Decidable and Undecidable in Quantum Mechanics

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This work shows that the ontic-epistemic dichotomy is insufficient to capture the different levels of ignorance and their implications for probability theories. It proposes an essentially epistemic interpretation of quantum mechanics, built on an operational basis firmly anchored to experimental data and scientific methods. This approach enables a rigorous treatment of numerical values obtained from experiments without resorting to unnecessary ontological or metaphysical assumptions.

I. INTRODUCTION

Classical probability theory (CPT) and quantum mechanics (QM) are two fundamentally different algebraic frameworks that yield strikingly similar outputs: real values in $[0, 1]$ representing the frequency or probability of a given physical phenomenon.

It is therefore natural to ask whether, despite their markedly different forms, the two theories are ultimately equivalent. The answer is negative, as definitively demonstrated by Bell[1–5], who derived inequalities establishing a quantitative boundary between the two frameworks: QM violates certain inequalities that remain unviolated within CPT.

Bell's theorems have been extensively analysed, reformulated from multiple perspectives, and subjected to thorough experimental verification[6]. Their profound significance was ultimately recognised with the 2022 Nobel Prize.

Bell's work has played, and continues to play, a central role in the debate surrounding ontology, realism, and, more broadly, the interpretation of quantum mechanics. In particular, it appears to definitively rule out the possibility that a classical reality lurks behind the formal structure of quantum theory. If such a reality existed, it would yield experimental values consistent with CPT—yet this is not the case. Put differently, the peculiarities of QM, including its inherently probabilistic nature, cannot merely be attributed to epistemic ignorance. The theory necessarily describes a world with some intrinsically unusual ontological features.

An alternative stance is to reject the ontological debate—and, more generally, the philosophical discussion—altogether. QM is what it is; it works exceptionally well, and nothing more need be said. This radical position is encapsulated in the motto "*shut up and calculate*"[7], resonates in the aversion some physicists have expressed towards philosophy (Hawking's "*philosophy is dead*"[8] being a well-known example), and, beyond its provocative phrasing, has a rationale of its own.

Even within the most classical framework—Newtonian gravitation—profound ontological issues arise concerning the nature of the universal gravitational force, as already acknowledged by Newton himself in his famous "*hypotheses non fingo*"[9]. The prevailing ontology that emerged from this theory, further shaped by Faraday's visionary insights, conceives this force as being produced by a field—an intangible substance emitted, in some undefined manner, by massive bodies.[1]

This ontology, in light of Einsteins results, has been shown to be fundamentally incorrect. The same holds for electromagnetism in the light of quantum electrodynamics. Perhaps "interpreting" scientific theories is indeed a futile—if not counterproductive—endeavour. The physicist Valentine Telegdi expressed his disenchantment by stating: "*All physical theories are just recipes, nothing more!*"[11]

On the other hand, a significant fraction of physicists openly declare themselves driven by a profound desire to understand the world, rather than merely to produce new recipes. The tension between physics and philosophy remains as alive as ever.

The most direct ontological consequence of Bell's theorems is that quantum properties—for instance, the spin value of a particle—cannot possess a well-defined state of reality prior to measurement. If they did, statistical analysis of such values would follow a classical pattern, and no violation of Bell's inequalities would be observed. This implies the necessity of a non-local mediating mechanism correlating the values that quantum properties assume upon measurement.

The thesis of this work is that such deductions are weak. It will be shown, in contrast, that the violation of Bell's inequalities is fully compatible with the assignment of well-defined properties that pre-exist measurement. That is, it will be demonstrated that QM is fundamentally compatible with a notion of probability of an epistemic nature.

The path ahead is fraught with difficulties and will require a series of at times pedantic clarifications. Unfortunately, there seems to be no alternative, as the interpretative key proposed here lies precisely in such refinements.

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[1] Equally stimulating is Feynman's toy hypothesis[10], which envisions gravity as a consequence of collisions within an omnipresent gas of particles, with attraction arising from an imbalance in these collisions due to the shadow cones cast by one body upon another.

II. LOGIC AND PHYSICS

In the 1930s, in response to the problems posed by Hilbert[12] at the beginning of the century, Gödel produced a series of theorems concerning the consistency and completeness of formal logical systems[13, 14]. The most famous of these, known as the first incompleteness theorem, establishes that for any consistent formal system sufficiently powerful to include Peano arithmetic, there exists a syntactically well-formed formula that is unprovable within that system.

This highly technical result was independently rediscovered within months by von Neumann and later reformulated by Chaitin[15] in the study of universal Turing machines, leading to the development of algorithmic information theory.

During the same historical period, in the vastly different domains of logic, physics, and the emerging theory of computation, similar natural-language expressions began to resonate: undecidability, incompleteness, indeterminacy, the impossibility of assigning well-defined properties, and so forth.

However, Gödel's results, despite their profound implications, remain extraordinarily technical, addressing aspects that, in a sense, are marginal to the practice of mathematics and even perceived as irrelevant from an intuitionistic perspective[16]. With only a few pioneers exploring this connection—notably Benioff's work[17] and some notes by Wheeler[18]—physics, as a whole, has not seriously pursued the idea of a suggestive relation between Gödel and Heisenberg.

