzin 1936, Youschkevitch 1976, Kleiner 1993).

[28] Functional thinking was moving *beyond* causality, understood in the simple classical sense, of basic causal schemes linked to 'manipulative' causation.

rise of functional thinking would have been inextricably linked with scientific topics in astronomy, mechanics, and perhaps other fields of study.

An obvious example can be Galileo's study of accelerated motion, "the intimate relationship between time and motion" that he discovered (1638, 197) (which however was unclear to him even in 1604)[29]. Notice that we find here a clear conceptual understanding of dynamic interrelations between (physically significant) variable magnitudes, but without any explicit technical language for the concept of function. Let me quote from the Third Day of *Two New Sciences*:

> Some superficial observations have been made, as, for instance, that the free motion [*naturalem motum*] of a heavy falling body is continuously accelerated; but to just what extent this acceleration occurs has not yet been announced; for so far as I know, no one has yet pointed out that the distances traversed, during equal intervals of time, by a body falling from rest, stand to one another in the same ratio as the odd numbers beginning with unity. (Galileo 1638, 153)

Galileo manages to produce a precise mathematical correlation between times and spaces, expressed in the classical language of proportionality: if a mobile falls from rest, under gravity, in equal times the spaces are in the same proportion as the odd numbers 1, 3, 5… Also:

> This we readily understand when we consider the intimate relationship between time and motion; for just as uniformity of motion is defined by and conceived through equal times and equal spaces (thus we call a motion uniform when equal distances are traversed during equal time-intervals), so also we may, in a similar manner, through equal time-intervals, conceive additions of speed as taking place without complication; thus we may picture to our mind a motion as uniformly and continuously accelerated when, during any equal intervals of time whatever, equal increments of speed are given to it. (Galileo 1638, 197-198)

His studies allow him to associate clear geometrical patterns with physically significant phenomena: "It has been observed that missiles and projectiles describe a curved path of some sort; however no one has pointed out the fact that this path is a parabola." The Galilean method is controlled by experiment, but depends also on idealization; he relies on the hypothetical construction of analogical models (see Crombie 1994).

In the next phase of development, analytical geometry is formed by linking geometric practices with the general (still imprecise) idea of real numbers, via the Cartesian way of representing and studying geometric curves by means of algebraic equations. This adds a new layer to previous mathematical practices, incorporating innovative methods, and it also adds to the methodological tools available for the

[29] See the letter to P. Sarpi, Oct. 1604.

construction and study of mathematical models of astronomical systems, mechanical systems, optics, etcetera.[30]

The second generation of Cartesians will go beyond the finitary restrictions of the master, also deploying infinite series—a great conceptual and methodological innovation, in a context that continues to be a mixture of mathematical practices, mechanics, optics, astronomical practices. Notice that the context is *non-mathematical* in the 20th-century sense (i.e., not pure mathematics), and this is where the crucial step of explicit introduction of the function concept will be prepared. It's an "impure" background of *mixed* mathematics: astronomy being the context for the introduction of trigonometrical operations; mechanical or astronomical systems and their evolution being the frame for emergence of the dynamic conception of how variable magnitudes are interrelated; the dynamics of such systems being a key goal for those who introduced infinite series.

The dominant conception of functions, from their inception in the early eighteenth century, was as "analytical expressions." I already quoted Bernoulli, defining in 1718: "We call function of a variable magnitude, a quantity composed in any manner of this variable magnitude and of constants;" essentially similar definitions can be found in Euler and others, even in (Cauchy 1821). The quantity so composed is again a variable magnitude, whose variation can be computed from the variation of the independent magnitude. Thus, conceptually a function is just an interrelation of variable magnitudes, with one independent and one dependent variable. But the mathematical, operational aspect of things dominates the picture, leading authors to insist on analytical representability. Nevertheless, as is very well known, there is tension with alternative ideas inspired in physical images: the idea of free graph (to speak in later terms) is defended by Euler, against the ideas of D'Alembert and Bernoulli, in the celebrated controversy on the vibrating string.[31] Notice that Euler's intervention was saluted by Truesdell as "the greatest advance in scientific methodology in the entire [18th] century" (1957, 248).

The mixed or impure context of physical and mathematical practices intermingled continues to be decisive in the subsequent evolution of the function concept. The next

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[30] Perhaps we shouldn't forget engineering, artillery, and so on, which were branches of eighteenth century mathematics.

