Digital Analysis of Logical Equivalences

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Abstract

This paper introduces a digital method for analyzing propositional logical equivalences. It transforms the theorem-proof method from the complex statement-derivation method to a simple number-comparison method. By applying the digital calculation method and the expression-number lookup table, we can quickly and directly discover and prove logical equivalences based on the identical numbers, no additional operations are needed. This approach demonstrates significant advantages over the conventional methods in terms of simplicity and efficiency.

Keywords: logic, proposition, truth value, mathematical logic, symbolic logic, propositional logic, Boolean logic

1 Introduction

Since the mid-19th century, George Boole established Boolean algebra and laid the foundations for symbolic logic [1], and through the developments by Ludwig Wittgenstein and others in the last century, truth tables were introduced [2], despite some significant advancements have been made, the two conventional approaches have remained the dominant methods in propositional logical reasoning: The first approach is a statement-based method. It involves sequentially applying theorems, transforming symbols, and rewriting statements to reach a conclusion. The second approach is a table-based method: It involves using a table (called truth table) to reach a reasoning result.

Zhou Nongjian proposed a number-based reasoning method named "Digital calculation method" in his paper "A Digital Calculation Method for Propositional Logic" [3]. Zhou's method came from an idea: every logical expression has a truth value, and every truth value can be represented by a digital number. By calculating and comparing these numbers, we can transform the statement-based reasoning or the table-based reasoning into the number-based calculation. The digital calculation method demonstrates advantages in simplicity and efficiency over the two conventional approaches, including that it provides a solution to store calculation results in the expression-number lookup tables for data reuse and sharing.

Since each expression in the lookup tables has a truth value represented by a digital number, we can easily find and prove logical equivalences by comparing these numbers according to the principle of "two expressions that have the same value are equivalent".

In the following sections, we will apply the digital calculation method to analyze logical

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equivalences.

2 Find Equivalences from Expression-number Lookup Table

2.1 A-B Equivalence Formulas

Looking at "4-digit Expression-Number Lookup Table" in the Appendix, we observe that many items have identical numbers. For instance, in the A-B category columns, both expressions $(A \to B)$ and $(\neg A \vee B)$ share the numerical value 1011, demonstrating their equivalence:

1011: $(A \to B) \leftrightarrow (\neg A \vee B)$ //Both have the same truth value 1011

This is just one example among the numerous cases of sharing numbers across expressions. By analogy, we can find other seven equivalences. See the eight equivalences below:

1011: $(A \to B) \leftrightarrow (\neg A \vee B)$
0111: $(A \to \neg B) \leftrightarrow (\neg A \vee \neg B)$
1110: $(\neg A \to B) \leftrightarrow (A \vee B)$
1101: $(\neg A \to \neg B) \leftrightarrow (A \vee \neg B)$

0110: $(A \oplus B) \leftrightarrow (\neg A \oplus \neg B)$
1001: $(A \oplus \neg B) \leftrightarrow (\neg A \oplus B)$
1001: $(A \leftrightarrow B) \leftrightarrow (\neg A \leftrightarrow \neg B)$
0110: $(A \leftrightarrow \neg B) \leftrightarrow (\neg A \leftrightarrow B)$

Since in each of the above equivalences, the left-hand side (LHS) and the right-hand side (RHS) share the same number, the equivalence is logically proved according to the principle of "two expressions that have the same value are equivalent". No further proof is needed.

However, to verify that this method works and that no additional operations are needed, let's test it by randomly selecting one of the equivalences above. To prove an equivalence, let's use the digital proof method [3] (Section 4 Proof Method and Criteria) with the truth criterion number 1111 and the $(\leftrightarrow)$ calculation formulas:

$(1 \leftrightarrow 1) \leftrightarrow 1$
$(1 \leftrightarrow 0) \leftrightarrow 0$
$(0 \leftrightarrow 1) \leftrightarrow 0$
$(0 \leftrightarrow 0) \leftrightarrow 1$

Given: $(A \to B) \leftrightarrow (\neg A \vee B)$

1011 //$(A \to B)$
$\underline{1011 \leftrightarrow}$ //$(\neg A \vee B)$
1111 //Proved to be true

The result is 1111, indicating that the equivalence is proved to be true.

The digital proof method illustrates that, for any expressions, if they have the same digital number, then they are equivalent. Therefore, by looking up identical numbers in an expression-number lookup table, we can quickly and directly discover all equivalence formulas. No further proof is needed.