Another line of inquiry concerns the development of alternative logics to classical logic—a field explored by Gödel, von Neumann[19], the precursor Peirce[20], and, in a different context, Zadeh[21]. These logics typically reject or weaken the law of the excluded middle, allowing for propositions that are not necessarily true or false. This has a direct connection to QM, as it provides a natural framework for accommodating paradoxical superposition states—such as Schrödinger's cat[22] being neither alive nor dead.

However, the cost of abandoning the law of the excluded middle is immense: nearly every known theorem in mathematics depends, either directly or indirectly, on this principle. Such an approach undermines the mathematical foundations of QM (which are defined within classical logic) and, in a self-defeating loop, negates its own justification. The only viable resolution appears to be allowing these alternative logics to coexist with classical logic, applying them only in specific contexts. Yet, while they offer solutions to the dilemmas posed by QM in the real world, they simultaneously introduce new paradoxes within the domains of logic and mathematics.

On the other hand, the very definition of an undecidable proposition naturally—but also erroneously—leads to multivalued logics[23]. If undecidable propositions exist, then, in general, a proposition could be true, false, or undecidable, and this tripartite classification appears to negate the law of the excluded middle. However, this reasoning is fallacious, and no trace of such an argument can be found in Gödel's theorems. **Undecidable** is not an alternative state to **True** and **False**. Gödel, in his work, clarifies the subtle distinction between **truth** and **provability**.

A theorem provable in a system S is not merely true, but rather **true and provable** in S; we shall say that it is **manifestly true** in S. Similarly, a formula whose falsity can be proven in S is not merely false, but rather **manifestly false** in S. **Undecidable** is an alternative state to **manifestly true** and **manifestly false**, not to **true** and **false**. As we shall see, this distinction is the fundamental lever that will drive the arguments that follow.

In more recent years, the study of a possible connection between logic and quantum mechanics—particularly between logical undecidability and quantum undecidability—has been revitalised by the works of various authors[24–30], who are revisiting these ideas from a variety of promising new perspectives.

These works, despite their potential, often suffer from what could be termed an *excess of formalisation*. That is, there is a tendency to impose a forced coincidence between physical systems and formal logical systems (specifically, formal systems incorporating at least Peano arithmetic) in order to leverage logical incompleteness and project logical undecidability onto quantum undecidability.

More recently, however [24, 27], a way to avoid the trap of excessive formalisation has emerged—one that effectively disregards Gödel's and Chaitin's theorems and instead shifts the focus to a weaker form of undecidability within logical systems (including even the most radically intuitionistic ones), namely what is referred to as *indeterminacy* or *independence of propositions*.

Put simply, the premises "Socrates is a man" and "All men are mortal" do not allow us to determine the colour of Socrates' hair, and this is certainly not due to Gödel, but rather to an evident lack of logical connection. The proposition "Socrates has black hair" is **independent** of the given premises.

That quantum undecidability could be connected to this weaker form of logical independence is not usually given serious consideration because, once again, if the observer's ignorance were purely epistemic, classical probability theory would be expected to hold—and yet it does not, particularly in light of Bell's theorems.

However, it should be noted that the concept of epistemic ignorance has at least two subtly distinct shades:

- An observer is unaware of the outcome of a die roll because they do not know its initial conditions with sufficient precision, yet these conditions are, at least in principle, available.

- An observer is unaware of the outcome of a die roll because they do not know its initial conditions with sufficient precision, and these conditions are not available to him—not even in principle.

The first case is what is typically referred to as **epistemic ignorance**. The second is more subtle and delicate. If the die does not possess well-defined initial conditions—even in principle—then this is precisely what is meant by **ontological ignorance**: the die has a structurally fuzzy and peculiar nature, with an ontology that exhibits some intrinsic peculiarity. However, this reasoning contains a fallacy remarkably similar to the one discussed earlier. The claim is not that the die lacks certain properties, but rather that the observer does not have access to them.

The proposition "*The die has initial properties* $x, y, z, \ldots$" is not neither true nor false—which would negate the law of the excluded middle and trigger a host of logical paradoxes—but rather undecidable, meaning neither manifestly true nor manifestly false from the observer's perspective.

To clarify this distinction, let us define a **physical system** as a space-time neighbourhood. Specifically, we shall consider a neighbourhood containing an observer over a given time span. Let us call A a physical system that includes a scientist, Alice, and her laboratory over a working day. Within A, there are Alice, her memory, her notes, and her measuring instruments, spanning a temporal interval of several hours. The system A is not isolated—it interacts with its surroundings: the sun heats the walls of the laboratory, and external sounds enter through the windows.

Now, consider a second system, B, that is a distinct space-time neighbourhood, external to A. B can be conceived as a spatially separate physical system or as the future state of A, meaning A during the following workday. In this latter case, we shall also refer to such systems as **states**, with the time-evolved version A' of A being called the **final state** of A. In the example of the die, A can be identified with the observer and B with the die itself over a given time interval.

