Global Gauge Symmetries and Spatial Asymptotic Boundary Conditions in Yang-Mills theory

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February 25, 2025

Abstract

In Yang-Mills gauge theory on a Euclidean Cauchy surface the group of gauge symmetries carrying direct empirical significance is often believed to be $\mathcal{G}_{\mathrm{DES}} = \mathcal{G}^{\mathrm{I}}/\mathcal{G}^{\infty}_{0}$, where $\mathcal{G}^{\mathrm{I}}$ is the group of boundary-preserving gauge symmetries and $\mathcal{G}^{\infty}_{0}$ is its subgroup of transformations that are generated by the constraints of the theory. These groups are identified respectively as the gauge transformations that become constant asymptotically and those that become the identity asymptotically. In the Abelian case $\mathsf{G} = \mathsf{U}(1)$ the quotient is then identified as the group of global gauge symmetries, i.e. $\mathsf{U}(1)$ itself. However, known derivations of this claim are imprecise, both mathematically and conceptually. We derive the physical gauge group rigorously for both Abelian and non-Abelian gauge theory. Our main new point is that the requirement to restrict to $\mathcal{G}^{\mathrm{I}}$ does not follow from finiteness of energy only, but from the requirement that the Lagrangian of Yang-Mills theory be defined on a tangent bundle to configuration space. Moreover, we explain why the quotient consists precisely of a copy of the global gauge group for every homotopy class, even if the various gauge transformations apparently have different asymptotic rates of convergence. Lastly, we consider Yang-Mills-Higgs theory in our framework and show that asymptotic boundary conditions differ in the unbroken and broken phases.[1]

1 Introduction

The physical status of gauge symmetries is a central topic in contemporary physics, both in Yang-Mills theory and general relativity. The term "gauge" is sometimes used as a synonym for "unphysical" or "empirically insignificant," but gauge transformations can acquire a physical meaning in the presence of boundaries. A well-known example is the Josephson current flowing between two superconductors that are brought close together [2]. This current depends only on the relative difference between the *global* $\mathsf{U}(1)$ phases of the superconductors' Ginzburg-Landau order parameters, suggesting that global gauge symmetries are physical. Similarly, some gauge symmetries are physical on *asymptotic* boundaries. For instance, the asymptotic

[1]This article grew out of the master thesis of the corresponding author, supervised by the other author and by Sebastian de Haro [1].

symmetry group of gravity in asymptotically flat spacetimes is the well-known BMS group [3–5] and asymptotic symmetries of Yang-Mills fields on both the null and spatial conformal boundaries of Minkowski spacetime are studied in the context of celestial holography, see e.g. [6–9]. The general idea is that the asymptotic symmetry group consists of all "allowed symmetries" quotiented by all "trivial symmetries" [7]. Here "allowed" means those symmetries that respect the boundary conditions of the system and "trivial" means those symmetries that have no physical effect on the system. In this article we will identify the trivial symmetries as the gauge transformations that are generated by the Hamiltonian constraints of the theory.

Our aim is to rigorously derive the quotient of boundary-preserving gauge symmetries by trivial gauge symmetries for the specific case of Yang-Mills and Yang-Mills-Higgs theory on a Cauchy surface isomorphic to $\mathbb{R}^3$. Our motivation to do so comes from the desire to understand the physical content of the Higgs mechanism [10–13], which is thought to happen at a particular instant in time during the electroweak phase transition. This means that we introduce a 3+1 split of spacetime into $\Sigma \times \mathbb{R}$ and discuss *instantaneous* spatial asymptotic symmetries, for which the time t is held fixed and the radial coordinate r on Σ is taken to infinity. This means that we do not consider the asymptotic symmetry group of full spatial infinity of Minkowski spacetime.[2] It is sometimes said that asymptotic analyses are more of an art than a science [7, p. 34], but for the specific case of Yang-Mills theory on a Euclidean Cauchy surface we will present a fairly algorithmic and unambiguous method for deriving the physical gauge group. It is probable that this method can be extended at least to Yang-Mills theory on Cauchy surfaces in other spacetimes than Minkowski, and perhaps also to the gravitational field itself.

The case of Maxwell theory, possibly with a Higgs field, on Euclidean space has been studied extensively in the foundations of physics community, see e.g. [14–25]. The terminology used there to describe physical and trivial gauge symmetries respectively is that of *direct empirical significance* (DES) and *redundant* gauge transformations [18]. We will stick to this terminology. Redundant gauge transformations are contrasted with *formal* gauge transformations, which make up the full infinite-dimensional gauge group $\mathcal{G}$ without any regard for their physical status. For pure electromagnetism on Σ with spatial asymptotic boundary conditions, the group of gauge symmetries carrying DES has been identified as the asymptotic symmetry group

$$\mathcal{G}_{\mathrm{DES}} = \mathcal{G}^{\mathrm{I}}/\mathcal{G}_0^{\infty},$$

where $\mathcal{G}^{\mathrm{I}}$ denotes the subgroup of the formal gauge group $\mathcal{G}$ whose elements leave asymptotic boundary conditions invariant[3] (the "allowed" symmetries), and $\mathcal{G}_0^{\infty}$ is the subgroup of redundant gauge transformations that are generated by the primary first-class constraints of the theory (the "trivial" symmetries). Here the ∞-superscript stands for the trivial action of these transformations at infinity[4] and the subscript 0 denotes the identity component of $\mathcal{G}^{\infty}$. The identification of redundant gauge symmetries as the ones generated by the primary[5] first-class[6] constraints is based on the Dirac-Bergmann theory of constraints, in which one takes Poisson brackets of the primary first-class constraints with the fields of the theory to generate gauge transformations. For details see e.g. [26–29].

[2] Spatial infinity understood as the timelike boundary at which spacelike geodesics end connects the infinite past with the infinite future, and is therefore itself infinitely long in time and not instantaneous.

[3] Hence the notation I, which will be used throughout to denote classes of maps that leave the asymptotic boundary conditions invariant, i.e. which are constant at infinity (except in the broken phase of the Yang-Mills-Higgs theory, where boundary-preserving transformations must actually vanish at infinity, see Section 5).

[4] Throughout this article we use the *subscript* ∞ to denote certain conditions (usually the vanishing of classes of maps) at asymptotic infinity, which is not to be confused with the superscript denoting infinite differentiability (smoothness). Only for $\mathcal{G}_0^{\infty}$ have we used ∞ as a superscript since there we already have the 0 and there is no danger of confusion.

[5] Primary constraints are constraints that are obtained without using the equations of motion.

[6] First-class constraints are constraints whose Poisson bracket with any other constraint vanishes.

For electromagnetism (with structure group $U(1)$) on three-dimensional space Σ, the group $\mathcal{G}^I$ is identified as consisting of those gauge transformations $g\colon \Sigma \to U(1)$ that become asymptotically constant [18,20]. Furthermore, the subgroup $\mathcal{G}_0^\infty$ is identified as the one generated by the Gauss law constraint, consisting of all transformations $g\colon \Sigma \to U(1)$ that asymptotically approach the identity [18,30]. The quotient is then said to be isomorphic to $U(1)$ itself, i.e. the group of global (or rigid) gauge symmetries [17,18,20].

However, the derivations supporting these results are at the least shaky and sometimes simply incorrect. The common lore is that one must impose asymptotic fall-off boundary conditions on gauge fields, e.g.

$$A_i \to 0 + \mathcal{O}(r^{-2}), \quad i = 1,2,3,$$

to ensure finiteness of energy and/or action, and that the gauge group must preserve these conditions [17,31]. But this argument is problematic, since energy and action only depend on gauge-invariant quantities (the field strength tensor). Thus there is no need to require gauge fields to become zero asymptotically, but only that they become pure gauge. Any gauge transformation preserves this condition, so it would seem naively that one can always allow the full gauge group $\mathcal{G}$, instead of restricting to $\mathcal{G}^I$.[7] If true, this would greatly enlarge the group $\mathcal{G}_{\text{DES}}$, well beyond the group of global (rigid) gauge transformations. The aim of this article is to explain why we can in fact still only allow the subgroup $\mathcal{G}^I$, although this does not follow from finiteness of energy only but also from the requirement that *the Lagrangian be defined on the tangent bundle to configuration space*. For finite-dimensional systems this latter requirement is the foundation for proving the equivalence of the Euler-Lagrange equations and stationarity of the action in variational principles, see e.g. [32, Chapter 8] or [33, Chapter 19]. But gauge field theories are infinite-dimensional systems with infinite-dimensional symmetry groups, resulting in the added difficulty that the Lagrangian is degenerate (exhibits constraints) [26,34]. In that case, not all vectors in the tangent bundle to configuration space admit solutions to the Euler-Lagrange equations of which they are the initial datum [34, Section 6.4]. Still, the constraints are found in the first place through the Legendre transform $\mathbb{L} : TQ \to T^*Q$ from the tangent bundle to the cotangent bundle of the configuration space Q. The constraint surface $\mathcal{C}$ is the image $\mathbb{L}(TQ)$ of the tangent bundle under the Legendre transform [26,34]. Thus, even in gauge field theories, one always starts with a Lagrangian defined on the tangent bundle to configuration space.

