Relational Abstraction in the History of Mathematics

Rami Jreige*

Abstract

The philosophical literature on mathematical structuralism and its history has focused on the emergence of structuralism in the 19th century. Yet modern abstractionist accounts cannot provide an historical account for the abstraction process. This paper will examine the role of relations in the history of mathematics, focusing on three main epochs where relational abstraction is most prominent: ancient Greek, 17th and 19th centuries, to provide a philosophical account for the abstraction of structures. Though these structures emerged in the 19th century with definitional axioms, the need for such axioms in the abstraction process comes about, as this paper will show, after a series of relational abstractions without a suitable basis.

Keywords— Philosophy of Mathematics, History of Mathematics, Metaphysics of Relations, Mathematical Abstraction

1 Introduction

In any discussion on mathematical structuralism, whether historical and/or analytical, the sole focus has been on the status of mathematical objects. Problems pertaining to object definitions have been proliferous in the modern philosophical literature. Examples of these are the problem of identity of indiscernibles (Keränen 2001, Burgess 1999, Button 2006) and their responses (Ladyman 2005, Leitgeb and Ladyman 2007, Shapiro 2008), the structural properties that define the objects (Hellman 2001, Linnebo 2008, Korbmacher and Schiemer 2018) and implicit definitions (Giovannini and Schiemer 2021). Historical discussions themselves have also

*UMR Pays Germanique, École Normale Supérieure, 45 rue d'Ulm, Paris, 75005, France
Contact: rami.jreige@gmail.com

revolved around the notion of a mathematical objects while placing relations to the side as we will see throughout this paper.

Throughout these debates, though the notion of relations is alluded to, and sometimes mentioned, there is no discussion of mathematical structuralism that places relations themselves as the focus, or constituent elements of mathematical structures. Though Russell (1903), in his *Principles of Mathematics*, does discuss relations and their constituents quite thoroughly in mathematics, Russell takes one specific view of relations, what I am calling the intensional view[1], that becomes the basis for his view of mathematics, now called *post rem* structuralism.

This paper will present and take both, the intensional and extensional view of relations and place three periods of the history of mathematics where relational abstraction is most prominent within these two views: ancient Greece, 17th and 19th centuries. It will be shown how, though initially the extensional view of relations was dominant in mathematics, the slow shift towards the formalisation of objects meant that the intensional view necessarily became dominant, even if it remained implicit until the definitional axioms in the 19th century and Russell's book with discusses in a philosophical context explicitly.

The paper will proceed as follows: in section 2, we will present a distinction amongst relations that better highlights the shift and then present the three historical steps in relational abstraction. In section 3, we will make the claim that the first step in the process is well accounted for in ancient Greek mathematics, providing a discussion on the status of objects and relations. In section 4, we will show the second step is accounted for in seventeenth century mathematics and provide a discussion on mathematics of the era. In section 5, we discuss the emergence of mathematical structuralism as the final step in the process by providing the context that led to the emergence of structuralism as a response to previous mathematical advancements. We will also provide a brief discussion on the effect that had on the field and the resistance it received at the time.

2 Abstracting relations

In discussion the role of abstraction in the history of mathematics, the discussion present in Russell's *Principles of Mathematics* is important. A key point in his work that is imperative for our discussion but has not gained sufficient attention is the claim that "relations do not have instances but are strictly the same in all propositions" (Russell 1903, p. 52). This particular statement presents a specific view of relations, that I will call the intensional view. This view stands in contrast

[1]In the literature on the metaphysics of relations, Russell's view point has become know as Directionalism (Gaskin and Hill 2012, Ostertag 2019, Donnelly 2021). However, given that the discussions prevalent there is regarding the nature of relations in general, this is beyond the scope of the this article and his view will be called the intensional view given its purpose in for this paper.

to the traditional, or extensional, view of relations in mathematics that claims that "Mathematics is typically extensional throughout" (Fitting 2006). The notion of intensional meanings with regards to objects is well discussed (Fitting 2006). With regards to relations, however, it is less so.[2]

The quote by Russell above provides, however, a simple idea. If we take two propositions, 'A differs than B' and 'C differs than D', then the relation "differs than", that is present in both propositions, is the same irrespective of the objects it relates. In other words, the relation is understood irrespective of its extensions. Otherwise, if the relation is dependent on the objects it relates, on its extension, then the relation in each proposition would be different, a particular, leading to a problematic regress (Russell 1903, pp. 50-51). Simply, intensional relations can be given properties of their own and have their own identity irrespective of how they are used in propositions. By contrast, an extensional definition is one through which the relation can be defined solely in terms of their extensions or examples.[3]

What will be argued is that, taking solely the traditional view of relations misses a key elements in mathematical abstraction when looking towards history for examples: relations have been present in the history of mathematics both extensionally and intensionally. A discussion about mathematical abstraction throughout history can be better viewed by taking the abstraction of both objects and relations into account.

With this in mind, we will categorise the historical abstraction process into the following three steps:

- *Relational Abstraction* : With a physical state of affairs we abstract the objects and the relations, the latter of which are defined extensionally over the former. With several such states of affairs one can construe, in the abstract, a system of determinate objects whose properties are necessary to sustain the relations. Such systems' results is constrained by the nature of the pre-mathematical abstract objects and their corresponding assertoric axioms.

- *Relational Generalisation* : The determinate nature of the abstract objects is removed, rendering a system with objects whose only properties are those that are functional in the system. The relations are now defined intensionally and the methodology takes precedence over the nature of the objects. One can now obtain new objects from the methodology unconstrained by their nature.

[2]Fine (2000) presents a similar view of relations. He, however, contrasts it with Russell as being different. Their differences relates to the directional aspect of Russell's view that is irrelevant here.

[3]Though we will talk only in terms of relations in this paper, what will be argued can be easily extended to functions and operations given that they are also relations (Russell 1903, Tarski 1986).

- *Categorical Axiomatisation* : The intensional relations are used as axioms to create or define a structure where the objects are extensionally defined. In the abstraction process, these axioms are chosen so as to *post factum* attempt to recreate the results of the generalised systems.[4]

As I will attempt to show, these three steps are neither all-encompassing nor completely distinct. The historical account is a sequence of incremental steps most clearly divided into the three above when looking at the roles relations play.

3 Relational abstraction: ancient Greek mathematics

Relational abstraction necessitates an ontology of the objects in a system. The relations in states of affairs will be dependent on the objects to be meaningful. This means that the relations are to be defined extensionally over abstracted objects. The objects, in this case, necessarily need to be defined prior to the system in question so that the discussions on the abstract systems will be meaningful.

Ancient Greek mathematics exhibited such ontological commitments. The acceptance of mathematical results depended on the ontology of the objects under consideration, not the methods or operations used. This meant that the discussion of the period were regarding the nature of mathematical objects and what to admit into the field's discourse. The ontology of the objects came prior to the methodology and the axioms. As Mancosu points out, one cannot easily determine what the Greeks meant by magnitude since they "never gave an explicit axiomatization of the properties that magnitudes should satisfy. But their work implicitly assumes certain properties for magnitudes" (Mancosu 1996, p. 35). In other words, the definition of objects was not generally present, neither in mathematical practice nor the axioms, they came prior to them. The axioms were extensionally defined relations over the objects.

The discussions of object ontology, however, can be traced back to Plato and Aristotle. Though they have discussions regarding mathematical objects, they have very little mathematical work that is known. Later mathematical developments, such as in the works of Euclid, Diophantus and others can be linked to the these discussions (Kline 1990,Dunham 1990, p. 28, Klein 1992).