2.2 A-B / B-A Equivalence Formulas

In the same lookup table, when comparing the numbers of the A-B expressions with the numbers of the B-A expressions, we can find the following 20 equivalences:

1000: $(A \land B) \leftrightarrow (B \land A)$
0100: $(A \land \neg B) \leftrightarrow (\neg B \land A)$
0010: $(\neg A \land B) \leftrightarrow (B \land \neg A)$
0001: $(\neg A \land \neg B) \leftrightarrow (\neg B \land \neg A)$

1110: $(A \lor B) \leftrightarrow (B \lor A)$
1101: $(A \lor \neg B) \leftrightarrow (\neg B \lor A)$
1011: $(\neg A \lor B) \leftrightarrow (B \lor \neg A)$
0111: $(\neg A \lor \neg B) \leftrightarrow (\neg B \lor \neg A)$

1011: $(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$
0111: $(A \rightarrow \neg B) \leftrightarrow (B \rightarrow \neg A)$
1110: $(\neg A \rightarrow B) \leftrightarrow (\neg B \rightarrow A)$
1101: $(\neg A \rightarrow \neg B) \leftrightarrow (B \rightarrow A)$

0110: $(A \oplus B) \leftrightarrow (B \oplus A)$
1001: $(A \oplus \neg B) \leftrightarrow (\neg B \oplus A)$
1001: $(\neg A \oplus B) \leftrightarrow (B \oplus \neg A)$
0110: $(\neg A \oplus \neg B) \leftrightarrow (\neg B \oplus \neg A)$

1001: $(A \leftrightarrow B) \leftrightarrow (B \leftrightarrow A)$
0110: $(A \leftrightarrow \neg B) \leftrightarrow (\neg B \leftrightarrow A)$
0110: $(\neg A \leftrightarrow B) \leftrightarrow (B \leftrightarrow \neg A)$
1001: $(\neg A \leftrightarrow \neg B) \leftrightarrow (\neg B \leftrightarrow \neg A)$

2.3 A-B / ¬(A-B) Equivalence Formulas

When comparing numbers of the A-B expressions with numbers of the ¬(A-B) expressions in the lookup table, we can find the following 48 equivalences:

1000: $(A \land B) \leftrightarrow \neg(\neg A \lor \neg B)$
0100: $(A \land \neg B) \leftrightarrow \neg(\neg A \lor B)$
0010: $(\neg A \land B) \leftrightarrow \neg(A \lor \neg B)$
0001: $(\neg A \land \neg B) \leftrightarrow \neg(A \lor B)$

1110: $(A \lor B) \leftrightarrow \neg(\neg A \land \neg B)$
1101: $(A \lor \neg B) \leftrightarrow \neg(\neg A \land B)$
1011: $(\neg A \lor B) \leftrightarrow \neg(A \land \neg B)$
0111: $(\neg A \lor \neg B) \leftrightarrow \neg(A \land B)$

0110: $(A \oplus B) \leftrightarrow \neg(A \oplus \neg B)$
0110: $(A \oplus B) \leftrightarrow \neg(\neg A \oplus B)$
1001: $(A \oplus \neg B) \leftrightarrow \neg(A \oplus B)$
1001: $(A \oplus \neg B) \leftrightarrow \neg(\neg A \oplus \neg B)$

1001: $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$
1001: $(A \leftrightarrow B) \leftrightarrow \neg(\neg A \leftrightarrow B)$
0110: $(A \leftrightarrow \neg B) \leftrightarrow \neg(A \leftrightarrow B)$
0110: $(A \leftrightarrow \neg B) \leftrightarrow \neg(\neg A \leftrightarrow \neg B)$

0110: $(\neg A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow B)$
0110: $(\neg A \leftrightarrow B) \leftrightarrow \neg(\neg A \leftrightarrow \neg B)$
1001: $(\neg A \leftrightarrow \neg B) \leftrightarrow \neg(A \leftrightarrow \neg B)$
1001: $(\neg A \leftrightarrow \neg B) \leftrightarrow \neg(\neg A \leftrightarrow B)$

0110: $(\neg A \oplus B) \leftrightarrow \neg(A \oplus B)$
0110: $(\neg A \oplus B) \leftrightarrow \neg(\neg A \oplus \neg B)$
1001: $(\neg A \oplus \neg B) \leftrightarrow \neg(A \oplus \neg B)$
1001: $(\neg A \oplus \neg B) \leftrightarrow \neg(\neg A \oplus B)$

1011: $(A \rightarrow B) \leftrightarrow \neg(A \wedge \neg B)$
0111: $(A \rightarrow \neg B) \leftrightarrow \neg(A \wedge B)$
1110: $(\neg A \rightarrow B) \leftrightarrow \neg(\neg A \wedge \neg B)$
1101: $(\neg A \rightarrow \neg B) \leftrightarrow \neg(\neg A \wedge B)$

1000: $(A \wedge B) \leftrightarrow \neg(A \rightarrow \neg B)$
0100: $(A \wedge \neg B) \leftrightarrow \neg(A \rightarrow B)$
0010: $(\neg A \wedge B) \leftrightarrow \neg(\neg A \rightarrow \neg B)$
0001: $(\neg A \wedge \neg B) \leftrightarrow \neg(\neg A \rightarrow B)$