Besides the problem of finding $\mathcal{G}^I$, there is further obscurity in the literature when $\mathcal{G}_{\text{DES}}$ is identified with the group of global gauge symmetries. This pertains to the question of the appropriate rate at which transformations $g \in \mathcal{G}^I$ must become constant asymptotically, and the rate at which elements $g \in \mathcal{G}_0^\infty$ must become the identity. It is only when these rates are exactly equal that we can conclude that the quotient of these two subgroups of $\mathcal{G}$ is isomorphic to $U(1)$ (in the Abelian case). However, in the usual approach it is not obvious that these rates are the same. To see this, note that, in 3-dimensional space, the electric field must vanish asymptotically with order $\mathcal{O}(r^{-3/2-\epsilon})$ to guarantee that it is square-integrable,[8] where $\epsilon > 0$ is any (small) number. As we will explain later, this same rate is needed for the gauge field itself. It is then concluded that gauge transformations $g\colon \Sigma \to G$ must become constant asymptotically to preserve this boundary condition. But at what rate? In the Abelian case, we would need the gauge parameter $\lambda\colon \Sigma \to \mathbb{R}$ to be such that its derivative $\partial_i \lambda$ becomes constant with order $\mathcal{O}(r^{-3/2-\epsilon})$,

[7] There is another way to formulate this critique: the very statement $A_i \to 0$ is made in a specific gauge. What we call "zero" is therefore gauge-dependent. Thus, the fact that this asymptotic boundary condition is not preserved by most gauge transformations is not surprising - it is a consequence of our working in a gauge.

[8] Square-integrability is required because the energy carried by the electric field is the integral of the square of its norm, and this energy is required to be finite.

if it is to be boundary-preserving. But what does this imply for λ itself? It is not obvious that we can simply conclude that $\lambda \to 0 + \mathcal{O}(r^{-1/2-\epsilon})$, i.e. that λ falls off with one power of r fewer. Indeed, there are examples of functions which themselves vanish in a certain limit but whose derivative behaves very badly. Besides, the choice of asymptotic behavior of the fields has a great effect on what transformations $\mathcal{G}^I$ contains precisely, as already noted in [17, 18].

Similar issues arise when considering the order of asymptotic behavior for transformations $g \in \mathcal{G}_0^\infty$. In fact, in the argument by Balachandran [30], which formed the basis for Teh's derivation [18], the requirement that $g \to 1$ asymptotically is based on the need for a certain boundary term to vanish in the calculation of a specific Poisson bracket. But this boundary term contains the electric field, and so its vanishing could also be guaranteed simply by requiring rapid enough asymptotic fall-off of the electric field, such that gauge transformations do not need to go to the identity to make sure this Poisson bracket exists. We will run into this issue again in Section 4.1. At any rate, it is clear that quite a lot of fine-tuning of asymptotic behavior is needed to ensure that, in the end, the quotient $\mathcal{G}_{\text{DES}} = \mathcal{G}^I/\mathcal{G}_0^\infty$ corresponds precisely to the group of global gauge transformations. This arbitrariness is highly unsatisfactory.

These ambiguities contrast sharply with other characterizations of the special status of global gauge symmetries, from which it is obvious that it is precisely the global gauge group that stands apart from other gauge transformations. We mention three such characterizations.

Firstly, in the formalization of gauge theories using fiber bundles, connections live on a principal G-bundle $P \to \Sigma$. Gauge transformations correspond to bundle automorphisms $P \to P$. But in the Abelian case, there is clearly a special class of gauge transformations, namely the ones that are given by the global action $G\colon P \to P$, which forms part of the very definition of a principal bundle.[9] A connection on P is a choice of horizontal subspace at every point $p \in P$. Since the global action of G on P is, by definition, perfectly vertical, it is not felt by the connections.

Secondly, but relatedly, in the symplectic formulation of gauge theories, the global gauge group appears precisely as the obstruction to the possibility of a smooth symplectic reduction. To see this, recall that any Hamiltonian group action on a symplectic manifold can be used to define a momentum map (Definition 4.1) such that, if the group acts freely[10] and properly[11] on the zero set of this momentum map, one can take a symplectic quotient [32–34]. However, since global gauge transformations can be viewed as the constant maps $g\colon \Sigma \to G$, they do not act freely. In the Abelian case, a connection A transforms as

$$A \to A + g^{-1}dg,$$

so if g is constant then dg is zero, and *any* connection will be a fixed point of the global gauge group action. This prevents the possibility of a smooth symplectic reduction. The symplectic quotient will instead be a stratified space. In the non-Abelian case even constant gauge transformations act by conjugation $g^{-1}Ag$, because non-Abelian gauge bosons are charged under the force they themselves carry, but then the central global gauge transformations still do not act freely.[12]

[9] In the non-Abelian case the action $g\colon P \to P$ defined by $g(p) = pg$ does not necessarily define a bundle automorphism, since $g(ph) = phg$, which is not necessarily the same as $g(p)h = pgh$, as g and h need not commute. Yet the central elements of G do define a bundle automorphism this way.

[10] The action of a group H on a set X is called free if $h \cdot x = x$ for some $x \in X$ implies that g is the identity.

[11] The action of a topological group H (such as a Lie group) acting by homeomorphisms on a topological space X (such as a manifold) is called proper if the map $H \times X \to X \times X$ is proper. A map between topological spaces is called proper if the inverse image of a compact set is itself compact.

[12] Another possibility would be to consider the group $\mathcal{G}_*$ of *pointed* gauge transformations, i.e. those transformations that are the identity at some arbitary fixed point $x_0 \in \Sigma$. Then the only global transformation is the trivial one and the action of $\mathcal{G}_*$ is free, so that the symplectic reduction is a smooth space. This approach is pursued in [35]. We could also

Thirdly, Gomes has identified the global gauge group in electromagnetism as the one carrying empirical significance [21–23, 37]. This is achieved by means of horizontal symplectic geometry, in which the dressing

$$h[\mathbf{A}] = \int_\Sigma \frac{dy^3}{4\pi} \frac{\partial^i A_i}{|\mathbf{x} - \mathbf{y}|}$$

singles out the gauge-invariant component of the gauge field **A** on 3-dimensional space Σ. This dressing corresponds to a projection onto the Coulomb gauge and is insensible precisely to the global gauge transformations, as these do not change **A**. Clearly this is related to the previous point: the common idea is that (central) global gauge transformations do not change the gauge field, whereas these do change the global phase of matter fields.[13]

Thus, we arrive at the central goal of this article: unifying the various approaches to deriving precisely the global gauge group as the one carrying DES, by carefully considering the configuration space of Yang-Mills fields and their spatial asymptotic boundary conditions. Our approach is as follows. We first construct the configuration space of gauge fields in Section 2, without working in a particular gauge. In Section 3 we then use this construction to define boundary conditions that are necessary to ensure finiteness of energy, and we examine their consequence for the structure of the configuration space of gauge fields. Subsequently, we find the redundant gauge symmetries, i.e. those generated by the Gauss law constraint, in Section 4, finally giving us the quotient of transformations with DES. Lastly, we study what happens when a Higgs field is added in Section 5, in which case we find different boundary conditions for the unbroken and broken phases.

2 The configuration space of gauge fields

In this Section we explain what the configuration space of Yang-Mills theories, on whose tangent bundle the Yang-Mills Lagrangian is defined, looks like. We do this without working in a particular gauge. This is of paramount importance for conceptual clarity because, if we impose boundary conditions such as $A_i \to 0$, then we are already working in a specific gauge. It is therefore not surprising that gauge transformations change this boundary condition. However, it is not clear whether this violation of the boundary condition is really problematic or just an artifact of our choice to work in a specific gauge, and we should avoid this ambiguity.

The results of this Section are a necessary prerequisite for understanding the main point of Section 3: that the need to restrict the gauge group to $\mathcal{G}^I$, i.e. the subgroup of transformations that leave the boundary conditions invariant, comes not directly from the boundary conditions themselves, but rather from the requirement that the domain of the Lagrangian be a tangent bundle.

Throughout this article we assume a 3+1 split of flat spacetime into $\Sigma \times \mathbb{R}$, where $\Sigma \cong \mathbb{R}^3$, and work in the temporal gauge, thus setting $A_0 = 0$. This means that we do not consider gauge transformations in the temporal component of the gauge field, but only in its spatial components. We do this because we are ultimately interested in understanding the breaking of spatial gauge transformations in the Higgs mechanism. For details on the relation between such a 3+1 split and covariant formulations of Yang-Mills theory, we refer the reader to Section 8.3 of [34] and to [40, 41].

consider so-called *irreducible connections*, i.e. connections for which the holonomy group acts irreducibly. The gauge group does act freely on the space of irreducible connections [36].

[13] For this reason they are used in so-called 't Hooft beam splitter [38] constructions, see e.g. [39].

We consider a principal G-bundle $P \to \Sigma$, where the structure group G is some compact matrix Lie group such as $U(1)$ or $SU(N)$, with Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$. The structure group should not be confused with the gauge group $\mathcal{G} = \mathrm{Aut}(P)$ of all gauge transformations. A gauge field in Yang-Mills theory is a connection on this bundle P, i.e. a choice of horizontal distribution in the tangent bundle TP. Equivalently a gauge field can be viewed as a Lie algebra-valued 1-form on P, i.e. an element $A \in \Omega^1(P, \mathfrak{g})$, that is both G-equivariant and reproduces the Lie algebra generators of the fundamental vector fields[14] [42]. G-equivariance means that $r_h^* \circ A = \mathrm{Ad}_{h^{-1}} \circ A$ for all $h \in G$, where $\mathrm{Ad} : G \to GL(\mathfrak{g})$ denotes the adjoint representation[15] and $r_h^* \colon \mathfrak{g} \to \mathfrak{g}$ the pullback of the right multiplication $r_h \colon G \to G$ by $h \in G$. Such a connection 1-form A can be pulled down to Σ if we have a gauge, i.e. a section $s \colon \Sigma \to P$, in which case it is acted upon by the gauge group $\mathcal{G}$ in the usual way:

$$s^* \tilde{g} A = \tilde{g}^{-1} s^* A \tilde{g} + \tilde{g}^{-1} d\tilde{g}, \qquad \tilde{g} \in C^\infty(\Sigma, G).$$

Here $s^* \colon \Omega^1(P, \mathfrak{g}) \to \Omega^1(\Sigma, \mathfrak{g})$ denotes the pullback of the gauge s, and we have used the isomorphism $\mathcal{G} = \mathrm{Aut}(P) \cong C^\infty(\Sigma, G)$, which sends $g \mapsto \tilde{g}$, induced by s.[16] Henceforth we drop the $\sim$, meaning that we freely switch between the gauge-invariant definition $\mathcal{G} = \mathrm{Aut}(P)$ and the gauge-dependent definition $\mathcal{G} = C^\infty(\Sigma, G)$.

