Einstein Algebras and Relationalism Reconsidered

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Abstract

This paper reconsiders the metaphysical implication of Einstein algebras, prompted by the recent objections of Chen (2024) on Rosenstock *et al.* (2015)'s conclusion. Rosenstock *et al.*'s duality theorem of smooth manifolds and smooth algebras supports a conventional wisdom which states that the Einstein algebra formalism is not more "relationalist" than the standard manifold formalism. Nevertheless, as Chen points out, smooth algebras are different from the relevant algebraic structure of an Einstein algebra. It is therefore questionable if Rosenstock *et al.*'s duality theorem can support the conventional wisdom. After a re-visit of John Earman's classic works on the program of Leibniz algebras, I formalize the program in category theory and propose a new formal criterion to determine whether an algebraic formalism is more "relationalist" than the standard manifold formalism or not. Based on the new formal criterion, I show that the conventional wisdom is still true, though supported by a new technical result. I also show that Rosenstock *et al.* (2015)'s insight can be re-casted as a corollary of the new result. Finally, I provide a justification of the new formal criterion with a discussion of Sikorski algebras and differential spaces. The paper therefore provides a new perspective for formally investigating the metaphysical implication of an algebraic formalism for the theory of space and time.

Contents

1 Introduction

In a recent paper, Lu Chen (2024) advocates for several potential algebraic formalisms for the theory of space and time, and for an accompanying philosophical thesis named *algebraic relationalism*. According to Chen's algebraic relationalism, the algebraic formalism for the theory of space and time should be a genuine implementation of relationalism (Chen, 2024, p2). However, as Chen reports, the *conventional wisdom* has it that, the algebraic formalism for the theory of space and time that is discussed the most by philosophers, i.e. the Einstein algebra formalism, is not more "relationalist" than the standard manifold formalism for the theory of space and time, which is usually associated with substantivalism. This conventional wisdom is supported by the categorical equivalence of Einstein algebras and relativistic spacetimes established by Rynasiewicz (1992) and Rosenstock *et al.* (2015). Chen hence casts serious doubts on this conventional wisdom.

This paper aims to provide a new perspective for drawing philosophical implications on the substantivalism versus relationalism debate from the Einstein algebra formalism, and from algebraic formalisms for the theory of space and time in general. Philosophers of physics have worked with Robert Geroch (1972)'s formalism of Einstein algebras for decades, starting with classic works of John Earman (1977, 1979, 1986, 1989) proposing his program of Leibniz algebras. In response to Earman, Rynasiewicz (1992) and Rosenstock *et al.* (2015) show various senses in which Einstein algebras are equivalent to relativistic spacetimes. They both point out that the structure of a smooth manifold and the relevant algebraic structure of the collection of smooth scalar fields on a manifold turn out to be mutually constructible from each other. Rosenstock *et al.* identify the relevant algebraic structure to be the smooth algebra introduced by Jet Nestruev (2003), and illustrate the equivalence of smooth manifolds and smooth algebras by establishing a categorical duality. On top of that, they show another categorical duality that holds for relativistic spacetimes and Einstein algebras. The philosophical takeaway, as Rosenstock *et al.* (2015) suggest, is that insofar as one wants to associate the two formalisms with the two metaphysical views about the nature of space and time, the Einstein algebra formalism is as "substantivalist" as — and, for the same reason, as "relationalist" as — the standard manifold formalism (p17). Chen (2024) raises several challenges against Rosenstock *et al.*'s argument. This paper takes on these challenges, reformulates Rosenstock *et al.*'s equivalence result, and defends the conventional wisdom, i.e. that the Einstein algebra formalism is not more "relationalist".

Here I give a more detailed plan of the paper. Section 2 provides the technical background. I first introduce the standard manifold formalism, the Einstein algebra formalism, and Rosenstock *et al.* (2015)'s categorical duality theorem of smooth manifolds and smooth algebras. Then I present Chen's arguments against the conventional wisdom. To respond, I take a step back to reconsider Rosenstock *et al.*'s work. Among Chen's objections, I agree with her that smooth algebras that Rosenstock *et al.* work with have more constraints than the Einstein algebras originally formulated by Geroch. Nevertheless, I believe that Rosenstock *et al.*'s result can be reformulated to still support their philosophical conclusion. This is shown in section 3. In this section, I formalize under the categorical framework Earman's program of Leibniz algebras, which is a forerunner of Chen's idea of algebraic relationalism. Inspired by this formalization, I propose a new formal criterion which states that the appropriately defined representation functor from the category of smooth manifolds to the category of algebraic models in consideration has to fail to be full and faithful for the algebraic formalism to be more "relationalist" than the standard manifold formalism. Then I show that the Einstein algebra formalism is not more "relationalist" than the standard manifold formalism based on this criterion and re-cast Rosenstock *et al.*'s duality theorem. In section 4, I further illustrate the idea that only the fullness and faithfulness of the representation functor matter, with the example of Sikorski algebras (to be defined in section 4) and differential spaces. Finally, the paper concludes with re-affirming the conventional wisdom and defending the new formal criterion to determine the metaphysical implications of algebraic formalisms for the theory of space and time on the nature of space and time.

2 Two Formalisms and a Duality Theorem

2.1 Two Formalisms

In the standard presentation of general relativity, one starts with the notion of *smooth manifolds*[1]. An n-dimensional smooth manifold (M, C) consists of a set M and an *atlas* C of *n-charts* on M, which defines a topology that is Hausdorff and second-countable, and introduces a smoothness structure on M, that allows us to identify *smooth maps* from M to another smooth manifold. Two manifolds (M, C) and (M', C') are called *diffeomorphic* to each other if there is a bijective smooth map between M and M' whose inverse is also smooth — this map is called a *diffeomorphism.* A *tensor field* on M is an assignment of a tensor to each point of the manifold. A *relativistic spacetime* is defined to be a Lorentzian manifold, which is a four-dimensional connected manifold with a smooth metric (tensor) field of Lorentzian signature. Relevant physical fields, such as the electromagnetic field, can be defined as tensor fields on the background spacetime. This is the *standard manifold formalism* for the theory of space and time.

On the other hand, alternative *algebraic formalisms* strive to define fields by their algebraic structures only, without referring to an underlying manifold. The mostly discussed algebraic formalism by philosophers of physics is the *Einstein algebras*, formulated by Robert Geroch (1972). Geroch observes that, for any smooth manifold M, the collection of all smooth scalar fields on M forms a commutative ring with pointwise addition and multiplication. Denote this commutative ring by $C^\infty(M)$. The collection of all constant functions on M forms a subring of $C^\infty(M)$ that is isomorphic to $\mathbb{R}$. He then illustrates how the mathematical notions, including tensor fields, needed for general relativity can be introduced by just constructing relevant algebraic structures based on $C^\infty(M)$, instead of defining them on the points of the underlying smooth manifold. Geroch therefore defines an *Einstein algebra* to consist of a commutative ring $\mathscr{F}$ which has a subring $\mathscr{R}$ isomorphic to $\mathbb{R}$ and a metric defined algebraically on $\mathscr{F}$[2] (Geroch, 1972, p274).

[1] This brief presentation follows Malament (2012)'s and Wald (1984)'s definition of smooth manifolds. Refer to these textbooks for the full definitions of the notions presented in this paragraph.

[2] According to Geroch, we can define a contravariant vector field on M as a derivation on $C^\infty(M)$. The collection $\mathscr{D}$ of all smooth contravariant vector fields forms a module over the commutative ring $C^\infty(M)$. The dual module $\mathscr{D}^*$ of $\mathscr{D}$ is the collection of all smooth covariant vector fields. A metric g is then defined to be an isomorphism from $\mathscr{D}$ to $\mathscr{D}^*$ that satisfies the symmetry condition: for any ξ and η in $\mathscr{D}$, $g(\xi, \eta) = g(\eta, \xi)$, where $g(\xi, \eta)$ is defined to be $g(\xi)(\eta)$. This is equivalent to the symmetry condition in the standard manifold formalism. The fact that a metric is defined to be an isomorphism

For simplicity, this paper does not deal with the metrics in either the standard manifold formalism or the Einstein algebra formalism.[3] This simplification should not compromise the arguments of the paper, as Chen makes the same simplification (Chen, 2024, footnote 10). Call a commutative ring which has a subring isomorphic to $\mathbb{R}$ an *Einstein ring*. The following discussions will focus on smooth manifolds and Einstein rings exclusively, from which we draw conclusions about the standard manifold formalism and the Einstein algebra formalism.

