The identification of numbers with operators

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Abstract

This article describes confirmation of the proposition that numbers are identified with operators in the following three steps. 1. The set of operators to construct finite cardinals satisfies Peano Axioms. 2. Accordingly, the natural numbers can be identified with these operators. 3. From the operators, five kinds of operators are derived, and on the basis of the step 2, the integers, the fractions, the real numbers, the complex numbers and the quaternions are identified with the five kinds of operators respectively. These operators stand in a sequential inclusion relationship, in contrast to the embedding relationship between those kinds of numbers defined as sets.

Keywords: numbers; operators; cardinals; structures of sets; iteration; activation.

1. Introduction

Our intuitive conception of a natural number would be the number of elements of a finite set, such as the number of the people in a room, namely, finite cardinals. Then, a cardinal is a common property to the sets with the same number of elements, where the same number is defined using a bijection between the sets. That is, a cardinal is the set of the sets related by bijections. Thus, a natural number is defined as the set of these sets. Russell (1919) and Holmes (1998) give this definition of the natural numbers.

Then, 'the set of the sets with the same number of elements' in this definition can be replaced with a different formulation. Since the sets with a cardinal have a common structure preserved under bijections between them, this structure is identified with the cardinal. For instance, three persons in a room sitting on three chairs respectively make a set of cardinal 3. Then, any combination of three persons is a set of cardinal 3. These sets have a common structure regardless of the combinations of three persons, i.e., the room with the three chairs. The chairs represent spaces where persons occupy in the room. The spaces differ each other and any person or objects in general can occupy any space one to one. These spaces are denoted by s. Then, the structure of the sets of cardinal 3 is represented by $\{s, s, s\}$. These three ss are spaces different each other to situate different objects, but are not differentiated as components with the same capacity in the structures, like chairs in a room or bricks in a brick wall. By assigning any three objects to the three ss in $\{s, s, s\}$, sets of cardinal 3 are acquired, for example, {Socrates, the Earth, Eiffel Tower}, and this structure keeps constant between the sets related by bijections. This structure is equivalent with the set of the sets with three elements or cardinal 3. Thus, the above definition of cardinals is equivalent with that based on the structures stated above.

To define the natural numbers, Russell (1919) constructs the natural numbers from the null set and a successor function. The successor function is the operator to add a new element to each of the sets with a finite cardinal, say n, to construct the sets with the cardinal $n+1$. Then, the set of finite cardinals satisfies Peano Axioms. Thus, the natural numbers can be defined as the set of the cardinals related by the successor function with the null set as the beginning of the relation.

In the case of cardinals defined as set structures, the successor function is the operator to add a new space to a set structure with, say *n* spaces, to construct the set structure with *n+1* spaces. As the beginning of the construction of cardinals, let the set structure with no space introduce into the set of set structures, which defines the cardinal 0, and is denoted by $\emptyset$. The operator 'addition of a new space *s* to a set structure η' is denoted by $P(\eta)$. For example, $\mathrm{P}(\emptyset) = \{s\}$, $\mathrm{P}\big(\mathrm{P}(\emptyset)\big) = \mathrm{P}(\{s\}) = \{s, s\}$.
The set of all set structures constructed in this way is supposed to satisfy Peano Axioms. If this is the case, the natural numbers can also be defined as the set of the set structures constructed in this way.

Corresponding to the set of these set structures, the operators to construct them, which are iterations of P operating on $\emptyset$, form a set. If the set of the set structures satisfies Peano Axioms, this set of operators should also satisfies Peano Axioms. Then, there is a possibility that the natural numbers are identified with these operators. In general, operators are components of the world different from sets. They transform objects and cause various phenomena. Addition, for instance, is an action that creates an object from objects, but sets are not action or do not possess action as their attribute. In other words, the extension of an operator, such as the set of ordered pairs, do not have the capacity of the operation to create something in the world. Therefore, I try to construct operators based on the operator P with which the natural numbers are identified. If this is possible, we can further suppose that there may exist certain extensions of the operators with which the integers, the fractions, or others are identified. These anticipations are examined in the following sections.

2. The natural numbers N

The set of all the set structures constructed from iterations of P on $\emptyset$ is denoted by *[N]*. For this construction, it is postulated that whenever P operates on a set structure, a new space not in the set structure exists. Then, it can be shown that [N] with P as a successor function satisfies the following Peano Axioms:

1.0 $\in$ N;
2. For any $n \in$ N, there is only one $n' \in$ N, where $'$ is a successor function, ;
3. For any $n, m \in$ N, if $n \neq m$, then, $n' \neq m'$;
4. There is no $n \in$ N, such that $n' = 0$;
5. If S $\subseteq$ N such that 0 $\in$ S, and for all $n \in$ S , $n' \in$ S , then S = N.