If we write $\mathrm{Conn}(P)$ for the space of all connection 1-forms on P, then the space of "coordinates" and "velocities" of Yang-Mills theory naively consists of the tangent bundle $T\mathrm{Conn}(P)$ to $\mathrm{Conn}(P)$. However, as we will see in Section 3, asymptotic boundary conditions are required on the tangent vectors (electric fields) in this tangent bundle, thereby complicating the construction. But (asymptotic) boundary conditions are usually imposed on fields on the space Σ and not on the bundle P, so we need to bring down our fields to Σ. We could do this by working in a gauge as above, but we have just explained that it is vital to work gauge-invariantly. Fortunately it is also possible to work gauge-invariantly on Σ, using the following definitions and results.

A k-form $\omega \in \Omega^k(P, \mathfrak{g})$ is called *horizontal* if it vanishes whenever at least one vector it eats is vertical, i.e. if for all $p \in P$ we have $\omega_p(X_1, ..., X_k) = 0$ whenever $X_i \in V_p P = \ker(\pi_*)$ for some $1 \leq i \leq k$. Here $V_p P = \ker(\pi_*)$ denotes the space of vertical vectors at the point p, which should be thought of as the vectors that lie along the fibers (which are isomorphic to G) of P. Furthermore, we say a k-form is *of type Ad* if $r_h^* \circ \omega = \mathrm{Ad}_{h^{-1}} \circ \omega$ for any $h \in G$. We denote the set of horizontal k-forms of type Ad by $\Omega^k_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$. We have the following result [42].

Proposition 2.1. Let $P \to \Sigma$ be a principal G-bundle. If $A, A' \in \Omega^1(P, \mathfrak{g})$ are two connection 1-forms then $A - A' \in \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$ and for any $\omega \in \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$ we have that $A + \omega$ is a connection 1-form. For the curvature we have $F(A) \in \Omega^2_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$.

In other words: differences of connections as well as curvatures are horizontal forms of type Ad. This is extremely useful because of the following well-known theorem [42].

Theorem 2.2. Let $\pi \colon P \to \Sigma$ be a principal G-bundle. Then $\Omega^k_{\mathrm{hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$ and $\Omega^k(\Sigma, \mathrm{Ad}(P))$ are canonically isomorphic as vector spaces through the pullback π^*.[17]

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[14] That is: $A(X_\xi) = \xi$ for all $\xi \in \mathfrak{g}$, where X_ξ denotes the fundamental vector field in $\mathfrak{X}(P)$ generated by ξ through the right action of G on P.

[15] Defined by $\mathrm{Ad}_h(X) = hXh^{-1}$, where $h \in G, X \in \mathfrak{g}$.

[16] The isomorphism between the two groups is as follows. If we have a G-valued map $g \colon \Sigma \to G$, then we can produce a bundle automorphism $f \colon P \to P$ using the section $s \colon \Sigma \to P$. We simply define $f(p) = p \cdot s(\pi(p))$.

[17] Recall that for a fiber bundle $E \to N$ any map $f \colon M \to N$ induces a pullback bundle $f^*E \to M$. In this case the pullback (of the adjoint bundle) is the trivial vector bundle $P \times \mathfrak{g}$.

Here $\mathrm{Ad}(P)$ denotes the adjoint bundle.[18] Thus, if we choose a basis connection A_{ref}, we can view the space of connection 1-forms $\mathrm{Conn}(P)$ as the vector space $\Omega^1(\Sigma, \mathrm{Ad}(P))$. In other words: we can view differences of connections as well as curvatures as forms on Σ instead of on P *in a gauge-invariant manner*, as long as we remember that it is in reference to the basis connection A_{ref}. For an Abelian structure group the adjoint bundle $\mathrm{Ad}(P)$ is even trivial, i.e. just $\mathrm{Ad}(P) = \Sigma \times \mathfrak{g}$, so that the space of connections becomes simply $\Omega^1(\Sigma, \mathfrak{g})$.

Now, we know what the tangent space to a vector space looks like: it is just isomorphic to the original vector space. This allows us to obtain the tangent bundle to the space of connections. We find that the enigmatic space $T\mathrm{Conn}(P)$ is just $T\Omega^1(\Sigma, \mathrm{Ad}(P)) \cong \Omega^1(\Sigma, \mathrm{Ad}(P)) \times \Omega^1(\Sigma, \mathrm{Ad}(P))$. In electromagnetism $\mathfrak{g} = i\mathbb{R}$, so that $T\mathrm{Conn}(P)$ reduces to $\Omega^1(\Sigma) \times \Omega^1(\Sigma)$.

3 Asymptotic boundary conditions and the gauge group

This far we have not considered any boundary conditions on the connection 1-forms or the tangent vectors in $T\Omega^1(\Sigma, \mathrm{Ad}(P))$, nor on the curvature, even though this is essential for ensuring finiteness of the Lagrangian and/or Hamiltonian and/or action. The Lagrangian is an integral over $\Sigma \cong \mathbb{R}^3$, so terms that appear in it must fall off asymptotically with order at least $\mathcal{O}(r^{-3-\epsilon})$ in order to make this integral well-defined. Let us now see what these terms are and what imposing boundary conditions implies for the group of gauge transformations $\mathcal{G}$.

Our goal is to identify a subspace $Q \subset \Omega^1(\Sigma, \mathrm{Ad}(P))$ of the space of all gauge fields which is such that its tangent bundle TQ consists only of fields that satisfy the asymptotic boundary conditions dictated by the Lagrangian. To know what these boundary conditions are, we need to find the Lagrangian of Yang-Mills theory on Σ, which is a map $\mathcal{L}\colon TQ \to \mathbb{R}$. Elements of TQ consist of pairs $(A, \alpha) \in Q \times T_A Q$ of gauge fields and tangent vectors. We think of α as the electric field, but it is entirely independent of A as long as we do not impose the equations of motion, which is why we have chosen not to use the symbol E. The tangent vectors α_A are the "velocities" at the "coordinate" A. The Lagrangian of Yang-Mills theory in temporal gauge is then [43]

$$\mathcal{L}(A, \alpha) = \frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|F(A)\|^2 . \tag{1}$$

Here $F(A)$ denotes the curvature 2-form of the connection 1-form A, which is the magnetic field, and $\|\cdot\|$ is the usual norm on forms:

$$\|\omega\|^2 = \int_\Sigma \mathrm{Tr}(\omega \wedge *\omega),$$

where $*$ denotes the Hodge star operator. We can derive this expression for the Lagrangian from the usual covariant action on spacetime $M = \Sigma \times \mathbb{R}$:

$$\mathcal{S}(\tilde{A}) = -\frac{1}{2}\int_M \mathrm{Tr}\, F(\tilde{A}) \wedge *F(\tilde{A}) = -\frac{1}{2}\int_{\mathbb{R}}\int_\Sigma \mathrm{Tr}\, F(\tilde{A}) \wedge *F(\tilde{A}).$$

Here we have written $\tilde{A}$ to stress that this is a gauge field on spacetime M instead of on space Σ. Denoting coordinates on Σ by x^i and the coordinate on $\mathbb{R}$ by $t = x^0$, it is not difficult to show

[18] The adjoint bundle is the associated real vector bundle $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g}$ constructed through the adjoint representation $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$. Here the product $P \times_\rho \mathfrak{g}$ signifies that we quotient $P \times \mathfrak{g}$ by the equivalence relation $(p, X) \sim (ph, \mathrm{Ad}_{h^{-1}}(X))$ for $h \in G$.

that the action in coordinates becomes the usual [42]

$$\mathcal{S}(A_\mu) = -\frac{1}{4}\int_{\mathbb{R}} dt \int_\Sigma d^3x \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\int_{\mathbb{R}} dt \int_\Sigma d^3x \mathrm{Tr}\, \left(2F_{0i}F^{0i} + F_{ij}F^{ij}\right),$$

where $\mu = 0, 1, 2, 3$, $i = 1, 2, 3$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the antisymmetric field strength tensor (which clearly satisfies $F_{00} = 0$). The term Tr $F_{0i}F^{0i}$ is (minus) the energy of the electric field (the "kinetic" energy) and the term Tr $F_{ij}F^{ij}$ twice the energy of the magnetic field (the "potential" energy).

