Can Unitary Gauge Provide a Local and Gauge-Invariant Explanation of the Aharonov-Bohm Effect?

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Abstract

How to explain the Aharonov-Bohm (AB) effect remains deeply controversial, particularly regarding the tension between locality and gauge invariance. Recently Wallace argued that the AB effect can be explained in a local and gauge-invariant way by using the unitary gauge. In this paper, I present a critical analysis of Wallace's intriguing argument. First, I show that the unitary gauge transforms the Schrödinger equation into the Madelung equations, which are expressed entirely in terms of local and gauge-invariant quantities. Next, I point out that an additional quantization condition needs to be imposed in order that the Madelung equations are equivalent to the Schrödinger equation, while the quantization condition is inherently nonlocal. Finally, I argue that the Madelung equations with the quantization condition can hardly explain the the AB effect, even if in a nonlocal way. This analysis suggests that the unitary gauge does not resolve the tension between locality and gauge invariance in explaining the AB effect, but highlights again the profound conceptual challenges in reconciling the AB effect with a local and gauge-invariant framework.

1 Introduction

The Aharonov-Bohm effect (AB effect hereafter) is a quantum effect that demonstrates the significance of electromagnetic potentials in quantum mechanics (Aharonov and Bohm, 1959). Although it has been confirmed by experiments (Tonomura et al, 1986), how to explain the effect remains an unsolved, controversial issue (see, e.g. Earman, 2024). It is widely thought that an explanation of the AB effect must be either local but gauge-dependent or gauge-invariant but nonlocal. For example, in the AB effect, a charged particle confined to a region without electromagnetic field may be affected by the local gauge-dependent potentials or by the gauge-invariant fields in other regions nonlocally. But each of these two explanations is not fully satisfactory. In this background, Wallace (2024, 2025) suggested a very intriguing way to resolve this dilemma. He argued that when considering the gauge-invariant features of the wave function of the charged particle and the electromagnetic potentials jointly, the AB effect can be explained in a local and gauge-invariant way by choosing the unitary gauge. After all, the AB effect arises because the wave function and the potentials are coupled in a certain way in the Schrödinger equation. This is an important insight, and the result deserves to be carefully examined.

In this paper, I will present a detailed analysis of Wallace's explanation of the AB effect. In Section 2, I first introduce the widely-discussed magnetic AB effect, and explain why the effect can hardly be explained. In Section 3, I show that the unitary gauge transforms the Schrödinger equation into the Madelung equations, which are expressed entirely in terms of local and gauge-invariant quantities. This result is consistent with Wallace's expectation. Then, in Section 4, I point out that the Madelung equations are not mathematically equivalent to the Schrödinger equation, and an additional quantization condition needs to be imposed to re-establish its equivalence with the Schrödinger equation. In Section 5, I argue that the quantization condition is not local but inherently nonlocal. This raises a serious objection to Wallace's argument that unitary gauge may provide a local and gauge-invariant explanation of the AB effect. In Section 6, I further argue that the Madelung equations with the quantization condition can hardly explain the the AB effect, even if in a nonlocal way. Conclusions are given in the last section.

2 The AB effect

Let me first introduce the magnetic AB effect, which has been widely discussed in the literature. Its basic setup is as follows (Aharonov and Bohm, 1959). A coherent beam of electrons is split into two parts, each going on opposite sides of a solenoid. After the beams pass by the solenoid, they are re-combined to interfere coherently. By means of an electric current flowing through the solenoid, a magnetic field, B, which is essentially confined within the solenoid, can be generated. However, the (time-independent) vector potential, A, cannot be zero everywhere outside the solenoid, since the total magnetic flux through every circuit enclosing the solenoid is equal to a nonzero constant $\Phi = \oint B \cdot ds = \oint A \cdot dr$.

The Hamiltonian of the electron is[1]

$$H = \frac{(p - eA)^2}{2m}, \tag{1}$$

where e and m are respectively the charge and mass of the electron, and $H_0 = p^2/2m$ is the free Hamiltonian when the current is absent in the solenoid.