0110: $(A \oplus B) \leftrightarrow \neg(A \leftrightarrow B)$
0110: $(A \oplus B) \leftrightarrow \neg(\neg A \leftrightarrow \neg B)$
1001: $(A \oplus \neg B) \leftrightarrow \neg(A \leftrightarrow \neg B)$
1001: $(A \oplus \neg B) \leftrightarrow \neg(\neg A \leftrightarrow B)$

1001: $(A \leftrightarrow B) \leftrightarrow \neg(A \oplus B)$
1001: $(A \leftrightarrow B) \leftrightarrow \neg(\neg A \oplus \neg B)$
0110: $(A \leftrightarrow \neg B) \leftrightarrow \neg(A \oplus \neg B)$
0110: $(A \leftrightarrow \neg B) \leftrightarrow \neg(\neg A \oplus B)$

0110: $(\neg A \leftrightarrow B) \leftrightarrow \neg(A \oplus \neg B)$
0110: $(\neg A \leftrightarrow B) \leftrightarrow \neg(\neg A \oplus B)$
1001: $(\neg A \leftrightarrow \neg B) \leftrightarrow \neg(A \oplus B)$
1001: $(\neg A \leftrightarrow \neg B) \leftrightarrow \neg(\neg A \oplus \neg B)$

1001: $(\neg A \oplus B) \leftrightarrow \neg(A \leftrightarrow \neg B)$
1001: $(\neg A \oplus B) \leftrightarrow \neg(\neg A \leftrightarrow B)$
0110: $(\neg A \oplus \neg B) \leftrightarrow \neg(A \leftrightarrow B)$

0110: $(\neg A \oplus \neg B) \leftrightarrow \neg(\neg A \leftrightarrow \neg B)$

2.4 A-A Equivalence Formulas

In the A-A category of the lookup table, if we convert the 4-digit numbers to A, ¬A, 1 or 0, we can obtain the following 20 formulas:

1100: $(A \wedge A) \leftrightarrow A$
0000: $(A \wedge \neg A) \leftrightarrow 0$
0000: $(\neg A \wedge A) \leftrightarrow 0$
0011: $(\neg A \wedge \neg A) \leftrightarrow \neg A$

1100: $(A \vee A) \leftrightarrow A$
1111: $(A \vee \neg A) \leftrightarrow 1$
1111: $(\neg A \vee A) \leftrightarrow 1$
0011: $(\neg A \vee \neg A) \leftrightarrow \neg A$

1111: $(A \rightarrow A) \leftrightarrow 1$
0011: $(A \rightarrow \neg A) \leftrightarrow \neg A$
1100: $(\neg A \rightarrow A) \leftrightarrow A$
1111: $(\neg A \rightarrow \neg A) \leftrightarrow 1$

0000: $(A \oplus A) \leftrightarrow 0$
1111: $(A \oplus \neg A) \leftrightarrow 1$
1111: $(\neg A \oplus A) \leftrightarrow 1$
0000: $(\neg A \oplus \neg A) \leftrightarrow 0$

1111: $(A \leftrightarrow A) \leftrightarrow 1$
0000: $(A \leftrightarrow \neg A) \leftrightarrow 0$
0000: $(\neg A \leftrightarrow A) \leftrightarrow 0$
1111: $(\neg A \leftrightarrow \neg A) \leftrightarrow 1$

2.5 A-1 Equivalence Formulas

In the A-1 category of the lookup table, when converting the 4-digit numbers to A, ¬A, 1 or 0, we can obtain the following 20 formulas:

1100: $(A \wedge 1) \leftrightarrow A$
1100: $(1 \wedge A) \leftrightarrow A$
0011: $(\neg A \wedge 1) \leftrightarrow \neg A$
0011: $(1 \wedge \neg A) \leftrightarrow \neg A$

1111: $(A \vee 1) \leftrightarrow 1$
1111: $(1 \vee A) \leftrightarrow 1$
1111: $(\neg A \vee 1) \leftrightarrow 1$
1111: $(1 \vee \neg A) \leftrightarrow 1$

1111: $(A \rightarrow 1) \leftrightarrow 1$
1100: $(1 \rightarrow A) \leftrightarrow A$

1111: $(\neg A \to 1) \leftrightarrow 1$
0011: $(1 \to \neg A) \leftrightarrow \neg A$

0011: $(A \oplus 1) \leftrightarrow \neg A$
0011: $(1 \oplus A) \leftrightarrow \neg A$
1100: $(\neg A \oplus 1) \leftrightarrow A$
1100: $(1 \oplus \neg A) \leftrightarrow A$