The question we pose is: what can be deduced about B given A (or *within* A)? That is, what can Alice, during her workday, rationally infer about system B? More specifically, what form should the best scientific theory adopted by Alice take in order to predict the properties of B?

An extreme possibility, which we shall refer to as **local determinism**, is that every neighbourhood external to A is, in some way, holographically projected into A. That is, there exists some form of universal law—which we may assume Alice knows—that allows one to extract from A all possible information concerning B. In logical terms, this means that every proposition about B is decidable within A, except for possible self-referential or degenerate propositions constrained by the limits of formal systems. To avoid the trap of excessive formalisation, we shall state that every proposition about B that is provable in B is also provable in A.

It is instructive to consider how a classical theory such as Newtonian mechanics relates to the notion of local determinism. Without claiming complete rigour, let us imagine A within a Newtonian universe governed solely by the forces of universal gravitation and Coulomb interaction. Within A, Alice can study in detail the motion of a falling object, which will depend on a Lagrangian involving all masses and positions of all bodies in the universe, consisting of the sum of gravitational potentials, electric potentials, and all kinetic energy terms. Since the forces involved are analytic functions, the Lagrangian itself will be analytic. This implies that the state of the entire universe, throughout its entire history, will be an analytic extension of A. The motion of the falling object in A thus contains all the information necessary to reconstruct the entire history of the universe: the universe is entirely holographically projected into A. This represents an extreme form of Laplace's demon[31], in which every arbitrarily small neighbourhood contains a holographic copy of the whole. Such a universe would be a kind of vast fractal, a Mandelbrot-like structure in which one part (every part) replicates every detail of the whole.

A second instructive observation concerns the relationship between local determinism and the very notion of experimental science. The LIGO experiment[32], to cite a striking example, demonstrates how, within the vicinity of a large interferometer, for a duration of several minutes, traces persist that allow scientists to reconstruct information about a second system extraordinarily distant in both time and space. This experiment was remarkable for the scientists' ability to read these microscopic traces and uncover their logical consequences.

Clearly, the successes of science do not in any way imply that local determinism must be a necessary property of our universe. On the contrary, it appears to be an artificial imposition dictated more by the human desire to describe the world using elegant and simple formulas. It is entirely natural to hypothesise its negation, assuming that B is not holographically projected into A, that is, that there exists a property of B that is undecidable within A, or, equivalently, that there exists a proposition provable in B but not in A.

The negation of local determinism does not entail the negation of classical determinism. Consider once again the purely Newtonian universe discussed above, but now modify the law of universal gravitation so that it is no longer an analytic function. For instance, suppose that this force instantaneously vanishes beyond a certain (possibly very large) distance. The resulting system remains deterministic: Laplace's demon can still construct the Lagrangian, populate it with initial conditions, and compute the present and future of that universe in every detail. However, within A, the motion of the falling object may no longer depend on masses that are too distant, and information concerning those masses may no longer be available there. In this case, although the universe remains deterministic in the classical sense, it is no longer an analytic extension of A—that is, the universe is not holographically projected into A.

The existence of this type of undecidable propositions within A is not purely epistemic—i.e., it does not concern Alice's knowledge, memory, or awareness—but rather reflects a deep, objective lack of information within A. We shall refer to this form of ignorance as **onto-epistemic ignorance**.

Despite full awareness that the topics discussed so far traverse the individually complex domains of the foundations of physics, the foundations of mathematics, the foundations of logic, and the philosophy of science—and acknowledging that the risk of slipping on conceptual pitfalls is consequently high—the definition of local determinism and its negation appears sufficiently weak and, in particular, independent of the technical peculiarities of formal logical systems. This makes it a solid and broadly acceptable starting platform.

This chapter has examined various details of the vast debate concerning physics, logic, and the philosophy of science. This was done not out of mere pedantry, but because within these very details may lie (and indeed lies!) the solution to the foundational issues raised by QM. In particular, the following points have been clarified:

1. The problem of logical undecidability is relevant to physics even if one completely disregards the intricate incompleteness theorems of formal systems. Even adopting a radically intuitionistic stance, the issue of indeterminacy arising from the logical independence of propositions remains.

2. The inevitable concept of undecidability is always relative to a system or an observer and is not an alternative to true and false, but rather to manifestly true and manifestly false. This concept does not imply a violation of the law of the excluded middle and does not require any form of alternative logic.

3. The notion of local determinism is central because it further clarifies that what is at stake here pertains strictly to physics—not to abstract logical, mathematical, or philosophical constructions projected in some Platonic manner onto the real world. From these powerful disciplines, only the bare minimum necessary is extracted to analyse what Alice, a real physicist conducting real measurements, can rationally infer from the numerical relationships among her measurements.

4. The categories of ontic ignorance and epistemic ignorance are overly simplistic and fail to capture the nuances outlined in the previous points. The form of ignorance arising from the negation of local determinism has distinct characteristics, necessitating a specific definition: **onto-epistemic ignorance**.

5. Bell's inequalities are crucial. Well before any speculation concerning ontology (such as the absence of pre-measurement ontological properties or nonlocality), they establish a strictly technical point: CPT and QM are incompatible insofar as QM violates certain inequalities that are unviolable within CPT.