There are two main metaphysical views about the nature of space and time, namely *substantivalism* and *relationalism*[4]. Relationalism is the view that only material bodies exist, standing in various relationships with each other. Substantivalism, on the other hand, is the view that there exists a fundamental spatio-temporal structure in addition to and independent of material bodies. The difference of the two metaphysical views can be illustrated by the different takes they have on the representational goal of the spacetime manifold in the standard manifold formalism. For substantivalists, the mathematical structure of the spacetime manifold *is supposed to* represent the fundamental spatio-temporal structure[5]. Relationalists, on the other hand, deny that very assumption of the representational goal for the spacetime manifold, as they do not believe that there exists a fundamental spatio-temporal structure. That is to say, relationalists should expect the spacetime manifold in the standard manifold formalism to contain *excess structures*, in the sense that these excess structures do not characterize any relationalist physical reality, but are only posited to support other mathematical structures in the formalism that do characterize something real.

The job for relationalists is therefore to establish the presence of excess structures in the standard manifold formalism. Here the Einstein algebra formalism enters the picture: if the standard manifold formalism can to shown to contain excess structures compared to the Einstein algebra formalism, then these excess structures can potentially be what relationalists are looking for. Moreover, if this turns out to be the case, Einstein algebras can also potentially be argued to characterize only the relational physical reality — hence a formal implementation of relationalism. Note that we don't need to commit to the idea that the standard manifold formalism is an implementation of substantivalism. We only need to assume that the standard manifold formalism is not the most desirable implementation of relationalism, due to excess structures manifested by the representational redundancy of the spacetime manifold. A formalism that gets rid of these excess structures can thus potentially be a better implementation of relationalism than the standard manifold formalism, or, in fewer words, more "relationalist". The flip side of finding excess structures is to establish in-equivalence: if two formalisms are not equivalent in a relevant sense, then we can argue that one of the two posits excess structures compared to the other.

guarantees that it is non-degenerate. This algebraic definition of a metric therefore corresponds to the definition of a metric in the standard manifold formalism.

[3] Despite of the importance of the metric field, we can still draw useful insights from comparing simpler mathematical structures without metrics, as demonstrated recently by Wu & Weatherall (Forthcoming).

[4] For an overview of the two metaphysical views, see (Pooley, 2013).

[5] I do not assume whether or not the spacetime manifold does successfully represent *the* fundamental spatio-temporal structure that substantivalists may have in mind. It is the representational *goal* that is important here.

2.2 The Duality Theorem

As surveyed by Weatherall (2019), philosophers of physics have discussed widely formal criterions of identifying theoretical equivalence. One prevalent criterion in the recent literature is the *categorical criterion of equivalence*. According to the categorical criterion of equivalence, two theories are equivalent if the categories of models of the two theories are equivalent as categories. This equivalence of categories is made precise by well-behaved functors between two categories[6]. Let **C** and **D** be two categories, and let F be a contravariant functor from **C** to **D**. F is said to be *faithful* if and only if for any two objects A and B in **C**, the induced map $(f : A \to B) \mapsto (F(f) : F(B) \to F(A))$ taking arrows between A and B in **C** to arrows between $F(A)$ and $F(B)$ in **D** is injective. F is said to be *full* if and only if for any two objects A and B in **C**, the induced map $(f : A \to B) \mapsto (F(f) : F(B) \to F(A))$ taking arrows between A and B in **C** to arrows between $F(A)$ and $F(B)$ in **D** is surjective. F is said to be *essentially surjective* if and only if for any D in **D**, there is an object A in **C** such that $F(A)$ is isomorphic to D in **D** — that is to say, there are arrows $f : F(A) \to D$ and $f^{-1} : D \to F(A)$ in **D** such that $f^{-1} \circ f = 1_{F(A)}$ and $f \circ f^{-1} = 1_D$. If F fails to be faithful, we say that F *forgets stuff*. If F fails to be full, we say that F *forgets structure*. If F is faithful, full, and essentially surjective, then F is said to realize a *categorical duality* of **C** and **D**, and the categories **C** and **D** are said to be *dual* to each other. Dual categories are *equivalent* to each other, as they can be viewed as "mirrored copies" of each other in the sense that the direction of their arrows is systematically reversed.

The category of models for smooth manifolds in the standard manifold formalism is defined by Rosenstock *et al.* (2015) as **SmoothMan** with the following:

- objects: smooth manifolds $M, N, \cdots$, and
- arrows: smooth maps $\varphi : M \to N$ where M and N are smooth manifolds.

On the algebraic side, Rosenstock *et al.* work with the notion of *smooth algebras* introduced by Jet Nestruev (2003). To give the definition of smooth algebras, some preliminaries are needed. We start with $\mathbb{R}$-*algebras*:

Definition 1 ($\mathbb{R}$-algebras)**.** *An* $\mathbb{R}$*-algebra* $\mathcal{A}$ *is a vector space over* $\mathbb{R}$ *with an additional associative and commutative vector multiplication and a multiplicative identity.*

It is not hard to see that an Einstein ring is an $\mathbb{R}$-algebra. We call the collection of $\mathbb{R}$-algebra homomorphisms — which preserve the vector space operations, products, and the multiplicative identity — from an $\mathbb{R}$-algebra $\mathcal{A}$ to $\mathbb{R}$ (which is also an $\mathbb{R}$-algebra) the *dual algebra* of $\mathcal{A}$, denoted by $|\mathcal{A}|$. Elements in the dual algebra of $\mathcal{A}$ are called *points* of the algebra $\mathcal{A}$. If $\mathcal{A}$ has only the zero element in the intersection of kernels of all the points of $\mathcal{A}$, then elements in $\mathcal{A}$ can be canonically identified with functions taking points of $\mathcal{A}$ to $\mathbb{R}$ by a bijective map. Such an algebra $\mathcal{A}$ is said to be *geometric* (Nestruev, 2003, p23). Define the coarsest topology on $|A|$ that makes every element of $\mathcal{A}$, canonically identified in the way described just now continuous. An $\mathbb{R}$-algebra $\mathcal{A}$ is said to be *complete* if it contains all the maps on $|\mathcal{A}|$ that are *locally equivalent* to elements of $\mathcal{A}$, in the sense that if $f : |\mathcal{A}| \to \mathbb{R}$ agrees with some $g \in \mathcal{A}$ restricting to some neighborhood of p for any $p \in |\mathcal{A}|$, then $f \in \mathcal{A}$ (Nestruev, 2003, p30-31). Restricting to a subset $U \subset |\mathcal{A}|$, $\mathcal{A}|_U$

[6]The following presentation of the categorical criterion of equivalence follows Weatherall (2019) and Rosenstock *et al.* (2015). It assumes familiarity with definitions of categories and functors, which can be found in standard textbooks on category theory such as (Awodey, 2010).

is defined to be the $\mathbb{R}$-algebra containing all the functions $f : U \to \mathbb{R}$ that are locally equivalent to some element of $\mathcal{A}$. Now we give the definition of smooth algebras:

Definition 2 (Smooth algebras). *A complete, geometric algebra $\mathcal{A}$ is called a smooth algebra if there is an at most countable open covering $\{U_k\}_{k\in\mathbb{N}}$ of $|\mathcal{A}|$ such that all the algebras $\mathcal{A}|_{U_k}, k \in \mathbb{N}$, are isomorphic to $C^\infty(\mathbb{R}^n)$ for some fixed natural number n.*

Given a smooth manifold M, the collection $C^\infty(M)$ of all its smooth scalar fields is not just an $\mathbb{R}$-algebra, but a smooth algebra (Nestruev, 2003, 7.5 & 7.6). The category of models for smooth algebras, **SmoothAlg**, is defined to consist of:

- objects: smooth algebras $\mathcal{A}, \mathcal{B}, \cdots$, and
- arrows: $\mathbb{R}$-algebra homomorphisms[7] $f : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are smooth algebras.