The proof is as follows.
1. $\emptyset \in$ [N], by the definition of $\emptyset$.
2. There is only one $\mathrm{P}(\eta) \in$ [N] for any $\eta \in$ [N]. This is obvious by the definition of [N].
3. Let $\alpha, \beta \in$ [N] and $\alpha \neq \beta$, then, $\mathrm{P}(\alpha) = \{the\ spaces\ in\ \alpha, s\}$, and $\mathrm{P}(\beta) = \{the\ spaces\ in\ \beta, s\}$. Since there is no bijection between α and β, so is no bijection between $\mathrm{P}(\alpha)$ and $\mathrm{P}(\beta)$. Thus, $\mathrm{P}(\alpha) \neq \mathrm{P}(\beta)$.
4. There is no such η that $\mathrm{P}(\eta) = \emptyset$. This is obvious by the definition of $\emptyset$.
5. Let $S \subseteq$ [N], $\emptyset \in$ S, and for any $\eta \in$ S , $\mathrm{P}(\eta) \in$ S. If S $\neq$ [N], there exists $\chi \in$ [N] such that $\chi \notin S$. Let the first element with respect to P within $\chi \notin$S be β. Then there exists $\alpha \in$ S such that $\mathrm{P}(\alpha) = \beta$, which is followed by $\mathrm{P}(\alpha) \in$S. Hence, there is no such β. That is, S = [N].

P(P(P ⋯ (η) ⋯)) is denoted by $P * P * P * \cdots (\eta)$. Iteration of P, $P * P * P \cdots$, is called a *connection of Ps*, and the iteration times of P in the connection is called *it-times*, and P is called *it-unit* of the connection.

[N] is constructed by operating connections of Ps on ∅ without limit. Then, [N] do not include the set structure with infinitely many elements, which is the limit that the construction of operators cannot reach.

Since construction of [N] follows corresponding construction of operators, i.e. connections of Ps, these operators also form a set. This set plus the operator that adds no space to a set structure, denoted by P^0, is denoted by $[N^o]$. Then, it is obvious that $[N^o]$ also satisfies Peano Axioms with $P *$ as the successor function.

1. $P^0 \in [N^o]$;
2. If $a \in [N^o]$, then $P * a \in [N^o]$;
3. If $a \neq b$ for $a, b \in [N^o]$, then, $P * a \neq P * b$;
4. There is no $b \in [N^o]$ such that $P * b = P^0$;
5. If $S \subseteq [N^o]$, $P^o \in S$ and for all $n \in S$, $P * n \in S$, then $S = [N^o]$.

The definition of addition between elements of [N] follows addition between elements of $[N^o]$. It would be natural that the latter addition is connection of the elements with $*$ to form a connection of Ps.

• Definition of addition on $[N^o]$.

For $a, b \in [N^o]$,
$a + b = a * b$.

This addition satisfies Peano Axioms on addition:
1. $\forall x(x + 0 = x)$;
2. $\forall x \forall y(x + y' = (x + y)')$.

By the definition of addition stated above,
1. For any $x \in [N^o]$, $x + P^0 = x * P^0 = x$;
2. For any $x,\ y \in [N^o]$, $x + P * y = x * (P * y) = x * P * y = P * x * y = P * (x * y)$,

because *x* is a connection of Ps.
Thus, this addition satisfies those axioms.

Since every element of $[N^o]$ is a simple iteration of Ps by $*$, there is no order of Ps in the connection. That is, addition of elements of $[N^o]$ is irrelevant to the order of connection of Ps in the elements. Therefore, the commutative law and associative law hold for this addition. Then, P^o is the additive identity for this addition.

An operator *a*, an iteration of P it-times in *a*, is denoted as $\Sigma^a P$. Then,
$a + b = a * b = \Sigma^a P * \Sigma^b P = \Sigma^{a+b} P$.
This is the connection of Ps it-times of P in *a* plus those in *b*.

Addition between elements of [N] follows from addition in $[N^o]$.
For $\alpha, \beta \in [N]$, there are $a, b \in [N^o]$ such that
$\alpha = a(\emptyset)$ and $\beta = b(\emptyset)$.

Then, addition of α and β is defined as
$\alpha + \beta = a * b(\emptyset)$.
Naturally, Peano Axioms on addition hold for [N] with this addition.
The associative law and commutative law also hold for [N], where ∅ is the additive identity.

• The conception of multiplication $a \times b$ for $a, b \in$ [N^o], is iteration of a it-times of P in b, i.e. a is regarded as the it-unit of $a \times b$. Then, multiplication on [N^o] is defined as
$a \times b = \Sigma^b a$

Then, Peano Axioms on multiplication
1. $\forall x(x \times 0 = 0)$;
2. $\forall x \forall y(x \times y' = xy + x)$;
are satisfied by this definition in the following way.

