Equivalence, reduction, and sophistication in teleparallel gravity

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Abstract

We discuss the (in)equivalence of various formulations of teleparallel gravity, building upon recent work by Weatherall and Meskhidze (2024). We then think about these different versions of teleparallel gravity from the point of view of reduction/sophistication—a distinction drawn by Dewar (2019) in the context of philosophical literature on symmetries—and along the way introduce and scrutinise the resources of Cartan geometry and of higher gauge theory.

Keywords: Teleparallel gravity; general relativity; theoretical equivalence; categorical equivalence; reduction; sophistication; Cartan geometry; higher gauge theory

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1. Introduction

In line with the orthodoxy in contemporary philosophy of physics, let the symmetries of a physical theory be maps from the dynamical possibilities of that theory to dynamical possibilities which preserve certain salient structure (see e.g.

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(Belot, 2013)). Without wishing to comment on whether this is part of the *definition* of a symmetry transformation (see (Dasgupta, 2016) for discussion on that front), the symmetry transformations which will be relevant for our purposes are those which preserve empirical content (for more on this, see (Read and Møller-Nielsen, 2020)).

Given a theory with symmetries of this kind—symmetries which, to be clear, needn't act as isomorphisms (more on this below)—how to articulate the ontological commitments of symmetry-related models? In a seminal article addressing this question, Dewar (2019) proposes three distinct means of tackling the issue:[1]

Reduction: Map orbits of symmetry-related models of the original theory to unique models of some new, 'symmetry-reduced' theory.

External sophistication: Treat the symmetry-related models of the original theory 'as if' they are isomorphic. Then, apply anti-haecceitism/anti-quidditism in one's interpretation of those models in order to justify their representing the same physical states of affairs.

Internal sophistication: Mathematically reformulate the original theory in order to 'forget' about structure such that symmetries now act as isomorphisms. Then, apply anti-haecceitism/anti-quidditism in one's interpretation of those models in order to justify their representing the same physical states of affairs.

For more on this threefold distinction, see (Martens and Read, 2020), in particular on the second interpretative step of both external and internal sophistication which involves anti-haecceitism/anti-quidditism, which we won't go into in any further detail here. One would be perfectly within one's rights to find puzzling (and perhaps 'metaphysically unperspicuous') external sophistication as presented above (again, see (Martens and Read, 2020) for discussion); in our view, following March (2024d), the correct way to understand this approach is in terms of inserting 'extra' morphisms into a theory understood categorically; we'll return to this below.[2]

Here, let's jump to an illustration of the difference between reduction and sophistication: the well-known case of electromagnetism. Consider in particular the following four formulations of source-free electromagnetism:[3]

EM1: Kinematical possibilities given by $\langle M, \eta_{ab}, F_{ab}\rangle$, where M is a differentiable manifold diffeomorphic to $\mathbb{R}^4$, η_{ab} is a flat Lorentzian metric on M, and F_{ab} is a 2-form on M.[4] Dynamical possibilities given by $d_a F_{bc} = 0$ and $\star d_a \star F_{bc} = 0$,

[1] Dewar (2019) in fact presents a few different ways of understanding the sophistication/reduction distinction—the way in which we cash out the distinction below, however, has by now become canonical, and we'll set aside other ways of understanding the distinction (e.g. in terms of 'changing the syntax' versus 'changing the semantics').

[2] We don't mean to suggest here that this way of understanding external sophistication affords it the resources to evade the metaphysical unperspicuity charge.

[3] Nomenclature here is chosen to match that of Weatherall (2016c). Note also that Weatherall does not mention the dynamics of EM3; we take this to be an unproblematic oversight given that he mentions the dynamics of the other formulations of electromagnetism.