3.1 The objects of ancient Greek mathematics

Within the context of mathematics, the Greeks had an important distinction amongst the objects: the continuous geometry and the discrete arithmetic. Numbers

[4] Once one has a structure, there is no need to, or possibly nothing to attempt to recreate. The discussions here are about the abstraction process, not the manipulation of structures in the abstract and/or applicability.

were in the realm of the latter, line segments in the former. In either of these domains one could preform calculations both concretely, e.g. as was used by merchants and architects, and in the abstract. The theory dictating admissible methods in the abstract, called theory of proportion, was about the relations between the objects.[5] This theory was not independent of the objects; it necessitated an object ontology for it to be meaningful. Different ontologies, e.g. that of geometry or arithmetic, meant that the theory was applicable in the fields to varying degrees. This theory plays a crucial role insofar as for the Greeks, it was consistently used as a fruitful geometric tool, whereas in arithmetic, though playing a similar role, provided only limited results, thereby exhibiting the strong ontological background necessary for any talk of relations in a system.

3.1.1 The monad

The abstract numbers were never thought of in isolation; they were multitudes of monads, indivisible units. Klein cites the use of monad and multitude in relation to arithmetic by various Greek authors (Klein 1992, pp. 51-52). Naturally, the status and nature of fractions and ratios was a discussion given an indivisible unit.

For Plato, the monads were detached from the world as a platonic form (Klein 1992, p. 70). As such, the indivisibility of monad lead to a major problem: the nature of fractions. If one attempts to partition the unit of calculation then all the mathematicians "would laugh at him and would not allow it, but whenever you were turning it into small change, they would multiply it, taking care lest the one should ever appear not as one, but as many parts" (Plato's Republic 525 E, quoted from Klein 1992, p. 59).

Aristotle, on the other hand, had no such limitation. For him a monad has no independent existence outside of its examples: "For a thing to be in number is for there to be some number of the object, and for its being to be measured by the number in which it is" (Aristotle 1983, Book IV, Chapter 12, 221b7). A monad was attained via abstraction. This allows one, to have an unlimited field of 'pure' numbers from which one can choose multitudes, which is the domain of arithmetic (Klein 1992, p. 51). This homogeneity of units further becomes a contentious issue in in light of Aristotle's kind-crossing prohibition.[6]

[5] Klein (1992, Ch.4-5) connects the theory of proportion in the Neoplatonist Euxodus' work and subsequent discussions to theoretical logistic in Plato. The main idea was that theoretical logistic was supposed to provide a foundation for the art of calculation in arithmetic. A distinction between theory of relations of numbers and theory of numbers becomes hard to maintain, leading to Klein's comments on the correspondence between theory of proportion in arithmetic and theoretical logistic. In fact, he notes that theoretical logistic, given that it is about relations in arithmetic, becomes a part of the theory of proportions, which is more general in Euclid. Therefore, for brevity, we will omit talk of logistic and simply talk of theory of proportion, only mentioning the former where appropriate.

[6] See Cantù (2010a) for a discussion of the effects of Aristotle's kind-crossing prohibition

A multitude is abstracted from the concrete. When dividing theoretically, one can always alter the monad, so that we have, loosely speaking, a smaller one. Aristotle's monad abstraction is what provides a sufficiently strong ontological basis for the use of the theory of proportions in arithmetic.

3.1.2 The magnitude

The continuous was "the main and essential characteristic of magnitudes" (Karasmanis 2011, p. 389). Continuous line segments and geometrical diagrams constructed from them were the basis of geometry. Similarly to monads, magnitudes for Aristotle were abstracted from the concrete measure, called sensible magnitude (Klein 1992, p. 102). The Aristotelian conception is also evident in the geometrical works of Euclid and subsequently Diophantus; their usage of magnitude exhibits a change in measure when dealing with fractions (Klein 1992, p. 157). However, this definition is not uncontested, because, as mentioned, Mancosu points out that one cannot easily determine what the Greeks meant by magnitude from mathematical practice. At this level of abstraction, the definition and identity of the objects come prior to the axioms, prior to mathematical practice, given that the latter are extensionally defined relations.

The nature of the magnitude, however, was not in question in mathematical practice. Netz (1999) argues that diagrams in ancient Greek geometry could not be understood separately, but were interdependent with texts and symbols. Though these symbols could specify unknowns, the nature of the unknown object was understood; there was no symbol used for an indeterminate object in Euclid and Diophantus' works (Klein 1992, Bos 2001). What is meant by that is that all symbols represented determinate objects, objects whose particular values or sizes was essential for the proof of a theorem and therefore, their nature was known. Though the lack of indeterminate symbols is not unanimously accepted, Netz writes that his "semiotic hypothesis shows why [the symbol] A must be determinate: because it was never a symbol to begin with. It is a signpost, and signposts are tied to their immediate object" (Netz 1999, p. 50).[7]

Even definitions in Euclid's Elements show signs of these ties to the concrete. Netz mentions that the nouns defined in the Elements are "mostly things in space" (Netz 1999, p. 92). Dunham even points to this in contrast with modern definitions, in that "For modern geometers, then, the notions of 'point' and 'straight line' remain undefined [in Euclid's elements]. Statements such as Euclid's may serve to convey some image in our minds, and this is not without merit; but as precise, logical definitions, these first few items are unsatisfactory" (Dunham 1990, pp. 32-

into the nineteenth century.

[7]For further readings about the discussions around symbolism in Greek mathematics, see Stenlund (2014), who discusses the rejection of indeterminate symbols in ancient Greek Mathematics by some philosophers, most notably, Klein (1992), Unguru (1975) and the subsequent reactions to those authors.

33). Another example is Euclid's fourth axiom which states that "Two right angles are equal to one another". The objects, angles, are defined prior to the system, thereby rendering the relation in the axiom meaningful. The relation itself is asserted as an indisputable truth about the world, i.e. derived from experience. In other words, the relations are defined extensionally over and depend on the objects, in contrast to modern axiomatisations which would render the axiom vacuous.

3.2 The theory of proportion

The theory of proportion, as already mentioned, was problematic in arithmetic. In practical calculations, "all meaningful operations on number presuppose knowledge of the relations which connect the single numbers. This knowledge, which we acquire in childhood and which we use in every calculation, although it is not always present to us as a whole, is logistike" (Klein 1992, p. 19). Yet any talk of theoretical calculations necessitates talk of theoretical arithmetic, of pure units.[8]. Given the discrete nature of arithmetic, however, limitations were in place so as not to obtain unintelligible numbers; a multitude in arithmetic had to be just that, a multitude. The results of ancient Greek arithmetic "were severely restricted to ensure that the answers were acceptable — that is to say, that the answers were positive integer numbers" (Malet 2006, p. 64). A multitude of monads was necessarily, in the modern vernacular, a positive integer. Negative multitudes made no sense ontologically and ratios were not multitudes, they were merely relations between them, the existence of which was dependent on the multitude and by extension, the monad (Bos 2001, p. 121).

In geometry, however, the theory of proportions provided substantial results. By taking the Aristotelian notion of magnitude as abstracted from measure, one can have ratios and proportions between line segments, no matter how one divides. The existence of incommensurable magnitudes was unproblematic as magnitudes are continuous.

The usage of this theory in geometry and Euclid's treatment of it, based on Euxodus' mathematics, renders its usage in arithmetic as a special case of the general one. This can also be seen from Aristotle

> Again, the law that proportionals alternate might be supposed to apply to numbers qua numbers, and similarly to lines, solids and periods of time; as indeed it used to be demonstrated of these subjects separately. It could, of course, have been proved of them all by a single demonstration, but since there was no single term to denote the common quality of numbers, lengths, time and solids, and they differ in species from one another, they were treated separately; *but now the law is proved universally; for the property did not belong to*

[8] It is important to note here that there is also the issue of incommensurable magnitudes from Geometry. For further discussion on this see Klein (1992, Ch.5)

> *them qua lines or qua numbers, but qua possessing this special quality which they are assumed to possess universally.* Hence, even if a man proves separately — whether by the same demonstration or not — of each kind of triangle, equilateral, scalene and isosceles, that it contains angles equal to the sum of two right angles, *he still does not know, except in the sophistical sense, that a triangle has its angles equal to the sum of two right angles, or that this is a universal property of triangles, even if there is no other kind of triangle besides these*; for he does not know that this property belongs to a triangle qua triangle, nor that it belongs to every triangle, except numerically; for he does not know that it belongs to every triangle specifically, even if there is no triangle which he does not know to possess it. ((My emphasis) Aristotle 1960, Posterior Analytics, Book I, Chapter 5, 74a)

The theory of proportion, in its generality, can be, in theory, applied to all mathematical fields. Yet this knowledge is only sophistical. In practice, the ontology becomes a restriction. Yet, it was shown in the arithmetical books of Euclid that this theory has "no separate existence" from the arithmetical tradition it is based on (Klein 1992). In other words, this theory was a tool to be used within its respective field. Therefore some caution is required when discussing the generality of the theory of proportion ontologically and as method.