1. For any $a \in$ [N^o], $a \times \mathrm{P}^o = \Sigma^{Po} a = \mathrm{P}^o$;
2. $a \times (P + b) = \Sigma^{P*b} a = \Sigma^P a * \Sigma^b a = a * \Sigma^b a = \Sigma^b a * a = a \times \mathrm{b} + a$.
Since
$a \times P = \Sigma^P a = a,$
P is the unit element for this multiplication.

The associative law and commutative law of this multiplication and the distributive law for [N^o] can be proved by simply comparing the it-times of P in the right side and the left side of each of the following equations.
1. $a \times b = b \times a$;
2.$a \times (b \times c) = (a \times b) \times c$;
3.$a \times (b + c) = (a \times b) + (a \times c)$.
I omit the proofs of these equations.

For $\alpha, \beta \in$ [N], let $\alpha = a(\emptyset)$ and $\beta = b(\emptyset)$), where $a, b \in$ [N^o]. Then, multiplication of α and β is defined as
$\alpha \times \beta = (a \times b)(\emptyset)$.

Naturally this multiplication satisfies Peano Axioms on multiplication. The commutative law, associative law and distributive law also hold for [N], according to those that hold for [N^o].

The elements of [N] formed according to the construction of the set structures are ordered by the numbers of spaces in the set structures, i.e. $\emptyset < \{s\} < \{s,\ s\} < \{s, s, s\}\ldots$, which is a total order. The elements of [N^o] , which correspond to set structures, are also ordered by their it-times of P.

As a result of the discussion stated above, N is defined as [N]. At the same time, N can be identified with the set of operators [N^o]. This would be another viewpoint on the natural numbers, in contrast to that based on the numbers of elements in sets. The following discussion is constructed on this operator-based viewpoint on numbers.

3. The integers Z

There is an operator that operates on P to reverse the direction of its operation, that is, from addition of a space to a set structure to subtraction of a space from a set structure. This operator is denoted by –. For $a \in$ [N^o], $-a(\eta)$ is the operator that subtract spaces one by one the it-times of P in a from the set structure η. Since
$a * -a = P^o,$
–a is the inverse element of a. The operator that does not vary the direction of P is denoted by $+'$: $+'a = a$. The parameter that ranges over the set $\{+', -\}$ is denoted by Δ. Then the set of Δa, for $a \in$ [N^o] is denoted by [ΔN^o].

• Addition on $[\Delta N^{o}]$ is defined by extending that on $[N^{o}]$.
For $a, b \in [\Delta N^{o}]$,
$a + b = a * b$.

The additive identity is P^{0}: $a + P^{o} = a * P^{o} = a$.
The inverse element of $a \in [\Delta N^{o}]$ is $-a$.

An operator that is a connection of it-unit operator ΔP is irrelevant to the order of it-unit operators in the connection. Hence, this addition satisfies the associative law and commutative law. There are cases where the results of the operations by operators of $[\Delta N^{o}]$ on set structures do not exist in [N], e.g. $-P(\emptyset)$. However, operators themselves can be constructed from operators in $[N^{o}]$, which have results of their operations.

• Multiplication on $[\Delta N^{o}]$ is also defined by extending that on $[N^{o}]$.
For $a, b \in [N^{o}]$, $\Delta a \times \Delta b$ is iteration of Δa it-times of P in b in the direction of the Δ in Δ*b*, which is denoted as
$\Delta a \times \Delta b = \Sigma^{\Delta b} \Delta a$.

For example,
$\Delta a \times (+'P) = \Sigma^{P} \Delta a = \Delta a$
i.e. The operator $+'P = P$ is the unit element for this multiplication;
$\Delta a \times (-P) = \Sigma^{-P} \Delta a = -\Delta a$.
Then, the following equations hold.
$\Delta P^{o} = P^{o}$;
$\Delta a \times P^{o} = P^{o}$;
$a \times b = (+'a) \times (+'b) = (-a) \times (-b) = +'(a \times b)$;
$(-a) \times (+'b) = (+'a) \times (-b) = -(a \times b)$.