If we now impose temporal gauge $A_0 = 0$ we obtain $F_{0i} = \partial_0 A_i = \dot{A}_i$. We can then rewrite the action as

$$\mathcal{S}(A_\mu) = \frac{1}{2}\int_{\mathbb{R}} dt \int_\Sigma d^3x \mathrm{Tr}\, \left(-\dot{A}_i\dot{A}^i - F_{ij}F^{ij}\right) = \int_{\mathbb{R}} dt \mathcal{L}(A_i, \dot{A}_i).$$

But F_{ij} is just the curvature of the connection A_i on three-dimensional space Σ, so in coordinate-free notation we find[19]

$$\mathcal{S}(A_\mu) = \frac{1}{2}\int_{\mathbb{R}} dt \int_\Sigma \mathrm{Tr}\, \left(\dot{A} \wedge *\dot{A} - F(A) \wedge *F(A)\right) = \frac{1}{2}\int_{\mathbb{R}} dt \left(\|\dot{A}\|^2 - \|F(A)\|^2\right),$$

where it is understood that $A \in \Omega^1(\Sigma, \mathrm{Ad}(P))$ should now be viewed as only the spatial part of A_μ. Writing $\alpha = \dot{A}$ and realizing that these velocities are tangent vectors we obtain the Lagrangian in Eq. 1.

Now, we want the Lagrangian to be well-defined as an integral over Σ, and so we require both $\|\alpha\|$ and $\|F(A)\|$ to be separately finite, anticipating also that the energy is the sum of these. As these norms are just integrals over 3-dimensional space, square-integrability requires that α and $F(A)$ fall-off sufficiently quickly towards spatial asymptotic infinity. It is enough to require that they approach 0 asymptotically with order $r^{-3/2-\epsilon}$, where $\epsilon > 0$ is a small number. Thus, we find two asymptotic boundary conditions:

(i) $\alpha \to 0 + \mathcal{O}(r^{-3/2-\epsilon})$ as $r \to \infty$;
(ii) $F(A) \to 0 + \mathcal{O}(r^{-3/2-\epsilon})$ as $r \to \infty$.

That is: the gauge field A must become flat at asymptotic infinity sufficiently quickly and the tangent vector "electric field" α must vanish at infinity. We note that there is NO requirement for the gauge field itself to vanish at infinity, since it does not appear in the Lagrangian directly. It only needs to become flat [20]. This raises the question: do the above boundary conditions produce a proper tangent bundle TQ? That is: if we take Q to consist of those connections that become flat asymptotically at the right rate, will its tangent space $T_A Q$ at a point $A \in Q$ then consist precisely of those α that approach zero asymptotically at that same rate? The answer is no. To see this, we consider the space of flat connections at infinity and examine its tangent space. It should consist of the zero vector only, since we require α to vanish at infinity. In other words: the tangent space at infinity should be 0-dimensional, which in turn implies that the space of flat connections at infinity should be 0-dimensional, i.e. a discrete space. For simplicity we take it to consist of a single point, i.e. some *fixed* boundary choice of flat connection at infinity. This is why gauge transformations must become constant at infinity: they must leave this fixed, flat connection invariant.

Let us be more precise about this. As explained in Sections 8.4 and 8.5 of [34], the conformal invariance of Yang-Mills theory allows us to make use of a conformal embedding of Minkowski

[19] We use $(-,+,+,+)$ signature for the metric, which explains the minus sign in $-\dot{A}_i\dot{A}^i$.

spacetime (M, η) into a Lorentzian manifold $(\hat{M}, \hat{\eta})$ with compact Cauchy surfaces. Such an embedding is map $f\colon (M, \eta) \to (\hat{M}, \hat{\eta})$ which sends M into the interior of $\hat{M}$ and which is such that $f^*\hat{\eta} = K^2\eta$ for some positive function K.[20] The Cauchy surface $\Sigma \cong \mathbb{R}^3$ is then mapped into the interior of a compact space $\hat{\Sigma}$ with boundary $\partial\hat{\Sigma} \cong S^2$ (the celestial sphere of directions at infinity), so we can view asymptotic infinity of Σ as S^2. This conformal embedding is very useful for making precise statements about the asymptotic behavior of fields as well as gauge transformations.

It can straightforwardly be seen that the space of flat connections on $\partial\hat{\Sigma}$ is large enough to have a tangent space which is not zero-dimensional. Indeed, if we choose some flat connection $\hat{A}_\infty$ on $\partial\hat{\Sigma}$, then any connection obtained from this connection by a gauge transformation at infinity, i.e. by an element of $\mathrm{Aut}(\hat{P}_{\partial\hat{\Sigma}})$, will also yield a flat connection. After all, gauge transformations do not change the flatness of a connection. Thus, the space of flat connections at infinity contains at least the orbit of $\hat{A}_\infty$ under the action of the gauge group at infinity. The tangent space to this orbit is $C^\infty(\partial\hat{\Sigma}, \mathfrak{g})$, i.e. the $\mathfrak{g}$-valued maps on S^2. But this tangent space is clearly not 0-dimensional.

Thus we see that we must restrict Q to those connections that, through the conformal embedding, approach some fixed choice of flat connection on the asymptotic boundary, denoted $\hat{A}_\infty$. Then the space of connections at infinity will be 0-dimensional (consisting only of this one fixed connection) and its tangent space too. However, such a choice of a fixed flat connection at infinity obviously *breaks gauge invariance*, but in a trivial sense: any gauge transformation that is not constant at infinity will change $\hat{A}_\infty$. In the Abelian case, the group of gauge transformations that do preserve this fixed choice of flat connection consists precisely of all transformations that are constant at infinity. In the non-Abelian case one has to take into account the fact that even constant transformations may change the fixed flat connection at infinity by means of a conjugation. The orbit of a flat connection under the conjugation action of the group of constant gauge transformations is itself a smooth manifold (whose dimension depends on G), with tangent vectors which are nonzero unless the connection takes a value that is invariant under $\mathrm{Ad}(G)$. Intuitively, this corresponds to the fact that non-Abelian gauge fields carry currents even in the absence of matter fields. Avoiding such currents with infinite energy at infinity forces us to pick a connection which is invariant under $\mathrm{Ad}(G)$, e.g. zero. Then the asymptotically constant gauge group will still leave this boundary choice invariant. This choice of picking the zero connection at infinity so that the full asymptotically constant gauge group is allowed, rather than allowing for any flat connection but only the central constant transformations, harmonizes with Doplicher-Haag-Roberts superselection theory in algebraic quantum field theory, in which the global gauge group gives rise to observable superselection sectors and can in turn be reconstructed from such a superselection structure [44–49].

In this way, we again arrive at the familiar fact that the group of boundary-preserving "allowed" gauge transformations $\mathcal{G}^I$ consists of those that become constant at infinity at the appropriate rate. We need not worry anymore about what this rate is precisely, since it does not play a role when working on the compact space $\hat{\Sigma}$, where there is only one simple condition on the transformations in $\mathcal{G}^I$, namely that they are constant on $\partial\hat{\Sigma}$. It is also clear why gauge fields A, when viewed on $\hat{\Sigma}$, automatically approach the fixed flat connection $\hat{A}_\infty \in \Omega^1(\partial\hat{\Sigma}, \mathfrak{g})$ at the same rate that tangent vectors α approach zero. This follows from the requirement that the space of "electric fields" is precisely the tangent space to Q. Since $T_A Q \cong Q$, any choice of asymptotic behavior for elements in $T_A Q$ automatically translates this behavior onto to Q itself.

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[20] In more detail, we can take $\hat{M}$ to be $\mathbb{R} \times S^3$ with the metric $g_{\hat{M}} = -d\tau^2 + g_{S^3}$. Using standard angular coordinates (α, β, γ) for S^3 and spherical coordinates (r, θ, φ) for $\mathbb{R}^3$, the embedding $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times S^3$ is explicitly given by $\tau = \arctan(t + r) + \arctan(t - r)$, $\alpha = \arctan(t + r) - \arctan(t - r)$, $\beta = \theta$ and $\gamma = \varphi$ [34, Equation 8.4.5].

4 Redundant gauge symmetries and constraints

Having reproduced the result that the subgroup of boundary-preserving gauge transformations $\mathcal{G}^I$ consists of those transformations that become constant at infinity - interpreted properly as the boundary of the compact space $\hat{\Sigma}$ - it is time we turn to the question of the redundancy or "triviality" of these gauge transformations. That is: which elements of $\mathcal{G}^I$ are generated by the Gauss law constraint, which is the primary first-class constraint of Yang-Mills theory,[21] and can therefore be interpreted to be unphysical, i.e. to not have DES?

In constrained Hamiltonian analysis, *gauge orbits* are null directions[22] of the symplectic form pulled back to the constraint surface $\mathcal{C}$ [26]. These null directions give a clear definition of "gauge" in the redundant sense: they are not felt by the symplectic form, which is the central object in the classical structure of the theory. It was Dirac's great insight that these gauge orbits are generated by the primary first-class constraints. In symplectic geometry, this idea is made precise by means of the momentum map, which formalizes infinitesimally generated symmetries. Indeed, in Yang-Mills theory the constraint surface equals the inverse image of zero under the momentum map for the group of redundant, trivial gauge symmetries [34]. Thus, in order to discover precisely which transformations in $\mathcal{G}^I$ are redundant, we pursue the following strategy: we calculate for which infinitesimal gauge transformations the momentum map is given by the Gauss law constraint. This approach can be seen as a precise version of the argument from [30], and we will follow it now to highlight how local and global gauge symmetries obtain a different physical status: only the former have the Gauss law constraint as their momentum map and should therefore be viewed as redundant.