The objective of the following chapters will be to determine the properties that a scientific theory developed by Alice must possess in order to make predictions about undecidable properties in the sense discussed here. In other words, we shall ask what form a theory must take in order to handle and manage onto-epistemic ignorance.

III. LOCAL INDETERMINISM AND PROBABILITY

The scientist Alice might be interested in studying undecidable properties such as "*the diameter of B*", "*the mass of B*", "*the velocity of B*", and so forth. For simplicity, and without loss of generality, we shall reduce this complexity to propositions of the form "*The mass of B is greater than 10kg*" or "*The height of B is 174cm*", thereby simplifying the outputs and associated difficulties to a straightforward binary yes/no choice.

Moreover, we shall focus on concretely verifiable propositions. That is, the system A can evolve into a state A' in which the previously undecidable proposition p becomes decidable—generally through direct or indirect interaction with B. In other words, Alice can conduct suitable experiments such that in the future state of A, p becomes manifestly true or false, allowing Alice to apply the scientific method by comparing her theoretical predictions, formulated in A, where p is undecidable, with the measurement results available in A', where p is decidable—indeed, decided. The act of measurement, the scientific experiment, can thus be entirely identified with this state transition [2], the acquisition of the necessary information for a certain property to become decidable.

By construction, it is not possible within A to assign a definite truth value to p. Nevertheless, the type of onto-epistemic ignorance under discussion is entirely compatible with the realist assumption that the truth value of p is well-defined, particularly within B. Alice's theory regarding p will, at best, be probabilistic.

It becomes necessary to determine what type of probability—CPT? QM? Something else?

An instinctive response might be that, given the epistemic nature of the ignorance involved, such a theory should adhere to CPT. However, the onto-epistemic scenario delineated here is subtly different from the more common notion of epistemic ignorance, and this conclusion requires, at the very least, further examination.

[2] which, it should be noted, involves neither Alice nor her consciousness but solely the definition of local determinism

We shall denote the classical probability of a proposition p by the symbol $|p|$, while the probability observed by Alice—the actual frequency she measures—will be denoted by $[p]_A$, or, where no confusion arises, simply by $[p]$. This probability will be referred to as **Probability Relative to A** or, more concisely, **Relative Probability**.

Classical probability is governed by the theorem of conditional probability, from which we have:

$$|p \wedge q| = |p|\,|q|_p = |q|\,|p|_q \tag{1}$$

where $|q|_p$ and $|p|_q$ are new algebraic symbols associated with the "*probability of p, given q*" and the "*probability of q, given p*", respectively.

Equation 1 is particularly critical in relation to the discussion at hand. In A, p is undecidable. However, Alice can interact directly or indirectly with B, conducting appropriate experiments such that in the future state A' (the evolved state of A), the truth value of p becomes manifest.

In A', by construction, not only is p finally true or false, but necessarily the following proposition also holds:

$$\bar{p} := \text{"}p \text{ is decidable"} \tag{2}$$

That is, assuming CPT can serve as a suitable theory for Alice, the probability measured by Alice—the actual frequencies she observes—are, by construction:

$$[p] = |p|_{\bar{p}} \tag{3}$$

This same result can be obtained from a different perspective.

The frequency value of an event, as actually measured by Alice in A, will be the number of cases in which Alice obtained a positive result divided by the number of cases in which she conducted the experiment. However, the very act of conducting the experiment is nothing other than bringing A into a state A' in which p is manifest. Thus, what Alice measures is necessarily:

$$[p] = \frac{|\text{p is manifestly true}|}{|\text{p is manifest}|} \tag{4}$$

which can be expressed in terms of CPT as:

$$[p] = \frac{|p \wedge \bar{p}|}{|\bar{p}|} \tag{5}$$

From which, using 1, we obtain once again:

$$[p] = |p|_{\bar{p}} \tag{6}$$

The subtle difference between $|p|$ and $[p]$ disappears in the presence of local determinism but emerges in the presence of onto-epistemic ignorance. As we have seen, this holds even in scenarios involving classical determinism.

The most evident consequence of 3 is that 1 is inapplicable in A. However, even within relative probability, it is possible to define in a semi-classical manner the concept of $[p]_q$, that is, the *relative probability of p, given q*.

Let T, F, U be symbols denoting conditions of manifest truth, manifest falsehood, and undecidability, respectively. A pair of such symbols (e.g., TT, TF,...) will represent the joint state of p and q, respectively. The relative probability of p can then be defined, in terms of these joint states, as:

$$[p] = \frac{|TT| + |TF| + |TU|}{|TT| + |TF| + |TU| + |FT| + |FF| + |FU|} \tag{7}$$

and $[p]_q$ can be defined as the relative probability of p in states where q is manifestly true, i.e.,

$$[p]_q = \frac{|TT|}{|TT| + |FT|} \tag{8}$$

from which we directly obtain:

$$[p][q]_p \neq [q][p]_q \tag{9}$$

That is, unlike in CPT, relative probability is subject to a fundamental law of **non-commutativity**.