[4] Our index conventions here roughly follow (Weatherall, 2016b): all indices are abstract unless stated otherwise; we use lowercase Roman indices for tensor fields valued in the tangent/cotangent spaces to a spacetime manifold, lowercase Greek indices for tensor fields valued in the total space of a bundle, capital Fraktur indices for tensor fields valued in a Lie algebra, and capital Roman indices for other vector spaces.

where '$\star$' denotes the Hodge dual with respect to η_{ab}.[5]

EM2: Kinematical possibilities given by $\langle M, \eta_{ab}, A_a \rangle$, where M and η_{ab} are as before and A_a is a 1-form on M. Dynamical possibilities given by $\star d_a \star d_b A_c = 0$.[6]

EM2′: Kinematical possibilities given by $\langle M, \eta_{ab}, [A_a] \rangle$, where M and η_{ab} are as before and $[A_a]$ is an equivalence class of 1-forms on M (as in EM2) related by $A_a \mapsto A_a + \varphi_a$, where φ_a is an arbitrary closed one-form. Dynamical possibilities given by $\star d_a \star d_b A_c = 0$, for each element of the equivalence class.

EM3: Kinematical possibilities given by $\langle M, P, \omega, \eta_{ab} \rangle$, where P is the total space of the (unique, trivial) principal bundle $U(1) \to P \xrightarrow{\pi} M$ over Minkowksi spacetime $\langle M, \eta_{ab} \rangle$ and ω is a principal connection on P. Defining the curvature of the connection as $\Omega^{\mathfrak{A}}{}_{\alpha\beta} := D_\alpha \omega^{\mathfrak{A}}{}_\beta$, dynamical possibilities are given by $\star D_\alpha \star \Omega^{\mathfrak{A}}{}_{\beta\gamma} = 0$, where D is the exterior covariant derivative associated with ω.

To each of these theories, there is at least one associated category, in which the objects are the dynamical possibilities of the theory and the morphisms are chosen in line with Table 1.[7] In that table, $\chi : M \to M'$ denotes a spacetime diffeomorphism and Ψ denotes a principal bundle diffeomorphism (Ψ, χ), $\Psi : P \to P'$, $\chi : M \to M'$ where χ is again a spacetime diffeomorphism.[8] Importantly, note that we said above 'at least one' because, when understood categorically, one theory (say EM2) might give rise to many distinct theories *qua* categories, depending upon how morphisms are chosen (witness in particular the difference between **EM2** and $\overline{\textbf{EM2}}$ depending upon whether morphisms include gauge transformations of the vector potential—transformations which, to be clear, don't act as isomorphisms of the objects of the theory).

For our purposes, there are two key points to note about these theories. The first has to do with the (in)equivalence of the above categories.[9] There is a hierarchy here: **EM2** has more structure than any of the other categories, and so is inequivalent to them. **EM1**, $\overline{\textbf{EM2}}$, and **EM2′**, on the other hand, are all categorically equivalent to each other and therefore have the same amount of structure.[10] **EM3** has the least amount of structure and is categorically inequivalent to any of the other theories.[11]

[5]See e.g. (Burke, 1985) for background on differential forms and the Hodge dual.

[6]The second Maxwell equation, $d_a d_b A_c = 0$, is a mathematical (Bianchi) identity in EM2. We return to this in §6.2.

[7]Here, again, we follow the terminology of Weatherall (2016a,c). For a justification of these being the morphisms of **EM2′**, see Weatherall (2016a, lemma 5.3); cf. (Nguyen et al., 2018), which we'll discuss further briefly below. Note also that dynamical equations are suppressed in Table 1 for clarity.

[8]See (Weatherall, 2016c, p. 1046).

[9]In §4 we rehearse the definition of categorical equivalence, following the lead of Weatherall (2016a,c). For now, we just recall the standard verdicts in the literature on the categorical (in)equivalence of these various formulations of electromagnetism.

[10]Note that these equivalence claims can break down on different manifold topologies; see Chen (2024) for a detailed theorem of categorical inequivalence given different topologies.