However, we will then explain the weakness of such an approach: we run into the same issues about the appropriate rates of asymptotic behavior as those highlighted in the introduction. Thus we will be forced to revert back to the compact space $\hat{\Sigma}$ related to Σ through a conformal embedding. We will then see that the redundant gauge transformations $\hat{\mathfrak{g}}$ are the ones that equal the identity on the conformal boundary $\partial\hat{\Sigma}$, even though it is not a priori clear that, on a compact space, it is only these transformations that can be generated by the Gauss law constraint. We will clarify this confusion by explaining the notion of an *infinitesimal localizable symmetry*, which in the mathematical literature are the symmetries that yield Noether's second theorem and the resulting constraints, and are therefore redundant. Only global gauge symmetries are not localizable, so these should be viewed as carrying a different empirical status than local gauge symmetries. They are symmetries that do not lead to constraints, similar to e.g. rotational symmetry for a point particle moving in Euclidean space.[23]

4.1 The momentum map for the gauge group

Let $Q \subset \Omega^1(\Sigma, \mathrm{Ad}(P))$ denote the space of connections on $P \to \Sigma$ satisfying the boundary conditions we arrived at in the previous Section, i.e. approaching a fixed choice of flat connection invariant under $\mathrm{Ad}(G)$ at infinity at the right rate. Let us, for simplicity, assume P has now been trivialized, i.e. that we work in a specific gauge. Then $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g} \cong \Sigma \times \mathfrak{g}$, so that $Q \subset \Omega^1(\Sigma, \mathfrak{g})$. To study the momentum map for Yang-Mills theory, we need to know what the phase space looks like. In Section 2 we already found the domain of the Lagrangian, namely

[21] Besides the $\Pi^0 = 0$ constraint that tells us that the time-component A_0 of the gauge field is a Lagrange multiplier, but which is excluded in our analysis because we are working in temporal gauge $A_0 = 0$ from the beginning.

[22] A symplectic form is required to be non-degenerate only on the full phase space and not on the constraint surface.

[23] The Lagrangian for such a particle is $\mathcal{L}(q, v) = \frac{1}{2} g_q(v, v) - V(q)$, where $g_q \colon T_q\mathbb{R}^3 \times T_q\mathbb{R}^3 \to \mathbb{R}$ is a metric. The symmetries of the system are the isometries that leave the potential V invariant. If the potential is rotationally symmetric, then rotations are symmetries. But the Legendre transform $L \colon T\mathbb{R}^3 \to T^*\mathbb{R}^3$ is given by $v_q \to g_q(v, \cdot)$, which is clearly a diffeomorphism. This means that there are no constraints.

the tangent bundle to configuration space $TQ \cong Q \times Q$. The phase space is a dense subspace[24] $\mathcal{P} := Q \times \Omega^2_\infty(\Sigma, \mathfrak{g}) \subset T^*Q$ of the cotangent bundle [32]. It consists of pairs (A, E) with $A \in Q$ and $E \in \Omega^2_\infty(\Sigma, \mathfrak{g}) \subset T^*_A Q$. Here $\Omega^2_\infty(\Sigma, \mathfrak{g})$ denotes the space of 2-forms that approach zero asymptotically at the appropriate rate. These 2-forms can indeed be viewed as elements of the cotangent space $T^*_A Q$, which consists of covectors $T_A Q \to \mathbb{R}$, through their action on an element $\alpha \in T_A Q \cong Q \subset \Omega^1(\Sigma, \mathfrak{g})$ by means of the conjugate pairing [43]

$$E(\alpha) = \langle \alpha, E \rangle = \int_\Sigma \mathrm{Tr}\, \alpha \wedge E.$$

The constraint for Yang-Mills theory is the Gauss law [50]

$$D_A E := dE + [A \wedge E] = 0\,.$$

The action of the boundary-preserving gauge group $\mathcal{G}^I$ lifts to phase space in the obvious way:

$$\forall g \in \mathcal{G}^I : \qquad g \cdot (A, E) = (g^{-1} A g + g^{-1} dg, g^{-1} E g)\,.$$

The Lie algebra $\mathrm{Lie}(\mathcal{G}^I)$ is isomorphic to $C^\infty_I(\Sigma, \mathfrak{g})$, i.e. the space of smooth gauge transformation parameters that become constant towards infinity at the "right rate" (we recall that this rate could be found by reverse-engineering the conformal embedding from the previous Section). We equip $\mathcal{P} \subset T^*Q$ with the canonical symplectic form $\omega = \int_\Sigma \mathbb{d}A \wedge \mathbb{d}E$, where the $\mathbb{d}$ symbol is used to stress that this is the derivative operator on the infinite-dimensional phase space of fields and not the d on 3-space Σ. Henceforth we will occasionally use double-slashed symbols to stress that these objects are defined on an infinite-dimensional phase space $\mathcal{P}$.

We should like to check that, with this symplectic form, the Gauss law constraint generates gauge transformations, i.e. check for which gauge parameters $\xi \in \mathrm{Lie}(\mathcal{G}^I)$ the momentum map equals the Gauss law. Let us recall the definition of the momentum map [33].

Definition 4.1. Let $(\mathcal{P}, \omega)$ be a symplectic manifold and H a Lie group that acts on $\mathcal{P}$ by symplectomorphisms.[25] Let $\mathfrak{h}$ denote the Lie algebra of H with dual $\mathfrak{h}^*$, and write $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{R}$ for the pairing of the algebra and its dual. Then a *momentum map* for the H-action on $\mathcal{P}$ is an equivariant[26] map $\mu : \mathcal{P} \to \mathfrak{h}^*$ such that, for all $\xi \in \mathfrak{h}$, we have:

$$\mathbb{d}\langle \mu, \xi \rangle = \iota_{\mathbb{X}_\xi} \omega = \omega(\mathbb{X}_\xi, \cdot)\,.$$

Here, $\mathbb{X}_\xi$ denotes the fundamental vector field[27] generated by ξ, and $\langle \mu, \xi \rangle$ is understood as a function $\langle \mu, \xi \rangle : \mathcal{P} \to \mathbb{R}$, defined as follows: $\langle \mu, \xi \rangle(x) = \langle \mu(x), \xi \rangle$.

The idea behind this definition is that the fundamental vector field $\mathbb{X}_\xi$ infinitesimally generates the H-action with parameter ξ, while the values $\langle \mu, \xi \rangle$ of the momentum map for specific ξ provide constants of motion. The required relation $\mathbb{d}\langle \mu, \xi \rangle = \omega(\mathbb{X}_\xi, \cdot)$ can then be viewed in the light of Noether's theorem: it relates the conservation of the constants of motion to the symmetry of the theory.[28]

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[24] The full cotangent bundle would include distribution-like functionals that are not smooth and which we want to exclude. One could of course also consider restricting Q further and allow for the full cotangent bundle T^*Q. For instance, one could considering taking Q to consist of only Schwarz functions, so that T^*Q consists of tempered distributions. The power of our argument in this article is that such alterations would not change the main result that the asymptotic symmetry group is the global gauge group.

[25] I.e. the action of H preserves the symplectic form ω.

[26] With respect to the H-action on $\mathcal{P}$ and the coadjoint action on $\mathfrak{h}^*$.

[27] In this definition we use the double-slashed notation because this agrees with our subsequent calculations, but of course this definition of the momentum map is also valid for finite-dimensional symplectic manifolds.

[28] For technical details see [32–34], for a conceptual exposition see [51].

We will now check that the Gauss law constraint is indeed the momentum map for the gauge group. We find that, by partial integration, the smeared Gauss constraint splits into a "bulk" term corresponding to the infinitesimally generated gauge symmetries and a boundary term. The boundary term must vanish, leading to a condition on the gauge transformation parameters. We will present our derivation on the Cauchy surface $\Sigma \cong \mathbb{R}^3$, interpreting $\partial\Sigma$ as an asymptotic boundary, but the exact same derivation would work on the compact space $\hat{\Sigma}$ with actual boundary $\partial\hat{\Sigma}$.

The momentum map $\mu\colon \mathcal{P} \to \Omega^3_I(\Sigma, \mathfrak{g})$ for the action of the gauge group $\mathcal{G}^I$ on $\mathcal{P} \subset T^*Q$ is supposed to be the Gauss law[29] constraint $\mu(A, E) = D_A E$ [34, 50]. Here we identify $\eta \in \Omega^3_I(\Sigma, \mathfrak{g})$ as an element in the dual $C^\infty_I(\Sigma, \mathfrak{g})^*$, through the pairing $\langle \eta, \xi \rangle = \int_\Sigma \mathrm{Tr}\, \xi \wedge \eta$, similar to the pairing defined above. Thus, for any $\xi \in C^\infty_I(\Sigma, \mathfrak{g})$ (and using Stokes' theorem/partial integration), we have:

$$\begin{aligned}\langle \mu, \xi \rangle(A, E) &= \int_\Sigma \mathrm{Tr}\, D_A E \wedge \xi = \int_\Sigma \mathrm{Tr}\, (dE + [A, E]) \wedge \xi = \int_\Sigma \mathrm{Tr}\, (dE \wedge \xi - [E, A] \wedge \xi) \\ &= -\int_\Sigma \mathrm{Tr}\, E \wedge d\xi + \int_{\partial\Sigma} \mathrm{Tr}\, E \wedge \xi - \int_\Sigma \mathrm{Tr}\, E \wedge [A, \xi] = -\int_\Sigma \mathrm{Tr}\, E \wedge D_A \xi + \int_{\partial\Sigma} \mathrm{Tr}\, E \wedge \xi\,,\end{aligned} \tag{2}$$

where we have used the ad-invariance of the trace, i.e. $\mathrm{Tr}\, [E, A] \wedge \xi = \mathrm{Tr}\, E \wedge [A, \xi]$. Note that for consistency we have used the $\wedge$ symbol even on $\xi \in C^\infty_I(\Sigma, \mathfrak{g})$, even though it is a 0-form.