[31] See (Truesdell 1957, Bos 1980). This is described in any good exposition of the evolution of the function concept, e.g. (Kleiner 1993). Apparently the first time that this episode was emphasized, thanks to Dirichlet's advice, was in Riemann's exposition written in 1854 for his *Habilitation* thesis.

crucial steps came with the triumph of trigonometric series in the hands of Fourier and his followers. P. G. Lejeune Dirichlet, who studied in Paris and was a young member of Fourier's circle, made a decisive contribution with his rigorous study of the representation of functions by trigonometric series, and his clear but challenging conception of "arbitrary functions." This was introduced in an 1829 paper published in Crelle's *Journal*, but also (and the detail is worth notice) in a chapter of Dove's *Repertorium der Physik* (1837).[32] A function was now to be just any (many-to-one) correspondence of real values, which is exactly the idea expressed by the modern notation f: $\mathbf{R} \to \mathbf{R}$:

> it is not at all necessary that y depend on x in the whole interval according to the same law, indeed one does not need to think of a dependence that is expressible by mathematical operations. … This definition does not prescribe a common law [*Gesetz*] to the different parts of the curve; one can think of it as composed of the most heterogeneous parts or even totally lawless [*gesetzlos*]. ... As long as one has determined a function for only a part of the interval, the manner of its continuation over the remaining interval will be totally left to arbitrariness [*ganz der Willkür überlassen*].

The idea of arbitrary function, defined in a purely 'logical' way, is meant seriously here.

In effect, Dirichlet's 1829 result established that any *arbitrary* function that is piecewise continuous (and complies also with another condition) will be representable by a trigonometric series. Thus, a conceptual determination of the functions, by very general properties, leads to the existence of an analytical representation. That throws light on some of the deep reasons for the sharp *conceptual turn* that Dirichlet promoted, reasons behind his famous remark on "the ever more prominent tendency in modern analysis to replace calculations by thoughts."[33]

To summarize, we have reviewed some of the main mathematical and scientific practices that were involved in the first explicit emergence of the function concept. They include:

– Euclidean geometry, proportion theory, real numbers, and algebra, on the mathematical side;

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[32] The only paper in the section Mathematical Physics: 'On the representation of completely arbitrary functions by series of sinus and cosinus' (Dirichlet 1837).

[33] See (Ferreirós 2007, 28-31). Similar remarks were offered by Eisenstein in 1843, Riemann in 1851, Dedekind in 1871, Hilbert in 1897, and others. Riemann in particular (1851, sec. 1) offered a very clear reflection on the import of Dirichlet's contribution. Eisenstein remarked on the strong contrast between the older school and the "fundamental principle of the new school, founded by Gauss, Jacobi and Dirichlet" (full quote in Wussing 1984, 270).

– and astronomical models, mechanics (e.g. vibrating string), theory of heat (Fourier), etc. on the side of physics.

All of these elements allowed for the articulation of a basic general conception of function, in the sense of a dynamic interconnection between variable magnitudes, that had been ripening in contexts such as astronomy and mechanics—e.g., in Galileo's work, where geometrically represented physical magnitudes appear as functions of time.[34] This underlays the earliest explicit definitions of 'function' as analytical expressions (Bernoulli in 1718, Euler in 1748), which however emphasize algebraic features, and which were already incorporating new ingredients from the emerging infinitesimal calculus (e.g. infinite series).

We have seen how the process starts with a 'dynamic' conception, couched first in algebraic language, which leads to an 'analytic' notion of function. And also that this early conception evolved into the modern 'logical' one of Dirichlet, again in connection with models of physical situations and with innovative mathematical methods such as the trigonometric series of Fourier.

6. Basic, low-level practices in the case of functions.

Leaving aside the further development of the function concept, I'd like to say a few more words on its early stages of development and its connections with very basic practices. In my view, the prehistory of this concept (from geometric models in astronomy, until about 1650) would be particularly interesting and promising as a field of future studies. But this prehistory is particularly understudied because it *requires* us to go beyond 'mathematical' history of mathematics: to analyse it in detail, we have to enter into transdisciplinary considerations. We'll need to carefully explore cases in which the mathematical ideas are emerging in the context of astronomy, geometry of intensions (Oresme), the rise of mechanics (making explicit of interrelations between inhomogeneous variable magnitudes, e.g., time and distance and speed), and so on.