Let $\psi(r,t) = \psi_1(r,t) + \psi_2(r,t)$ be the wave function of the electron, where ψ_1 represents the beam on one side of the solenoid and ψ_2 the beam on the other side. Since each of these beams stays in a simply connected region R_i (i=1,2) that does not include the solenoid, where $B = \nabla \times A = 0$, we can use two gauges for calculating ψ_i (i=1,2). The first gauge is $A_0 = 0$, and the second gauge is $A_i \neq 0$ (where A_i satisfies the relation $\int_{R_1} A_1 \cdot dr + \int_{R_2} A_2 \cdot dr = \phi_0$). They are related with each other by a gauge transformation $A_i = A_0 + \nabla\chi_i$, where χ_i is the gauge function. In the first gauge, the Hamiltonian will be $H_0 = p^2/2m$, which is the same as the free Hamiltonian for the case of no current being in the solenoid, for which the solutions to the Schrödinger equation are supposed to be ψ_1^0 and ψ_2^0 for the two beams, respectively. In the second gauge, the Hamiltonian will be given by (1). By the gauge transformation of the wave function, the corrsponding solutions of the Schrödinger equation will be

$$\psi_1 = \psi_1^0 e^{ie\chi_1}, \psi_2 = \psi_2^0 e^{ie\chi_2}, \tag{2}$$

where $\chi_1 = \int A_1 \cdot dr$ and $\chi_2 = \int A_2 \cdot dr$, being equal to $\int A \cdot dr$ along the paths of the first and second beams, respectively (according to the above gauge transformation of A_i). The interference between the two beams will then depend on the phase difference: $e\chi_1 - e\chi_2 = e\oint A \cdot dr = e\Phi$. This is the magnetic AB effect, which will exist even if there are no magnetic forces acting in the places where the electron beam passes.[2]

Here it is worth pointing out that although we can choose the gauge $A = 0$ in each half of the whole region enclosing the solenoid, we cannot choose the gauge $A = 0$ in the whole region, since, as noted above, the integral of the magnetic vector potential A around every circuit enclosing the solenoid is not zero. As a result, although the solutions of the Schrödinger equation for each beam are gauge-equivalent to the free solutions (when $B = 0$ everywhere), the solutions for the two beams are not gauge-equivalent to the free solutions. As we will see later, this bizarre situation will pose a great difficulty for explaining the AB effect.

[1]Units are chosen so that $\hbar = c = 1$ throughout this paper, unless otherwise stated.

[2]For a more rigorous analysis of the AB effect see Aharonov and Bohm (1959) and Ballesteros and Weder (2009).

3 Equations of motion in the unitary gauge

In this section, I will analyze Wallace's derivation of the equations of motion in the unitary gauge (Wallace, 2025). Moreover, I will show that the equations of motion in the unitary gauge are actually the Madelung equations.

Wallace (2025) considered the motion of a charged particle (without spin) interacting with a background magnetic field. The Schrödinger equation in the position representation is

$$i\frac{\partial\psi(r,t)}{\partial t} = -\frac{1}{2m}\left(\nabla - ieA(r,t)\right)^2 \psi(r,t), \tag{3}$$

where $\psi(r,t)$ is the wave function of the particle, $A(r,t)$ is the magnetic vector potential, and e is the charge of the particle. The Schrödinger equation is invariant under the following gauge transformation:

$$A \longrightarrow A + \nabla\chi, \psi \longrightarrow \mathrm{e}^{ie\chi}\psi, \tag{4}$$

for an arbitrary smooth function χ. When decomposing the wave function ψ into its magnitude and phase: $\psi = Re^{iS}$, we can see that there are two local gauge-invariant quantities: R and $v \equiv (\nabla S - eA)/m$.