• The associative law of this multiplication is derived using mathematical induction. For $a, b \in [\Delta N^{o}]$,
1. $a \times (b \times \Delta P) = a \times \Delta b = \Delta(a \times b) = (a \times b) \times \Delta P$ … (1)
2. To the next step of the proof, the distributive law
$b \times (c + d) = (b \times c) + (b \times d)$ for $c, d \in [\Delta N^{o}]$,
is necessary. Let $d = \Delta P$, then,
$b \times (c + \Delta P) = \Sigma^{c+\Delta P} b = (b \times c) + (b \times \Delta P)$ … (2).
By the induction hypothesis,
$b \times (c + (d + \Delta P)) = b \times ((c + d) + \Delta P) = b \times (c + d) + \Delta b = (b \times c) + (b \times d) + \Delta b = (b \times c) + (b \times (d + \Delta P))$ … (3).
Thus, the distributive law follows from the equations (2) and (3). Hence,
$a \times (b \times (c + \Delta P)) = a \times ((b \times c) + \Delta b) = a \times (b \times c) + \Delta(a \times b)$ … (4).
On the other hand.
$(a \times b) \times (c + \Delta P) = (a \times b) \times c + \Delta(a \times b)$ … (5).
By the induction hypothesis, the formula (5) is equivalent with the formula (4). Hence, the associative law follows from the equations (1), (4), and (5); $(a \times b) \times c$ can be denoted as $a \times b \times c$.

• The commutative law
$a \times b = b \times a$
is proved in the similar way as the associative law.

1.$a \times \Delta\mathrm{P} = \Delta\mathrm{P} \times a \ldots$ (6).
2.$a \times (b + \Delta\mathrm{P}) = a \times b + \Delta a \ldots$ (7).
3. To prove $(b + \Delta P) \times a = b \times a + \Delta a \ldots$ (8),
The distributive law, $(a + b) \times c = a \times c + b \times c$, is necessary in advance.
In the first place,
$(a + b) \times \Delta\mathrm{P} = a \times \Delta\mathrm{P} + b \times \Delta\mathrm{P}$.
By the induction hypothesis,
$(a + b) \times (c + \Delta\mathrm{P}) = ((a + b) \times c) + \Delta(a + b) = a \times c + b \times c + \Delta a + \Delta b.$
On the other hand,
$a \times (c + \Delta\mathrm{P}) + b \times (c + \Delta\mathrm{P}) = a \times c + \Delta a + b \times c + \Delta b.$
Thus,
$(a + b) \times (c + \Delta\mathrm{P}) = a \times (c + \Delta\mathrm{P}) + b \times (c + \Delta\mathrm{P}) = a \times c + \Delta a + b \times c + \Delta b.$
Accordingly, the distributive law, hence, equation (8), holds. Therefore, the commutative law follows from the equations (6), (7), and (8).

• The distributive law has been proved in the course of the above two proofs.

The order relation in $[\mathrm{N}^{\mathrm{o}}]$ can be extended to $[\Delta\mathrm{N}^{\mathrm{o}}]$ by ordering $-a$ for $a \in [\mathrm{N}^{o}]$, as inverse order of $a \in [\mathrm{N}^{\mathrm{o}}]$ and combining this order with that in $[\mathrm{N}^{\mathrm{o}}]$. Thus, the set of integers Z is identified with $[\Delta\mathrm{N}^{\mathrm{o}}]$, which is written by *[Z]*.

4. The fractions Q

When $a = b \times c = \Sigma^{c} b$ for $a, c \in$ [Z], an operator to construct b from a and c can be defined as a divided equally it-times of P in c. This operator is denoted by $\div\,(a, c)$ or $a \div c$. When $\mathrm{P} = b \times c$, $\mathrm{P} \div c$ is the inverse element of c, which is denoted by P_c: P divided equally it-times of P in c, so it is a part of P. Thus, P_c is associative and commutative with P. Then,

$\mathrm{P} \div \mathrm{P}_c = c$;
$\mathrm{P} \div \mathrm{P} = \mathrm{P}_{\mathrm{P}} = \mathrm{P}$ by definition;
$a \div c = (a \times (\mathrm{P}_{\mathrm{c}} \times c)) \div c = ((a \times \mathrm{P}_c) \times c) \div c = \Sigma^{c}(a \times \mathrm{P}_c) \div c = a \times \mathrm{P}_{\mathrm{c}}$.

Except for the case $x = \mathrm{P}$, the inverse element of $x \in$ [Z], P_x, is not a member of [Z]. The set of operators $\div\,(x, y)$ ($= \mathrm{P}_y \times x$, for $x, y \in$ [Z]) is written by *[Q]*. It should be noted that $\div\,(x, \mathrm{P}^0)$ cannot be constructed. By the definition,
$\div\,(a, \mathrm{P}) = a$, for $a \in$ [Z].
That is, $\div(a, \mathrm{P})$ and a are the same operators. Therefore,
$[Z] \subset$ [Q].

• Since P_x is the same kind of operator as P, addition of $\mathrm{P}_y \times x$ is defined as $*$.
$\mathrm{P}_a \times m + \mathrm{P}_{\mathrm{b}} \times n = \mathrm{P}_a \times m * \mathrm{P}_{\mathrm{b}} \times n = \mathrm{P}_{a\times b} \times (m \times b) * \mathrm{P}_{b\times a} \times (n \times a) =$
$\Sigma^{m\times b * n\times a}\mathrm{P}_{a\times b} = \mathrm{P}_{a\times b} \times (m \times b + n \times a)$, for $a, b, m, n \in [Z]$.