The negation of local determinism implies the existence of undecidable propositions in a weak, epistemic sense—one that does not involve the completeness theorems of formal logical systems. On the other hand, the ignorance arising from these propositions is not linked to what a sentient agent knows or does not know, but rather to an objective, ontological condition of missing information—an onto-epistemic ignorance.

A scientist subject to this kind of ignorance will, at best, be able to develop probabilistic theories to make predictions about undecidable propositions or properties. On the experimental front, they will collect concrete data—that is, frequency values of a given event or measurement. The numerical relationships among these collected values will exhibit a fundamental **non-commutativity**, implying that the theoretical framework this physicist must develop cannot be CPT, but rather an algebra that is structurally non-commutative.

The following chapters will be dedicated to demonstrating that this algebra coincides with the foundational postulates of quantum mechanics.

IV. RELATIVE PROBABILITY AND QUANTUM MECHANICS

In this chapter, we shall demonstrate that the relative probability defined in A coincides with the first postulates of quantum mechanics. By this expression, we refer to the axiomatic foundations of QM that define its algebraic structure—Hilbert spaces, state vectors, Hermitian operators, and the Born rule—while excluding the more explicitly dynamical postulate, the Schrödinger equation, which is understood here as the quantum formulation (within the algebra defined by the preceding postulates) of the Hamiltonian formalism.

For simplicity, and as done previously, the demonstration will be restricted to observables with only two possible values (0–1, true–false, yes–no, etc.), namely the subset of QM known as *quantum logic*.

The proof will proceed in three stages:

- In the first stage, we shall bridge CPT and QM by reformulating CPT within a **pseudo-quantum algebra**, that is, an algebraic structure composed of vector spaces, state vectors, and linear operators.
- This algebra will then be used as a stepping stone to provide an analogous representation, in terms of operators, for relative probability.
- Finally, we shall demonstrate the identity between this algebra and QM.

A. Geometric Representation of Classical Probability Theory

Equation 1, by introducing a new algebraic symbol, highlights a typical semantic issue in classical probability theory. The probability of "*p and q*" is not an algebraic function of the probabilities of p and q, but rather depends on the semantic relationship between the propositions.

Conversely, given the probabilities $|p \wedge q|$, $|p \wedge \neg q|$, $|\neg p \wedge q|$, and $|\neg p \wedge \neg q|$, it is possible to reconstruct the probabilities of p and q. In particular:

$$|p| = |p \wedge q| + |p \wedge \neg q| \tag{10}$$

In other words, the semantic relationships between p and q are captured by the quadruple:

$$v = \{|p \wedge q|, |p \wedge \neg q|, |\neg p \wedge q|, |\neg p \wedge \neg q|\} \tag{11}$$

A powerful geometric representation of these objects can be provided by considering the vectors:

$$|s\rangle = (\pm\sqrt{|p \wedge q|}, \pm\sqrt{|p \wedge \neg q|}, \pm\sqrt{|\neg p \wedge q|}, \pm\sqrt{|\neg p \wedge \neg q|})$$

These vectors are subject to a form of the excluded middle[3]:

$$\langle s|s\rangle = 1 \tag{12}$$

The propositions p and q can now be associated with diagonal projectors that extract only the components of $|s\rangle$ in which p (or respectively q) is true:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

Applying Born's rule:

$$|p| = \langle s|P|s\rangle \tag{14}$$

This geometrisation process has the advantage of eliminating the additional symbols such as $|p|_q$ and reducing the probability algebra to a fundamentally Boolean form, specifically:

$$\begin{cases} \neg P = I - P \\ P \wedge Q = PQ = QP \\ P \vee Q = P + Q - PQ \end{cases} \tag{15}$$

Extending this reasoning to n propositions p, q, r, ..., we find that:

- The semantic relationships among n propositions [4] are captured by a state vector $|S\rangle$, i.e., a direction, a one-dimensional subspace in $\mathbb{R}^{2^n}$.
- A generic proposition p, formed from a combination of the involved propositions, is associated with an appropriate diagonal projector P.
- The probability $|p|$ is given by Born's rule $|p| = \langle s|P|s\rangle$

Through this quick procedure, classical probability theory has been given a **pseudo-quantum** geometric form, where Hilbert spaces are replaced by real vector spaces, and general Hermitian projectors are replaced by diagonal real projectors—or at least by projectors that are all simultaneously diagonalizable via an appropriate change of basis.

[3] Since there are no conjugates in $\mathbb{R}$, $\langle s|$ can simply be considered equivalent to $|s\rangle$.

[4] If the n propositions are not independent, the space may have a lower dimension, but we shall avoid unnecessary technical complications here.

B. Geometric Representation of Relative Probability

Let p, q be two generic propositions, associated with the operators P and Q.

Let $\bar{P}$ be the operator associated with the proposition $\bar{p} := p$ *is decidable*.