1100: $(A \leftrightarrow 1) \leftrightarrow A$
1100: $(1 \leftrightarrow A) \leftrightarrow A$
0011: $(\neg A \leftrightarrow 1) \leftrightarrow \neg A$
0011: $(1 \leftrightarrow \neg A) \leftrightarrow \neg A$

2.6 A-0 Equivalence Formulas

And in the A-0 category of the lookup table, when converting the 4-digit numbers to A, ¬A, 1 or 0, we can obtain the following 20 formulas:

0000: $(A \wedge 0) \leftrightarrow 0$
0000: $(0 \wedge A) \leftrightarrow 0$
0000: $(\neg A \wedge 0) \leftrightarrow 0$
0000: $(0 \wedge \neg A) \leftrightarrow 0$

1100: $(A \vee 0) \leftrightarrow A$
1100: $(0 \vee A) \leftrightarrow A$
0011: $(\neg A \vee 0) \leftrightarrow \neg A$
0011: $(0 \vee \neg A) \leftrightarrow \neg A$

0011: $(A \to 0) \leftrightarrow \neg A$
1111: $(0 \to A) \leftrightarrow 1$
1100: $(\neg A \to 0) \leftrightarrow A$
1111: $(0 \to \neg A) \leftrightarrow 1$

1100: $(A \oplus 0) \leftrightarrow A$
1100: $(0 \oplus A) \leftrightarrow A$
0011: $(\neg A \oplus 0) \leftrightarrow \neg A$
0011: $(0 \oplus \neg A) \leftrightarrow \neg A$

0011: $(A \leftrightarrow 0) \leftrightarrow \neg A$
0011: $(0 \leftrightarrow A) \leftrightarrow \neg A$
1100: $(\neg A \leftrightarrow 0) \leftrightarrow A$
1100: $(0 \leftrightarrow \neg A) \leftrightarrow A$

2.7 1-0 Equivalence Formulas

And more, in the 1-0 category of the lookup table, if we convert the 4-digit numbers to 1 or 0, we can obtain the following 20 formulas:

1111: $(1 \wedge 1) \leftrightarrow 1$

0000: $(1 \wedge 0) \leftrightarrow 0$
0000: $(0 \wedge 1) \leftrightarrow 0$
0000: $(0 \wedge 0) \leftrightarrow 0$

1111: $(1 \vee 1) \leftrightarrow 1$
1111: $(1 \vee 0) \leftrightarrow 1$
1111: $(0 \vee 1) \leftrightarrow 1$
0000: $(0 \vee 0) \leftrightarrow 0$

1111: $(1 \rightarrow 1) \leftrightarrow 1$
0000: $(1 \rightarrow 0) \leftrightarrow 0$
1111: $(0 \rightarrow 1) \leftrightarrow 1$
1111: $(0 \rightarrow 0) \leftrightarrow 1$

0000: $(1 \oplus 1) \leftrightarrow 0$
1111: $(1 \oplus 0) \leftrightarrow 1$
1111: $(0 \oplus 1) \leftrightarrow 1$
0000: $(0 \oplus 0) \leftrightarrow 0$

1111: $(1 \leftrightarrow 1) \leftrightarrow 1$
0000: $(1 \leftrightarrow 0) \leftrightarrow 0$
0000: $(0 \leftrightarrow 1) \leftrightarrow 0$
1111: $(0 \leftrightarrow 0) \leftrightarrow 1$

We may notice that the above 1-0 formulas are exactly the same as the five sets of digital calculation formulas [3] (Section 2 Digital Calculation Formulas and Calculation Method).

3 Group Expressions and Equivalences by Digital Numbers

3.1 Group Basic Expressions by Sixteen Digital Numbers

Although we can discover hundreds of equivalences from the lookup table based on numerical values, the total numerical values in two-element expressions are only sixteen. Let's group all the basic expressions by the digital numbers:

Table 3.1: Basic Expressions Listed in Sixteen Number Groups

Num	+/¬	A/B	∧	∨	→	⊕	↔
0000	0		(A ∧ ¬A), (A ∧ 0), (¬A ∧ 0)			(A ⊕ A), (¬A ⊕ ¬A)	(A ↔ ¬A)
0001			(¬A ∧ ¬B)	¬(A ∨ B)	¬(¬A → B), ¬(¬B → A)		
0010			(¬A ∧ B)	¬(A ∨ ¬B)	¬(¬A→ ¬B), ¬(B → A)		
0011		¬A	(¬A ∧ ¬A), (¬A ∧ 1)	(¬A ∨ ¬A), (¬A ∨ 0)	(A → ¬A), (1 → ¬A), (A → 0)	(¬A ⊕ 0), (A ⊕ 1)	(¬A ↔ 1), (A ↔ 0)
0100			(A ∧ ¬B)	¬(¬A ∨ B)	¬(A → B), ¬(¬B → ¬A)		
0101		¬B					
0110						(A ⊕ B), (¬A ⊕ ¬B), ¬(A ⊕ ¬B), ¬(¬A ⊕ B)	(A ↔ ¬B), (¬A ↔ B), ¬(A ↔ B), ¬(¬A↔ ¬B)
0111			¬(A ∧ B)	(¬A ∨ ¬B)	(A → ¬B), (B → ¬A)		
1000			(A ∧ B)	¬(¬A ∨ ¬B)	¬(A → ¬B), ¬(B → ¬A)		
1001						(A ⊕ ¬B), (¬A ⊕ B), ¬(A ⊕ B), ¬(¬A ⊕ ¬B)	(A ↔ B), (¬A ↔ ¬B), ¬(A ↔ ¬B), ¬(¬A ↔ B)
1010		B					
1011			¬(A ∧ ¬B)	(¬A ∨ B)	(A → B), (¬B → ¬A)		
1100		A	(A ∧ A), (A ∧ 1)	(A ∨ A), (A ∨ 0)	(¬A → A), (1 → A), (¬A → 0)	(¬A ⊕ 1), (A ⊕ 0)	(A ↔ 1), (¬A ↔ 0)
1101			¬(¬A ∧ B)	(A ∨ ¬B)	(¬A → ¬B), (B → A)		
1110			¬(¬A ∧ ¬B)	(A ∨ B)	(¬A → B), (¬B → A)		
1111	1			(A ∨ ¬A), (A ∨ 1), (¬A ∨ 1)	(A → A), (¬A → ¬A), (A → 1), (¬A → 1), (0 → A), (0 → ¬A)	(A ⊕ ¬A)	(A ↔ A), (¬A ↔ ¬A)

Note, this table can be downloaded from: *https://doi.org/10.7910/DVN/D8XHSP*

In the table above, expressions with identical numbers are in the same row, indicating that all expressions within a row are equivalent. This allows us to identify equivalences directly by their numbers. For instance, expressions (A ∧ 1) and (¬A ↔ 0) are in Row 1100, indicating they are equivalent:

(A ∧ 1) ↔ (¬A ↔ 0) //Both have the same truth value 1100

No additional operations are needed for proving it. By this way, we can directly find equivalences without performing complex statement-based reasoning or truth table analysis.

It should be noted that there are three omissions in the table above:

1. Omission of B-A expressions: Except for the "→" relations, in all other relations, the B-A expressions are omitted because in those relations, the numerical value remains the same when interchanging the positions of A and B.

2. Omission of negative expressions in columns A, B, +/¬, A-A, A-1, A-0 and 1-0: This involves 6 numbers: 1111, 0000, 1100, 1010, 0011 and 0101.

3. Omission of expressions for B and ¬B: In the table above, there is no equivalence expression for B and ¬B. They are omitted. Because when the object is a one-element expression, A is sufficient as a representation, making B redundant. Using B instead of A would yield the same result. For instance, if we replace "A" in the formula (A ↔ A) ↔ 1 with "B", the formula becomes (B ↔ B) ↔ 1. The result remains the same. Another example, if we replace "A" in the formula (A ∧ A) ↔ A with "B", the formula becomes (B ∧ B) ↔ B. and the essence would remain unchanged.

3.2 Group Basic Equivalences by Fourteen Distinct Truth Values

In the table above, the basic expressions are grouped by numbers. Now let's group all basic equivalences (instead of basic expressions). In the following table, we will discover all basic equivalences of two-element relations.

Note, since there is no equivalence expression for B and ¬B in the table above, we will have fourteen number groups in the following list:

1111:

(A ∨ ¬A) ↔ 1 //Complement Law
(A ∨ 1) ↔ 1 //Annulment Law
(¬A ∨ 1) ↔ 1
(A → A) ↔ 1
(¬A → ¬A) ↔ 1
(A → 1) ↔ 1
(¬A → 1) ↔ 1

$(0 \rightarrow A) \leftrightarrow 1$
$(0 \rightarrow \neg A) \leftrightarrow 1$
$(A \oplus \neg A) \leftrightarrow 1$
$(A \leftrightarrow A) \leftrightarrow 1$
$(\neg A \leftrightarrow \neg A) \leftrightarrow 1$
$(1 \wedge 1) \leftrightarrow 1$
$(1 \vee 1) \leftrightarrow 1$
$(1 \vee 0) \leftrightarrow 1$
$(1 \rightarrow 1) \leftrightarrow 1$
$(0 \rightarrow 1) \leftrightarrow 1$
$(0 \rightarrow 0) \leftrightarrow 1$
$(1 \oplus 0) \leftrightarrow 1$
$(1 \leftrightarrow 1) \leftrightarrow 1$
$(0 \leftrightarrow 0) \leftrightarrow 1$