[11]See (Weatherall, 2018a). Note that the situation changes here if the morphisms of $\overline{\textbf{EM2}}$ are instead defined as pairs (χ, Θ) preserving the metric and (gauge-transformed) potential $A_a + d_a\Theta$;

	Ob	Mor
EM1	$\langle M, \eta_{ab}, F_{ab} \rangle$	$F_{ab} \mapsto \chi_* F_{ab}$
EM2	$\langle M, \eta_{ab}, A_a \rangle$	$A_a \mapsto \chi_* A_a$
$\overline{\textbf{EM2}}$	$\langle M, \eta_{ab}, A_a \rangle$	$A_a \mapsto \chi_* (A_a + \varphi_a)$
EM2′	$\langle M, \eta_{ab}, [A_a] \rangle$	$[A_a] \mapsto [\chi_* A_a]$
EM3	$\langle M, P, \omega, \eta_{ab} \rangle$	$\omega \mapsto \Psi_* \omega$

Table 1: Objects and morphisms for various formulations of electromagnetism, understood categorically. $\chi : M \to M'$ denotes a spacetime diffeomorphism and (say) '$F_{ab} \mapsto \chi_* F_{ab}$' is shorthand for a morphism between models which preserves the metric η_{ab} and Faraday tensor F_{ab} in the sense that the models $\langle M, \eta_{ab}, F_{ab} \rangle$ and $\langle M', \eta'_{ab}, F'_{ab} \rangle$ related by this morphism are such that $\eta'_{ab} = \chi_* \eta_{ab}$ and $F'_{ab} = \chi_* F_{ab}$ (*mutatis mutandis* for preservation of the relevant geometric objects in the models of the other categories); Ψ denotes a principal bundle diffeomorphism (Ψ, χ), $\Psi : P \to P'$, $\chi : M \to M'$ preserving η_{ab} and ω (in the same sense as above), and φ_a is an arbitrary closed one-form.

The second point to note here comes back to reduction and sophistication: **EM1** and **EM2′** are both reduced theories associated with **EM2** (for, recall, classes of models of **EM2** are mapped to unique models of **EM1** and **EM2′**), although the former is an 'intrinsic' formulation of electromagnetism (i.e., one which doesn't formulate its models in terms of equivalence classes) whereas the latter is not (for more on intrinsic versus extrinsic formulations of physical theories, see (March, 2024d)); **EM3** is an internally sophisticated theory associated with **EM2** (for, recall, moving from **EM2** to **EM3** mathematically reformulates the former theory such that that the symmetries act as isomorphisms in **EM3**, but not in **EM2**).[12] $\overline{\textbf{EM2}}$ is best understood as a theory which is externally sophisticated since it merely inserts more morphisms into the category without any mathematical reformulation of the objects.[13] Hence we see that (a) reduction might or might not involve taking equivalence classes—this point was already made by March (2024d), who points out that taking equivalence classes is in fact *orthogonal* to reduction/sophistication; (b) reduction and sophistication needn't be unique;[14] (c) a sophisticated theory might or might not be categorically equivalent to a reduced theory—in this case, **EM3** is in fact *not* categorically equivalent to (say) **EM1**.

All of this by way of introduction. In this article, we'll show that there are analogous theories in the case of relativistic spacetime physics; moreover, verdicts on sophistication and categorical equivalence broadly (but not exactly) carry over

in that case, the resulting category is categorically equivalent to **EM3** (see Bradley and Weatherall (2020)).

[12]For more on this, see (Jacobs, 2023).

[13]$\overline{\textbf{EM2}}$ therefore counts as a version of electromagnetism which is not 'literally interpreted'—see (March, 2024b).