From equation 1, we have:

$$|q|_p = \frac{|p \wedge q|}{|p|} \tag{16}$$

In terms of operators:

$$|q|_p = \frac{\langle s|PQ|s\rangle}{|p|} \tag{17}$$

which leads to:

$$[p] = |p|_{\bar{p}} = \frac{\langle s|\bar{P}P|s\rangle}{|\bar{p}|} \tag{18}$$

The effect of the operator

$$\hat{\bar{P}} = \frac{\bar{P}}{|\bar{p}|} \tag{19}$$

is nothing more than projecting $|s\rangle$ onto the subspace of $\bar{p}$ and rescaling it to 1. That is, $\hat{\bar{P}}$ represents a rotation and thus an orthonormal change of basis.

The operator:

$$\hat{P} = \frac{\bar{P}P}{|\bar{p}|} \tag{20}$$

is therefore a projector that is not necessarily diagonal.

Extending this approach to n propositions p, q, r, ..., it is possible to construct a geometry embedding relative probability, entirely analogous to what was done for classical probability. However, in this case, the following differences emerge:

- The reference space is $\mathbb{R}^{2^{2n}}$ instead of $\mathbb{R}^{2^n}$, in order to accommodate the $2n$ propositions p, $\bar{p}$, q, $\bar{q}$, r, $\bar{r}$, ...
- The projectors associated with the propositions are not necessarily diagonal nor simultaneously diagonalizable. Thus, in general—as expected from 9—they do not commute.

Equations 19 and 20 are crucial and critical. Different relationships between p and $\bar{p}$ correspond to different rotation angles, meaning that 19 does not define a basis transformation valid for every $|s\rangle$. However, $|p|$ and $|\bar{p}|$ are not measurable within A—by construction, A can only measure:

$$[p] = \frac{|p \wedge \bar{p}|}{|\bar{p}|}$$

Thus, in A, it is observed that, de facto, the numerical relationships among the measured frequencies are mediated, in the geometric representation, by operators rotated by a certain angle α. Moreover, even the value of this angle is inaccessible within A [5]. This angle has a factual role for Alice only when composing multiple propositions. The only values concretely measurable in A are the differences between the angles α_i associated with different propositions.

This geometry in $\mathbb{R}^{2^{2n}}$ is clearly hybrid and impure. It represents probability from the perspective of an observer who has more information than Alice—specifically, a system that includes both A and B. From the perspective of this observer, p is decidable, and they can apply CPT geometry to determine what is observed within A—i.e., the relative probability measured in A.

However, within A, this geometry is not directly usable. Alice will only observe specific subspaces of $\mathbb{R}^{2^{2n}}$, and the operators associated with undecidable propositions will behave *as if* they were ordinary CPT operators, mutually rotated with respect to one another.

C. Relative Probability and Quantum Mechanics

The state of a single proposition p is represented as a direction in a four-dimensional space, $\mathbb{R}^4$. This direction encodes two pieces of information:

1. The classical probability of p.
2. The classical probability of p *is decidable*.

Let α be the angle associated with 2. Once α is fixed, the state vector $|p\rangle$ exists only in the two-dimensional subspace $\mathbb{R}^2$, which is a rotation of the subspace of p by an angle α.

From the perspective of A, this space must then collapse into a complex two-dimensional space, $\mathbb{C}^2$, such that, for a single proposition:

$$|s\rangle = (ae^{i\vartheta}, be^{i\sigma}) \tag{21}$$

with the conditions:

$$\begin{cases} a^2 + b^2 = 1 \\ \vartheta - \sigma = \pm\alpha \end{cases} \tag{22}$$

[5] indeed, a change of basis has no practical effect on the geometric representation

Within this space, the proposition p will be associated with a generic projector, that is, a Hermitian operator with eigenvalues in $(0, 1)$. Denoting by H_p the space associated with p, a generic combination of propositions p, q, r, ... will be embedded in a Hilbert space of the form:

$$H = H_p \otimes H_q \otimes H_r \dots \tag{23}$$

It is thus possible to formulate a geometric representation of relative probability such that:

- The semantic relationships between n propositions—including undecidable ones—are captured by a state vector $|S\rangle$, i.e., a direction, a one-dimensional subspace in $\mathbb{C}^{2^n}$.
- A generic proposition p, formed from a combination of the involved propositions, is associated with an appropriate projector P.
- The probability $[p]$ is given by Born's rule $[p] = \langle s|P|s\rangle$

Or, more directly:

- The fundamental principles of quantum mechanics define the algebra of semantic relationships between propositions, including undecidable ones.

In the geometry constructed by Alice within A, in order to correctly capture the numerical relationships actually observed, there is a clear **loss of information** relative to CPT. This loss corresponds precisely to the onto-epistemic ignorance that necessitated the construction of this framework.

In this geometry, there is no preferred basis in which all operators are diagonal, and the complex values it must handle possess a norm with a clear physical interpretation: they represent actually measured frequencies. However, their phase lacks any direct physical significance and can be considered entirely arbitrary. This phase becomes measurable in A only in the relationships between multiple propositions or measurements—that is, ultimately, in relation to the theorem of conditional probability.