Similarly to Aristotle, the generality of the theory is evident in the works of the neoplatonist philosopher Proclus. Cantù (2010a) claims that for Proclus, there are certain theorems that are common to all mathematical objects, namely proportions[9], due to the common nature of all mathematical forms. However she states, though Proclus unifies number and magnitude under the concept of quantity, their nature remains uncertain. Bos, through a terminological discussion further highlights this point by noting that "rational fractions were not multitudes of units" (Bos 2001, p. 120) because fractions are relations between multitudes, not multitudes themselves.

One can note that there is some underlying algebraic thought here; discussing theory of proportions in its generality while leaving their nature undecided implies at least a functional algebra. However, it was neither pronounced nor explicit and it becomes difficult at times to separate what was used heuristically and what was considered rigorous. The theory of proportion, though not separable from arithmetic does exhibit algebraic features (Kline 1990). Though the existence of algebraic thought in Greek mathematics is now widely rejected following Unguru (1975), proponents of this mode of thought still remain, highlighting heuristic approaches that never made it into final publications due to their conception of rigour. Blåsjö (2016), in critiquing this rejection notes that though we cannot outright claim

[9]"proportion, namely, the rules of compounding, dividing, converting, and alternating; the theorems concerning ratios; the theorems about equality and inequality in their most general and universal aspects" (Cantù 2010a, p. 228).

algebra existed in ancient greek geometry, one can neither outright say that there was none. We can see certain elements of their heuristic approaches which could indicate what was rigorous and acceptable. The later usage of infinitistic techniques (and appeal to the infinite) was heuristic for the Greeks; appeal to the infinite never made it into publications, yet Archimedes' method provides evidence that it was used heuristically (Mancosu 1996, p. 35).

Nonetheless, in looking at ancient Greek mathematics, one can clearly see an abstraction from the concrete with a view of relations as extensionally defined. The strong ontological commitments which arose from the consideration of mathematical objects as abstraction was necessary to make sense of the relations and theory of proportion. Though methodologically one can generalise from the theory, the results and applicability were nonetheless ontologically restricted. As such, any properties mathematical objects possessed were from abstraction; they were properties that the concrete objects possessed in the states of affairs that rendered the relations meaningful. Therefore any relations in the abstract system are extensionally defined between mathematical objects. Fractions, relations and the theory of proportions were secondary; they were only useful insofar as they relate numbers of monads or magnitudes of line segments. In addition arithmetic and geometry had separate domains, one was the discrete and the other continuous. The former could never describe the latter given the ontological difference between them. At this level of abstraction, epistemology is of no concern given that the methodology is restricted by the predefined ontology. All objects in Greek mathematics were determinate and the relations are meaningful given the nature of the objects.

4 Relational generalisation: seventeenth century mathematics

The next step of the abstractionist approach is the generalisation of the systems. In a system, relations are extensionally defined and ontologically dependent on the objects that they relate. The next step in abstraction removes such ontological constraints, giving priority to the method, and with that one can have new conceptions of mathematical objects that can be applied to more state of affairs. The admittance of negative and imaginary numbers to the rigorous discourse in the seventeenth century is a testament to this. This wide applicability provides support and possible reliability to the continued use of such methods despite their ontological ambiguity.

Whereas in the previous step, the objects used in calculations were determinate, no such restrictions is present here. This lack of restriction brought about the advent of symbolic mathematics in the seventeenth century, the most important advancement of mathematical practice of the time (Mancosu 1996, Mahoney 1980).

This step meant that Aristotle and Proclus' common nature in the theory of proportion became central and the algaebrisation of the field into a *mathesis* *universalis* meant that the separation between arithmetic and geometry became ill-defined, leading to various philosophical debates (Sasaki 1985, Mancosu 1996, Klein 1992).

This indicates that “The ‘material’ of this universal and fundamental science is no longer furnished by ‘pure’ units whose mode of being may be subject to dispute [...][but] is now rather constituted by - ‘numbers’” (Klein 1992, p. 223). Mahoney (1980, p. 147) even pushes the idea further by claiming that mathematics in the seventeenth century became about relations, as opposed to objects. There is an agreement among historians of mathematics that the seventeenth century, though not without controversy, saw a loss in the background ontology that was imperative for the Greeks.

4.1 Brief background: The origin of the confusion

The field in that period was in a precarious position: on the one hand, the rediscovery of Greek geometry and its methods provided a strong basis from which to advance. On the other, the loss of heuristic and background ontology meant that there was uncertainty as to why Greek mathematicians did what they did: its status as a science, ontological standing of the objects and the use of sense perception were in question (Mancosu 1996, Molland 1976, Mahoney 1980).

The recovery and translations of Greek mathematical texts in the sixteenth century and the fact that it had come to Europe via the Arabic tradition further obfuscated this background ontology, causing mathematicians of the time to attempt and read between the lines of Greek mathematics to extract their methods. For example, Francois Vieta attempted to rid Greek mathematics of its, as he called it, Arab filth, claiming at the end that he rediscovered the hidden Greek gold (Klein 1992, pp. 153-154). Mahoney quotes Descartes Meditations of 1641 as saying that “The ancient mathematicians used to employ only synthesis [as opposed to algebraic analysis] in their writings, not because they were simply ignorant of the other, but, as I see it, because they made so much of it that they reserved it as a secret for themselves alone” (Mahoney 1980, p. 149). The progress of mathematics in the seventeenth century was marred by misunderstanding, which was ubiquitous in the development of mathematics in that period.

In addition, the differing translations and interpretations of the texts that were present at the time led to an ambiguity regarding the subject-matter. Given that the definition of mathematical objects, as noted, came prior to mathematical practice, there were attempts to makes sense of the use of mathematical objects and the restrictions placed by Greek geometers on the field. Peter Ramus in the sixteenth century, for example, assumed that algebra was implicit in Greek mathematics and that definitions in Euclid’s elements were defective(Mahoney 1980, pp. 148-149). This led him to misconstrue “every quantity and ratio as one-dimensional” (Sasaki 1985, p. 110), as opposed to a relation between two one-dimensional magnitudes, as the Greeks thought of them. This superficial interpretation of Greek mathematical texts later became the basis for the Wallis-Barrow debate concerning the primacy of geometry or arithmetic (Sasaki 1985).

4.2 The algebraeisation of mathematics

Regardless of how that came about, the existence of the algebraeisation of mathematics during that period, though found in various degrees, was a fact (Mancosu 1996, p. 85). Mahoney (1980) characterises it with three main ideas: symbolism, reliance on relations, and lack of (or freedom from) ontological commitments.

Klein traces this algabraeisation to Vieta. He notes that his interpretation of Greek mathematics via Diophantus "can arise only on the basis of an insufficient distinction between the generality of the method and the generality of the object of investigation" (Klein 1992, p. 123). The importance of the object ontology for the ancient Greek and the constraints that placed on the relations was lost to Vieta. In Klein's reconstruction, Vieta misunderstood the Greeks' use of the word 'eidos' and translated it to 'species', thus rendering the concept more general. Whereas the word 'eidos' literally means 'looks' it could also be interpreted as 'idea' or 'figure', terms that carry with them ontological or epistemological gravitas. Vieta's use of the word 'species' however, had no such analogue: "[it] is to be understood neither as independent in the Pythagorean or Platonic sense nor as attained 'by abstraction' [. . .] [they are] in themselves symbolic formations [. . .] comprehensible only within the language of symbolic formalism" (Klein 1992, p. 175).