The two categories **SmoothMan** and **SmoothAlg** turn out to be equivalent, as shown by the following duality theorem:

Theorem 1. ***SmoothMan*** *is dual to* ***SmoothAlg****. (Rosenstock* et al.*, 2015, Theorem 3.5)*

According to the categorical criterion of theoretical equivalence, smooth manifolds and smooth algebras are equivalent. That is to say, the standard manifold formalism does not posit excess structures compared with the smooth algebra formalism. Therefore, smooth algebras cannot help relationalists single out any structure in the standard manifold formalism that does not characterize relational physical reality. The smooth algebra formalism is hence not more "relationalist" than the standard manifold formalism. Rosenstock *et al.* then identify an Einstein algebra as a 4-dimensional smooth algebra with additional structures defined on it (Rosenstock *et al.*, 2015, Section 4). They show that the Einstein algebra formalism and the relativistic spacetime formalism are equivalent, by establishing another categorical duality (Rosenstock *et al.*, 2015, Theorem 4.5), and conclude that the two formalisms "encode precisely the same physical facts about the world, in somewhat different languages" (Rosenstock *et al.*, 2015, p316). The conventional wisdom is therefore established that the Einstein algebra formalism is not more "relationalist" than the standard manifold formalism.

2.3 Chen's Objections

In her paper, Chen gives three objections to the conventional wisdom presented above and to the Einstein algebra formalism itself. Chen's first objection concerns the algebraic purity of smooth algebras. Recall that in Definition 2, an $\mathbb{R}$-algebra is defined be complete, geometric, and smooth for it to be a smooth algebra. Chen raises concerns for each of the three requirements: geometric-ness, completeness, and smoothness. For the geometric-ness, she claims that this requirement rules out nilpotent algebras for no good reason other than that Rosenstock *et al.* want to establish a categorical duality:

> Why should we rule these out for algebraicism? No rationale is given by the authors other than the apparent reason that without this condition, we wouldn't be able to recover standard manifolds through categorical duality. (Chen, 2024, p10)

[7] An $\mathbb{R}$-algebra homomorphism is a map that preserves the vector space operations, the product, and the multiplicative identity; a bijective algebra homomorphism is an $\mathbb{R}$-algebra isomorphism.

For the completeness, Chen argues that, to make sense of the idea of local equivalence of maps, we have to treat a neighborhood of $|\mathcal{A}|$ as a set of points on $|\mathcal{A}|$ so that restricting an $\mathbb{R}$-algebra homomorphism to a neighborhood makes sense. Hence she claims that "to require algebras to be complete, we make reference to geometric objects, which suggests this requirement as a disguised geometric discourse" (Chen, 2024, p10). Finally, for smoothness, Chen states that geometric concepts are directly invoked as the smoothness requirement is stated with topological vocabularies like open coverings. To sum up, Chen believes that the smooth algebra formalism is not properly algebraic, for the reasons that parts of its definition invoke geometric concepts and that the only motivation to work with smooth algebras seems to be recovering the standard manifold formalism.

The second objection that Chen raises concerns the metaphysical interpretation of the Einstein algebra formalism. She argues that the collection of smooth scalar fields in an Einstein algebra, i.e. the Einstein ring, is "essentially a surrogate manifold that represents spacetime" (Chen, 2024, p11). This is because, she explains, if one interpret the Einstein algebra formalism realistically, then one has to make an ontological commitment to scalar fields before any other type of fields. However, she points out that there is no fundamental real-valued scalar field acknowledged by current physics supporting this ontological fundamentality of scalar fields[8]. Therefore, she claims that, from a relationalist's perspective, the Einstein ring essentially plays a role equivalent to the one that the spacetime manifold plays in the standard manifold formalism, as both lack metaphysical support. As a result, algebraic relationalists should not be content with the Einstein algebra formalism:

> All these indicate that the scalar field behaves just like substantival spacetime. If algebraic relationalists are unhappy with the ghostly arena of physical ongoings, they should be equally unhappy about a ghostly background scalar field that must be posited in addition to those acknowledged by standard physics. (Chen, 2024, p12)

Finally, Chen claims that the Einstein algebra formalism is not sufficiently interesting for physicists, for the reason that it is not clear how much the Einstein algebra formalism can contribute to physics (Chen, 2024, p13). She then presents three different potential algebraic formalisms for the theory of space and time: nilpotent algebras, natural operator algebras, and non-commutative algebras. She argues that each of these algebraic formalism possesses conceptual advantages over the standard manifold formalism. Furthermore, she claims that they are also "demonstratively not equivalent to manifold substantivalism" (Chen, 2024, p22), because neither of them satisfy the requirements of smooth algebras (Chen, 2024, p13, p19).

Chen concludes that her paper defends algebraic relationalism, which states that the algebraic formalism for the theory of space and time is a genuine implementation of relationalism and "not equivalent to substantivalism" (Chen, 2024, p2, p22). To properly understand her formulation of algebraic relationalism, we note that Chen endorses the substantivalist reading of the standard manifold formalism, i.e. that the standard manifold formalism is an implementation of substantivalism. For the algebraic formalism for

[8]Chen claims that one cannot avoid this issue by interpreting the Einstein ring abstractly, since "elements of $C^\infty(M)$ are invoked to describe the spacetime coincidence of fields"(Chen, 2024, p12).

the theory of space and time to be not equivalent to substantivalism, it hence has to be not equivalent to the standard manifold formalism. But what is *the* algebraic formalism for the theory of space and time in consideration? Chen didn't give a definite answer, but according to her, it should not be the Einstein algebra formalism, since she thinks it is not sufficiently interesting for physicists or relationalists. Furthermore, the examples of new algebraic formalisms she gives in the paper, despite that she acknowledges that they are preliminary, are argued by her to have greater conceptual advantages for physicists and relationalists than the Einstein algebra formalism does. Therefore, Chen calls for an exploration of new algebraic formalisms for the theory of space and time, and defends an optimistic prospect for algebraic relationalism.

2.4 Some Preliminary Responses

Chen's arguments are more ambitious and wide-ranging than the scope of this paper, i.e. the relationship between the Einstein algebra formalism and relationalism. In this subsection, I give some preliminary responses to her objections and single out the relevant parts for the purpose of this paper.

Chen's first objection is the only one of the three that concerns the smooth algebra formalism, hence the only one that directly concerns Theorem 1 and its philosophical implications. To start, I agree with Chen that a crucial motivation behind the smooth algebra formalism is to recover the standard manifold approach, as it is clearly stated by Nestruev[9] and recognized by Rosenstock *et al.*. I agree with Chen that smooth algebras have more stringent requirements than Einstein rings do. As stated before, it is not hard to see that an Einstein ring is an $\mathbb{R}$-algebra, but there is no indication in Geroch's original paper that Einstein rings must be geometric, complete, and smooth as defined in Definition 2. Therefore, it is unclear whether the categorical duality between smooth algebras and smooth manifolds shown in Theorem 1 is able to lead to any philosophical conclusion about the Einstein algebra formalism and relationalism. If Rosenstock *et al.* were to establish a conventional wisdom concerning the smooth algebra formalism and relationalism, then they would have succeeded with their duality theorem, but again in that case, as Chen pointed out, it would seem redundant to do so, as smooth algebras are defined to be equivalent to smooth manifolds in the first palce. This casts a doubt on the philosophical significance of Rosenstock *et al.*'s result, hence the conventional wisdom that the Einstein algebra formalism is not more "relationalist" than the standard manifold formalism — is it still true?

Nevertheless, I do not share Chen's concerns about the presence of so-called "geometric discourse" and geometric concepts in the definition of smooth algebras. Chen believes that they make smooth algebras not algebraic. It is not clear from her paper what definition of "algebraic-ness" she has in mind. But regardless of what it may be, it is not compatible with mathematical practice to regard any mathematical notion that involves concepts and terminologies like "neighborhood of points" or "open coverings" as not algebraic. To give a trivial example, there is nothing stop anyone from defining a trivial topology on a given algebraic structure, and in this way one can meaningfully talk

[9]The two definitions of smooth manifold (in which the algebraic approach and the coordinate approach result) are of course equivalent. ... Essentially, this book is a detailed exposition of these two approaches to the notion of smooth manifold and their equivalence (Nestruev, 2003, p11)

about neighborhood of points and open coverings. Numerous existing mathematical notions also deny the possibility of any clean division of "algebraic" versus "not algebraic" that Chen might have in minds. C^* algebras is a Banach space, Boolean algebras admit stone space representations, and Lie algebras are closely related to Lie groups and hence smooth manifolds. All of them are algebraic and at the same time involve "geometric discourse" and geometric concepts. Furthermore, it is not clear how being free from "geometric discourse" and geometric concepts is helpful for relationalism either. Relationalists, as explained in subsection 2.1, are motivated to get rid of excess structures of the spacetime manifold, for it doesn't represent anything in the relationalist's physical reality. However, that motivation doesn't necessarily extend to getting rid of everything non-algebraic. Therefore, I believe that smooth algebras are still algebraic, and that it is not necessary to insist that an algebraic formalism for the theory of space and time cannot involve "geometric discourse" or geometric concepts at all.