[14]As further illustration of this, note that (i) the holonomy interpretation of electromagnetism (endorsed by Healey (2007)) constitutes another reduced version of electromagnetism, and (ii) yet another sophisticated theory (without equivalence classes) could be found by availing oneself of the 'bundle of connections'. For discussion of both of these, see again (Jacobs, 2023).

to this new context.[15] To be specific, we consider in this article the relationship between general relativity (GR) on the one hand, and 'teleparallel gravity' (TPG) on the other: the latter is a spacetime theory dynamically equivalent to GR, but in which curvature degrees of freedom are traded for torsion.[16] Recently, Weatherall and Meskhidze (2024) have argued that GR and TPG are not categorically equivalent; while we don't disagree with their verdict for the version of TPG which they consider, in fact in the physics literature there is a very diverse variety of different formulations of TPG which are not obviously (in)equivalent to each other; in our view continuing this project with respect to the broader space of TPG formulations is a worthwhile exercise. In addition to this, we have a number of further aims in carrying out the work of this article:

1. To show that it is possible to construct theories in the relativistic spacetime context which are both sophisticated and which are reduced, and to show that the pattern of reduced/sophisticated theories broadly carries over from other contexts (specifically, that of electromagnetism discussed above).[17]

2. To enrich philosophers of physics' toolkits with the powerful equipment of both Cartan geometry and of higher gauge theory. The former constitutes the mathematical wherewithal needed to formulate TPG in the manner of Le Delliou et al. (2020a); Huguet et al. (2021a); Le Delliou et al. (2020b); Huguet et al. (2021b); the latter constitutes the wherewithal needed to formulate TPG in the manner of Baez and Wise (2015).

3. To connect up the literature on theoretical equivalence in electromagnetism/Yang–Mills with that on theoretical equivalence in spacetime physics. Links between these fields have already been noted by e.g. Weatherall (2016a), but we contend that they in fact run much deeper than has been appreciated hitherto.

Our article is structured as follows. In §2, we provide a self-contained introduction to mathematics underlying the different formulations of TPG which we consider in this article. In §3, we remind readers of the basics of both GR and TPG, the latter in its various formulations as they appear in the physics literature. For each of these theories, we present the relevant category (as done above for versions of electromagnetism) in terms of its objects and morphisms. In §4, we prove a number of propositions regarding the (in)equivalence of these categories, securing thereby a map of their relative amounts of structure. In §5, we consider whether these spacetimes theories can be regarded as reduced/sophisticated versions

[15]The story here is likely to be complicated further once one brings into the picture considerations of e.g. boundaries: this issue is discussed by Wolf and Read (2023). We'll set such complications aside in this article.

[16]For background on TPG, see (Aldrovandi and Pereira, 2013; Bahamonde et al., 2023).

[17]The issue of reduction/sophistication in the context of TPG is also addressed briefly by Weatherall and Meskhidze (2024, §6); again, we don't disagree with their verdicts, but rather intend our work here to add to and further the discussion. (As an aside, Weatherall and Meskhidze (2024) maintain that the structure of GR is the structure 'common' to GR and TPG—we agree with this verdict too (at least for the version of TPG which they consider), for the reasons given by March et al. (2024); these reasons are somewhat different to the reasons given by Knox (2011) for the same conclusion—Knox's reasons have been called into question by Wolf et al. (2024); Mulder and Read (2023).)

of one another, and whether the interaction between these verdicts and categorical (in)equivalence results carries over from the case of electromagnetism (broadly but not exactly speaking, it does). In §6, we step back a little, and assess the virtues of formulating physical theories as Cartan or higher gauge theories.

2. Mathematical preliminaries

In this section, we provide the relevant mathematical background to (i) 'Palatini TPG' (§2.1), (ii) TPG as a 'standard' gauge theory (§2.2),[18] (iii) TPG as a Cartan gauge theory (§2.3), and (iv) TPG as a higher gauge theory (§2.4). The full details of all of these version of TPG will follow in §3.