V. CONCLUSIONS

The subtle distinction between *true* and *provable* introduces a concrete difficulty in probability theory. This theory assigns a weight to the possibility or expectation that a given proposition is true, but the moment one carries out an actual verification, it simultaneously becomes provable. The proof of the formula r also proves "*r is provable.*"

This means that, when proceeding constructively and concretely with verifications, one obtains the actual frequencies of the conditioned proposition *p, given p is decidable.*

Quantum mechanics is not a fundamental property of the world; it is a mathematical tool, entirely analogous to classical probability theory. More precisely, quantum mechanics **is** classical probability [see 3] from the perspective of a concrete observer.

Quantum mechanics does not presuppose any ontology, and conversely, constructing extravagant ontologies to account for the peculiarities of the theory—such as many-worlds[33, 34], many-minds[35], spontaneous collapse[36, 37], quantum non-locality, or quantum logic (understood as an alternative logical framework)—is entirely superfluous.

Quantum mechanics is a logical tool that does not violate any principle of classical logic.

One may wonder how Heisenberg was able to develop this framework without, as far as we know, being guided by the notion of *undecidability*. Yet, perhaps this is not so surprising. Heisenberg explicitly abandoned the ontology of electron orbits, physical positions, and metaphysical truth, focusing solely on the algebra of real measurements. Heisenberg, perhaps for the first time in history, conducted physics from Alice's perspective rather than from that of Laplace's demon.

Heisenberg and Bohr repeatedly emphasised that quantum mechanics concerns what is actually measurable, not what is metaphysically real [38–40]; however, the principle of complementarity remains, to me, so obscure and impenetrable that I cannot say to what extent Bohr was referring to something analogous to what has been described here.

Paradoxically, QM does not introduce a **measurement problem**—it resolves one. In a reversal of perspectives, it is actually classical physics that faces a measurement problem, as it describes an abstract, ontological world observed by a deus ex machina, without adequately addressing the fundamental necessity of defining what a real observer within that world can measure and what numerical relationships hold among those measurements. However, the scientific method demands that this part of the problem be addressed with the same depth and rigour—if not more—than the rest.

The violation of Bell's inequalities does not allow for any ontological conclusions, as QM carries no ontological content. The only rational deduction that can be drawn from the empirical confirmation of such violations is that the real world is not universally subject to local determinism—a constraint on possible representations of reality that is so weak that it does not even exclude classical determinism.

Finally, probability theory straddles the boundary between formal mathematical logic and the practical reality of measurement. Long before QM, it already posed foundational problems. In particular, the classical axiomatization of probability holds only in the absence of onto-epistemic ignorance. In the presence of undecidable measurements, in the specific sense described here, CPT fails, and the actual numerical relationships among frequencies observed by a real observer—a strict frequentist—violate the theorems of CPT and instead adhere to the theorems of QM.

This work excludes ontological interpretations of quantum mechanics. Conversely, it engages in a constructive dialogue with non-ontological interpretations, particularly the relational interpretation[41] and the so-called QBism[42].

More specifically, the approach adopted here leads to a direct and measured reading of the relational interpretation—one that, once again, avoids ontological issues such as whether the relationship possesses an ontological status that precedes or follows the ontology of related entities. Instead, the logical lever used here is the rather elementary fact that the scientific method itself requires physics to be a relational theory, given that the outputs constituting its ultimate verification can only arise from an observer–observed relationship.

Moreover, like the relational interpretation, the undecidability-based perspective has the advantage of being symmetric, respecting the first principle of relativity[43]: in general, B will be a quantum object relative to A just as much as A will be relative to B—or, more poetically, the cat is as much both alive and dead from the observer's perspective as the observer is from the cat's, and as both are from Wigner's friend's perspective[44].

The Bayesian interpretation is also clearly connected to this work—not only because of the central role played by the theorem of conditional probability, which underpins Bayes' theorem, but more generally due to the epistemic nature of both approaches.

Once again, the undecidability-based perspective provides a more direct reading of QBism, allowing it to entirely avoid the need to define complex and highly debatable notions such as *agent*, *intentionality*, and *expectation.*