But, if $\mu(A, E) = D_A E$ really is to define the momentum map for the action of $\mathcal{G}^I$, then by definition it must satisfy the property

$$\mathbb{d}\langle \mu, \xi \rangle = \iota_{\mathbb{X}_\xi} \omega := \omega(\mathbb{X}_\xi, \cdot), \qquad \xi \in C^\infty_I(\Sigma, \mathfrak{g}), \tag{3}$$

where $\mathbb{X}_\xi \in \mathfrak{X}(\mathcal{P})$ denotes the fundamental vector field on $\mathcal{P} \subset T^*Q$ generated by the Lie algebra element ξ. We will now check what assumption on the asymptotic behavior of the gauge transformation parameter is required for the above condition to hold.

To this end we first calculate the right- and left-hand sides of Eq. (3) separately and then compare them. We begin with the right-hand side, i.e. $\omega(\mathbb{X}_\xi, \cdot)$. By definition, for any function $\mathbb{F} \in C^\infty(\mathcal{P})$, we have:

$$\mathbb{X}_\xi(\mathbb{F})(A, E) = \frac{d}{dt}\bigg|_{t=0} \mathbb{F}\left(e^{t\xi} \cdot (A, E)\right) = \frac{d}{dt}\bigg|_{t=0} \mathbb{F}\left(e^{-t\xi} A e^{t\xi} + e^{-t\xi} d(e^{t\xi}), e^{-t\xi} E e^{t\xi}\right).$$

For the functions $\mathbb{F} = A$ and E, this simply gives:

$$\begin{aligned}\mathbb{X}_\xi(A) &= \frac{d}{dt}\bigg|_{t=0} \left(e^{-t\xi} A e^{t\xi} + e^{-t\xi} d(e^{t\xi})\right) = -\xi A + A\xi + d\xi = [A, \xi] + d\xi = D_A \xi\,, \\ \mathbb{X}_\xi(E) &= \frac{d}{dt}\bigg|_{t=0} \left(e^{-t\xi} E e^{t\xi}\right) = -\xi E + E\xi = [E, \xi]\,.\end{aligned}$$

Thus if we put $\mathbb{X}_\xi$ in the first slot of the symplectic form $\omega = \int_\Sigma \mathrm{Tr}\, \mathbb{d}A \wedge \mathbb{d}E$, i.e. the right-hand side of Eq. (3), we get:

$$\begin{aligned}\omega_{(A,E)}(\mathbb{X}_\xi, \cdot) &= \int_\Sigma \mathrm{Tr}\, \left(\mathbb{d}A(\mathbb{X}_\xi) \wedge \mathbb{d}E - \mathbb{d}E(\mathbb{X}_\xi) \wedge \mathbb{d}A\right) = \int_\Sigma \mathrm{Tr}\, \left(\mathbb{X}_\xi(A) \wedge \mathbb{d}E - \mathbb{X}_\xi(E) \wedge \mathbb{d}A\right) \\ &= \int_\Sigma \mathrm{Tr}\, \left(([A, \xi] + d\xi) \wedge \mathbb{d}E - [E, \xi] \wedge \mathbb{d}A\right) = \int_\Sigma \mathrm{Tr}\, \left(D_A \xi \wedge \mathbb{d}E - [E, \xi] \wedge \mathbb{d}A\right).\end{aligned} \tag{4}$$

[29] If we consider Maxwell theory, then the momentum map μ applied to an element $\xi \in C^\infty_I(\Sigma, \mathfrak{g})$ is just the familiar Gauss law $\nabla \cdot \mathbf{E}$ smeared with ξ. This can be seen by switching to the physicists' convention $\xi = i\lambda$ and writing $D_A E = \nabla \cdot \mathbf{E}$, yielding $i \int_\Sigma d^3x\, \lambda(\mathbf{x})\, \nabla \cdot \mathbf{E}(\mathbf{x})$.

The left hand-side of Eq. (3) gives:

$$\mathbb{d}\langle\mu,\xi\rangle = \mathbb{d}\int_\Sigma \mathrm{Tr}\, D_A E \wedge \xi = \int_\Sigma \mathrm{Tr}\,\left(\mathbb{d}(D_A E)\wedge\xi - D_A E\wedge \mathbb{d}\xi\right).$$

However, we cannot immediately see how this agrees with the expression in Eq. (4), because Eq. (4) contains a term linear in $D_A\xi$, while the above result has a term that is linear in ξ. Thus we need to do the partial integration in Eq. (2), which gives:

$$\mathbb{d}\langle\mu,\xi\rangle = -\int_\Sigma \mathrm{Tr}\,(\mathbb{d}E\wedge D_A\xi - E\wedge \mathbb{d}(D_A\xi)) + \mathbb{d}\int_{\partial\Sigma}\mathrm{Tr}\, E\wedge\xi\,. \tag{5}$$

The second term in the first integral can be rewritten as:

$$\begin{aligned} E\wedge\mathbb{d}(D_A\xi) &= E\wedge\mathbb{d}(d\xi+[A,\xi]) = E\wedge\mathbb{d}[A,\xi] = E\wedge[\mathbb{d}A,\xi] = E\wedge\mathbb{d}A\xi - E\wedge\xi\mathbb{d}A \\ &= -E\xi\wedge\mathbb{d}A + \xi E\wedge\mathbb{d}A - \xi E\wedge\mathbb{d}A + E\wedge\mathbb{d}A\xi = -[E,\xi]\wedge\mathbb{d}A + [E\wedge\mathbb{d}A,\xi]\,. \end{aligned}$$

Thus, the first integral in Eq. (5) equals:

$$\int_\Sigma \mathrm{Tr}\,(D_A\xi\wedge\mathbb{d}E + E\wedge\mathbb{d}(D_A\xi)) = \int_\Sigma \mathrm{Tr}\,(D_A\xi\wedge\mathbb{d}E - [E,\xi]\wedge\mathbb{d}A + [E\wedge\mathbb{d}A,\xi])\,.$$

But the trace of the full commutator term gives zero,[30] so we obtain precisely the final expression in Eq. (4)! This implies that from requiring that the Gauss constraint $\mu(A,E) = D_A E$ is the momentum map for the action of the gauge group, it follows that the boundary term in Eq. (5) must be zero. To guarantee this, we require that ξ vanishes asymptotically. Since we originally demanded $E \to 0 + \mathcal{O}(r^{-3/2-\epsilon})$, we must have $\xi \to 0 + \mathcal{O}(r^{-1/2})$ to guarantee that there is no boundary term, for then the integrand $E\wedge\xi$ on the 2-dimensional boundary $\partial\Sigma$ has fall-off behavior of order $\mathcal{O}(r^{-2-\epsilon})$. However, if we require only slightly stronger asymptotic fall-off conditions on the electric field, e.g. $E \to 0 + \mathcal{O}(r^{-2-\epsilon})$, then there is no longer any need for asymptotic requirements on ξ. Thus, a "Balachandran-like approach" [30], even if formalized in this way, does not provide a completely unambiguous and satisfactory answer to the question of precisely what asymptotic behavior of gauge transformations is required to be able to call them redundant. Moreover, even if we do conclude that we must have $\xi \to 0+\mathcal{O}(r^{-1/2})$, it is not very clear that the quotient $\mathcal{G}^I/\mathcal{G}_0^\infty$ will be precisely the group of global gauge transformations, even though this global group can be pristinely deduced from other approaches, as was explained in Section 1.

Still, there is a useful conclusion to be drawn from the above derivation. By partial integration the momentum map naturally falls into two parts, i.e. two integrals, viz. $\int_\Sigma \mathrm{Tr}\, E\wedge D_A\xi$ and the boundary term $\int_{\partial\Sigma}\mathrm{Tr}\, E\wedge\xi$. The first corresponds precisely to the symmetries that are infinitesimally generated by the fundamental vector fields $\mathbb{X}_\xi$, whereas the second does not. The boundary term must therefore vanish. If we allow the most liberal asymptotic behavior on E that is still consistent with finiteness of energy, then this requirement that the boundary term vanishes in turn leads to the requirement that gauge transformations vanish asymptotically. This means that only gauge transformations vanishing at infinity, i.e. local transformations, are associated to the Gauss law constraint through Noether's second theorem [34, Proposition 7.2.6]. Only these should be viewed as unphysical. Global gauge transformations, which do act at infinity, are not included and only appear in Noether's first theorem [52].

[30] Or, since the trace is ad-invariant, we could also immediately have rewritten $\mathrm{Tr}\, E\wedge[\mathbb{d}A,\xi] = -\mathrm{Tr}\,[E,\xi]\wedge\mathbb{d}A$.

4.2 Infinitesimal localizable symmetries

As we have just explained, the problem with the above conclusion is that it seems to depend on the choice of asymptotic boundary conditions for E. If we choose stronger conditions than $E \to 0+\mathcal{O}(r^{-3/2-\epsilon})$, e.g. $E \to 0+\mathcal{O}(r^{-3})$ or that E is a Schwarz function, then the boundary term in Eq. 2 automatically vanishes, regardless of the asymptotic behavior of the gauge parameter ξ. The goal of this Section is to explain why global gauge symmetries still never play a role in Noether's second theorem, i.e. why they do not give rise to constraints, and should therefore not be considered to be redundant even if the boundary term in Eq. 2 vanishes due to stricter boundary conditions.