For $x \in$ [Q],
$x + \mathrm{P}^0 = x$, and $x + (-x) = \mathrm{P}^0$.
Thus, additive identity and the inverse element of x for the addition are P^0 and $-x$ respectively.

• The associative law

$\mathrm{P}_a \times m + (\mathrm{P}_b \times n + \mathrm{P_c} \times k) = \mathrm{P}_a \times m + \mathrm{P_{b\times c}} \times (n \times c + k \times b) = \mathrm{P}_{a\times b\times c}(m \times b \times c + a \times (n \times c + k \times b)) = \mathrm{P}_{a\times b\times c} \times (m \times b \times c + a \times n \times c + a \times k \times b)$ … (1).
On the other hand,
$(\mathrm{P}_a \times m + \mathrm{P}_b \times n) + \mathrm{P}_c \times k = \mathrm{P}_{a\times b} \times (m \times b + n \times a) + \mathrm{P}_c \times k = \mathrm{P}_{a\times b\times c} \times ((m \times b + n \times a) \times c + a \times b \times k) = \mathrm{P}_{a\times b\times c}(m \times b \times c + n \times a \times c + a \times b \times k)$ … (2).
The associative law follows from equations (1) and (2).

• The commutative law
By the commutative law for [Z],
$\mathrm{P}_a \times m + \mathrm{P}_b \times n = \mathrm{P}_{a\times b} \times (m \times b + n \times a) = \mathrm{P}_{b\times a} \times (n \times a + m \times b) = \mathrm{P}_b \times n + \mathrm{P}_a \times m.$

• Multiplication on [Q] is also defined by extending that on [Z].
$(\mathrm{P}_a \times m) \times (\mathrm{P}_b \times n) = (\mathrm{P}_{a\times b} \times m \times b) \times (\mathrm{P_{b\times a}} \times n \times a) = \mathrm{P}_{a\times b} \times \mathrm{P}_{a\times b} \times m \times b \times n \times a = \mathrm{P}_{a\times b} \times (m \times n).$
Then,
$(\mathrm{P}_a \times m) \times (\mathrm{P_m} \times a) = \mathrm{P}_{a\times m} \times (m \times a) = \mathrm{P}$, and
$(\mathrm{P}_a \times m) \times \mathrm{P} = \mathrm{P}_a \times m.$
Thus,
$\mathrm{P_m} \times a$ is the inverse element of $\mathrm{P}_a \times m$, and P is the identity element for the multiplication.

• The associative law follows from the equations:
$(\mathrm{P}_a \times m) \times \big((\mathrm{P_b} \times n) \times (\mathrm{P}_c \times k)\big) = (\mathrm{P}_a \times m) \times \big(\mathrm{P}_{b\times c} \times (n \times k)\big) = \mathrm{P}_{a\times b\times c} \times \big(m \times (n \times k)\big) = \mathrm{P}_{a\times b\times c} \times (m \times n \times k).$
$((\mathrm{P}_a \times m) \times (\mathrm{P}_b \times n)) \times (\mathrm{P}_c \times k) = (\mathrm{P}_{a\times b} \times (m \times n)) \times (\mathrm{P}_c \times k) = \mathrm{P}_{a\times b\times c} \times ((m \times n) \times k) = \mathrm{P}_{a\times b\times c} \times (m \times n \times k).$

• The commutative law
$(\mathrm{P}_a \times m) \times (\mathrm{P_b} \times n) = \mathrm{P}_{a\times b} \times (m \times n) = \mathrm{P}_{b\times a} \times (n \times m) = (\mathrm{P}_b \times n) \times (\mathrm{P}_a \times m).$

• The distributive law
$(\mathrm{P}_a \times m) \times ((\mathrm{P_b} \times n) + (\mathrm{P}_c \times k)) = (\mathrm{P}_a \times m) \times (\mathrm{P_{b\times c}} \times (n \times c + k \times b)) = \mathrm{P}_{a\times \mathrm{b}\times \mathrm{c}} \times (m \times (n \times c + k \times b)) = \mathrm{P}_{a\times b\times c} \times (m \times n \times c + m \times k \times b)$ … (3).
On the other hand,
$(\mathrm{P}_a \times m \times \mathrm{P}_b \times n) + (\mathrm{P}_a \times m \times \mathrm{P}_c \times k) = (\mathrm{P}_{a\times b} \times (m \times n)) + (\mathrm{P}_{a\times c} \times (m \times k)) = \mathrm{P}_{a\times b\times a\times c} \times (m \times n \times a \times c) + \mathrm{P}_{a\times b\times a\times c} \times (m \times k \times a \times b) = \mathrm{P}_{a\times b\times c} \times (m \times n \times c + m \times k \times b)$ … (4).
The distributive law follows from the equations (3) and (4).

