Generalized Aharonov-Bohm Effect: Its Derivation, Theoretical Implications and Experimental Tests

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Abstract

The Aharonov-Bohm (AB) effect highlights the fundamental role of electromagnetic potentials in quantum mechanics. While extensively studied in the static case, the impact of a time-varying magnetic flux on the electron's phase shift remains an open and debated question. In this paper, we derive the AB phase shift for a time-dependent magnetic vector potential and show that it is proportional to the time average of enclosed magnetic flux. Our analysis reveals that the AB phase is continuously accumulated as the electron traverses its path, challenging the conventional view that it emerges instantaneously at the point of interference. This generalized AB effect may provide deeper insight into the role of gauge-dependent potentials in quantum mechanics and also suggest novel experimental tests using alternating or pulsed magnetic flux.

1 Introduction

The Aharonov-Bohm (AB) effect demonstrates the fundamental role of electromagnetic potentials in quantum mechanics, even in regions where the electromagnetic fields vanish [1,2]. In its conventional form, an electron moving around a solenoid acquires a phase shift due to the enclosed magnetic flux inside the solenoid. Recent studies have explored how a time-varying magnetic flux affects the motion of the electron, but it is still an unsolved and controvertial issue whether the resulting phase shift is time-dependent and what the exact formula of the AB phase shift in this case is [3-12].

In this paper, we analyze the time-dependent extension of the AB effect by considering the angular velocity variation of electrons moving around the solenoid under the influence of a time-varying magnetic flux. We rigorously derive the resulting AB phase shift, incorporating the effects of an induced electric field. Unlike the static case, the time evolution of the system plays a crucial role in the final phase shift. We also discuss theoretical implications and experimental tests of this generalized AB effect.

2 Phase Shift Derivation

The Schrödinger equation for a charged particle such as an electron in the presence of an electromagnetic potential (in units where $\hbar = c = 1$) is given by:

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\left(\nabla - e\mathbf{A}\right)^2 \psi + eA_0\psi, \tag{1}$$

where e and m are respectively the charge and mass of the electron, and $\mathbf{A}$ is the magnetic vector potential and A_0 is the electric scalar potential. The Schrödinger equation is gauge invariant, namely it is invariant under a gauge transformation:

$$A_\mu(x) \;\rightarrow\; A'_\mu(x) = A_\mu(x) - \partial_\mu \Lambda(x), \tag{2}$$

$$\psi(x) \;\rightarrow\; \psi'(x) \;=\; e^{-ie\Lambda(x)}\,\psi(x), \tag{3}$$

where $A_\mu = (-A_0, \boldsymbol{A})$, and $\Lambda(x)$ is an arbitrary smooth gauge function. When choosing the gauge function as

$$\Lambda(x) \;=\; \int_{x_0^\mu}^{x^\mu} A_\mu(x')\,dx'^\mu, \tag{4}$$

where x_0^μ is a certain initial 4-position, we can gauge away the electromagnetic potential and turn the above Schrödinger equation into a free Schrödinger equation in a simply connected space-time region.[1] In other words, we have

$$\psi(\boldsymbol{x}, t) \;=\; e^{ie\Lambda(x)}\,\psi_0(\boldsymbol{x}, t), \tag{5}$$

where $\psi(\boldsymbol{x}, t)$ is the solution of the Schrödinger equation with an electromagnetic potential $A_\mu(\boldsymbol{x}, t)$, and $\psi_0(\boldsymbol{x}, t)$ is the solution of the free Schrödinger equation. This means that the phase change of the wave function along a path L due to the existence of the electromagnetic potential is given by

$$\Delta\phi = e\Lambda(x) \;=\; \int_L A_\mu(x)dx^\mu. \tag{6}$$

Consider the magnetic AB effect. A beam of electrons emitted from a source is split into two parts, each going on opposite sides of a solenoid. After

[1] The simply-connectedness ensures that the gauge function $\Lambda(x)$ can be defined as a unique function of space-time point x^μ, independently of the space-time path connecting x_0^μ and x^μ [12].