In the unitary gauge where $\psi = |\psi|$, the Schrödinger equation becomes

$$i\frac{\partial R}{\partial t} = -\frac{1}{2m}\left(\nabla^2 R - e^2 A\cdot AR - 2ieA\cdot\nabla R - ie(\nabla\cdot A)R\right). \tag{5}$$

After separating the real and imaginary parts we get

$$(\nabla^2 - e^2 A\cdot A)R = 0, \tag{6}$$

$$2m\frac{\partial R}{\partial t} = 2eA\cdot\nabla R + e(\nabla\cdot A)R. \tag{7}$$

Since $\psi = |\psi|$, we have $\nabla S = 0$ and thus $v = -eA/m$. Then, we can replace eA with $-mv$ in the above equations to get

$$(\nabla^2 - m^2 v^2)R = 0, \tag{8}$$

$$2\frac{\partial R}{\partial t} + 2v\cdot\nabla R + (\nabla v)R = 0. \tag{9}$$

This set of equations is expressed entirely in terms of *local* gauge-invariant quantities R and v, and it does not depend on the unitary gauge. According to Wallace (2024, 2025), it may provide a gauge-invariant, local, and separable description of the AB effect.

In the following, I will analyze Wallace's two equations. It can be seen that the second equation (9) is just the usual continuity equation, and it can be written as follows:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(v\rho) = 0, \tag{10}$$

where $\rho = R^2$. But the first equation cannot be the right one even when the magnetic field strength B vanishes or $\nabla \times A = 0$. First, this equation does not admit simple solutions of (nonzero) momentum eigenstates of the particle, for which $\nabla^2 R = 0$ but $v^2 R \neq 0$. Second, this equation does not specify how v evolves over time. Finally, Wallace's two equations do not specify how ρ and v evolve under a given electromagnetic field either. Then, where does it go wrong?

The key is to realize that we must also consider the electric scalar potential ϕ in the unitary gauge. The complete Schrödinger equation which includes the electric scalar potential is

$$i\frac{\partial \psi}{\partial t} = [-\frac{1}{2m}(\nabla - ieA)^2 + e\phi]\psi. \tag{11}$$

Then the first equation (8) should be

$$(\nabla^2 - m^2 v^2 - 2me\phi)R = 0, \tag{12}$$

from which we can obtain $\phi = (\nabla^2 R/R - m^2 v^2)/2me$. Moreover, since $E = -\nabla\phi - \frac{\partial A}{\partial t}$, we have $\partial A/\partial t = -E - \nabla\phi$ or $m\partial v/\partial t = eE + e\nabla\phi$ in the unitary gauge. Thus Wallace's first equation will become

$$m\frac{\partial v}{\partial t} = e(E + v \times B) - mv \cdot \nabla v - \nabla U, \tag{13}$$

where $U = -\nabla^2 R/(2mR)$ is the so-called quantum potential. Here the relation $v = -eA/m$ is used. Note that v is a vector and $\nabla v^2 = 2[v \times \nabla \times v + v \cdot \nabla v]$. This equation, together with the continuity equation (10), are in fact the Madelung equations (Madelung, 1927).

4 Inequivalence between the Madelung equations and the Schrödinger equation

It has been known (and was first shown by Erwin Madelung in 1927) that by differentiating the Schrödinger equation with $\psi = Re^{iS}$ and separating the real and imaginary parts, one can directly obtain the above equations, the Madelung equations, without choosing the unitary gauge. Since the Madelung equations, unlike the Schrödinger equation, involve only fields and not potentials, it appears to be the case that these equations can provide a local and gauge-invariant explanation of the AB effect as Wallace thought. However, there is a subtle issue here. That is, the Madelung equations and the Schrödinger equation are not mathematically equivalent.

In order to see the inequivalence between these two equations, let's first derive the Madelung equations from the Schrödinger equation and then try to recover the latter from the former (Here I mainly follow Wallstrom

(1994b)). Consider again the Schrödinger equation (11). Let the wave function be $\psi = Re^{iS}$, insert it into the equation, divide the equation by ψ, and then separate the equation into real and imaginary parts.[3] This yields the following two coupled nonlinear equations (valid wherever $\psi \neq 0$):

$$\frac{\partial R^2}{\partial t} = -\frac{1}{m}\nabla[R^2(\nabla S - eA)], \tag{14}$$

$$\frac{\partial S}{\partial t} = -\frac{1}{2m}(\nabla S - eA)^2 - e\phi + \frac{1}{2m}\nabla^2 R. \tag{15}$$

The first equation is the continuity equation (10) when we write $\rho = R^2$. The gradient of the second equation yields Madelung's second equation (13).