Chen's second and third objections concern Einstein algebras instead of smooth algebras. In a word, they raise the following question: why do we grapple with Einstein algebras at all, given that, as Chen believes, they are metaphysically (second objection) and physically (third objection) not desirable?

Recall that Chen argues that, since an Einstein ring has to be posited before other fields are defined, it enforces an ontological fundamentality of scalar fields. She then claims that it shows that an Einstein ring plays the role of a "ghostly background scalar field" (Chen, 2024, p12), which, she argues, is equivalent to that of the spacetime manifold in the standard manifold formalism and hence not desirable for relationalists. However, it is not clear that relationalists will share Chen's metaphysical worry. The thesis of relationalism is compatible with a possible hierarchy of ontological fundamentality of fields, describing a hierarchy of metaphysical fundamentality of relationships that exist among material bodies. Whether there is a legitimate hierarchy of metaphysical fundamentality of either can be an open question, which doesn't have to be settled for us to investigate questions concerning substantivalism and relationalism. Similarly, relationalists do not have to be unhappy about a "ghostly background scalar field" as they are unhappy about a spacetime manifold, as Chen suggests. The reason why relationalists want to get rid of the spacetime manifold is that it doesn't directly represent anything that *can* be real according to the relationalist's view of physical reality. Rather, it is usually supposed to represent a fundamental spatio-temporal structure whose existence is denied by relationalists. It is not clear that a similar remark can be made that scalar fields don't represent anything that *can* be real according to relationalism. Therefore, there is no sufficient reason why relationalists should reject the Einstein algebra formalism for the reasons that Chen gives.

Finally, I respond to her third objection. I echo Chen's call for exploring alternative algebraic formalisms, as there are sufficient interests of physicists and mathematicians to do so. This call for exploration, I believe, can be adequately justified without any metaphysical consideration. Nevertheless, it doesn't imply that it is not meaningful for philosophers to engage with Einstein algebras. The reason why it appears meaningless to do so is because that, in Chen's formulation of algebraic relationalism, it states that *the* algebraic formalism for the theory of space and time is a genuine implementation of relationalism. Therefore, it seems that, to establish or refute algebraic relationalism, one should first find out what *the* algebraic formalism for the theory of space and time is, then figure out whether it is a genuine implementation of relationalism in an appropriate sense. I doubt whether it is feasible to establish or refute algebraic relationalism in this formulation at all. Unless we can determine *the* algebraic formalism for the theory of space and time once and for all, or at least determine fairly many properties that it must have, algebraic relationalism in this formulation can only remain indeterminate. Nevertheless, it is unclear how one can determine what is *the* algebraic formalism for the theory of space and time. It is therefore questionable whether algebraic relationalism in this formulation is fruitful for philosophers to pursue, at least at the current stage. If we do not have algebraic relationalism in the form formulated by Chen as our goal, then her third objection against Einstein algebras cannot dissuade philosophers from engaging with Einstein algebras. Given that there is existing literature in philosophy of physics spanning several decades concerning Einstein algebras and relationalism, there is evidently enough interest for philosophers of physics to still investigate questions about Einstein algebras, despite that they might very well turn their attention to new algebraic formalisms in the future.

Therefore, we are left with only one objection of Chen, namely that Rosenstock *et al.*'s categorical duality established between smooth algebras and smooth manifolds doesn't seem to inform us anything about the relationship between the Einstein algebra formalism and the standard manifold formalism, for Einstein rings are defined much less stringently than smooth algebras are. It is not clear whether we should still trust their philosophical claim that the Einstein algebra formalism is not more "relationalist" than the standard manifold formalism. In the next section, I show that we should, via a re-visit to classic works of John Earman.

3 Re-cast the Categorical Duality

In this section, I take a step back to reflect on the role that Einstein algebras can be expected to play for relationalism. I will start with illustrating and formalizing the ideas behind Earman's classic works of his program of Leibniz algebras, for it is the first introduction of Einstein algebras to the substantivalism and relationalism debate. The formalization results in a new formal criterion for an algebraic formalism for the theory of space and time to be more "relationalist" than the standard manifold formalism. We will see that, based on the new criterion, the conventional wisdom that the Einstein algebra formalism is not more "relationalist" than the standard manifold formalism still holds, and the idea behind Rosenstock *et al.*'s categorical duality result can be preserved.

3.1 Einstein Algebras and the Program of Leibniz Algebras

In a series of publications, Earman (1977, 1979, 1986, 1989) introduces Einstein algebras as he explicates Leibniz' relational view on the spatio-temporal structure, which is extrapolated from Leibniz' writings on the nature of space and motion. Earman calls any model of the spatio-temporal structure of the form $\langle M, O_1, O_2, \cdots \rangle$ a *substantivalist world model*, where M is a smooth manifold and $O_i, i \in \mathbb{N}$ are geometric object fields on M, which indicates that he believes that the spatio-temporal structure written in the standard manifold formalism carries a substantivalist interpretation[10]. Given two substantivalist world models $\langle M, O_1, O_2, \cdots \rangle$ and $\langle M', O'_1, O'_2, \cdots \rangle$, if there is a diffeomorphism $\varphi : M \to M'$ such that $O'_i = \varphi^*(O_i), i \in \mathbb{N}$ — in Earman's words, φ^* denotes "dragging along" by the mapping φ (Earman, 1979, p268) — a relationalist like Leibniz must take the two substantivalist models as giving different descriptions of the same physical reality, instead of different physical realities. After all, space-time points are "descriptive fluff" (Earman, 1989, p170) for relationalists. In other words, one can say that these substantivalist world models are *Leibnizian equivalent*, which consequently gives rise to equivalence classes of substantivalist world models. According to Earman, an equivalence class of substantivalist world models corresponds to a single physical reality in the relationalist's sense[11]. However, it is not sufficient for relationalists to stop there, according to Earman. They need to complete an additional program, later called by Rynasiewicz (1992) "the program of Leibniz algebras", which consists of the following steps: (1) give a direct characterization of the relationalist's physical reality, i.e. the equivalence classes of substantivalist world models, without referencing to smooth manifolds. Earman (1977, 1979) calls such a direct characterization a *Leibniz world model*; (2) show that the laws of physics can be expressed directly in terms of Leibniz world models; (3) explain how Leibniz-equivalent substantivalist world models arise as different but equivalent representations of the same Leibniz world model; (4) show that the Leibniz world models are not subject to the hole argument.[12]

The motivation for first two steps can be attributed to the substantivalist interpretation that Earman attaches to the standard manifold formalism and the Quine-Putnam style indispensability argument for substantivalism. One notable Quine-Putnam style indispensability argument is given by Hartry Field, who states that, since space-time points seems indispensable to positing physical fields, relationalists have to come up a different way of describing fields to avoid a realist's commitment to space-time points. We see from the following quote that Earman shares Field's worry:

> But drawing circles around groups of space-time models and labeling them equivalence classes does not show that there is a viable alternative to substantivalism. To show that one would have to show how to do all the physics we did before without treating the fields O_j as residing in M; in effect, one would have to show how to do differential geometry without the differential manifold. (Earman, 1986, p237)

But more importantly, it is also evident that Earman expects more from relationalists than merely coming up with an alternative formalism for the theory of space and time that is manifold-free. Immediately after the previous quote, he states the following:

[10] The notation of a substantivalist world model varies in different pieces of Earman's writings. Here we follow (Earman, 1989, p171). He also calls certain substantivalist world models by certain names. For example,Earman (1977) defines a *Leibnizian pre-model*, which consists of a so-called *intermediate Leibnizian space-time* and a momentum field on the intermediate Leibnizian space-time (p100). The precise definition of the intermediate Leibnizian space-time does not concern us here. What's important for the purpose of this paper is that it is formulated in the standard manifold formalism, hence a substantivalist world model.

[11] See (Earman, 1977, p101), (Earman, 1979, p268), (Earman, 1986, p236-237), and (Earman, 1989, p171).