In the mathematical physics literature the symmetries that give rise to constraints through Noether's second theorem are the so-called *infinitesimal localizable symmetries*. These form an ideal (under the Lie bracket) $\mathfrak{G} \subset \mathrm{Lie}(\mathcal{G}^I)$ of the Lie algebra of the full symmetry group, and the constraint surface is the zero set of the momentum map for the infinitesimal localizable symmetries (see Section 7.5 of [34]), i.e.

$$\mathcal{C} = \mu_{\mathfrak{G}}^{-1}(0).$$

For this reason the infinitesimal localizable symmetries should be identified as the redundant, "trivial" ones. When exponentiated, they generate the minimal symmetry group that must be called *gauge* in the sense of "unphysical" in order to guarantee an appropriate form of determinism. These infinitesimal localizable symmetries are introduced in Definition 7.2.5 of [34], but we will not reproduce that definition here, since it is based on the formalism of jet bundles and the De Donder equations. However, when adapted to our case at hand, it reads as follows:

Definition 4.2. An infinitesimal symmetry $\xi \in \mathrm{Lie}(\mathcal{G}^I) = C_I^\infty(\Sigma, \mathfrak{g})$ is called *localizable* if it vanishes on the asymptotic boundary of Σ and if for any pair of open sets $U, V \subset \Sigma$ with disjoint closures, there exists a $\xi' \in \mathrm{Lie}(\mathcal{G}^I)$ such that

$$\begin{aligned} \xi(x) &= \xi'(x), \qquad x \in U; \\ \xi'(x) &= 0, \qquad x \in V. \end{aligned}$$

In other words: an infinitesimal symmetry is localizable if it is zero at asymptotic infinity and for any two disjoint open regions we can always find another infinitesimal symmetry that is equal to the original one on the one region, but zero on the other. That is: we can always localize the infinitesimal symmetry to some open region of space.

Clearly, global gauge transformations are not localizable since they do not vanish at asymptotic infinity, or more precisely, at the boundary $\partial\hat{\Sigma}$ of the compact space $\hat{\Sigma}$ from Section 3. The question, then, is whether *all other* infinitesimal symmetries in $\mathrm{Lie}(\mathcal{G}^I)$ are localizable. If this is so, then the quotient $\mathcal{G}_0^I/\mathcal{G}_0^\infty$ equals precisely the global gauge group G, where $\mathcal{G}_0^I$ is the identity component of $\mathcal{G}^I$ and $\mathcal{G}_0^\infty$ denotes the group generated by all e^ξ with $\xi \in \mathfrak{G}$.

Let us therefore check that all gauge symmetries except the global ones are localizable. This is done most easily by working on the compact space $\hat{\Sigma}$. There $\mathrm{Lie}(\mathcal{G}^I)$ consists of all maps $\hat{\xi}\colon \hat{\Sigma} \to \mathfrak{g}$ that are constant on $\partial\hat{\Sigma}$. We note that $\mathrm{Lie}(\mathcal{G}^I)/\mathfrak{g}$, with $\mathfrak{g}$ viewed as the constant maps in $C_I^\infty(\Sigma, \mathfrak{g})$, consists of all maps $\hat{\xi}\colon \hat{\Sigma} \to \mathfrak{g}$ that vanish on $\partial\hat{\Sigma}$. We denote the algebra of these latter maps by $\mathfrak{G}_\infty$ and check that $\mathfrak{G}_\infty = \mathfrak{G}$.

There are two situations to consider: if U, V are the open subsets from the above definition, such that $\hat{\xi} \in \mathrm{Lie}(\mathcal{G}^I)$ must be localized on U relative to V, then either U could lie in the interior of $\hat{\Sigma}$ or contain (part of) the boundary $\partial\hat{\Sigma}$. In the first case it is obvious that $\hat{\xi}$ can be localized: we just use a $\mathfrak{g}$-valued bump function $\hat{f}$ that is the identity on U and becomes zero very quickly outside of U, in particular on V. It is then clear that $\hat{f} \cdot \hat{\xi}$ will be the required element of $\mathrm{Lie}(\mathcal{G}^I)$

that agrees with $\hat{\xi}$ on U and is zero on V. In the case in which U contains part of the boundary it is not immediately clear whether $\hat{f} \cdot \hat{\xi} \in \mathrm{Lie}(\mathcal{G}^I)$. But since $\hat{\xi}$ is zero on $\partial\hat{\Sigma}$, so is the product $\hat{f} \cdot \hat{\xi}$. This means $\hat{f} \cdot \hat{\xi} \in \mathfrak{G}_\infty \subset \mathrm{Lie}(\mathcal{G}^I)$. We conclude that the algebra of infinitesimal localizable symmetries $\mathfrak{G}$ is indeed $\mathfrak{G}_\infty$. Thus we find that, in Yang-Mills theory, localizability effectively reduces to just the condition of vanishing at infinity.[31] Of course, this is not a surprising result, since gauge symmetries are meant to be localizable. But we clearly see that if a field theory contains only global (rigid) symmetries, then no infinitesimal transformation is ever localizable, in which case $\mathfrak{G}$ would be zero and there would be no constraints.

We finally arrive at the result we were aiming to derive all along. The subalgebra $\mathfrak{G}_\infty$ generates (through the exponential map) the subgroup $\mathcal{G}_0^\infty$ of gauge transformations that become the identity at asymptotic infinity at the appropriate rate and lie in the identity component of $\mathcal{G}$ (i.e. can be obtained by exponentiating Lie algebra elements). The quotient of physical gauge transformations

$$\mathcal{G}_{\mathrm{DES}} = \mathcal{G}^I / \mathcal{G}_0^\infty$$

then looks like a copy of the global gauge group G for every homotopy class. which is what we wanted to show. These homotopy classes are determined by the fundamental group $\pi_3(G)$ in three dimensions, since gauge transformations on Σ that are constant at asymptotic infinity can be viewed as maps $S^3 \to G$. For $G = U(1)$ this homotopy group is trivial,[32] but for $G = SU(2)$ we have $\pi_3(SU(2)) \cong \pi_3(S^3) \cong \mathbb{Z}$.

5 Adding the Higgs field

Over the past two decades there has been a substantial conceptual debate about the Higgs mechanism [17, 53–57]. Much of this debate centers around the physical status of gauge symmetries in relation to gauge symmetry breaking. As has been pointed out in [17, 20], a key point is that the unbroken and broken phases of the Higgs model exhibit differing asymptotic boundary conditions. However, the derivation of this point has not been performed rigorously. We can now do this in the framework developed in the previous Sections.

To include a Higgs field, we must enlarge the configuration space Q of Yang-Mills fields to $Q \times \tilde{Q}$, where $\tilde{Q}$ is the space of Higgs fields, which are sections of an associated vector bundle $P \times_\rho V \to \Sigma$ through a representation $\rho\colon G \to GL(V)$, where V is the Higgs vector space.[33] If we equip V with an inner product $\langle \cdot, \cdot \rangle$, then we can define a norm on the sections in $\Gamma(P \times_\rho V)$ in a similar way as for the gauge fields: by integrating the absolute value of such a section over all of Σ.

The tangent space $T_\varphi \tilde{Q}$ at a point $\varphi \in \Gamma(P \times_\rho V)$ is itself just a copy of $\tilde{Q}$. However, we need to restrict both $\tilde{Q}$ and $T\tilde{Q}$ with appropriate asymptotic boundary conditions. These follow from the Yang-Mills-Higgs Lagrangian,[34] which is given by

$$\mathcal{L}_{\mathrm{YMH}}(A, \alpha, \varphi, \psi) = \frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|F(A)\|^2 + \frac{1}{2}\|\psi\|^2 - \frac{1}{2}\|D_A\varphi\|^2 - \int_\Sigma V(\varphi)\mathrm{dVol},$$

[31] Note that this result is quite independent of the precise form of Q and $\mathcal{G}^I$. No matter what asymptotic conditions on the fields are required, we always find that the redundant gauge transformations are all elements of $\mathcal{G}^I$ which vanish at infinity.

[32] In one dimensions we do have an interesting topology for electromagnetism since $\pi_1(S^1) \cong \mathbb{Z}$.

[33] It is $\mathbb{C}$ for $G = U(1)$, and $\mathbb{C}^2$ for both $G = SU(2)$ and $G = U(1) \times SU(2)$ [42].

[34] For the well-posedness of the Yang-Mills-Higgs initial value problem see [58].

for $A \in Q, \alpha \in T_A Q, \varphi \in \tilde{Q}, \psi \in T_\varphi \tilde{Q}$. Here $V(\varphi)$ is the well-known Higgs potential, $D_A \varphi$ is the covariant derivative of the Higgs field and $\psi \in T_\varphi \tilde{Q}$ must be thought of as the velocity of $\varphi \in \tilde{Q}$.