This misunderstanding meant that algebra was promoted by Vieta as the proper method in the treatment of geometry and arithmetic (Bos 2001, p. 146). His interpretation of Diophantus' usage of unknowns and his misconstrual of both the ontological concerns and the determinate nature of the object was key to the algebraic method. It was to Vieta "the essential tool [. . .] as a method of symbolic calculations [. . .] applying to magnitudes irrespective of their nature (number, geometrical magnitude, or otherwise - note that he considered number to be a kind of magnitude)" (Bos 2001, p. 147). This naturally poses the question of whether, for Vieta, there exists a common nature to these various magnitudes, in a similar manner to Proclus, or rather, these various magnitudes each possess a unique nature that symbolic calculations can be equally applied to. Given that these assumptions arise on the basis of misunderstanding of Greek ontology, this question becomes more ambiguous.[10]

Due to this confusion, this new algebra in Vieta's work was symbolic: for abstract magnitudes (which includes numbers) multiplication had no definite meaning (Bos 2001, p. 148). Whereas ancient Greek mathematics specified actions for the mathematician to take, Vieta's program only had a symbolic representation

[10] It is important to note that though Vieta's work presents a turning point in the history of mathematics, it was not in isolation, as Klein implies, but rather part of continuous advancements in the field. For further criticisms of Klein's treatment of Vieta's work, see Malet 2006.

of a transformation, irrespective of the actions needed to preform them. This means that though there was multiplication in algebra, its meaning changed depending on the background ontology. In both arithmetic and geometry, multiplication had the Greek meaning. Specifically, in geometry, the multiplication of two lines forms a square. This means that though, with Vieta, one now has algebra, the sense of the symbols is still contingent on the respective ontology. Ratios themselves remained relations for Vieta. [11]

Later, Descartes, often considered the father of symbolic mathematics, further spurred the algebraisation of the field, relying on his philosophy to justify the use of mathematical objects and/or techniques.[12] Descartes' use of the theory of proportion in arithmetic by introducing the notion of a unit as a line segment marks a drastic shift in the correspondence of the two mathematical fields (Mancosu 1996, p. 67). Arithmetical calculations could now be used in geometry and the geometrical advances in the theory of proportion are now applicable in arithmetic. It is, however, uncertain whether for Descartes algebra was simply a tool, akin to the theory of proportion, or whether it was the core of his program (Mancosu 1996, p. 82).

Nonetheless, this has an important consequence. The theory of proportion takes precedence as it allowed treatment of both arithmetic and geometry under one umbrella. Both Descartes and Vieta conceived the theory of proportions as something more general than both arithmetic and geometry but common to both (Cantù 2010a). Though mirroring Proclus and Aristotle, this marks a drastic shift in the philosophical underpinnings of mathematics given the breadth of application that arose from this idea. Importantly, one can no longer view relations as extensionally defined given that new mathematical objects are being defined by the relations themselves and as such, discussions regarding the ontology of mathematical objects took place after their introduction.

4.3 Object ontology as secondary

For Descartes, though using magnitudes, multitudes and measures in varying degrees, the domain of mathematics was that of relations and proportions among those concepts (Cantù 2010a). Vieta's misconstrual of the Greek notions of objects with a more generalised one, meant that debates of the period were regarding the ontologies of the objects used, one that now came second to methodology. The ontology of mathematical objects became contentious and, as Cantù (2010a) points out, different translations and notions presented the objects and their ties with the theory of proportions differently. One can still see that though the domain

[11]For other attempts of the period to develop symbolic algebra see Bos (2001).

[12]The influence of Vieta on Descartes is uncertain: "Although Descartes insisted several times that he had read Viete's Introduction only after the publication of his own *Geometry*, he may still be viewed as having developed Viete's new ideas farther." (Mahoney 1980, p. 144)

is shifting and the focus is becoming more on relations, mathematical objects, nonetheless, possessed ontology based on 'magnitudes', however construed.

The pervasive misunderstanding of the what are multitudes and magnitudes can be encapsulated by the following quote

> the terms 'quantitas', 'quantity', 'quantité', 'quantita' on the one hand and 'magnitudo', 'magnitude', 'grandezza', 'grandeur' on the other hand were often used ambiguously and because the same definition was adopted by different authors to convey distinct conceptions of mathematics. Moreover the emergence of that definition was strictly connected to the development of the concept mathesis universalis and to the preference for one or other of the words 'quantitas' and 'magnitudo' to translate the Euclidean term 'méghezos'. (Cantù 2010a, p. 226)

It's difficult to disentangle exactly the meaning of each of these words for each author of the time. Objects as abstractions were still present. Guldin, in his 1622 lecture, states that "Mathematics is the science that considers quantity abstracted from sensible matter" (Mancosu 1996, p. 56) . Another example is that of the seventeenth century mathematician, Cavalieri. In his works, he had to " show that the O's are magnitudes in the Greek sense" (Mancosu 1996, p. 41).[13]

In addition, some mathematicians, such as Stevin and Wallis rejected the Greek conception and considered algebraic numbers as the focus of mathematics, but they were called quantities (Klein 1992, Cantù 2010a).[14] The former criticised the Greek concept of multitude by saying that

> la matiere de multitude d'unitez est nombre, Doncques la matiere d'unité est nombre. Et qui le nie, fait comme celui, qui nie qu'une piece de pain soit du pain.[15] (Stevin 1958, p. 496)

The latter noted that number is no different from magnitude.

A prime instance that exemplifies this debate is the use of infinitesimals. They became part of mathematical vernacular at the time, though not without opposition due to their ambiguous ontology. Descartes, among others, though possessing the technical resources, never used infinitesimals in his publications due to his philosophical stance (Mancosu 1996, p. 5).

Leibniz, despite being the author of integral calculus, makes the claim that infinitesimals are just a tool, much to the dismay of his followers in the academy of

[13]The Os in Cavalier's work were defined as a collection of lines. This gave rise to the problem of showing that the collections were also magnitudes so that geometrical operations were well defined when applied to them. Mancosu (1996, p.40-41) points to Cavalieri's second book of *Geometria* as doing exactly that.

[14]For a critique of Klein's treatment of Stevin, see Malet (2006, pp. 65-66)

[15]The matter of a multitude of a unity is number. Therefore, the matter of unity is number. And he who denies it will be like someone who denies that a piece of bread is bread.

Paris who argued for infinitesimals as ontologically existent objects: "There is no need to take the infinite in a rigorous way, but only in the way in one says in optics that the rays of the sun come from an infinitely distant point and are therefore taken to be parallel" (Leibniz 1701, quoted from Mancosu 1996, p. 171). He further notes that his calculus was not to be justified by its ontological commitments, but rather, by its reliability in application, which was very much present at the time.

To stress the importance of these words further, in another letter in 1716, Leibniz writes that

> Quand ils disputèrent en France avec l'Abbé Gallois , le Père Gouge & d'autres, je leur témoignai , que je ne croyois point qu'il y eut des grandeurs veritablement infinies ni véritablement infinitésimales, que ce n'étoient que des fictions , mais des fictions utiles pour abréger et pour parler universellement comme les racines imaginaires.[16] (Leibniz 1768, p. 500)

Even the standing of ratios underwent a shift. Eudoxus, whose works on the theory of proportions became the basis for book V and XII of Euclid's elements, treated ratios as relations, not numbers. Yet seventeenth century discussion were fueled by "systematic misreading of ratios as numbers (fractions) that the analytic techniques allowed" (Mancosu 1996, p. 36).

4.4 Relational Generalisation

In the seventeenth century, the generality of the method took precedence and the unification of arithmetic and geometry under the umbrella of algebra shifted the balance to a focus on relations and a new theory of proportions. As such, there was a need to re-evaluate the definition of mathematical objects. However, for some, such as Leibniz, these discussions seemed to have come second to utility and applicability. The admissibility of fictional elements, such as in optics, became permissible in mathematics. The debates around the metaphysics of mathematical objects did not subside, though it came after utility, with the various translations and definitions further obfuscating the landscape.