The derivation of the Schrödinger equation from the Madelung equations (10) and (13) is supposed to be as follows. We need first to assume that v is a gradient. Then by substituting $v = \nabla S$ and integrating (13), we obtain (15). Note that the integration constant can be set equal to zero, since it only contributes a global phase to the wave function. Then subtract (15) from $i/(2R^2)$ times (10), multiply by Re^{iS}, and replace Re^{iS} with ψ. The result will be the Schrödinger equation.

However, the above derivation of the Schrödinger equation from the Madelung equations cannot go through. The main reason is that S being the phase of the wave function is a many-valued function in general, e.g. for wave functions with angular momentum,[4] and thus v cannot be expressed as the gradient of a globally defined single-valued function. Once S is allowed to be many-valued, nothing in the Madelung equations constrains $\psi = Re^{iS}$ to be single-valued, which is inconsistent with the usual requirement of the single-valuedness of the wave function in quantum mechanics. In other words, the solutions of the Madelung equations are not necessarily the solutions of the Schrödinger equation, and thus the two equations are not equivalent.

Wallstrom (1994b) gave an explicit example to show the inequivalence between the Madelung equations and the Schrödinger equation. Consider the solution of the Schrödinger equation for a particle without spin in a well-behaved two-dimensional central potential $V(r)$. This problem can be solved by separation of variables, and the solutions assume the form $\psi(r,\theta) = R_l(r)e^{il\theta}$, where (r,θ) are polar coordinates, and l denotes the angular momentum. The single-valuedness of the wave function requires that $\psi(r,\theta) = \psi(r,\theta + 2n\pi)$ (n is an integer), and this implies that l must be

[3] Note that given ψ, the ansatz $\psi = Re^{iS}$ does not uniquely determine R and S, since $S' = S+n\pi$ and $R' = (-1)^n R$ (n is an integer) give the same ψ. If R is required to be non-negative, then R is uniquely determined, but S still has arbitrariness within $S' = S + 2n\pi$ (n is an integer) (Takabayasi, 1983).

[4] For example, for the wave functions with angular momentum which contain a factor like $e^{il\theta}$, where l is an integer and θ is an azimuthal angle, $S = l\theta$ assumes different values for the azimuthal angles $\theta + 2n\pi$ (n is an integer).

an integer or the angular momentum of the particle must be quantized. On the other hand, all of the solutions $\rho_l = R_l(r)^2$ and $v_l = (l/r)\theta$ satisfy the Madelung equations for the potential $V(r)$, regardless of whether l is an integer, and (ρ_l, v_l) corresponds to a single-valued solution of the Schrödinger equation only when l is an integer.

It can be seen that in order that the Madelung equations are equivalent to the Schrödinger equation (in which the wave function of a particle without spin is single-valued), an additional quantization condition must be imposed. The wave function being single-valued requires that $S(\theta+2\pi)-S(\theta)=2n\pi$ (n is an integer) for any θ. Then, in terms of $v = \nabla S/m$, the quantization condition will be

$$m \oint v \cdot dr = 2n\pi, \tag{16}$$

where $\oint$ denotes integral along any closed loop in space, since we have $m\oint v \cdot dr = \oint dS = S(2\pi) - S(0)$. When including the electromagnetic interaction, we have $mv = \nabla S - eA$, and the quantization condition will be

$$m \oint v \cdot dr = 2n\pi - e\Phi, \tag{17}$$

where $\Phi = \oint A \cdot dr$ is the magnetic flux through the closed loop. This condition may re-establish the formal equivalence of the Madelung equations with the Schrödinger equation.

5 The quantization condition is nonlocal

In this section, I will further analyze the quantization condition for the Madelung equations, namely $m\oint v(r,t) \cdot dr = nh - e\Phi$. Here I explicitly include the Planck constant h.

First of all, the quantization condition is arguably indispensable. One might think that since the quantization condition for the Madelung equations results from the single-valuedness of the wave function in the Schrödinger equation, it may be dropped if the wave function is permitted to be multi-valued. However, there are convincing arguments for the result that the wave function of a particle without spin must be single-valued (see, e.g. Merzbacher, 1962; Davidson, 2020). Moreover, the quantization condition is also required by the agreement with experiments. For example, experiments show that the orbital angular momentum of the electron in the hydrogen atom is quantized, and this requires the quantization condition for the Madelung equations, as argued in the last section.