0000:

$(A \wedge \neg A) \leftrightarrow 0$ //Complement Law
$(A \wedge 0) \leftrightarrow 0$ //Annulment Law
$(\neg A \wedge 0) \leftrightarrow 0$
$(A \oplus A) \leftrightarrow 0$
$(\neg A \oplus \neg A) \leftrightarrow 0$
$(A \leftrightarrow \neg A) \leftrightarrow 0$
$(1 \wedge 0) \leftrightarrow 0$
$(0 \wedge 0) \leftrightarrow 0$
$(0 \vee 0) \leftrightarrow 0$
$(1 \rightarrow 0) \leftrightarrow 0$
$(1 \oplus 1) \leftrightarrow 0$
$(0 \oplus 0) \leftrightarrow 0$
$(1 \leftrightarrow 0) \leftrightarrow 0$

1100:

$(A \wedge A) \leftrightarrow A$ //Idempotent Law
$(A \wedge 1) \leftrightarrow A$ //Identity Law
$(A \vee A) \leftrightarrow A$ //Idempotent Law
$(A \vee 0) \leftrightarrow A$ //Identity Law
$(\neg A \rightarrow A) \leftrightarrow A$
$(1 \rightarrow A) \leftrightarrow A$
$(\neg A \rightarrow 0) \leftrightarrow A$
$(\neg A \oplus 1) \leftrightarrow A$
$(A \oplus 0) \leftrightarrow A$
$(A \leftrightarrow 1) \leftrightarrow A$
$(\neg A \leftrightarrow 0) \leftrightarrow A$
$\neg(\neg A) \leftrightarrow A$ //Double Negation Law

0011:

$(\neg A \wedge \neg A) \leftrightarrow \neg A$
$(\neg A \wedge 1) \leftrightarrow \neg A$
$(\neg A \vee \neg A) \leftrightarrow \neg A$
$(\neg A \vee 0) \leftrightarrow \neg A$
$(A \rightarrow \neg A) \leftrightarrow \neg A$
$(1 \rightarrow \neg A) \leftrightarrow \neg A$
$(A \rightarrow 0) \leftrightarrow \neg A$
$(A \oplus 1) \leftrightarrow \neg A$
$(\neg A \oplus 0) \leftrightarrow \neg A$
$(A \leftrightarrow 0) \leftrightarrow \neg A$
$(\neg A \leftrightarrow 1) \leftrightarrow \neg A$

0110:

$(A \oplus B) \leftrightarrow (\neg A \oplus \neg B)$
$(A \oplus B) \leftrightarrow \neg(A \oplus \neg B)$
$\neg(A \oplus \neg B) \leftrightarrow \neg(\neg A \oplus B)$
$(A \leftrightarrow \neg B) \leftrightarrow (A \oplus B)$
$(A \leftrightarrow \neg B) \leftrightarrow (\neg A \leftrightarrow B)$
$(A \leftrightarrow \neg B) \leftrightarrow \neg(A \leftrightarrow B)$
$\neg(A \leftrightarrow B) \leftrightarrow \neg(\neg A \leftrightarrow \neg B)$

1001:

$(A \oplus \neg B) \leftrightarrow (\neg A \oplus B)$
$(A \oplus \neg B) \leftrightarrow \neg(A \oplus B)$
$\neg(A \oplus B) \leftrightarrow \neg(\neg A \oplus \neg B)$
$(A \leftrightarrow B) \leftrightarrow (A \oplus \neg B)$
$(A \leftrightarrow B) \leftrightarrow (\neg A \leftrightarrow \neg B)$
$(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$
$\neg(A \leftrightarrow \neg B) \leftrightarrow \neg(\neg A \leftrightarrow B)$

1000:

$(A \wedge B) \leftrightarrow (B \wedge A)$ //Commutative Law
$(A \wedge B) \leftrightarrow \neg(A \rightarrow \neg B)$
$(A \wedge B) \leftrightarrow \neg(\neg A \vee \neg B)$

0100:

$(A \wedge \neg B) \leftrightarrow (\neg B \wedge A)$
$(A \wedge \neg B) \leftrightarrow \neg(\neg A \vee B)$
$(A \wedge \neg B) \leftrightarrow \neg(A \rightarrow B)$

0010:

$(\neg A \wedge B) \leftrightarrow \neg(A \vee \neg B)$
$(\neg A \wedge B) \leftrightarrow \neg(\neg A \rightarrow \neg B)$

0001:

$(\neg A \wedge \neg B) \leftrightarrow \neg(A \vee B)$ //de Morgan's Law
$(\neg A \wedge \neg B) \leftrightarrow \neg(\neg A \rightarrow B)$

0111:

$(\neg A \vee \neg B) \leftrightarrow \neg(A \wedge B)$ //de Morgan's Law
$(\neg A \vee \neg B) \leftrightarrow (A \rightarrow \neg B)$
$(A \rightarrow \neg B) \leftrightarrow \neg(A \wedge B)$
$(A \rightarrow \neg B) \leftrightarrow (B \rightarrow \neg A)$

1011:

$(\neg A \vee B) \leftrightarrow \neg(A \wedge \neg B)$
$(\neg A \vee B) \leftrightarrow (A \rightarrow B)$
$(A \rightarrow B) \leftrightarrow \neg(A \wedge \neg B)$
$(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$

1101:

$(A \vee \neg B) \leftrightarrow (\neg B \vee A)$
$(A \vee \neg B) \leftrightarrow \neg(\neg A \wedge B)$
$(A \vee \neg B) \leftrightarrow (\neg A \rightarrow \neg B)$
$(\neg A \rightarrow \neg B) \leftrightarrow \neg(\neg A \wedge B)$
$(\neg A \rightarrow \neg B) \leftrightarrow (B \rightarrow A)$

1110:

$(A \vee B) \leftrightarrow (B \vee A)$ //Commutative Law
$(A \vee B) \leftrightarrow \neg(\neg A \wedge \neg B)$
$(A \vee B) \leftrightarrow (\neg A \rightarrow B)$
$(\neg A \rightarrow B) \leftrightarrow \neg(\neg A \wedge \neg B)$
$(\neg A \rightarrow B) \leftrightarrow (\neg B \rightarrow A)$

The table above divides all equivalences into fourteen groups based on identical numerical values.

Using the digital analysis method and the expression-number lookup table, we can quickly and directly discover all the equivalence formulas, including those theorems of Boolean logic and propositional logic.

It should be noted that, this paper focuses on analyzing the basic expressions that are no more than two elements (AB), including the expressions in A-B, B-A, ¬(A-B), ¬(B-A), A-A, A-1, A-0, and 1-0 categories and their 16 truth values. Due to space limitations, the 256 truth values and their relations of the three-element (ABC) expressions are not addressed.