• Addition of $m \in [Z]$ and $\mathrm{P}_a \times n \in$ [Q], for $n \in [Z]$.
Since $m = \mathrm{P}_a \times a \times \mathrm{P_P} \times m = \mathrm{P}_a \times (a \times m)$,
$m + \mathrm{P}_a \times n = \mathrm{P}_a \times (a \times m) + \mathrm{P}_a \times n = \mathrm{P}_a \times (a \times m + n).$
As a result of the discussion stated above, the fractions Q is identified with [Q].

5. The real numbers R

The real numbers are infinite decimals that have their respective convergences. Then, the positive real numbers, denoted by R^+, are series that have the structure:

$$R^+(x_n, x_{n-1}, x_{n-2}, \cdots) = 10^n x_n + 10^{n-1} x_{n-1} + 10^{n-2} x_{n-2} + 10^{n-3} x_{n-3} + \cdots,$$

for any natural number n, where $x_{i,}\ i \leq n,$ is a variable that ranges over the set of natural numbers, $0 \leq x_i \leq 9$. The negative real numbers are represented by $-R^+(x_n, x_{n-1}, x_{n-2}, \cdots)$. Each real number is the convergence of an infinite series that is obtained by substituting the sequence of variables with a sequence of natural numbers satisfying the above condition.

Then, replacement of the sequence of the natural numbers with corresponding operators in [N^0] forms an infinite series of the operators in [Q]. These series of operators correspond to real numbers one to one. When a series is finite, it is also an operator as the addition of operators in [Q]. Then, introduction of an infinite series of operators that have convergences is an extended operator of a finite series of operators in [Q].

These operators correspond to the real numbers one to one. They are denoted by *[R]*. Since these extended operators are constructed from [Q], addition and multiplication of the extended operators are defined according to the construction. Accordingly, the associative law, commutative law and distributive law also hold for [R]. As a result, the real numbers R can be identified with [R].

6. The complex numbers C

Many physical objects that have operation on something go through levels of activation until they activate as operators. For instance, motors do not move until the electricity reaches to a certain voltage, or digestive enzymes begin to operate on foods when the temperature of them rises to a certain degree. By analogy with the levels of activation of these objects or operators, from dormant level to active level, I introduce three levels of activation of the operator $-$: The level of full activation, the level of null activation, the level of half a full activation (the level of half activation), which does not activate $-$, but reaches the level of full activation by adding more half a full activation. I try to extend the operators [R] by introducing these three levels to $-$, which are denoted by $-^1$, $-^0$, $-^{1/2}$, respectively. Then, $-^1 a = -a$, $-^0 a = a$, $-^{1/2}(-^{1/2})a = (-^{1/2} * -^{1/2})a = -^1 a = -a$, for $a \in$ [R].

Because a and $-^{1/2}b$ for $a, b \in$ [R] are both certain levels of operators, which have a capacity or potential to operate on set structures, they can be connected with $*$ to form an operator that can operate on set structures,

$a + (-^{1/2})b = a * (-^{1/2})b$.

It is obvious that

$a + \left(-^{1/2}\right)b = a' + \left(-^{1/2}\right)b' \leftrightarrow a = a' and\ b = b'$.

Naturally, they are associative and commutative for $*$.

The set of operators, $\{x + (-^{1/2})y |\ x, y \in$ [R]$\}$ is written as *[C]*.

• Addition on [C] is defined naturally as

$\left(a + (-^{1/2})b\right) + \left(a' + (-^{1/2})b'\right) = (a + (-^{1/2})b) * (a' + (-^{1/2})b') = (a + a') + (-^{1/2})(b + b')$, for $a,\ b,\ a',\ b' \in$ [R].

Since connection of operators does not depend on its order, the associative law and commutative law hold for [C].

Since

$(a + (-^{1/2})b) + (P^o + (-^{1/2})P^o) = a + (-^{1/2})b$,

$P^o + (-^{1/2})P^o$ is the additive identity.