the beams pass by the solenoid, they are combined to interfere coherently. For an infinitely-long solenoid with time-dependent magnetic flux $\Phi(t)$, we can choose a gauge in which $A_0(\boldsymbol{r}, t) = 0$ and

$$\mathbf{A}(\boldsymbol{r}, t) = \frac{\Phi(t)}{2\pi r}\hat{\boldsymbol{\theta}} \tag{7}$$

for the region outside the solenoid, where $\hat{\boldsymbol{\theta}}$ is a unit vector in the angular direction. Then, based on the above analysis of the gauge invariance of the Schrödinger equation, we can obtain the AB phase shift:

$$\phi_{AB} = e\int_{L_1} \mathbf{A}(\mathbf{r}, t) \cdot d\mathbf{r} - e\int_{L_2} \mathbf{A}(\mathbf{r}, t) \cdot d\mathbf{r} = e\oint_C \mathbf{A}(\mathbf{r}, t) \cdot d\mathbf{r}, \tag{8}$$

where L_1 and L_2 are the paths of the two electron beams respectively, each of which is in a simply-connected region, and C is the whole closed path around the solenoid. In the time-independent case where $\Phi(t) = \Phi_0$, this simplifies to:

$$\phi_{AB} = e\Phi_0. \tag{9}$$

However, when $\Phi(t)$ varies with time, we must consider the motion of the electron around the solenoid in order to calculate the AB phase shift.[2] This is the key idea of this paper. Substituting (7) in the phase shift integral (8) we have:

$$\phi_{AB} = e\oint_C \frac{\Phi(t)}{2\pi r}\hat{\boldsymbol{\theta}} \cdot d\mathbf{r}. \tag{10}$$

Since $\hat{\boldsymbol{\theta}} \cdot d\mathbf{r} = \omega(t) r dt$, we obtain:

$$\phi_{AB} = \frac{e}{2\pi}\int_0^T \Phi(t)(\omega_1(t) + \omega_2(t))dt, \tag{11}$$

where $\omega_1(t)$ and $\omega_2(t)$ are the angular velocities of the two beams respectively, $t = 0$ is the time when the two beams begin to move around the solenoid, and $t = T$ is the time when the two beams overlap and re-interfere. We have the relation $\int_0^T (\omega_1(t) + \omega_2(t))dt = 2\pi$.

Here it is worth noting that $\omega_i(t)$ (i=1,2) should be determined by the motion of the electron under the influence of the magnetic flux, not by the motion of the free electron. As we will see later, due to the existence of the induced electric field, one beam will be accelerated and the other beam will be decelerated, and thus the overlaping region will be in general different from the overlaping region for the static case, although the meeting time T are the same for both cases.

Now we need to calculate the angular velocity of each electron beam. When $\Phi(t)$ varies with time, the motion of the electron will be changed

[2]This can be seen more clearly when using the path integral formulation, in which the AB phase shift is $\oint_C \mathbf{A}(\mathbf{r}, t) \cdot \mathbf{v}(\mathbf{r}, t)dt$.

by the induced electric field. For a time-dependent magnetic flux $\Phi(t)$, the induced electric field at radius r is:

$$\mathbf{E} = E_\theta \hat{\boldsymbol{\theta}} = -\frac{1}{2\pi r}\frac{d\Phi}{dt}\hat{\boldsymbol{\theta}}. \tag{12}$$

This field will exert a force on the electron, changing its angular velocity. The equation of motion for the angular velocity $\omega(t)$ is:

$$mr\frac{d\omega}{dt} = eE_\theta = -\frac{e}{2\pi r}\frac{d\Phi}{dt}. \tag{13}$$

Suppose the electron moves in a circular path with radius R. Integrating from 0 to t, the angular velocities of the two electron beams are

$$\omega_1(t) = \omega_1(0) - \frac{e}{2\pi m R^2}(\Phi(t) - \Phi(0)), \tag{14}$$

$$\omega_2(t) = \omega_2(0) + \frac{e}{2\pi m R^2}(\Phi(t) - \Phi(0)). \tag{15}$$

Substituting these two formulae in (11) we obtain the AB phase shift:

$$\phi_{AB} = \frac{e}{2\pi}\int_0^T \Phi(t)(\omega_1(0) + \omega_2(0))dt = \frac{1}{T}\int_0^T e\Phi(t)dt. \tag{16}$$

Note that $\int_0^T (\omega_1(0) + \omega_2(0))dt = 2\pi$. When $\Phi(t) = \Phi_0$, this result reduces to the usual result for the static case $\phi_{AB} = e\Phi_0$.

There is also another way to derive the above result. As noted above, due to the existence of the induced electric field, one beam will be accelerated and the other beam will be decelerated. When the two beams overlap, the difference betweem the travelling distance of each beam and that of the free beam is

$$\Delta l_1 = \int_0^T (\omega_1(t) - \omega_0)Rdt \tag{17}$$

$$= \int_0^T (\omega_1(0) - \omega_0)Rdt - \frac{e}{2\pi m R}\int_0^T (\Phi(t) - \Phi(0))dt \tag{18}$$

$$= -\frac{eA(R,0)T}{m} + \frac{e\Phi(0)T}{2\pi m R} - \frac{e}{2\pi m R}\int_0^T \Phi(t)dt \tag{19}$$