[12] My presentations of the program of Leibniz algebras is a combination of what Earman writes in (Earman, 1979) and (Earman, 1989). Step (4) is only explicit in (Earman, 1986, 1989), though a concern about indeterminism is visible in (Earman, 1977). See (Weatherall, 2020) for details.

> one would need to show how the old space-time models can be regarded as representations of the new models and prove that under the representation relation a single new model corresponds precisely to an equivalence class of old models. (Earman, 1986, p237)

That is to say, Leibnizian world models should correspond to Leibnizian-equivalent classes of substantivalist world models in the sense that a *representation* relation[13] between substantivalist world models and Leibniz world models should be defined, and that Leibniz-equivalent substantivalist world models represent one and the same Leibniz world models. This requires the step (3) of the program. In other words, Leibniz world models are expected to get rid of the "descriptive fluff" in the substantivalist world models, via the "many-to-one" representation relation from substantivalist world models to Leibniz world models.

Finally, Earman claims that "the desire for the possibility of determinism ... provides an independent motivation for a program like the above" (Earman, 1989, p172)[14]. The worry that the standard manifold formalism and the substantivalist interpretation of it imply the impossibility of determinism is most notably spelled out in (Earman & Norton, 1987), in the form of the hole argument. To briefly summarize, the hole argument is based on the *Hole Corollary* that's proven in the same paper, which states that given a substantivalist world model $\langle M, O_1, O_2, \cdots \rangle$ and a neighborhood $U \subseteq M$, there exists arbitrarily many $\langle M, O'_1, O'_2, \cdots \rangle$ which differ from $\langle M, O_1, O_2, \cdots \rangle$ only within U and is identical to $\langle M, O_1, O_2, \cdots \rangle$ on the boundary and outside of U. The neighborhood U is called a hole. If we place a hole U in the future of a time slice, then, for substantivalists, all the history up to that time slice is unable to determine the future, as all substantivalist world models which only differ within U match the history up to the given time slice. As Earman & Norton assume that substantivalists think distinct substantivalist world models are distinct physical realities, they argue that substantivalists have to deny any possibility of determinism for the theory of space and time[15]. The hole argument leads Earman to believe that the standard manifold formalism for the theory of space and time has excess structures and to suggest the Einstein algebra formalism as a suitable modification of it, as (Weatherall, 2020, p86) points out. That is to say, Earman expects relationalists to get rid of the excess structures in substantivalist world models, which he believes are shown to exist by the hole argument. This is what step (4) is for.

To sum up, the program of Leibniz algebras aims to ward off the indispensability argument and, more importantly, to get rid of excess structures in the standard manifold formalism. This is compatible with the role of Einstein algebras in the substantivalism versus relationalism debate explained in subsection 2.1, which is the conceptual basis for this paper. More specifically, the reasons why Earman believes that there are excess structures in the standard manifold formalism are that, on the one hand, relationalists should regard Leibniz-equivalent substantivalist world models as mere different descriptions of the same physical reality, and on the other hand, he believes that the standard

[13]Earman also calls a *realization* relation in several other places.

[14]Similar remarks can be found in (Earman, 1977) and (Earman, 1986).

[15]The hole argument receives great attention in the subsequent philosophical literature, including objections by Weatherall (2018) and Halvorson & Manchak (2024). This paper does not discuss the validity of the hole argument. The purpose of presenting the hole argument is solely to explain Earman's program of Leibniz algebras, for which it plays an important role (Weatherall, 2020, p81).

manifold formalism is subject to the hole argument. As a result, Leibniz algebras are expected to get rid of excess structures by accomplishing step (3) and (4).

3.2 Formalizing the Program of Leibniz Algebras

The program of Leibniz algebras specifies what it takes for a formalism for the theory of space and time to be more "relationalist" than the standard manifold formalism in Earman's view. We can formalize his ideas in the framework of categorical criterion of theoretical equivalence, and therefore connect them with the contemporary literature on theoretical equivalence. The program of Leibniz algebras concerns two kinds of world models: substantivalist world models and Leibniz world models. Substantivalist world models are connected by smooth maps that preserve geometric fields. Similarly, Leibniz world models are connected by algebraic homomorphisms that preserve fields defined algebraically. Suppose that we can describe the two kinds of world models by two categories — call them the *substantivalist category* and the *Leibniz category* respectively. Since substantivalist world models are expected to represent Leibniz world models, in the language of category theory, we can think of the representation relation as a functor to be defined from the substantivalist category to the Leibniz category. Call this the *representation functor*. If the representation functor, appropriately defined, can show the ways in which Earman expects Leibniz world models to get rid of the excess structures in substantivalist world models, then there are good reasons to believe that the Leibniz world models, i.e. the Leibniz algebras, provide a more "relationalist" formalism for the theory of space and time.

Returning to smooth manifolds and Einstein rings, the category **SmoothMan** takes the place of the substantivalist category here. For the algebraic side, we define the category **EinRings** as follows:

- objects: Einstein rings $\mathscr{E}, \mathscr{F}, \cdots$, and
- arrows: Einstein ring homomorphisms $h : \mathscr{E} \to \mathscr{F}$ where $\mathscr{E}$ and $\mathscr{F}$ are Einstein ringsn, and let $\mathscr{R}_{\mathscr{E}}$ be the subring of $\mathscr{E}$ that is isomorphic to $\mathbb{R}$ and $\mathscr{R}_{\mathscr{F}}$ be the subring of $\mathscr{F}$ that is isomorphic to $\mathbb{R}$, then $h|_{\mathscr{E}}$ is a ring isomorphism from $\mathscr{R}_{\mathscr{E}}$ to $\mathscr{R}_{\mathscr{F}}$.

We denote a representation functor from **SmoothMan** to **EinRings** by R. Step (3) and (4) of the program of Leibniz algebras can be interpreted as the following two expectations of the behavior of the representation functor R.

According to step (3), an Leibniz algebra should correspond to a Leibniz-equivalent class of substantivalist world models. Therefore, in terms of **SmoothMan** and **EinRings**, diffeomorphic smooth manifolds should be mapped by the representation functor R to one and the same Einstein ring. Moreover, as an Leibniz algebra is expected by Earman to directly characterize what an Leibniz-equivalent class of substantivalist world models represent, it should characterize the mathematical structure that is shared by substantivalist world models in one Leibniz-equivalence class. Hence a structure-preserving map defined between two Leibniz algebras should be expected to preserve less structure than a structure-preserving map of substantivalist world models does. As a result, for a morphism from one Leibniz algebra to another, there might not be a corresponding morphism from the the substantivalist world model that represents the first Leibniz algebra to the substantivalist world model that represents the second. Therefore, in terms of

SmoothMan and **EinRing**, we can interpret that the program of Leibniz algebras expects the representation functor R to be *not full*. Figure 1 illustrates this interpretation.

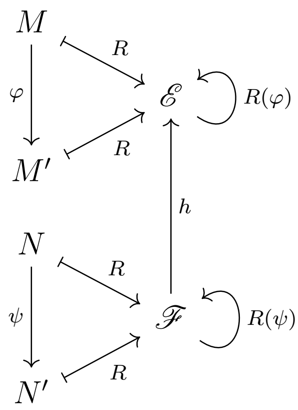

Figure 1:

Let diffeomoprhic smooth manifolds M and M' be mapped by the representation functor R to Einstein ring $\mathscr{E}$, the diffeomoprhism $\varphi : M \to M'$ be mapped to $R(\varphi) : \mathscr{E} \to \mathscr{E}$. Another pair of diffeomorphic smooth manifolds N and N' are mapped by the representation functor R to Einstein ring $\mathscr{F}$, the diffeomorphism $\psi : N \to N'$ is mapped to $R(\psi) : \mathscr{F} \to \mathscr{F}$. Let $h : \mathscr{F} \to \mathscr{E}$ be an arrow in **EinRings**. Based on the interpretation, the program of Leibniz algebras expects that there might not always be a morphism from M to N (or, similarly, a morphism from M' to N') that is mapped to h by R. That is to say, the fullness of the functor is not expected to hold.