Though the view taken at the time was of extensionally defined relations, as is evident by discussions surrounding the status of mathematical objects, the relations must be taken to have been intensionally defined. For example, if positive and negative integers are to have the same ontology, but the latter is arrived at via operations on the former, then the relations and operations have to be intensionally defined. Otherwise, the results would be meaningless given that only one of the relata, the positive integer, is ontologically defined, whereas the other, the negative

[16] When [our friends] debated in France with Abbé Gallois, Père Gouge and others, I told them that I did not believe that there were truly infinite or truly infinitesimal magnitudes, that they were only fictions, but useful fictions to abbreviate and to speak universally like imaginary roots.

one, derives its nature from the operation on the positive integer. Yet, both integers, being numbers, must have the same ontology, which implies that it cannot be a multitude, and the argument would be circular. Whereas, in ancient Greek mathematics, the definition of mathematical objects came prior to the axioms and results were rejected if they made no ontological sense, in the seventeenth century, though dependent on assertoric axioms, these debates often came after methodological advancements.

In the transition from Greek mathematics to the seventeenth century, there is an important epistemological element to note: the focus on relations comes only after object abstraction and the prior establishment of certain results. Though, as noted by many historians, this was fuelled by a systematic misunderstanding of Greek mathematical ontology, the generalised methodologies nonetheless found great success in application. This shows that only after verifying that a relation holds based on abstracted object properties, can one generalise the system and treat the relations as central. This is crucial and historically important, because, as Brouwer points out

> The statements in [geometry] were drawn from experience; of course they could hold only approximately, but they held good again and again with unlimited precision, and this was the case even for all the properties derived from them by logical reasoning (Brouwer 1975, p. 112)

Indeed, without Greek mathematical results, it is difficult to imagine what would constitute reliability. With a sufficiently robust methodology however, one can consider relations in the abstract. By viewing similarities between abstract relations and patterns one can then attempt to apply it to various other examples.

5 Categorical axiomatisation: nineteenth and early twentieth century

Given the generalisation of the system and the loss of objects ontology, the third step is finding a suitable basis from which one can recreate the results of the generalised system while simultaneously retaining the generality of the method in a way that ensures reliability. From a structural perspective, this is done via incorporation of abstract intensionally defined relations into definitional axioms that can define the objects. This is the hallmark of nineteenth century mathematics which admits talk of a conceptual approach to mathematics (Ferreirós and Reck 2020). In doing so, mathematical entities are no longer abstracted objects, but rather concepts whose properties are derived from the axioms, intensionally defined relations that come prior to the axiomatisation process.

5.1 A broad picture of the landscape

Already, as noted, in the seventeenth century, relations were becoming more central. This extends, mathematically and philosophically, to the eighteenth and nineteenth century. Gauss argued that "mathematics is, in the most general sense, 'the science of relations'" (Gauss 1917, quoted from and translated by Ferreirós and Reck 2020, p. 60). Though relational in nature, and with objects already becoming secondary, there was still no proper foundation. In fact, the axioms were still justified inductively for objects, which needed to be defined prior to them. Which meant that critiques and philosophical debated at the turn of the nineteenth century turned towards the status of mathematical objects, and can be traced to Berkeley (Folina 2012).

The use of ambiguous concepts in mathematics, or fictitious ones, such as infinitesimals, became the basis for his critiques against the status of arithmetic and use of such techniques. The impact of this was that the dominant philosophical debate of the late eighteenth and early nineteenth century was that of realism vs. anti-realism, with a particular emphasis on the importance of the reference of imaginary or irrational numbers (Folina 2012). The main question is "whether or not the idea of 'number' could be expanded to include new kinds of number that did not seem to denote any quantity" (Folina 2012, p. 134).

Furthermore, the confusion surrounding the notion of the science of quantity persisted until the nineteenth century and Cantù (2010a) ties it directly to the generalisation of the theory of proportion. She notes that criticisms of this seventeenth century definition of mathematics was prolific in German circles in the nineteenth century: "Kant, Bolzano, Hegel, and Grassmann all criticized the definition of mathematics as Grössenlehre and the confusion between numerical quantities and extensive magnitudes" (Cantù 2010a, p. 226).

The mathematical advancements of the time triggered the emergence of many foundational programs to attempt and provide a basis for mathematics. Until then, Euclidean geometry retained a special status given that its objects were tied to the concrete via abstraction. As the sole representation of physical space it was considered "the paradigm of epistemic certainty" (Torretti 2021). With the advent of non-Euclidean geometry, however, it lost that status and "The sudden shrinking of Euclidean geometry to a subspecies of the vast family of mathematical theories of space shattered some illusions and prompted important changes in the philosophical conception of human knowledge" (Torretti 2021).

With a traditional, extensional view of relations, rigour and reliability were based on the ontology of objects. Whereas algebra and arithmetic were heading in the conceptual direction due to its abstract objects, geometry, in an attempt to re-establish it epistemic certainty, was veering towards empiricism.

5.2 Back to an empirical foundation?

Geometry's connection to the concrete, coupled with an extensional view of relations was the source of its epistemic certainty: there was only one model of physical space where the objects are defined prior to the mathematics. Though there was also one model that dealt with numbers, arithmetic, geometry's clear abstractions provided it with reliability than the latter's ambiguous 'science of quantity'. The emergence of non-Euclidean geometry threatened that reliability. With competing geometrical theories, empiricism about geometry was brewing: there was no *a priori* way to tell which one is the true representation. Arithmetic had no such conflict. This led claims to an epistemic separation between the two fields: the primacy debate was re-emerging (Folina 2012).

The detachment of analysis from geometry in the late eighteenth century was driven by this. If the former was applied to the latter, then analysis would inherit geometry's empirical status, which was unacceptable for 'pure mathematics'. An prime example of this is to be found in Riemann's *Habilitationsvortrag* as analysed by Folina: "[Riemann's] view is that mathematicians explore logical connections between concepts: there is no logical connection between the concept of space and physical space, so experience alone can choose between the various options provided by mathematics" (Folina 2012, pp. 140-141). This mode of thought was so pervasive that Bolzano noted that

> it is an intolerable offence against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely geometry [...] it is self-evident that the strictly scientific proof or the objective reason, of a truth which holds equally for all quantities, whether in space or not, cannot possibly lie in a truth which holds merely for quantities which are in space. On this view it may, on the contrary, be seen that such a geometrical proof is, in this as in most cases, really circular. (Bolzano 1817 in Ewald 2007, p. 228)

One can clearly see the extensional view of relations which bases the validity and identity of relations solely on the relata. If, as Bolzano notes, that "a truth which holds equally for all quantities, whether in space or not, cannot possibly lie in a truth which holds merely for quantities which are in space", then a truth's holding depends only on the objects: if the objects are quantities in space, then the relations is different than that for all quantities.

Even Hilbert himself, prior to the development of his geometrical axioms, noted in 1891 that

> Geometry is the science that deals with the properties of space. It differs essentially from pure mathematical domains such as the theory of numbers, algebra, or the theory of functions [...] I can never penetrate the properties of space by pure reflection, much as I can never recognize

> the basic laws of mechanics, the law of gravitation or any other physical law in this way. Space is not a product of my reflections. Rather, it is given to me through the senses. (Hilbert 1891 quoted from and translated by Corry 2004, p. 84)

The former was an applied science, whose truths can only be grounded via experience. The latter's truths arrives from "pure reflection".[17] These concerns led to various empirical foundational geometrical programs of the time. Cantù (2010b) presents the geometrical programs of both Veronese and Battozzi, noting that the former called geometry a 'mixed' science, whereas the latter was about abstraction from the physical world.

This necessitates a traditional view of relations, and platonism regarding objects: if relations are intensionally defined, then the truth of a proposition holding for quantities in space must hold for all quantities which possess properties necessary for the truth of the proposition. Otherwise, one must supply a relation of difference between the relation holding for quantities in space and for all quantities. If that differences relies on the relata, then this argument would fall prey to Russell's regress (Russell 1903, pp. 50-51).