Since

$(a + (-^{1/2})b) + (-(a + (-^{1/2})b)) = P^o + (-^{1/2})P^o$,

$-(a + (-^{1/2})b)$is the inverse element of $a + (-^{1/2})b$

• Multiplication on [C] is defined by extending that on [R].
At first, by the definition of $-^{1/2}$,
$a \times (-^{1/2})b = \Sigma^{-^{1/2}b} a = -^{1/2}(a \times b)$,
that is, iteration of *a* it-times of $-^{1/2}$P in *b*, where $a \times -^{1/2}$P makes *a* with the stage $-^{1/2}$.
Then,
$(a + (-^{1/2})b) \times (a' + (-^{1/2})b') = \Sigma^{a'+(-^{1/2})b'}(a + (-^{1/2})b) = \Sigma^{a'}(a + (-^{1/2})b) + \Sigma^{-^{1/2}b'}(a + (-^{1/2})b) = (a + (-^{1/2})b) \times a' + (a + (-^{1/2})b) \times (-^{1/2})b' = (a \times a') + (-^{1/2})b \times a' + (-^{1/2})a \times b' + (-b \times b') = (a \times a') + (-b \times b') + (-^{1/2})(b \times a' + a \times b')$.
In particular,
$(-^{1/2}\mathrm{P})^2 = -^{1/2}\mathrm{P} \times (-^{1/2}\mathrm{P}) = -\mathrm{P} \times P = -P$.
Since P is the unit element for multiplication, $-^{1/2}$P is identical with the imaginary unit *i*.

Because of the associative law and commutative law of multiplication on [R], they also hold for [C].
Since
$(a + (-^{1/2})b) \times \mathrm{P} = a + (-^{1/2})b$,
P is the unit element for the multiplication.

• The distributive law
$((a + (-^{1/2})b) + (a' + (-^{1/2})b')) \times (c + (-^{1/2})c') = ((a + a') + (-^{1/2})(b + b')) \times (c + (-^{1/2})c') = \Sigma^{c+(-^{1/2})c'}((a + a') + (-^{1/2})(b + b')) = ((a + a') + (-^{1/2})(b + b')) \times c + ((a + a') + (-^{1/2})(b + b')) \times (-^{1/2})c' = (a + a') \times c + (-^{1/2})(b + b') \times c + (-^{1/2})((a + a') \times c') + (-(b + b') \times c') = (a + a') \times c + (-(b + b') \times c') + (-^{1/2})((b + b') \times c + (a + a') \times c')$. … (1),
On the other hand,
$(a + (-^{1/2})b) \times (c + (-^{1/2})c') + (a' + (-^{1/2})b') \times (c + (-^{1/2})c') = a \times c + (-b \times c') + (-^{1/2})(a \times c' + b \times c) + a' \times c + (-b' \times c') + (-^{1/2})(a' \times c' + b' \times c) = (a + a') \times c + -(b + b') \times c' + (-^{1/2})((b+b') \times c + (a + a') \times c')$ … (2).
Thus, the distributive law follows from the equations (1) and (2).

• Inverse element for the multiplication
Let $(a + (-^{1/2})b) \times x = \mathrm{P}$. Then,
$x = \mathrm{P} \div (a + (-^{1/2})b) = \mathrm{P}_{a+(-^{1/2})b} = ((a + -(-^{1/2})b) \div (a + -(-^{1/2})b)) \times \mathrm{P}_{a+(-^{1/2})b} = P_{a+-(-^{1/2})b} \times (a + -(-^{1/2})b) \times \mathrm{P}_{a+(-^{1/2})b} = \mathrm{P}_{(a+-(-^{1/2})b)\times(a+(-^{1/2})b)} \times (a + -(-^{1/2})b) = \mathrm{P}_{a\times a+b\times b} \times (a + -(-^{1/2})b)$.
This is the inverse element of $a + (-^{1/2})b$.

As stated above, [C] has a unit element, inverse elements, and satisfies the associative law, commutative law and distributive law with respect to addition and multiplication respectively. Therefore, the complex numbers C is identified with [C].

7. The quaternions H

The quaternions are defined by introducing the three kinds of imaginary units, i, j, k into the real numbers that satisfy the following conditions:

1. $i^2 = j^2 = k^2 = -1$, and quaternions have the form, $a + bi + cj + dk$, where a, b, c, d are real numbers.
2. $a + bi + cj + dk = a + b'i + c'j + d'k \leftrightarrow a = a', b = b', c = c', and\ d = d'$.
3. $(a + bi + cj + dk)+(a' + b'i + c'j + d'k) = (a + a') + (b + b')i + (c + c')j + (d + d')k$.
4. $ij = -ji = k, jk = -kj = i, ki = -ik = j$.
5. The distributive law.

Introduction of i, j, and k into R corresponds to introduction of three kinds of half activation levels of $-P$ into [R]. The three kinds of half activation levels of – are denoted by $-^{h1}, -^{h2}, and\ -^{h3}$, respectively. Then,
$(-^{h1}\mathrm{P})^2 = -^{h1}(-^{h1}\mathrm{P}) = -^{h1+h1}\mathrm{P} = -^{1}\mathrm{P} = -\mathrm{P}$
In the same way,
$(-^{h2}\mathrm{P})^2 = (-^{h3}\mathrm{P})^2 = -\mathrm{P}$.
That is, $-^{h1}\mathrm{P}, -^{h2}\mathrm{P}, and\ -^{h3}\mathrm{P}$ correspond to i, j, and k respectively in the condition 1.