As Earman believes that the hole argument is an important indicator that the substantivalist world models have excess structures, the Einstein algebra formalism — a candidate of a more "relationalist" formalism — is expected to get rid of the excess structures that lead to the hole argument. Step (4) of the program of Leibniz algebras is stipulated for this purpose. Recall that, according to the hole argument, the impossibility of determinism is a result of the existence of substantivalist world models that are identical except within a neighborhood of the spacetime manifold (the hole). These substantivalist world models have the same smooth manifold, and they can be derived from one another with a diffeomorphism of the smooth manifold to itself which leaves all the fields in the substantivalist world model unchanged except within the hole. Call these diffeomorphisms the *hole diffeomorphism*[16]. Since a relationalist formalism is expected by Earman to not be subject to the hole argument, we can understand that as expecting no counterpart of hole diffeomorphisms for Einstein algebras to exist. In terms of **SmoothMan** and **EinRings**, we can interpret this expectation to be that the representation functor R would fail to be faithful. To illustrate, suppose that, as in Figure 2, a smooth manifold M is mapped to an Einstein ring $\mathscr{F}$ by the representation functor R. Let $\varphi : M \to M$ be a hole diffeomorphism and id_M be the identity arrow of M. Since the program expects Einstein algebras to be free from the hole argument, the hole diffeomorphism φ would not be mapped by R to an isomorphism of $\mathscr{F}$ that has the potential

[16]Recently, Halvorson & Manchak (2024) argue that no hole isomorphism exists if it is required to be a isometry of relativistic spacetimes. Since this paper focuses on only the smooth manifold structure, we do not comment on the implications of Halvorson & Manchak (2024)'s result here.

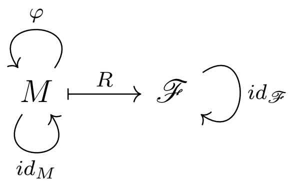

Figure 2:

to give rise to a hole argument on Einstein algebras. Instead, we expect φ to be mapped to the identity arrow $id_{\mathscr{F}}$ of $\mathscr{F}$, which would be the same as $R(id_M)$. Therefore, the representation functor R would be not faithful.

The formalization in this subsection shows that, according to the program of Leibniz algebras, the representation functor R from **SmoothMan** to **EinRings** has to fail to be full and faithful for the Einstein algebra formalism to be more "relationalist" than the standard manifold formalism. This criterion can be applied to any other algebraic formalism by substituting **EinRings** with the category of models of the algebraic formalism in consideration to investigate if it is more "relationalist" than the standard manifold formalism. This criterion is in a sense stricter than simply requiring the two categories to be not dual to each other, as it requires the duality to be spoiled specifically due to the failure of either full-ness or faithfulness of the representation functor. The requirement of essential surjectivity in the categorical criterion of theoretical equivalence is therefore not relevant to the objective of algebraic relationalism, according to the program of Leibniz algebras. In the next section, I defend this claim independently of the program of Leibniz algebras. For the rest of this section, we apply the new criterion to **SmoothMan** and **EinRings**.

3.3 Re-cast the Duality

Now we are ready to answer the question this paper started with: is the Einstein algebra formalism more "relationalist" than the standard manifold formalism? We adopt the formal criterion that the representation functor R from **SmoothMan** to **EinRings** has to fail to be full and faithful for a positive answer. We define the representation functor R from **SmoothMan** to **EinRings** as follows:

- Given a smooth manifold M, $R(M) = C^\infty(M)$.
- Given a smooth map $\varphi : M \to N$ from manifold M to manifold N, $R(\varphi) = \hat{\varphi} : C^\infty(N) \to C^\infty(M)$ where $\hat{\varphi}(f) = f \circ \varphi$ for all $f \in C^\infty(N)$ is an Einstein ring homomorphism[17].

The justification of this definition is straightforward. A smooth manifold M must be able to represent the Einstein ring $C^\infty(M)$. The mapping of arrows also makes intuitive sense. Then the following theorem is shown to be true:

Theorem 2. *The representation functor R : **SmoothMan** $\to$ **EinRings** is full and faithful.*

[17](Rosenstock *et al.*, 2015, Lemma 3.3) shows that, given a smooth map $\varphi : M \to N$, $\hat{\varphi} : C^\infty(N) \to C^\infty(M)$ defined as $\hat{\varphi}(f) = f \circ \varphi$ for all $f \in C^\infty(N)$ is an $\mathbb{R}$-algebra homomorphism. It is not hard to see that $\hat{\varphi}$ is also an Einstein ring homomorphism, as every Einstein ring is an $\mathbb{R}$-algebra.

Proof. To see that the representation functor R is faithful, let M and N be two smooth manifolds and $\varphi : M \to N$ and $\psi : M \to N$ be smooth maps such that $\varphi \neq \psi$. Then there must be some $p \in M$ such that $\varphi(p) \neq \psi(p)$. Since N is a Hausdorff manifold, there is a smooth map $f \in C^\infty(N)$ such that $f(\varphi(p)) \neq f(\psi(p))$. Hence $R(\varphi)(f) = \hat{\varphi}(f) \neq \hat{\psi}(f) = R(\psi)(f)$, which implies that $R(\varphi) \neq R(\psi)$.

To see that the representation functor F is full, let M and N be two smooth manifolds and $\theta : C^\infty(N) \to C^\infty(M)$ be an Einstein ring homomorphism. By (Nestruev, 2003, Theorem7.7), there is an atlas that can be defined on $|C^\infty(M)|$ such that $|C^\infty(M)|$ is a smooth manifold and $C^\infty(|C^\infty(M)|)$ is isomorphic to $C^\infty(M)$. Similarly, $|C^\infty(N)|$ can be endowed with a smoothness structure such that $C^\infty(|C^\infty(N)|)$ is isomorphic to $C^\infty(N)$. Furthermore, there is a bijective homeomorphism between $\theta_M : M \to |C^\infty(M)|$ defined by

$$\theta_M(p)(f) = f(p)$$

for all $p \in M$ and $f \in C^\infty(M)$, given by (Nestruev, 2003, Theorem 7.2). By (Rosenstock *et al.*, 2015, Theorem 3.5). θ_M is a diffeomorphism. Similarly, there is a diffeomorphism $\theta_N : N \to |C^\infty(N)|$. Define $|\theta| : |C^\infty(M)| \to |C^\infty(N)|$ as follows:

$$|\theta|(\gamma)(\theta_N(k)) = \theta(k)(\theta_M^{-1}(\gamma))$$

for all $\gamma \in |C^\infty(M)|$ and $k \in C^\infty(N)$. $|\theta|$ is a smooth map by the proof of (Rosenstock *et al.*, 2015, Lemma 3.4). Therefore, $\theta_N^{-1} \circ |\theta| \circ \theta_M : M \to N$ is a smooth map, i.e. an arrow in **SmoothMan**. Finally, $R(\theta_N^{-1} \circ |\theta| \circ \theta_M) = \theta$ because

$$R(\theta_N^{-1} \circ |\theta| \circ \theta_M)(g)(p) = g(\theta_N^{-1}(|\theta|(\theta_M(p)))) = \theta(g)(p)$$

for any $g \in C^\infty(N), p \in M$. □

Therefore, we conclude that the Einstein algebra formalism is not more "relationalist" than the standard manifold formalism. Theorem 2 does not involve any additional algebraic structure than the Einstein ring structure assumed in Geroch's original formulation of Einstein algebras. Hence it resolves the only remaining objection of Chen, which is that the conventional wisdom was supported by Rosenstock *et al.* (2015)'s duality theorem that involves smooth algebras instead of the less stringent Einstein rings. We therefore re-affirm the metaphysical status of the Einstein algebra formalism that Rosenstock *et al.* establish, which is that it is as "relationalist", and equivalently as "substantivalist", as the standard manifold formalism. The conventional wisdom established by Rosenstock *et al.* (2015) is still true, though the technical result that supports it takes a different form.