Next, it can be seen that the quantization condition is not of a local differential form (like the Madelung equations), but of a global integral form. It holds not for the value of $v(r,t)$ in a position r at an instant t, but for the values of $v(r,t)$ in all positions r on a closed loop at the same time t.

In particular, the physical quantity v along a closed loop in one region is determined or affected directly by the magnetic flux Φ in another region which may be sufficiently inside the loop (e.g. in the AB effect), without any mediating field to transmit the influence. This is similar to the case of Coulomb gauge fixing, in which the magnetic vector potential in one position is determined by the magnetic field in all positions nonlocally (Wallace, 2024). Thus the quantization condition is not local but nonlocal.

Third, the quantization condition, which results from a mathematical requirement, is lack of a plausible physical explanation. As argued above, the space of solutions of the Madelung equations is much larger than the space of solutions of the Schrödinger equation. Thus one must impose an additional condition, the quantization condition, to restrict the solutions of the Madelung equations so that they are in the same space of solutions as the solutions of the Schrödinger equation. However, as noted by Wallstrom (1994b), "this condition has not yet found any convincing explanation outside the context of the Schrödinger equation." Indeed, Takabayashi (1952), a strong proponent of the Madelung equations who first noticed the quantization condition, also admitted that it is "of ad hoc and compromising character for our formulation, just as the quantum condition for old quantum theory."

Finally, it is worth pointing out that the Madelung equations also need to be supplemented by another condition at the nodal boundary. The reason is that the initial-value problem for the Madelung hydrodynamic equations is not well-defined (Wallstrom, 1994a; Aharonov, Cohen and Rohrlich, 2016). If at any time the nodes of a wave function separate the wave function into two or more disjoint components,[5] there will be an infinite number of different solutions which all satisfy the Madelung equations wherever those equations are defined. Then some condition must be added to the Madelung equations in order to ensure a unique solution. This condition is arguably that the phase of the associated wave function $S(r,t)$, defined by $m\int^r v(r',t)\cdot dr'$ for the Madelung equations, be continuous across the nodal boundary. However, like the quantization condition, it is difficult to see how such a condition could be justified in terms of a theory in which the physical significance of both the wave function and its phase have been eliminated (Wallstrom, 1994b).

Note that the Madelung equations are not defined at the nodes of the wave function because they are derived from the Schrödinger equation by dividing ψ. This is related to the degeneracy problem of unitary gauge as discussed by Wallace (2024). The problem is that unitary gauge is not a gauge-fixing whenever $\psi = 0$ in a region. In this case, the phase of the wave

[5]For the electron in the hydrogen atom, for example, the radial part of its wave function will have nodes for energy eigenstates other than the ground state, which separate the wave function into many disjoint components.

function will be arbitrary, and thus we cannot recover the magnetic vector potential A from the (vanishing) covariant derivative of ψ. This suggests that unitary gauge will be not useful when applied to regions where the wave function of a charged particle is negligible.

6 Can Madelung explain the AB effect?

Now let's see whether the Madelung equations with the quantization condition can provide a plausible dynamical explanation of the AB effect, even though it must be nonlocal. So far no one has given such an explanation (see Takabayasi, 1983 for a relevant attempt). As I will argue below, the answer seems negative.

Consider again the magnetic AB effect. A beam of electrons is split into two parts, each going on opposite sides of a solenoid. After the beams pass by the solenoid, they are allowed to re-interfere. In the absence of any electric current through the solenoid (and hence of any induced magnetic field), the reinterference of the two electron beams will produce a usual interference pattern. When an electric current flows through the solenoid, there will be a shift in the interference pattern, and the magnitude of the shift will be proportional to the magnetic flux within the solenoid. Note that the solenoid can be constructed in a way that there is negligible magnetic field outside it, and thus the electron has been moving in a region where the magnetic field can be ignored.