By introducing them into [R], an extended [R] is constructed. Then, by replacing i, j, and k in the conditions 1~4 with these three operators, the conditions that the three operators must possess to identify the quaternion with this extended [R] are obtained. Multiplication of these operators is, for example,
$i \times j = k \rightarrow\ (-^{h1}\mathrm{P}) \times (-^{h2}\mathrm{P}) = -^{h2}(-^{h1}\mathrm{P}) = -^{h1+h2}\mathrm{P} = -^{h3}\mathrm{P}$
$j \times i = -k \rightarrow\ (-^{h2}\mathrm{P})(-^{h1}\mathrm{P}) = -^{h1}(-^{h2}\mathrm{P}) = -^{h2+h1}\mathrm{P} = -^{1+h1+h2}\mathrm{P} = -^{1+h3}\mathrm{P} = -(-^{h3}\mathrm{P})$
Multiplication of the other combinations of the three operators is obtained in the same way to satisfy the condition 4. Multiplication of operators in the extended [R] is defined so as to satisfy the condition 5, the distributive law.

As a consequence of the above construction, the quaternions H is identified with this extended [R], which is denoted by *[H]*.

8. Conclusion

In this article I have identified numbers with operators to construct finite cardinals and their derivatives in the following steps.
1. Finite cardinals are defined as structures of the sets with the cardinals.
2. The operator P to construct the set structure of a finite cardinal from the set structure of one smaller cardinal than the former is introduced. The iteration of the operation of P, which is made by connections of Ps, relates the set structures of the finite cardinals in a total order.
3. Furthermore, addition and multiplication are defined on the set structures on the basis of connections of Ps. Then, it has been shown that the set of the set structures with the addition and the multiplication satisfies Peano Axioms. Accordingly, the natural numbers are defined as the set of the set structures.
4. At the same time, it has been shown that the set of connections of Ps satisfies Peano Axioms, hence the natural numbers can be identified with this set.
5. The operator P is extended in five ways: (1) To reverse the direction of the operation of P, (2) To divide P into finite number of sub-operators, (3) To introduce infinite series of the sub-operators that converge, (4) To introduce the half activation level of the operator $-\mathrm{P}$, (5) To introduce two more half activation level of $-$Ps other than the former one. Addition and multiplication are defined on these five sets of the extended operators, respectively. Then, the integers, the fractions, the real numbers, the complex numbers, and the quaternions are identified with the sets of the operators extended in these five ways, respectively. This result

supports the possibility that numbers in general can be identified with such kinds of operators.

Since these operators are derivatives of P and the same kinds of objects with P, they are related in inclusion relationship. Naturally, these six sets of operators can be connected by addition or multiplication to form the same kind of operators as stated in this article. On the contrary, numbers defined as sets, e.g., one that proposed by Russell (1919) are related in embedding relationship. This is because the natural numbers, the integers, the fractions, the real numbers, and the complex numbers are all different kinds of sets. In this case, numbers form a complex system that consists of many kinds of objects. Moreover, addition of real part and imaginary part in complex numbers, that is, addition of different kinds of objects or sets, seems to be something different from addition of the same kind of numbers e.g. addition of two integers. Thus, to identify numbers with operators introduces a new viewpoint on numbers and would enlarge the concept of numbers.

Three problems concerning this construction of operators to identify numbers are left open.

1. Operators $[\mathrm{N^o}]$, [Z], [Q], [R], [C], and [H] exist on the basis of the existence of P, $*$, –, P_n, infinite series of P_n and $-^{h1}$, $-^{h2}$, $-^{h3}$, regardless of existence of the set structures that should be the result of the operations. $[\mathrm{N^o}]$ has [N] as the set structures constructed by $[\mathrm{N^o}]$. I wonder if the set structures constructed by [Z], [Q], [R], [C], or [H] exist.

2. Construction of [N] and $[\mathrm{N^o}]$ is limitless. Accordingly, the result of the limit is not included in [N] and $[\mathrm{N^o}]$. This limit is necessary to include infinity in numbers. What is the limit of the construction, i.e. limit of operators and set structures?

These problems will be the next steps to construct operators with which numbers are identified.

3. Finally, it will be natural to expect that provided that the operator $-$P has the half activation level, $+'$P also has the same kind of half activation level, that is, $+'^{1/2}$P and $(+'^{1/2}\mathrm{P})^2 = \mathrm{P}$. This characterization of $+'^{1/2}$P agrees with the definition of the imaginary unit of the split complex numbers. Then, in the same way as the case of the complex numbers, it is expected that the split complex numbers will be identified with [R] with $+'^{1/2}$P.

However, I wonder what these numbers or operators are in the world.

Reference

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