VI. DECLARATIONS

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[1] J. S. Bell, On the einstein podolsky rosen paradox, Physics Physique Fizika **1**, 195 (1964).
[2] J. S. Bell, Bertlmann's socks and the nature of reality, Le Journal De Physique Colloques **42**, 41 (1981).
[3] J. S. Bell, *Speakable and unspeakable in quantum mechanics: Collected papers on quantum philosophy* (Cambridge university press, 2004).
[4] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, Phys. Rev. **47**, 777 (1935).
[5] M. Cirillo, Il teorema di bell raccontato a modo mio (... più o meno ...) (2015).
[6] A. Aspect, P. Grangier, and G. Roger, Experimental tests of realistic local theories via bell's theorem, Phys. Rev. Lett. **47**, 460 (1981).
[7] N. D. Mermin, Could Feynman Have Said This?, Physics Today **57**, 10 (2004), https://pubs.aip.org/physicstoday/article-pdf/57/5/10/8317097/10_1_online.pdf.
[8] S. Hawking, *The Grand Design* (Transworld, 2017).
[9] I. Newton, *Philosophiae naturalis principia mathematica* (J. Societatis Regiae ac Typis J. Streater, 1687).
[10] R. Feynman, R. Leighton, M. Sands, and E. Hafner, *The Feynman Lectures on Physics; Vol. I*, Vol. 33 (AAPT, 1965) p. 750.
[11] R. A. Bertlmann, Real or not real that is the question..., The European Physical Journal H **45**, 205 (2020).
[12] D. Hilbert, *Ricerche sui fondamenti della matematica.* (Bibliopolis, Napoli, 1978).
[13] K. Gödel, *Opere, Volume 1 1929-1936* (Bollati Boringhieri, Torino, 1999).
[14] G. Lolli, *Incompletezza. Saggio su Kurt Gödel.* (Il Mulino, Bologna, 1992).
[15] G. J. Chaitin, *Information-Theoretic Incompleteness*, World Scientific Series in Computer Science, Vol. 10 (World Scientific, Singapore, 1992).
[16] J. Moschovakis, Intuitionistic logic, in *The Stanford Encyclopedia of Philosophy*, edited by E. N. Zalta and U. Nodelman (Metaphysics Research Lab, Stanford University, 2023) Summer 2023 ed.
[17] P. A. Benioff, Models of zermelo-fraenkel set theory as carriers for the mathematics of physics. i, Journal of Mathematical Physics **17**, 618 (1976).
[18] J. Wheeler, Add "participant" to "undecidable propositions" to arrive at physics, https://jawarchive.files.wordpress.com/2012/03/twa-1974.pdf (1974).
[19] G. Birkhoff and J. V. Neumann, The logic of quantum mechanics, Annals of Mathematics **37**, 823 (1936).
[20] S.-J. Shin, Peirce's Deductive Logic, `https://plato.stanford.edu/archives/sum2022/entries/peirce-logic/` (2022).
[21] L. A. Zadeh, Fuzzy sets, Information and Control **8**, 338 (1965).
[22] E. Schrodinger, Die gegenwartige situation in der quantenmechanik, Naturwiss. **23**, 807 (1935).
[23] S. Gottwald, Many-Valued Logic, `https://plato.stanford.edu/archives/sum2022/entries/logic-manyvalued/` (2022).
[24] Č. Brukner, Quantum complementarity and logical indeterminacy, Natural Computing **8**, 449 (2009).
[25] J. Szangolies, Epistemic horizons and the foundations of quantum mechanics, Foundations of Physics **48**, 1669– (2018).
[26] K. Landsman, Indeterminism and undecidability (2021).
[27] M. Poletti, On the strangeness of quantum mechanics, Foundations of Physics **52**, 1 (2022).
[28] M. Poletti, On the strangeness of quantum probabilities, Quantum Studies: Mathematics and Foundations **10**, 343 (2023).

[29] M. Poletti, Quanta iff discreteness, International Journal of Quantum Foundations **9**, 174 (2023).

[30] M. Poletti, Bertlmann's socks from a viennese perspective (2023), arXiv:2308.03341 [physics.hist-ph].

[31] P. Laplace, *Essai philosophique sur les probabilités* (Bachelier, Paris, 1840).

[32] B. P. Abbott, R. Abbott, T. D. Abbott, M. R. Abernathy, and et al. (LIGO Scientific Collaboration and Virgo Collaboration), Observation of gravitational waves from a binary black hole merger, Phys. Rev. Lett. **116**, 061102 (2016).

[33] H. Everett III, "relative state" formulation of quantum mechanics, Reviews of Modern Physics **29.3**, https://doi.org/10.1103/RevModPhys.29.454 (1957).

[34] L. Vaidman, Many-Worlds Interpretation of Quantum Mechanics, `https://plato.stanford.edu/archives/fall2021/entries/qm-manyworlds/` (2021).

[35] D. Albert and B. Loewer, Interpreting the many-worlds interpretation, Synthese **77**, 195 (1988).

[36] G. C. Ghirardi, *Un'occhiata alle carte di Dio: Gli interrogativi che la scienza moderna pone all'uomo* (Il Saggiatore, Milano, 1997).

[37] R. Penrose, *La strada che porta alla realtà. Le leggi fondamentali dell'universo* (Rizzoli, Milano, 2018).

[38] W. Heisenberg, *Lo sfondo filosofico della fisica moderna* (Sellerio, Palermo, 1999).

[39] F. Laudisa, *La realtà al tempo dei quanti. Einstein, Bohr e la nuova immagine del mondo* (Bollati Boringhieri, Torino, 2019).

[40] F. Laudisa, *Dalla fisica alla filosofia naturale* (Bollati Boringhieri, Torino, 2023).

[41] C. Rovelli, Relational quantum mechanics, International Journal of Theoretical Physics , 1637 (1996).

[42] C. G. Timpson, Quantum bayesianism: A study (postscript), Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics **39**, 579 (2008).

[43] A. Einstein, *Le due Relatività* (Bollati Boringhieri, Torino, 2015).

[44] E. P. Wigner, Remarks on the mind-body question, Philosophical Reflections and Syntheses , 247 (1995).

[45] J. VonNeumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, 1932).