Another concern was applicability. With mathematics becoming more conceptual in nature, applicability, or at least the possibility of conceptual mathematics being applied, was contentious. Leibniz's and Newton's calculus was imperative for advancements in the sciences. Its utility was also used to justify its reliability. At the conceptual level, with reliability already being under threat due to ontology, or lack thereof, justification via methodology was no longer secure.

For example, in the early twentieth century, even after the emergence of definitional axioms, an empirical geometrical program was proposed by Pasch . It was structural in spirit: for geometry to be genuinely deductive, he notes, the process must be independent of the sense of the geometrical figures and objects, and focus only on the relations between them (Schlimm 2020). Yet, citing applicability concerns, it retained an empiricist basis: the objects and initial propositions are based on observation and experience. Inductive axioms are to be followed by relational deductions independent of the objects involved. His insistence for such a basis was applicability because

> To apply mathematics, the basic concepts must refer to something that is present in the world of experience and for which the content of the basic propositions is meaningful and valid. We acknowledge this connection with experience as soon as we consider analysis to be something else than [...] an internally consistent construction [einen Bau von innerer Folgerichtigkeit]. (Pasch 1914, quoted from and translated by Schlimm 2020, p. 94)

[17] See Folina (2012) for the discussion in the context Kant's influence regarding intuition and reflection.

In this quote one can find the traditional attachment to object abstraction, with relations seemingly unable to be intensionally defined and provide meaning to the concepts.

5.3 The conceptual foundations

Whereas geometry lost its epistemic certainty, arithmetic and algebra were becoming more abstract and the notion of quantity became less relevant. In the early nineteenth century, both Bolzano and Cauchy developed more rigorous definitions of concepts for arithmetic. Both of them were concerned with epistemic certainty, e.g. avoiding geometric assumptions and diagrams. There was an attempt to separate arithmetic from geometry. Whereas previously, geometry's epistemic certainty was derived from its uniqueness and object abstraction, no such privilege was granted to arithmetic, algebra, and analysis. In light of new advancements, however, geometry's certainty became a matter of concern while arithmetic's use of concepts lent it further credibility. Yet, given the remaining reliance on inductive axioms, a different foundation was necessary.

There were various attempts to develop a sufficient basis. Grassmann, for example, treated arithmetical objects as a correspondence of two thought processes: the first about existent objects, the second about the first; a correspondence between concepts. In doing so, axioms can no longer be inductive, but must be definitional (Cantù 2020). Dedekind and Peano's foundational programs and axiomatic methods, however, are the dominant ones today in arithmetic. The former can be understood in terms of instantiation: "formal symbols acquire their full meaning when they are interpreted ('read as') ordinary mathematical language terms and sentences of a specific mathematical theory" (Cantù 2021, p. 225). In modern terminology, this can be seen as a quasi-nominalistic approach, that attaches meaning to symbols only in instantiations. The latter, often considered the father of modern structuralism, pushed the idea further by making relations central; objects have no internal nature, they were extensionally defined, and meaning in arithmetic is derived from the intensionally defined relations.[18]

Henri Poincaré, introducing Dedekind's axiomatisation, famously wrote that "Mathematicians do not study objects, but the relations between objects; to them it is a matter of indifference if these objects are replaced by others, provided that the relations do not change" (Poincaré 1905, p. 25). The last few words indicate a crucial element: the importance of the continuity of the relations between the objects. Whereas in ancient Greek mathematics the identity of the system depended on the continuity of the objects, in structural mathematics, it depends on the continuity of the relations. The nature of the objects is no longer what allows them to sustain these relations; it is the nature of the relations that define the objects, necessitating intensional ones.

[18] This was not in isolation, but rather an incremental methodological progress, driven by previous results. For more details on the background, see Ferreirós and Reck (2020).

Towards this foundational program, Dedekind began utilising concepts in his work and developed a general theory separate from the instantiations via abstractions: “With reference to this freeing elements from every other content (abstraction) we are justified in calling numbers free creations of the human mind” (Dedekind 2007, p. 33). The consequence of this, naturally, is the outright rejection of magnitudes. Epple (2003) notes that Dedekind was the first mathematician to intentionally avoid the use of the term magnitude. However, it was not obvious how to interpret the terms ‘abstraction’ and ‘creation’ (Ferreirós and Reck 2020). In fact, the latter term was and still is contentious.[19]

Regardless, with Dedekind, one now has a mathematical structure: a general theory based on relations and using concepts defined by these relations. The concepts of mathematics obtain their properties holistically via the axioms.[20] If there is an isomorphism between systems then the global properties and theorems proved about them would apply to both. This further allowed him to compare the real numbers with geometric lines (Dedekind 2007), yet it was not rigorous (Ferreirós and Reck 2020).

This conceptual program in arithmetic found support in Hilbert. In 1899, he opposed the ‘genetic’ approach to define the system of real numbers, rather insisted on the axiomatic one put forth by Dedekind (Corry 2004). This approach was then exported to geometry. Whereas initially there was a strong empiricist commitment, later, in his *Grundlagen de Geometrie*, that was no longer the case, at least in method. Influenced by Hertz’ foundation of mechanics, Hilbert notes:

> Nevertheless the origin [of geometrical knowledge] is in experience. The axioms are, as Hertz would say, pictures or symbols in our mind, such that consequents of the images are again images of the consequences, i.e. what we can logically deduce from the images is itself valid in nature. (Hilbert 1899 quoted from and translated by Corry 2004, p. 88)

We begin to see further abstraction in geometry, noting that images of images is more aligned to that of concepts. This becomes even more relevant in the context of an earlier quote

> Geometry is a science whose essentials are developed to such a degree, that all its facts can already be logically deduced from earlier ones. Much different is the case with the theory of electricity or with optics, in which still many new facts are being discovered. Nevertheless, with regards to its origins, geometry is a natural science. (Hilbert 1893 quoted from and translated by Corry 2004, p. 88)

[19]Philosophers have attempted to interpret the word ‘creation’ differently to either avoid constructivism or safeguard Platonism. See Ferreirós and Reck (2020) and Hellman and Shapiro (2018).

[20]See Giovannini and Schiemer (2021) for a discussion on structural implicit definitions and how they holistically define all the objects in a structure.

Geometry's origins are as a science, its methods, in contrast to other sciences, are logical deductions. Despite mirroring Pacsh in this approach, his methods were different: instead of relying on object abstraction as the basis for geometry, definitional axioms and their use of intensional relations become the basis for Hilbert. This axiomatic focus, according to Corry (2004), is what transforms geometry from a natural science, to a pure mathematical one.

It's important to explain here what the axioms meant, both in definition and importance for Hilbert. It seems that, in following Dedekind, Hilbert was attempting to find a set of mutually independent conditions[21] from which, formally, one can completely obtain all geometrical facts (Sieg 2020, p. 145). He states his goal in the *Grundlagen* as follows

> This present investigation is a new attempt to establish for geometry a complete, and as simple as possible, set of axioms and to deduce from them the most important geometric theorems in such a way that the meaning of the various groups of axioms, as well as the significance of the conclusions that can be drawn from the individual axioms, come to light. (Hilbert 1899 in Awodey and Reck 2002, p. 9)

Indeed, with over two millenia of established geometrical theorems, Hilbert's axioms had to necessarily have, as a consequence, these theorems. This was crucial: a definitional axiomatic structure is one that has to be based on, and tested against previous results, just as in the seventeenth century, relational generalisation was also tested against previous results.[22] Any resulting contradiction between a consequence of this structure and previous theorems has to have an explanation, which will necessarily be extra-structural (e.g. Euclidean and non-Euclidean contradictions).This is also emphasised by Corry who states that, for Hilbert,"definition of systems of abstract axioms and the kind of axiomatic analysis described above [regarding geometrical concerns of the nineteenth century and Euclidean geometry] was meant to be carried out always retrospectively, and only for 'concrete', well-established and elaborated mathematical entities" (Corry 2004, p. 99).