Moreover, the representation functor R is in fact almost the same as one contravariant functor that Rosenstock *et al.* use to prove the duality of **SmoothMan** and **SmoothAlg** (p312). Their duality theorem can therefore be re-stated as a corollary of Theorem 2:

Corollary. *The image of the representation functor R : **SmoothMan** $\to$ **EinRings** is equivalent to **SmoothAlg**.*

Proof. Since smooth algebras are Einstein rings and $\mathbb{R}$-algebra homomorphisms are Einstein ring homomorphisms, Theorem 2 shows that the image of the representation functor R is a full and faithful sub-category of **SmoothAlg**, in the sense that the inclusion functor from R[**SmoothMan**] to **SmoothAlg** which maps all objects and arrows to themselves is full and faithful. To see the equivalence, we need to show that the inclusion functor from R[**SmoothMan**] to **SmoothAlg** is essentially surjective. As pointed out by (Nestruev, 2003, Theorem 7.7), for any smooth algebra $\mathcal{A}$, there exists a smooth atlas on its dual space $|\mathcal{A}| =: M$ such that $\mathcal{A}$ is isomorphic to $C^\infty(M)$ as smooth algebras. That is to say, for any object $\mathcal{A}$ of **SmoothAlg**, there must be some object $R(M)$ of R[**SmoothMan**] where M is an object of **SmoothMan**, such that $\mathcal{A}$ is isomorphic to $R(M)$ in **SmoothAlg**. Therefore, the inclusion functor from R[**SmoothMan**] to **SmoothAlg** is essentially surjective. □

The corollary shows that the fundamental insight of Rosenstock *et al.* (2015)'s duality theorem is not as irrelevant to the Einstein algebra formalism and its metaphysical implication, as Chen's objection might have indicated. There are significant overlaps between the categorical criterion of equivalence, which Rosenstock *et al.* work with, and the stricter criterion this paper adopts to investigate the metaphysical implication of the Einstein algebra formalism. For one contravariant functor they use to establish categorical duality between **SmoothMan** and **SmoothAlg** is almost the same as the representation functor R, what Rosenstock *et al.* show is effectively a part of the picture that this paper presents.

4 The Irrelevance of Essential Surjectivity

We have given an answer to the question this paper started with. A crucial component of it is the new formal criterion for an algebraic formalism to be more "relationalist" than the standard manifold formalism. So far the only justification I have given for this formal criterion is the formalization of Earman's program of Leibniz algebras. In this section, I give another justification with the example of Sikorski algebras and differential spaces. The example shows that an algebraic formalism that spoils only the essential surjectivity of the representation functor from **SmoothMan** to a category of its models does not move us closer to relationalism. This is because a failure of essential objectivity has nothing to do with a failure of geometric reconstruction. The irrelevance of essential surjectivity supports the new formal criterion which requires only faithfulness and fullness of the representation functor.

To motivate the example, recall the categorical duality of **SmoothMan** and **SmoothAlg**. We are interested in the following question: if an algebraic formalism breaks the duality by only spoiling the essential surjectivity of the contravariant functor from **SmoothMan** to **SoomthAlg**, will that make the algebraic formalism more "relationalist" than the standard manifold formalism? We therefore relax the smoothness condition of smooth algebras and consider a less stringently defined algebraic structure, the Sikorski algebra, defined as follows:

Definition 3 (C^∞-closure)**.** *A geometric $\mathbb{R}$-algebra $\mathcal{A}$ is said to be C^∞-closed if for any finite collection of its elements $f_1, \cdots, f_k \in \mathcal{A}$ and any $g \in C^\infty(\mathbb{R}^n)$ for some n, there exists an element $f \in \mathcal{A}$ such that*

$$f(a) = g \circ (f_1(a), \cdots, f_k(a)), \text{ for all } a \in |\mathcal{A}|.$$

Note that the function $f \in \mathcal{A}$ here is uniquely determined, since $\mathcal{A}$ is geometric. (Nestruev, 2003, p33)

Definition 4 (Sikorski algebras)**.** *We call an* $\mathbb{R}$*-algebra* $\mathcal{A}$ *a Sikorski algebra if* $\mathcal{A}$ *is geometric, complete, and* C^∞*-closed.*

We note that the Sikorski algebras bear great similarities to C^∞-rings in the contemporary mathematics literature (for example, see (Joyce, 2012)). The rationale behind naming this algebraic structure "Sikorski algebras" is that, as Gruszczak *et al.* (1988) and Heller (1991) point out, Sikorski (1971) was the first who discussed this kind of algebraic structure. To see that Sikorski algebras have weaker requirements than smooth algebras do, we note the following facts:

Lemma 1. *Every smooth algebra is a Sikorski algebra. (Nestruev, 2003, Proposition 4.4)*

Lemma 2. *Not every Sikorski algebra is a smooth algebra.*

Proof. The collection of all real-valued continuous functions on $\mathbb{R}$, $C^0(\mathbb{R})$, is a Sikorski algebra but not a smooth algebra. □

We define the category **SikorskiAlg** as consisting of the following:

- objects: Sikorski algebras $\mathcal{A}, \mathcal{B}, \cdots$, and
- arrows: $\mathbb{R}$-algebra homomorphisms $i : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are Sikorski algebras.

Similarly, we define a representation functor R' from **SmoothMan** to **SikorskiAlg**, based on how Rosenstock *et al.* (2015) define the contravariant functor from **SmoothMan** to **SmoothAlg** (p312), as follows:

- Given a smooth manifold M, $R'(M) = C^\infty(M)$.
- Given a smooth map $\varphi : M \to N$ from manifold M to manifold N, $R(\varphi) = \hat{\varphi} : C^\infty(N) \to C^\infty(M)$ where $\hat{\varphi}(f) = f \circ \varphi$ for all $f \in C^\infty(N)$ is an $\mathbb{R}$-algebra homomorphism.

The representation functor R' also bears a great similarity to the representation functor R defined in the previous section. Moving from **SmoothAlg** to **SikorskiAlg** only spoils the essential surjectivity, as shown by the following theorem:

Theorem 3. *The representation functor* R' *from* ***SmoothMan*** *to* ***SikorskiAlg*** *is faithful and full, but not essentially surjective.*

Proof. By Lemma 1, a similar reasoning to the proof of Theorem 2 shows that the representation functor R' is faithful and full. Lemma 2 implies that the functor R' is not essentially surjective, for the reasons that $\mathbb{R}$-algebra isomorphisms, i.e. isomorphism arrows in **SikorskiAlg**, preserve smoothness of $\mathbb{R}$-algebras and that all objects in the range of R' are smooth algebras. □

Theorem 3 shows that Sikorski algebras satisfy the antecedent of the question we are investigating. In the rest of this section, I show that the Sikorski algebra formalism is not more "relationalist" than the standard manifold formalism, independent of the argument based on the program of Leibniz algebras in section 3. To establish this conclusion, we first note that the categorical duality of **SmoothMan** and **SmoothAlg** shows that the standard manifold formalism and the smooth algebra formalism have the same mathematical structure. Neither of them favors relationalism more than the other formalism does, for the reason that:

> Both encode precisely the same physical facts about the world, in somewhat different languages. (Rosenstock *et al.*, 2015, p315-316)

We follow this line of thought and show that Sikorski algebras encode precisely the same physical facts about the world as a generalized geometric structure from smooth manifolds — *differential spaces* (see Sikorski (1971), Heller (1992, 1991), and Gruszczak *et al.* (1988)) — by establish a categorical duality. We introduce the following definitions associated with differential spaces:

Definition 5 (Differential spaces)**.** *Let M be a set and D be a collection of $\mathbb{R}$-valued maps on M. Let τ_D be the coarsest topology on M such that all functions in D are continuous and that the following two hold:*

1. *if $f : M \to \mathbb{R}$ is a map such that, for every $p \in M$, there is a neighborhood U of p in the topological space (M, τ_D) and a map $g \in D$ such that $f|_U = g|_U$, then $f \in D$.*
2. *for any $n \in \mathbb{N}$ and any function $\omega \in C^\infty(\mathbb{R}^n)$, $f_1, \cdots, f_n \in D$ implies $\omega \circ (f_1, \cdots, f_n) \in D$.*

Then we say that D is a differential structure on M and that (M, D) is a differential space. (Gruszczak et al.*, 1988, Definition 3.1-3.4)*

Definition 6 (D-maps between differential spaces)**.** *Let (M, D_M) and (N, D_N) be differential spaces. A map $\Phi : M \to N$ is called a d-map from (M, D_M) to (N, D_N) if $h \circ \Phi \in D_M$, for every $h \in D_N$.*

Definition 7 (D-diffeomorphisms between differential spaces)**.** *Let (M, D_M) and (N, D_N) be differential spaces. A bijective map $\Phi : M \to N$ is called a d-diffeomorphism of (M, D_M) and (N, D_N) if $h \circ \Phi \in D_M$ and $g \circ \Phi^{-1} \in D_N$, for every $h \in D_N$ and $g \in D_M$. (Gruszczak* et al.*, 1988, Definition 4.1)*

According to Gruszczak *et al.* (1988) and Heller (1991), differential spaces are generalizations of smooth manifolds. An additional condition can be imposed on differential spaces to turn them into *differential manifolds*, which are equivalent to smooth manifolds defined in the standard way[18]. Generalizing from **SmoothMan**, we define a new category **D-Spaces** as consisting of the following:

- objects: differential spaces $(M, D_M), (N, D_N), \cdots$, and
- arrows: d-maps $\Phi : M \to N$ where (M, D_M) and (N, D_N) are differential spaces.