Beyond that, there's still the question about the 'technical' practices –in my sense– and the most basic cognitive experiences at the basis of the idea of function. This is a fascinating topic, still awaiting careful exploration, although much material on it has been amassed by educationalists. Here I'd like to present only a hypothesis, which I'll

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[34] Even if, lacking the conceptual and methodological resources that would be introduced along the following century, this is still expressed in the language of proportion theory.

introduce with some remarks on Cavaillès and Piaget. A phenomenon that is often found in the development of mathematics is what Cavaillès called *thematization* (Benis-Sinaceur 1994, 77), namely, progress by making explicit a notion that was implicit in a previous practice—by reflection on that practice and by making an aspect of it the thematic focus of a new symbolic framework and new methods (new concepts, a new kind of object). This is typically found in the "duality operation-object" (also "method-result") that Cavaillès emphasises in the development of mathematics (Benis-Sinaceur 1994, 81).

Let me offer two examples from nineteenth-century maths: At one stage we find groups of operations (e.g., permutations) as a method for the study of a certain body of mathematics, later we encounter a new generation of mathematicians studying groups themselves as their object, via higher-level methods. What was previously a tool in a particular theoretical setting, has become a new thematic object. Elsewhere, at one stage we find bijective correlations employed as a tool, be it in projective geometry or in Cantor's work, subsequently the general concept of mapping offers a thematic focus for a new theory that subsumes previous ideas of function, bijection, and so on. This idea of thematization seems to be intimately linked with that which Piaget called 'abstraction réfléchissante', *reflecting abstraction*: some aspect of, say, action is projected onto a higher level of representation, which involves its reconstruction and reorganisation on this higher plane.[35]

Thematization or reflecting abstraction seems to play an important role in the historical emergence of the notion of function, but also in the way students of mathematics grasp the function concept. (Paz & Leron 2009) argue that the formal function concept is intimately linked with "intuitions of actions on familiar objects;" more particularly, they identify an "algebraic" conception of functions, which they present as based on the image of an input-output machine—opposed to the "dynamic" conception that we discussed, based on the image of covariation of two magnitudes.[36] My own hypothesis is that the basis of this 'algebraic' conception emerges from thematization of some aspects of basic practices of calculation or reckoning.

[35] See selected quotes in (Montangero & Maurice-Naville 2013), citing the 1977 work: *Recherches sur l'abstraction réfléchissante*.

[36] There is a terminological problem, since Paz & Leron call this the "analytic" conception, but as we've seen historically the idea of function as an "analytic expression" is closer to their "algebraic" conception. I make my own terminological choice with a view to integrating history and cognitive analysis.

Children who learn mathematics are drilled into the practice of obtaining an output from a given input (e.g. to compute the square of a given number), in a kind of symbolic action or formal action that, naturally, we regard as intimately connected with mathematics. It's a significant part of "the mathematical experience." Such input-output aspects of our conscious behaviour can also be found in non-mathematical practices, to be sure—say, when a child is given a sentence and is assigned the task to parse it grammatically into its components. But usually, such situations are based on deployment of an explicit symbolic framework (be it numbers and calculation with them; or letters and writing), and thus the basis for thematization is not just action on familiar objects, but actions on familiar symbolic systems.

My hypothesis would be that the historical emergence of the function concept was a product of the combination of the two conceptions of function just described. On the one hand, based on scientific practices of modelling (astronomy, mechanics, optics), there emerged a conceptual understanding of dynamic interrelations, the "image of covariation of two magnitudes." On the other hand, the rise of algebra implied reflecting abstraction on reckoning operations, or their thematization, to become new focal concepts of algebraic practice. At some point, *cross-hybridisation* of these two lines of development led to the idea of *representing dynamical interrelations by means of algebraic operations*, with geometric models as a background and with time as the independent variable. This was the birth of the function concept, and it's less interesting to know the 'who' of this invention, than the 'what' was involved. Perhaps several generations of mathematicians (from Galileo to Bernoulli, say) were needed to distil the idea and fully elaborate the new conception.

Notice that the situation is mixed or impure, as I have stressed, and also that both conceptions had ties with geometric thinking. The birth of the function concept *was not purely mathematical*, unlike, say, the concept of set in the late nineteenth century. The result was a decisive new idea of the seventeenth-century 'New science', which would be articulated around the technical terms: *function* and *variable magnitude*.

The subsequent development, guided by refined methods of expression and study of functional interrelations, such as infinite series and trigonometric series, led in due course to the rise of a third key conception—the "logical" conception of functions. This is a more abstract, static conception, that came to be known as the "Dirichlet concept" of "arbitrary" function. Part of the complexity of the evolution of the concept of function, and the obstacles encountered by those who learn about functions (see, among many other works, Paz & Leron 2009), have to do with the conflicts between those different conceptions, which give rise to conflicting 'intuitions' about functions, variables and values.

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