$$= -\frac{e}{2\pi m R}\int_0^T \Phi(t)dt, \tag{20}$$

$$\Delta l_2 = \frac{e}{2\pi m R}\int_0^T \Phi(t)dt, \tag{21}$$

where ω_0 is the angular velocity of the free electron when the magnetic flux is absent inside the solenoid. Here we use the relation $\omega_1(0)R = \omega_0 R - eA(R,0)/m$ and (7). When considering the wavelength of the electron in each beam is $\lambda_i = 2\pi/m\omega_i(T)R$ (i=1,2) when the two beams overlap, the total phase shift will be

$$\phi_{AB} = 2\pi(\frac{\Delta l_2}{\lambda_2} - \frac{\Delta l_1}{\lambda_1}) \tag{22}$$

$$= \frac{1}{2\pi}(\omega_1(T) + \omega_2(T)) \int_0^T e\Phi(t)dt \tag{23}$$

$$= \frac{1}{T} \int_0^T e\Phi(t)dt. \tag{24}$$

Here we use $\int_0^T (\omega_1(0) + \omega_2(0))dt = 2\pi$ again. Then we obtain the same result as before. This analysis assures us that the derived AB phase shift for the dynamic case is gauge-invariant. Moreover, it also provides a novel way to calculate the usual AB phase shift for the static case where the magnetic flux is constant.

3 Theoretical implications

There are already several different derivations of the phase shift for the time-dependent case of the AB effect or the generalized AB effect. However, these derivations are arguably problematic and incomplete. For example, in the derivation of Singleton et al [6,7], the authors add the non-AB type phase shift due to the induced electric field (i.e. the phase shift coming from the last two terms of (19)) to the AB phase shift (16) and thus obtain the null result that there is no time-dependent AB phase shift but only static AB phase shift. This is not correct, since the AB phase shift (8) or the result (16) is already the total phase shift for the time-dependent case according to the gauge transformation (5), and it already takes into account the effects of the time variation of the magnetic vector potential according to the Schrödinger equation. On the other hand, a few authors argued that there is time-dependent AB phase shift, and the derived result is simply $e\Phi(t)$ [10,12]. But this cannot be right either, since the generation of the AB phase shift takes time, during which the magnetic flux changes, while their result is determined only by the magnetic flux at one instant. Finally, it is worth noting that Lee et al (1992)'s derivation is closest to ours. But their derivation is an approximation and also more complex, and their final result is not exact and general.

As we have demonstrated above, the key is to realize that the AB phase shift (8) is precisely the phase shift for the time-dependent case, and in order to calculate the phase shift integral one must consider the motion of the electron around the solenoid. The rest of the thing is a direct calculation. The result shows that the AB phase shift for the time-dependent case is proportional to the time average of the enclosed magnetic flux during the motion of the electron around the solenoid. Since the electron does not move in a field-free region due to the existence of the induced electric field, this is a hybrid AB effect with part of the phase shift coming from the potential and the other part coming from the field.

However, one can also make the effect of the field as small as possible. For example, one may put a contant magnetic flux inside the solenoid for a very short time such as half the travelling time of the electron around the solenoid, from $T/4$ to $3T/4$. Then, according to our result, the AB phase shift for this dynamic case is half of the AB phase shift for the static case where the same contant magnetic flux persists, and it almost all comes from the magnetic vector potential.

This result may have implications for a deeper understanding of the AB effect. For example, it strongly suggests that the AB phase shift is continuously generated during the traveling of the electron, not immediately generated when the electron beams overlap. When there is a constant magnetic flux inside the solenoid during the time interval $[T/4, 3T/4]$, no gauge-invariant quantities of the electron are affected by the magnetic flux or magnetic field inside the solenoid (Note that the effect of the induced electric field outside the solenoid can be ignored when the turn-on and turn-off times are arbitrarily short). While when the electron beams overlap, there is no magnetic flux inside the solenoid anymore, and thus the motion of the electron is not affected by the electromagnetic field either. This then supports a continuous, local potential explanation of the AB effect and disfavors a discontinuous, nonlocal field explanation of the AB effect.

4 Experimental tests

So far no experiments have been done to precisely test the generalized AB effect (see [3,4] for early attempts). Here we suggest two typical experimental tests of this new effect.

The first kind of test is to use an alternating current (AC) in the solenoid.[3] Suppose the resulting magnetic flux is $\Psi(t) = \Psi_0 \sin \omega_c t$, where Ψ_0 is the amplitude, and ω_c is the angular frequency of the current. Our result predicts that the AB phase shift will be

$$\phi_{AB} = \frac{1}{T}\int_0^T e\Phi(t)dt = \frac{\cos \omega_c t}{\omega_c T} e\Psi_0. \tag{25}$$

One key test is that when the angular frequency of the current is very large or the period of the current is very small relative to the traveling time of the electron, i.e. $\omega_c T \ll 1$, the AB phase shift will be close to zero, and there will be no AB effect.