In order to guarantee finiteness of action and energy, we require that each individual term in the above Lagrangian is finite. We already know that this requires $\alpha \to 0$ and $F(A) \to 0$, but now we also need $\psi \to 0$ and $D_A \varphi \to 0$, as well as a condition related to $V(\varphi)$. This last condition is ambiguous. If the Higgs potential has the familiar shape

$$V(\varphi) = -\mu \left\|\varphi\right\|^2 + \lambda \left\|\varphi\right\|^4 ,$$

then clearly the zero-point of $V(\varphi)$ lies at $\varphi = 0$, as well as some other manifold of roots at which $\varphi \neq 0$, if $\mu > 0$. Thus, we expect the boundary condition $\varphi \to 0$. However, we may instead want to think of the *minimum* of $V(\varphi)$ as the true vacuum, therefore requiring $\varphi \to \min$ instead. These two possibilities respectively correspond to the so-called unbroken and broken phases of the Higgs model. Let us now study what the group $\mathcal{G}^I$ of boundary-preserving gauge symmetries looks like in both cases.[35]

The unbroken phase. In the unbroken phase, we assume that $\varphi = 0$ is the vacuum for the Higgs field, i.e. that this state carries zero energy. We can either think of this state as lying in the symmetric middle of the "Mexican hat potential," or as the potential itself being such that it only has a minimum at $\varphi = 0$, e.g. by taking $\mu < 0$. Since $\varphi = 0$ corresponds to zero energy, we require the asymptotic boundary condition $\varphi \to 0$, besides the common boundary conditions $\psi \to 0$ and $D_A \varphi \to 0$ which are always needed. Note that this indeed gives the configuration space at infinity the right structure: the space of Higgs fields at infinity is zero-dimensional, since it consists only of $\varphi = 0$. The tangent space at infinity then also consists only of zero, which is what we want since we require $\psi \to 0$. Now, the conditions $\psi \to 0$ and $D_A \varphi \to 0$ are always preserved by any gauge transformation $g \colon \Sigma \to G$. This is obvious for the condition $D_A \varphi \to 0$, since the covariant derivative transforms covariantly via the linear Higgs representation $\rho \colon G \to GL(V)$. Similarly the condition $\psi \to 0$ is preserved since ψ also transforms covariantly.[36] This means that the conditions $\psi \to 0$ and $D_A \varphi \to 0$ are automatically preserved. The same goes for the condition $\varphi \to 0$, since zero is mapped to zero by any $\rho(g)$ with $g \in \mathcal{G}$. We already know that for pure Yang-Mills theory the group $\mathcal{G}^I$ consists of transformations that become constant asymptotically, so for the full Yang-Mills-Higgs theory we find the same.

The broken phase. In the broken phase things are different. The asymptotic conditions are now $\psi \to 0, D_A \varphi \to 0$ and $\varphi \to \min$. That is, the Higgs field must become a covariantly constant minimum of the potential $V(\varphi)$, and its velocity must become zero. At first sight, this seems to still allow $\mathcal{G}^I$ to contain all asymptotically constant transformations. After all, a gauge transformation maps a minimum of $V(\varphi)$ to another minimum. However, this is wrong, for the same reason as for pure Yang-Mills theory. Allowing for gauge transformations which act at infinity in this case gives rise to a nontrivial configuration space $\tilde{Q}_\infty$ at infinity. After all, if we let φ_∞ denote some covariantly constant minimum at infinity, then $\tilde{Q}_\infty$ will consist at least of an orbit of $\mathcal{G}^I$. But this means that the tangent space $T_{\varphi_\infty} \tilde{Q}_\infty$ is far from being 0-dimensional. In fact, it has the dimension of G, since we can think of it as the tangent space to an orbit of constant gauge transformations at infinity, i.e. as $\mathfrak{g}$. But we cannot allow $T_{\varphi_\infty} \tilde{Q}_\infty$ to contain nonzero vectors, since we required that the tangent vectors $\psi \in T_\varphi \tilde{Q}$ vanish at infinity! Like for pure Yang-Mills theory, the requirement that the Lagrangian be defined on the tangent

[35] We note that our ideas agree with [20], but fill in the missing argument, namely the Lagrangian must be defined on the tangent bundle to configuration space. Without this added argument one *cannot* deduce that the physical gauge group is different in the two cases.

[36] To see this, recall that, in covariant notation, we have $D_\mu \varphi \to \rho(g) \cdot D_\mu \varphi$, so in particular $D_0 \varphi \to \rho(g) \cdot D_0 \varphi$. In the 3+1 formalism we work in the temporal gauge $A_0 = 0$ and replace $D_0 \varphi = \partial_0 \varphi - eA_0 \varphi = \partial_0 \varphi$ by ψ.

bundle to configuration space forces us to put boundary conditions on the Higgs field itself, even though only the tangent vectors and derivatives appear in the Lagrangian.

Thus we are forced to require a stricter asymptotic boundary condition on the Higgs field: we need that $\varphi \to \varphi_0^\infty$, where φ_0^∞ denotes some *fixed*, covariantly constant minimum at infinity. This ensures that the configuration space at infinity is zero-dimensional, consisting only of φ_0^∞. The tangent bundle at infinity is therefore also zero-dimensional, consisting only of $\psi_\infty = 0$, as required for finiteness of energy. But clearly this stricter asymptotic boundary condition breaks gauge invariance, in the sense that it is not preserved by gauge transformations which act non-trivially at infinity. Only gauge transformations that are the identity at infinity preserve φ_0^∞, so we find that the groups of boundary-preserving and redundant gauge symmetries are equal up to connected components, i.e. $\mathcal{G}^I = \mathcal{G}^\infty$, where $\mathcal{G}^\infty$ denotes the group of gauge transformations that are constant at infinity (but only its identity component $\mathcal{G}_0^\infty$ is generated by the Gauss law constraint).

In this way we conclude that the physical gauge group $\mathcal{G}^I/\mathcal{G}_0^\infty$ equals (several copies of) the global gauge group in the unbroken phase and is discrete (trivial in the Abelian case) in the broken phase. This conclusion results from the difference of what we call the "vacuum" in the two cases: either $\varphi = 0$ or a minimum of the potential $V(\varphi)$. Gauge symmetry breaking in the Higgs mechanism must therefore be understood as an alteration in the vacuum itself, leading to different asymptotic boundary conditions.

6 Conclusion

In this paper we have given a rigorous derivation of the identification of (several copies of) the group of global gauge symmetries with the quotient of asymptotic symmetries

$$\mathcal{G}_{\text{DES}} = \mathcal{G}^I/\mathcal{G}_0^\infty$$

in Yang-Mills theory on a three-dimensional Euclidean Cauchy surface. Here $\mathcal{G}^I$ denotes the group of allowed or boundary-preserving transformations and $\mathcal{G}_0^\infty$ the group of transformations that are trivial in the sense that they yield the Gauss law constraint through Noether's second theorem, and must therefore be viewed as redundant. Global gauge symmetries thus correspond to the asymptotic symmetry group with DES. There were two main points to this derivation, corresponding to obtaining $\mathcal{G}^I$ and $\mathcal{G}_0^\infty$ respectively.

Firstly, we found that instantaneous spatial asymptotic boundary conditions on Yang-Mills fields that ensure finiteness of energy only lead to requirements on tangent vectors α and on the curvatures $F(A)$ of the connections A. However, as we want the domain of the Lagrangian to be the tangent bundle TQ to the configuration space Q of Yang-Mills fields, we need to also impose some asymptotic fall-off behavior on the gauge fields $A \in Q$ themselves. We have shown that it is not enough to require that gauge fields become flat at infinity, since this would still allow for non-zero tangent vectors α at infinity, which would spoil the finiteness of energy and action. Intuitively, this means gauge transformations acting at infinity create infinite energy, even if the energy depends only on gauge-invariant quantities. To counter this, we need to require that A approaches a *fixed* flat connection at infinity. In the non-Abelian case we must also choose this flat connection to be invariant under the adjoint action of the structure group G. The gauge transformations that leave this fixed flat connection at infinity invariant are then precisely the elements of $\mathcal{G}$ that are constant at infinity. Properly interpreted, this means that we consider the equivalent problem on a conformal compactification $\hat{\Sigma}$ of Σ, and require that gauge transformations be constant on $\partial\hat{\Sigma}$. This yields the group of boundary-preserving gauge symmetries $\mathcal{G}^I$.

Secondly, we explained that redundant gauge transformations in the Hamiltonian formulation of Yang-Mills theory must be understood as the infinitesimal localizable symmetries $\mathfrak{G}$. These give rise to the Gauss law constraint and must therefore be interpreted as unphysical if (an appropriate form of) determinism is to survive. All infinitesimal symmetries in $\mathrm{Lie}(\mathcal{G}^{\mathrm{I}})$ are localizable, except for the global ones. Thus $\mathcal{G}_0^\infty$ indeed consists precisely of all gauge transformations that are the identity at infinity and lie in the identity component of $\mathcal{G}^{\mathrm{I}}$. Again, properly interpreted this means we move to $\hat{\Sigma}$ from Σ and require elements of $\mathfrak{G}$ to be zero on $\partial\hat{\Sigma}$, so that elements of $\mathcal{G}_0^\infty$ are the identity on $\partial\hat{\Sigma}$. The quotient $\mathcal{G}_{\mathrm{DES}}$ then consists of a copy of the global gauge group G for every homotopy class in $\pi_3(G)$.

Subsequently, we applied these ideas to Yang-Mills-Higgs theory, where we derived that $\mathcal{G}^{\mathrm{I}}$ equals the group of asymptotically constant gauge transformations only in the unbroken phase. In the broken phase one can only permit asymptotically trivial transformations, for otherwise the action of the gauge group at infinity would create non-zero velocities of the Higgs field, carrying infinite energy.

In a future article [59] we aim to consider the implications of this last result for gauge symmetry breaking. We will argue that the Higgs mechanism must be understood as an instance of global gauge symmetry breaking, as has been proposed in the Abelian case [1, 17, 60–62]. In future research it would also be of interest to extend our results to spacetimes with a nonzero cosmological constant and to better understand the relation of our work to asymptotic symmetries of Yang-Mills fields on the full boundary of spacetime (e.g. in celestial holography), as well as to edge modes [63, 64] and boundaries which are not asymptotic [22, 37, 65]. Additionally, it may be conceptually cleaner to reformulate our derivation of the physical gauge group entirely on the cotangent bundle instead of (partly on) the tangent bundle.

6.1 Acknowledgements

The authors want to sincerely thank Klaas Landsman and Manus Visser for providing detailed feedback on manuscripts, suggesting technical and conceptual improvements and for their general involvement and guidance. They are also grateful to Sebastian de Haro for his collaboration on this and related work. This work is supported by the Spinoza Grant of the Dutch Science Organization (NWO) awarded to Klaas Landsman.

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