4 Appendix

4.1 The 4-digit Expression-Number Lookup Table

Table 4.1: The 4-digit Expression-Number Lookup Table

A		B		(+)		(¬)	
Expression	Num	Expression	Num	Expression	Num	Expression	Num
A	1100	B	1010	+	1111	¬	0000
¬A	0011	¬B	0101				
A-B		**B-A**		**¬(A-B)**		**¬(B-A)**	
Expression	Num	Expression	Num	Expression	Num	Expression	Num
(A ∧ B)	1000	(B ∧ A)	1000	¬(A ∧ B)	0111	¬(B ∧ A)	0111
(A ∧ ¬B)	0100	(¬B ∧ A)	0100	¬(A ∧ ¬B)	1011	¬(¬B ∧ A)	1011
(¬A ∧ B)	0010	(B ∧ ¬A)	0010	¬(¬A ∧ B)	1101	¬(B ∧ ¬A)	1101
(¬A ∧ ¬B)	0001	(¬B ∧ ¬A)	0001	¬(¬A ∧ ¬B)	1110	¬(¬B ∧ ¬A)	1110
(A ∨ B)	1110	(B ∨ A)	1110	¬(A ∨ B)	0001	¬(B ∨ A)	0001
(A ∨ ¬B)	1101	(¬B ∨ A)	1101	¬(A ∨ ¬B)	0010	¬(¬B ∨ A)	0010
(¬A ∨ B)	1011	(B ∨ ¬A)	1011	¬(¬A ∨ B)	0100	¬(B ∨ ¬A)	0100
(¬A ∨ ¬B)	0111	(¬B ∨ ¬A)	0111	¬(¬A ∨ ¬B)	1000	¬(¬B ∨ ¬A)	1000
(A → B)	1011	(B → A)	1101	¬(A → B)	0100	¬(B → A)	0010
(A → ¬B)	0111	(¬B → A)	1110	¬(A → ¬B)	1000	¬(¬B → A)	0001
(¬A → B)	1110	(B → ¬A)	0111	¬(¬A → B)	0001	¬(B → ¬A)	1000
(¬A → ¬B)	1101	(¬B → ¬A)	1011	¬(¬A → ¬B)	0010	¬(¬B → ¬A)	0100
(A ⊕ B)	0110	(B ⊕ A)	0110	¬(A ⊕ B)	1001	¬(B ⊕ A)	1001
(A ⊕ ¬B)	1001	(¬B ⊕ A)	1001	¬(A ⊕ ¬B)	0110	¬(¬B ⊕ A)	0110
(¬A ⊕ B)	1001	(B ⊕ ¬A)	1001	¬(¬A ⊕ B)	0110	¬(B ⊕ ¬A)	0110
(¬A ⊕ ¬B)	0110	(¬B ⊕ ¬A)	0110	¬(¬A ⊕ ¬B)	1001	¬(¬B ⊕ ¬A)	1001
(A ↔ B)	1001	(B ↔ A)	1001	¬(A ↔ B)	0110	¬(B ↔ A)	0110
(A ↔ ¬B)	0110	(¬B ↔ A)	0110	¬(A ↔ ¬B)	1001	¬(¬B ↔ A)	1001
(¬A ↔ B)	0110	(B ↔ ¬A)	0110	¬(¬A ↔ B)	1001	¬(B ↔ ¬A)	1001
(¬A ↔ ¬B)	1001	(¬B ↔ ¬A)	1001	¬(¬A ↔ ¬B)	0110	¬(¬B ↔ ¬A)	0110
A-A		**A-1**		**(A-0)**		**(1-0)**	
Expression	Num	Expression	Num	Expression	Num	Expression	Num
(A ∧ A)	1100	(A ∧ 1)	1100	(A ∧ 0)	0000	(1 ∧ 1)	1111
(A ∧ ¬A)	0000	(1 ∧ A)	1100	(0 ∧ A)	0000	(1 ∧ 0)	0000
(¬A ∧ A)	0000	(¬A ∧ 1)	0011	(¬A ∧ 0)	0000	(0 ∧ 1)	0000
(¬A ∧ ¬A)	0011	(1 ∧ ¬A)	0011	(0 ∧ ¬A)	0000	(0 ∧ 0)	0000
(A ∨ A)	1100	(A ∨ 1)	1111	(A ∨ 0)	1100	(1 ∨ 1)	1111
(A ∨ ¬A)	1111	(1 ∨ A)	1111	(0 ∨ A)	1100	(1 ∨ 0)	1111
(¬A ∨ A)	1111	(¬A ∨ 1)	1111	(¬A ∨ 0)	0011	(0 ∨ 1)	1111
(¬A ∨ ¬A)	0011	(1 ∨ ¬A)	1111	(0 ∨ ¬A)	0011	(0 ∨ 0)	0000
(A → A)	1111	(A → 1)	1111	(A → 0)	0011	(1 → 1)	1111
(A → ¬A)	0011	(1 → A)	1100	(0 → A)	1111	(1 → 0)	0000
(¬A → A)	1100	(¬A → 1)	1111	(¬A → 0)	1100	(0 → 1)	1111
(¬A → ¬A)	1111	(1 → ¬A)	0011	(0 → ¬A)	1111	(0 → 0)	1111
(A ⊕ A)	0000	(A ⊕ 1)	0011	(A ⊕ 0)	1100	(1 ⊕ 1)	0000

(A ⊕ ¬A)	1111	(1 ⊕ A)	0011	(0 ⊕ A)	1100	(1 ⊕ 0)	1111
(¬A ⊕ A)	1111	(¬A ⊕ 1)	1100	(¬A ⊕ 0)	0011	(0 ⊕ 1)	1111
(¬A ⊕ ¬A)	0000	(1 ⊕ ¬A)	1100	(0 ⊕ ¬A)	0011	(0 ⊕ 0)	0000
(A ↔ A)	1111	(A ↔ 1)	1100	(A ↔ 0)	0011	(1 ↔ 1)	1111
(A ↔ ¬A)	0000	(1 ↔ A)	1100	(0 ↔ A)	0011	(1 ↔ 0)	0000
(¬A ↔ A)	0000	(¬A ↔ 1)	0011	(¬A ↔ 0)	1100	(0 ↔ 1)	0000
(¬A ↔ ¬A)	1111	(1 ↔ ¬A)	0011	(0 ↔ ¬A)	1100	(0 ↔ 0)	1111

This dataset is sourced from *https://dataverse.harvard.edu/dataverse/digital-logic*:

Two-element Relational Expression-number Lookup Table for Propositional Logic *https://doi.org/10.7910/DVN/HWRWSR*

A-A, A-1, A-0 and 1-0 Relational Expression-number Lookup Table for Propositional Logic *https://doi.org/10.7910/DVN/OCN14M*

5 Conclusion

This paper introduces a digital method for analyzing propositional logical equivalences. It transforms the theorem-proof method from the complex statement-derivation method to a simple number-comparison method. By applying the digital calculation method and the expression-number lookup table, we can quickly and directly discover and prove logical equivalences based on the identical numbers, no additional operations are needed. This approach demonstrates significant advantages over the conventional methods in terms of simplicity and efficiency.

Conflict of Interest Statement

The author declares no conflicts of interest.

Data Availability Statement

The author confirms that all data generated or analysed during this study are included in this article. Furthermore, all sources and data supporting the findings of this study were all publicly available at the time of submission.

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