Finally, we show that Sikorski algebras and differential spaces have the same mathematical structure by establishing the following categorical duality:

Theorem 4. ***SikorskiAlg*** *is dual to* ***D-Spaces****.*

Proof. Define a contravariant functor G : **SikorskiAlg** $\to$ **D-Spaces** as follows:

- For each object $\mathcal{A}$, $G(\mathcal{A}) = (|\mathcal{A}|, \mathcal{A})$.
- For each arrow $i : \mathcal{A} \to \mathcal{B}$ between Sikorski algebras $\mathcal{A}$ and $\mathcal{B}$, $G(i) : |\mathcal{B}| \to |\mathcal{A}|$ is defined as $G(i)(p)(x) := i(x)(p)$, for all $p \in |\mathcal{B}|, x \in \mathcal{A}$.

We show that the functor G is well-defined. First, it is not hard to see that $G(\mathcal{A})$ is a differential space, since $\mathcal{A}$ is a complete and C^∞-closed $\mathbb{R}$-algebra. Secondly, we show that, for each $\mathbb{R}$-algebra homomorphism $i : \mathcal{A} \to \mathcal{B}$ between Sikorski algebras $\mathcal{A}$ and $\mathcal{B}$, $G(i) : |\mathcal{B}| \to |\mathcal{A}|$ defined as above is a d-map. Let $x \in \mathcal{A}$, we need show that $x \circ G(i) : |\mathcal{B}| \to \mathbb{R}$ is a continuous map. Since a composition of continuous maps is continuous, it is sufficient to show that $G(i)$ is a continuous map from $|\mathcal{B}|$ to $|\mathcal{A}|$. To see

[18]For a technical explanation of this equivalence, see (Gruszczak *et al.*, 1988, section 4).

that, note that for every $U \subseteq G(i)[|\mathcal{B}|]$ such that $U = x^{-1}(U_{\mathbb{R}})$ for some $x \in \mathcal{A}$, $U_{\mathbb{R}}$ an open set in $\mathbb{R}$, i.e. U is open given the subspace topology of τ_A on $G(i)[|\mathcal{B}|]$, we have

$$[G(i)]^{-1}[U] = [i(x)]^{-1}[U_{\mathbb{R}}],$$

which is an open set in $|\mathcal{B}|$ given τ_B. This is because that $[G(i)]^{-1}[U] = \{p \in |\mathcal{B}| : G(i)(p)(x) \in U\} = \{p \in |\mathcal{B}| : i(x)(p) \in U_{\mathbb{R}}\}$ and $i(x) \in \mathcal{B}$ by definition. Therefore $G(i)$ is continuous. Therefore, $G(i)$ defined as above is a d-map between $|\mathcal{B}|$ and $|\mathcal{A}|$. The contravariant functor G : **SikorskiAlg** $\to$ **D-Spaces** is well-defined.

Define a contravariant functor H : **D-Spaces** $\to$ **SikorskiAlg** as follows:

- For each object (M, D_M) in **D-Spaces**, $H((M, D_M)) = D_M$.
- For each arrow $\Phi : M \to N$, i.e. a d-map between (M, D_M) and (N, D_N), $H(\Phi) : D_N \to D_M$ is defined by $H(\Phi)(k) = k \circ \Phi$ for any $k \in D_N$.

To see that H is a well-defined contravariant functor, first note that for a differential space (M, D_M), there is a one-to-one correspondence between M and $|D_M|$, characterized by the map $\eta_M : M \to |D_M|$ defined as follows:

$$\eta_M(p)(f) = f(p)$$

for all $p \in M$, $f \in D_M$. Therefore, if (M, D_M) is a differential space, then D_M is a Sikorski algebra, by definition. Next, it is not hard to see that, for each arrow $\Phi : M \to N$, $H(\Phi) : D_N \to D_M$ is defined by $H(\Phi)(k) = k \circ \Phi$ for any $k \in D_N$ is an $\mathbb{R}$-algebra homomorphism, for it preserves the vector space operations, the product, and the multiplicative identity. Therefore, H is a well-defined contravariant functor.

Now we show that GH : **D-Spaces** $\to$ **D-Spaces** is naturally isomorphic to $1_{\textbf{D-Spaces}}$. We define a family of maps associated with GH, between objects (M, D_M) and $GH((M, D_M)) = (|D_M|, D_M)$ in **D-Spaces**, as follows:

$$\eta_M : M \to |D_M| \quad \text{s.t.} \quad \eta_M(p)(f) = f(p)$$

for all $p \in M$, $f \in D_M$. That η_M is a d-diffeomorphism follows from the fact that the two differential spaces have the same differential structure and that it is surjective by definition. Therefore, GH is naturally isomorphic to $1_{\textbf{D-Spaces}}$. One the other hand, we can see that HG : **SikorskiAlg** $\to$ **SikorskiAlg** is the same as $1_{\textbf{SikorskiAlg}}$ by definition of G and H. Therefore, **D-Spaces** is dual to **SikorskiAlg**. □

As Sikorski algebras and differential spaces have the same mathematical structure, to say that Sikorski algebras are more "relationalist" than the standard manifold formalism is equivalent to saying that differential spaces are more "relationalist" than the standard manifold formalism. Nevertheless, it doesn't make sense to say that the differential space formalism makes a weaker commitment to the existence of a fundamental spatio-temporal structure than the smooth manifold formalism does. If the smooth manifold formalism is considered to represent the fundamental spatio-temporal structure, then the differential space formalism merely disagrees with it about what this fundamental spatio-temporal structure precisely is. As differential spaces are generalizations of smooth manifolds, from a substantivalist's perspective, the differential space formalism simply stipulates a fundamental spatio-temporal structure that is different from the fundamental spatio-temporal structure that the standard manifold formalism stipulates. Differential spaces cannot get rid of the excess structures of the standard manifold formalism that are usually associated with substantivalism. Instead, the excess structures in the standard manifold formalism are re-defined in the formalism of differential spaces, which can be taken to represent a substantival reality that is different from the substantival reality that the standard manifold formalism can represent. In both case, the substantival realities and the representation apparatus in the formalisms should be denied by relationalists. The equivalence of Sikorski algebras and differential spaces therefore leads to the conclusion that the Sikorski algebra formalism are not "relationalist" than the standard manifold formalism. If anything, the Sikorski algebra formalism, and equivalently the differential space formalism, just tells a different substantivalist's story about the fundamental spatio-temporal structure than the standard manifold formalism does.

To sum up, the example of Sikorski algebras shows that the essential surjectivity of the representation functor is not relevant to the question of whether an algebraic formalism is more "relationalist" than the standard manifold formalism, for the reason that, despite that the representation functor R' from **SmoothMan** to **SikorskiAlg** is not essentially surjective, Sikorski algebras cannot be reasonably viewed as more "relationalist" because of the equivalence of them to differential spaces. This example justifies the proposed formal criterion which states that an algebraic formalism for the theory of space and time is more "relationalist" than the standard manifold formalism if the appropriately defined representation functor from **SmoothMan** to the category of algebraic models fails to be faithful and full, regardless of whether it is essentially surjective or not.

5 Conclusion

To conclude, this paper started with presenting Rosenstock *et al.* (2015)'s duality theorem of smooth manifolds and smooth algebras and the conventional wisdom which states that the Einstein algebra formalism is not more "relationalist" than the standard manifold formalism. Then I summarized and addressed Chen (2024)'s recent objections to the conventional wisdom. Among the objections, I agreed with Chen that Rosenstock *et al.* work with smooth algebras while the conventional wisdom concerns Einstein algebras instead. Nevertheless, after a re-visit of Earman's classic works on the program of Leibniz algebras, I proposed a new formal criterion to determine whether an algebraic formalism is more "relationalist" than the standard manifold formalism or not. Based on the new formal criterion, I showed that the conventional wisdom is still true, though supported by a different technical result. Finally, I provided another justification of the new formal criterion, which also provided additional support on the conclusion that the Einstein algebra formalism is not more "relationalist" than the standard manifold formalism.

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