[3] Alternatively, one can also generate a time-varying magnetic vector potential using a coherent light source [3].

The second kind of test is to use a pulsed direct current (DC) such as a square-pulse in the solenoid. One may input a square-pulse current to the solenoid when the electron is moving around the solenoid such as during the time interval $[T/4, 3T/4]$. Our result predicts that the AB phase shift will be half of the AB phase shift for the static case where the magnetic flux has the same amplitude. Note that Singleton et al's result predicts that the AB phase shift for this setup will be zero [6-9], since the initial magnetic flux is zero.

When incorporating the retardation effect demanded by special relativity, the AB phase shift for the generalized AB effect will be

$$\phi_{AB} = \frac{1}{T}\int_0^T e\Phi(t - R/c)dt, \tag{26}$$

where R is the radius of the circular path of the electron around the solenoid. Different from the static case, we can test the retardation effect of the AB phase shift for the dynamic case. For example, one may introduce a square-pulse magnetic flux with amplitude Ψ_0 during the time interval $[T-R/c, T]$. Our result predicts that there will be no AB phase shift, while a model without the retardation effect will predict that the AB phase shift is not zero but $e\Psi_0 R/(cT)$. The existence of the retardation effect in the dynamic case further supports the idea that the AB phase shift is continuously generated during the electron passing around the solenoid.

The generalized AB effect, if confirmed, could be useful in precision measurement devices where time-dependent phase control is essential, since it provides the possibility of controlling quantum interference by modulating the magnetic flux.

5 Conclusion

We have given a detailed derivation of the AB phase shift in the presence of a time-dependent magnetic vector potential. It turns out that the phase shift is proportional to the time average of the enclosed magnetic flux. Unlike the static AB effect, where the phase shift is proportional directly to the enclosed flux (by Stokes' theorem), the time-dependent case introduces an additional layer of complexity due to the dynamic angular velocity variations. This generalized AB effect highlights the interplay between time-dependent electromagnetic potentials and quantum mechanical phase accumulation, and in particular, it strongly suggests that the AB phase is locally and continuously generated via the action of gauge-dependent potentials. Future work may explore experimental verification and potential applications of this new effect in quantum technologies.

References

[1] Y. Aharonov and D. Bohm, Significance of Electromagnetic Potentials in Quantum Theory, Phys. Rev. 115(3), 485–491 (1959).

[2] A. Tonomura, N. Osakabe, T. Matsuda, T. Kawasaki, J. Endo, S. Yano, and H. Yamada, Evidence for Aharonov-Bohm effect with magnetic field completely shielded from electron wave, Phys. Rev. Lett. 56, 792-795 (1986).

[3] B. Lee, E. Yin, T. K. Gustafson, R. Chiao, Analysis of Aharonov-Bohm effect due to time-dependent vector potentials, Phys. Rev. A 45, 4319 (1992).

[4] A. N. Ageev, S. Yu. Davydov, and A. G. Chirkov, Magnetic Aharonov-Bohm effect under time-dependent vector potential, Technical Physics Letters 26, 392 (2000).

[5] B. Gaveau, A. M. Nounou, L. S. Schulman, Homotopy and path integrals in the time-dependent Aharonov-Bohm effect, Found. Phys. 41, 1462 (2011).

[6] D. Singleton and E. C. Vagenas, The covariant, time-dependent Aharonov-Bohm effect, Phys. Lett. B 723, 241-244 (2013).

[7] J. Macdougall and D. Singleton, Stokes' theorem, gauge symmetry and time-dependent Aharonov-Bohm effect, J. Math. Phys. 55, 042101 (2014).

[8] M. Bright and D. Singleton, Time-dependent non-Abelian Aharonov-Bohm effect, Phys. Rev. D 91, 085010 (2015).

[9] R. Andosca and D. Singleton, Time dependent electromagnetic fields and 4-dimensional Stokes' theorem, Am. J. Phys. 84, 848-857 (2016).

[10] J. Jing, Y. F. Zhang, K. Wang, Z. W. Long, and S. H. Dong, On the time-dependent Aharonov-Bohm effect, Phys. Lett. B 774, 87-90 (2017).

[11] S. R. Choudhury and S. Mahajan, Direct calculation of time varying Aharonov-Bohm effect, Phys. Lett. A 384, 2467-2471 (2019).

[12] M. Wakamatsu, Revisiting the controversy over the time-dependent Aharonov-Bohm effect. arXiv:2406.18046 (2024).