In stark contrast to the Euclidean inductive axioms, Hilbert's definitional ones are formulated for undefined objects: points, lines and planes. The axioms establish only relations that these objects must satisfy, with no reference to a domain or extra-mathematical definition. We here see the crucial difference: whereas the definition of mathematical objects for the Greeks was not found in mathematical practice, but rather came prior to the axioms, the definitions of relations and operations was not found in definitional axioms, but came prior to the axioms.

[21] In one of his responses to Frege, Hilbert states that "In my opinion, a concept can be fixed logically only by its relations to other concepts. These relations, formulated in certain statements, I call axioms, thus arriving at the view that axioms[...] are the definitions of the concepts" (Hilbert in Frege 1980, p. 51).

[22] It is important to note that this is only the case with abstraction. Once one is on the level of defining structures then this is no longer necessary.

Though genetically, this program is from “a logical analysis of our perception of space” (Hilbert 1971 in Awodey and Reck 2002, p. 9), it abstracts only the relations given that it is analysis and not the objects that plays a central role and is, according to Hilbert, what is applied in physics (Corry 2004, p. 156).

5.4 The Frege-Hilbert debate

These axiomatic methods, however, did not go uncontested. The most notable critic was Gottlob Frege. Though he had already criticised Dedekind's axiomatisation (Reck 2019), the exchange between Frege and Hilbert regarding *Grundlagen* exemplifies the shift from the extensionally to intensionally defined relations.[23] This debate highlights perfectly the shift from extensionally defined relations to intensionally defined ones.

Frege's attack on Hilbert's axiomatisation was twofold. First, the use of implicit definition did not sit well with him; objects should be defined explicitly and separately from the axioms. Second, the importance of truth that can be derived from the axioms was imperative. Hilbert's use of consistency and completeness as a test for truth and existence, for Frege, was fallacious; it could not guarantee the truth of propositions. For him, the nature of axioms is assertoric, stating truths about a fixed subject-matter, in other words, the axioms are there to define relations between perdefined objects. In contrast, for Hilbert, the axioms are relations that provide a basis for the concepts.

For Frege, definitions should in no way be part of the axioms, which are only supposed to express fundamental facts

> the meanings of the words “point”, “line”, “between” [in Grundlagen] are not given, but are assumed to be known in advance. At least it seems so. But it is also left unclear what you call a point [...] Here the axioms are made to carry a burden that belongs to definitions. To me this seems to obliterate the dividing line between definitions and axioms in a dubious manner, and beside the old meaning of the word ‘axiom’, which comes out in the proposition that the axioms express fundamental facts of intuition, there emerges another meaning but one which I can no longer grasp. (Frege 1980, pp. 35-36)

One can again see the prevalent mode of thought regarding a purely extensional view of relations. For Frege, the meaning of the words are assumed to be known in advanced and not defined via their relations.

Hilbert not only had no issue with this, for him, this was the strength of the axiomatisation process and exhibits succinctly what mathematics is actually about, the relations

> it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and

[23]Frege was not the only critic of these types of axiomatisations. See Reck (2019)

> that the basic elements can be thought of in any way one likes. If in speaking of my points, I think of some system of things, e.g. the system: love, law, chimney-sweep, [...] and then assume all my axioms are relations between things, then my propositions, e.g. Pythagoras' theorem, are also valid for these things. (Hilbert in Frege 1980, p. 40)

The relations between the objects take precedence over the objects themselves. A theory is a scaffolding that is held together by the axioms, relations. This, for Hilbert, means that a theorem can be applied to any system or state of affair, irrespective of their nature, granted that they possess the necessary properties.

This, however poses a problem to the truth of statements derived from the axioms and their reliability and is tied with Frege's second criticism. This is what necessitated Hilbert's addition of consistency and completeness. Awodey and Reck (2002) remark that Hilbert's use of these words was not rigorous. Completeness, however, is there to supposedly provide truth to the statements. To put it simply, if geometry's epistemic certainty was assured because it was the sole representation of physical space, then completeness provides the same certainty given that it shows that the "the system of axioms suffices to prove all geometric propositions" (Hilbert 1900a in Ewald 2007, p. 1093). Given Gödel's incompletess theorems, this is obviously false.[24]

Nonetheless, Hilbert's focus on axioms as relations and on theory as scaffolding or schema, exemplifies the final shift in the axioms themselves and provided a foundation for mathematical objects, giving us a mathematical structure as we now know it. In using implicit definitions, concepts become instantiated in the properties of the objects in their respective mathematical systems or concrete exemplifications. This allows for further generality in mathematics, placing relations and relational properties at the core of the field.

6 Conclusion

In the philosophy and history of mathematics, the focus has generally been on the objects: what they are and how they are abstracted. Though the notion of relations is often mentioned, it is nonetheless left as an opaque afterthought. This paper shed light on the notion of relations in the history of mathematics and how such a notion underwent a drastic historical change that is best understood in terms of intensional and extensional view of relations. This not only exemplifies how mathematics itself has a changed, but also provides a means to look at mathematical abstraction differently.

[24] For more on the truth of geometerical statements in Hilbert's work, see Sieg (2020), Hilbert (1922a) in Ewald (2007), Hilbert (1925) in Heijenoort (2002), Mancosu (1997, Ch. 15) and Paul Bernay's comments on his work with Hilbert on finitary mathematics (Bernays 1974 in Wang 1997).

References

Aristotle (1960), *Posterior Analytics Topica*, trans. by Hugh Tredennick and E. S. Forster, Harvard University Press, Cambridge, MA.

— (1983), *Physics Books III and IV*, trans. by Edward Hussey, 1st edition, Clarendon Press, Oxford Oxfordshire : New York, 274 pp.

Awodey, Steve and Erich Reck (2002), "Completeness and Categoricity. Part I: Nineteenth-century Axiomatics to Twentieth-century Metalogic", *History and Philosophy of Logic*, 23, 1, p. 1.

Blåsjö, Viktor (2016), "In defence of geometrical algebra", *Archive for History of Exact Sciences*, 70, 3, pp. 325-359, DOI: 10.1007/s00407-015-0169-5.

Bos, Henk J. M. (2001), *Redefining Geometrical Exactness*, 1st ed., Sources and Studies in the History of Mathematics and Physical Sciences, Springer New York, NY, XIX, 472.

Brouwer, Luitzen Egbertus Jan (1975), *L. E. J. Brouwer Collected Works*, ed. by A. Heyting, Elsevier Science Publishing, Amsterdam, vol. 1, 628 pp.

Burgess, John (1999), "Book Review: Stewart Shapiro. Philosophy of Mathematics: Structure and Ontology", *Notre Dame Journal of Formal Logic*, 40, 2, pp. 283-291, DOI: 10.1305/ndjfl/1038949543.

Button, Tim (2006), "Realistic Structuralism's Identity Crisis: A Hybrid Solution", *Analysis*, 66, 3, pp. 216-222.

Cantù, Paola (2010a), "Aristotle's prohibition rule on kind-crossing and the definition of mathematics as a science of quantities", *Synthese*, 174, 2, pp. 225-235, DOI: 10.1007/s11229-008-9419-2.

— (2010b), "The role of epistemological models in Veronese's and Bettazzi's theory of magnitudes", in *New Essays in Logic and Philosophy of Science*, ed. by Marcello D'Agostino, Federico Laudisa, Giulio Giorello, Telmo Pievani, and Corrado Sinigaglia, SILFS, College Publications, vol. 1, pp. 229-241.

— (2020), "Grassmann's Concept Structuralism", in *The Prehistory of Mathematical Structuralism*, ed. by Erich H. Reck and Georg Schiemer, Oxford University Press, pp. 21-58.

— (2021), "Peano's Philosophical Views between Structuralism and Logicism", in *Origins and Varieties of Logicism. On the Logico-Philosophical Foundations of Mathematics*, ed. by Francesca Boccuni and Andrea Sereni, 1, Routledge.

Corry, L. (2004), *David Hilbert and the Axiomatization of Physics*, Springer, Dordrecht ; Boston, 530 pp.

Dedekind, Richard (2007), *Essays on the Theory of Numbers*, trans. by Wooster Woodruff Beman.