Then, how to explain this AB effect? and in particular, what causes the shift of the interference pattern? According to the Madelung equations (10) and (13), since the electric and magnetic field can be ignored in the region where the electron moves, the equations will be the same as the free equations when the solenoid is turned off. In other words, since the magnetic field inside the solenoid does not appear in the Madelung equations, the solutions of these equations for the AB effect setup will be the same as those of the free equations, which are independent of the magnetic flux inside the solenoid. Thus the Madelung equations alone cannot explain the AB effect.

Can the quantization condition explain the AB effect? At first sight, the answer seems positive, since the quantization condition (17) contains the magnetic flux term that determines the shift of the interference pattern in the AB effect. However, a more detailed analysis suggests the opposite. Here one needs to first answer when the quantization condition (17), namely $m \oint_C v(r,t) \cdot dr = nh - e\Phi$ (where C denotes a loop that encloses the solenoid, and Φ denotes the magnetic flux inside the solenoid) can be imposed for the AB effect setup. Before the electrons are emitted from the source, or even before the electron beams overlap and re-interfere, it seems that the quantization condition cannot be defined, since the two electron beams have not formed a loop that encloses the solenoid (see also Zuchelli, 1984). Moreover, according to the Schrödinger equation, before the electron beams overlap, we can always choose the gauge $A = 0$, and thus $v = \nabla S - eA$ will be also independent of the magnetic flux inside the solenoid.[6]

On the other hand, once the two electron beams overlap and form a loop that encloses the solenoid, the quantization condition can be imposed, namely we immediately have $m \oint_C v(r,t) \cdot dr = nh - e\Phi$. Thus, the quantization condition not only introduces nonlocality (as argued in the last section), but also introduces discontinuity. Prior to the beam overlap, the two gauge-invariant quantities ρ and v reflects negligible, if any, information about the magnetic flux inside the solenoid. While immediately after the beam overlap, v, as well as ρ (via the continuity equation), will depend on the magnetic flux inside the solenoid. This change of ρ and v throughout space is instantaneous and discontinuous.[7] In this way, the quantization condition might be able to provide a nonlocal and discontinuous explanation of the AB effect.

But why? Why does and how can the magnetic field inside the solenoid influence the electrons outside the solenoid? And why does and how can the beam overlaping trigger the influence? If no physical mechanism is available, then the above explanation provided by the quantization condition seems vacuous. As noted before, the quantization condition is a mathematical condition for restricting the solutions of the Madelung equations so that they can be mathematically equivalent to the Schrödinger equation, but as a global and nonlocal condition, it can hardly be explained in physics.

Finally, it is worth noting that the effect of vacuum polarisation resulting from the magnetic field inside the solenoid is so small that it cannot explain the AB effect (cf. Wallace, 2025). In particular, at a distance greater than the Compton wavelength away from the solenoid, the effect is exponentially small (Serebryanyi, 1985; Gornicki, 1990).

7 Conclusions

In this paper, I have critically analyzed Wallace's argument that the AB effect can be explained in a local, gauge-invariant way using the unitary gauge. Through a detailed derivation, I have shown that the unitary gauge transforms the Schrödinger equation into the Madelung equations, which are expressed entirely in terms of local and gauge-invariant quantities ρ and v. However, an additional quantization condition needs to be imposed to ensure the equivalence of the Madelung equations with the Schrödinger equation, while this quantization condition is inherently nonlocal, lacking a

[6]Since ρ and v are gauge-invariant quantities that can be measured, this result is independent of the choice of a particular gauge and can also be confirmed by experiment.

[7]It is arguable that this is a general feature of all gauge-invariant explanations of the AB effect (Gao, 2025).

clear physical justification. Furthermore, I have argued that the Madelung equations, even when supplemented with the quantization condition, fail to provide a satisfactory explanation of the AB effect.

In conclusion, while the unitary gauge and the Madelung equations offer an intriguing perspective on the AB effect, they ultimately fall short of providing a local, gauge-invariant explanation of the AB effect. The necessity of a nonlocal quantization condition and the lack of a physical mechanism for the observed effects highlight the deep conceptual challenges in reconciling the AB effect with a local, gauge-invariant framework. The AB effect remains a deeply puzzling phenomenon for physicists and philosophers of physics, and its explanation continues to require a deeper understanding of the tension between locality and gauge invariance in quantum theory.

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