Donnelly, Maureen (2021), "Explaining the differential application of non-symmetric relations", *Synthese*, 199, 1, pp. 3587-3610, DOI: 10.1007/s11229-020-02948-x.
Dunham, William (1990), *Journey through Genius*, Wiley Science Editions, John Wiley & Sons.
Epple, Moritz (2003), "The End of the Science of Quantity: Foundations of Analysis", in *A History of Analysis*, ed. by Jahnke H., American Mathematical Society., Providence, RI, pp. 291-324.
Ewald, William Bragg (2007), *From Kant to Hilbert Volume 1: A Source Book in the Foundations of Mathematics*, Oxford University Press, Oxford, 678 pp.
Ferreirós, José and Erich Reck (2020), "Dedekind's Mathematical Structuralism: From Galois Theory to Numbers, Sets, and Functions", in *The Prehistory of Mathematical Structuralism*, ed. by Erich Reck and Georg Schiemer, Oxford University Press.
Fine, Kit (2000), "Neutral Relations", *The Philosophical Review*, 109, 1, pp. 1-33, DOI: 10.2307/2693553.
Fitting, Melvin (2006), "Intensional Logic" (July 6, 2006), Last Modified: 2015-04-02.
Folina, Janet (2012), *Some developments in the philosophy of mathematics, 1790-1870*, The Cambridge History of Philosophy in the Nineteenth Century (1790–1870).
Frege, Gottlob (1980), *Philosophical and Mathematical Correspondence of Gottlob Frege*, ed. by Brian McGuinness, trans. by Hans Kaal, University Of Chicago Press, Chicago, 234 pp.
Gaskin, Richard and Daniel J. Hill (2012), "On Neutral Relations", *Dialectica*, 66, 1, pp. 167-186, DOI: 10.1111/j.1746-8361.2012.01294.x.
Giovannini, Eduardo N. and Georg Schiemer (2021), "What are Implicit Definitions?", *Erkenntnis*, 86, 6, pp. 1661-1691, DOI: 10.1007/s10670-019-00176-5.
Heijenoort, Jean van (2002), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Fourth Printing edition, Harvard University Press, Cambridge, Mass, 680 pp.
Hellman, Geoffrey (2001), "Three Varieties of Mathematical Structuralism", *Philosophia Mathematica*, 9, 2, pp. 184-211.
Hellman, Geoffrey and Stewart Shapiro (2018), *Mathematical Structuralism*, Cambridge University Press.
Karasmanis, Vassilis (2011), "Continuity and Incommensurability in Ancient Greek Philosophy and Mathematics", in *Socratic, Platonic and Aristotelian Studies: Essays in Honor of Gerasimos Santas*, ed. by Georgios Anagnos-

topoulos, Philosophical Studies Series, Springer Netherlands, Dordrecht, pp. 389-399.
Keränen, Jukka (2001), "The Identity Problem for Realist Structuralism", *Philosophia Mathematica*, 9, 3, pp. 308-330, DOI: 10.1093/philmat/9.3.308.
Klein, Jacob (1992), *Greek Mathematical Thought and the Origin of Algebra*, Revised edition, Dover Publications, New York, 384 pp.
Kline, Morris (1990), *Mathematical Thought from Ancient to Modern Times, Vol. 1*, 1st edition, Oxford University Press, New York, 390 pp.
Korbmacher, Johannes and Georg Schiemer (2018), "What Are Structural Properties?", *Philosophia Mathematica*, 26, 3, pp. 295-323, DOI: 10.1093/philmat/nkx011.
Ladyman, James (2005), "Mathematical Structuralism and the Identity of Indiscernibles", *Analysis*, 65, 3, pp. 218-221.
Leibniz, Gottfried Wilhelm (1768), *Gothofredi Guillelmi Leibnitii, ... Opera omnia, nunc primum collecta, in classes distributa, praefationibus & indicibus exornata, studio Ludovici Dutens. Tomus primus sextus Tomus tertius, continens opera mathematica*, 782 pp.
Leitgeb, Hannes and James Ladyman (2007), "Criteria of Identity and Structuralist Ontology", *Philosophia Mathematica*, 16, 3, pp. 388-396, DOI: 10.1093/philmat/nkm039.
Linnebo, Øystein (2008), "Structuralism and the Notion of Dependence", *The Philosophical Quarterly (1950-)*, 58, 230, pp. 59-79.
Mahoney, Michael S. (1980), "The Beginnings of Algebraic Thought in the Seventeenth Century", in *Descartes: Philosophy, Mathematics and Physics*, ed. by Stephen Gaukroger, Barnes & Noble Imports, Sussex : Totowa, N.J, pp. 141-155.
Malet, Antoni (2006), "Renaissance notions of number and magnitude", *Historia Mathematica*, The Origins of Algebra: From al-Khwarizmi to Descartes, 33, 1, pp. 63-81, DOI: 10.1016/j.hm.2004.11.011.
Mancosu, Paolo (1996), *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*, 1st Paperback Edition, Oxford University Press, New York, 288 pp.
— (ed.) (1997), *From Brouwer To Hilbert: The Debate on the Foundations of Mathematics in the 1920s*, UK ed. edition, Oxford University Press, New York, 352 pp.
Molland, A. G. (1976), "Shifting the foundations: Descartes's transformation of ancient geometry", *Historia Mathematica*, 3, 1, pp. 21-49, DOI: 10.1016/0315-0860(76)90004-5.

Netz, Reviel (1999), *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*, 1st edition, Cambridge University Press, Cambridge, 352 pp.

Ostertag, Gary (2019), "Structured propositions and the logical form of predication", *Synthese*, 196, 4, pp. 1475-1499, DOI: 10.1007/s11229-017-1420-1.

Poincaré, Henri (1905), *Science and Hypothesis*, The Walter Scott Publishing, New York.

Reck, Erich (2019), "Frege's Relation to Dedekind: Basic Laws and Beyond.", in *Essays on Frege's: basic laws of arithmetic*, ed. by Philip A. Ebert and Marcus Rossberg, First edition, Oxford University Press, Oxford, pp. 264-284.

Russell, Bertrand (1903), *The Principles of Mathematics*, Cambridge University Press.

Sasaki, C. (1985), "The acceptance of the theory of proportion in the sixteenth and seventeenth centuries." *Historia Scientiarum*, 29, pp. 83-116.

Schlimm, Dirk (2020), "Pasch's Empiricism as Methodological Structuralism", in *The Prehistory of Mathematical Structuralism*, ed. by Erich H. Reck and Georg Schiemer, Oxford University Press.

Shapiro, Stewart (2008), "Identity, Indiscernibility, and ante rem Structuralism: The Tale of i and -i†", *Philosophia Mathematica*, 16, 3, pp. 285-309, DOI: 10.1093/philmat/nkm042.

Sieg, Wilfried (2020), "The Ways of Hilbert's Axiomatics: Structural and Formal", in *The Prehistory of Mathematical Structuralism*, ed. by Erich H. Reck and Georg Schiemer, Oxford University Press.

Stenlund, Sören (2014), *The origin of symbolic mathematics and the end of the science of quantity*, Uppsala philosophical studies, 59, Uppsala Universitet, Uppsala, 93 pp.

Stevin, Simon (1958), *The Principle Works of Simon Stevin*, ed. by D. J. Struik, C. V. Swets and Zeitlinger, Amsterdam, vol. 2.

Tarski, Alfred (1986), "What are logical notions?", *History and Philosophy of Logic*, 7, 2, ed. by John Corcoran, pp. 143-154.

Torretti, Roberto (2021), "Nineteenth Century Geometry", in *The Stanford Encyclopedia of Philosophy*, ed. by Edward N. Zalta, Fall 2021, Metaphysics Research Lab, Stanford University.

Unguru, Sabetai (1975), "On the need to rewrite the history of Greek mathematics", *Archive for History of Exact Sciences*, 15, 1, pp. 67-114, DOI: 10.1007/BF00327233.

Wang, Hao (1997), *A Logical Journey: From Gödel to Philosophy.*