2.1. Palatini TPG.

Just as kinematical possibilities of GR can be taken to be Lorentzian manifolds $\langle M, g_{ab}\rangle$,[19] so too can the kinematical possibilities of *Palatini TPG* be taken to be triples $\langle M, g_{ab}, \nabla\rangle$, where M is a differentiable manifold (assumed connected, Hausdorff, paracompact, and parallelizable), and ∇ is a flat, torsionful derivative operator compatible with g_{ab}. Unlike the Levi-Civita derivative operator (which is the unique torsion-free derivative operator compatible with g_{ab}), a torsionful derivative operator is not fixed by g_{ab}, and so needs to be specified in the models in addition to $\langle M, g_{ab}\rangle$.[20] The nomenclature 'Palatini TPG' here is chosen in order to allude to the 'Palatini approach' (on which see e.g. (Wald, 1984, pp. 454–5)), in which the connection is treated as a dynamical variable independent of g_{ab}.

2.2. TPG as a gauge theory.

Next, let's turn to TPG understood as a 'standard' gauge theory—one can find something resembling this presentation of the theory in classic sources such as (Aldrovandi and Pereira, 2013; Bahamonde et al., 2023; Krššák et al., 2019), although here we will be somewhat more explicit about the geometrical commitments of this approach to setting up the theory.[21]

Let M be as above, and let $\mathrm{Gl}(n,\mathbb{R}) \to LM \xrightarrow{\pi_L} M$ denote the frame bundle over M. Let V be an n-dimensional vector space, and fix a representation ρ of $\mathrm{Gl}(n,\mathbb{R})$ on V.[22] Given this structure, we can construct the associated bundle $LM \times_{\mathrm{Gl}} V$, which is isomorphic (as vector bundles) to TM. A coframe field (or solder form) e is just a choice of isomorphism $e : TM \to LM \times_{\mathrm{Gl}} V$. Such a choice of isomorphism can be represented as a smooth equivariant V-valued one-form e_a^A on LM,[23] with inverse e_A^a, which defines, at each $p \in LM$, a linear bijection between $T_{\pi_L(p)}M$ and V.[24]

[18] Here, 'standard' is to be taken with a pinch of salt, since as Wallace (2015) notes and as we'll discuss further below, TPG is *not* a 'standard' gauge theory in the sense of being a Yang–Mills gauge theory.

[19] In this article, we'll drop explicit reference to matter and its associated stress-energy tensor.

[20] For further recent discussion of this point, see (Weatherall and Meskhidze, 2024).

[21] We use textbook machinery of e.g. (Kobayashi and Nomizu, 1963).

[22] More generally, one can consider an arbitrary smooth n-manifold S equipped with a right $\mathrm{Gl}(n,\mathbb{R})$ action. Of course, one is always free to take $V = \mathbb{R}^n$ and ρ as the fundamental representation of $\mathrm{Gl}(n,\mathbb{R})$.

[23] Recall that the equivariance condition is the property that, for any $p \in LM$ any $g \in \mathrm{Gl}(1, n-1,\mathbb{R})$, $(e_n^A)_{xg} = \rho(g^{-1})^A_N (e_m^N)_x$.

[24] Alternatively (and equivalently), it can be represented by a V-valued one-form on M which

Now let η_{AB} be a Lorentzian metric on V. Together, e and η_{AB} induce a reduction of the structure group of LM to $\mathrm{O}(1, n-1, \mathbb{R})$. For this, note that we can pull back η_{AB} via e to a Lorentzian metric $\overset{e}{g}_{nm}$ on M as follows: for any $p \in LM$ and any ξ^a, $\kappa^a \in T_{\pi_L(p)}M$ we define

$$(\overset{e}{g}_{nm})_{\pi_L(p)}\xi^n\kappa^m = \eta_{NM}(e^N_n)_p(e^M_m)_p\xi^n\kappa^m. \tag{1}$$

Then, as usual, the bundle $\mathrm{O}(1, n-1, \mathbb{R}) \to LM_{\mathrm{O}} \overset{\pi_L}{\to} M$ of orthonormal frames with respect to $\overset{e}{g}_{nm}$ is a principal $\mathrm{O}(1, n-1, \mathbb{R})$ bundle and is a reduction of LM via the subspace embedding, which is equivariant in the required sense. Moreover, since M is parallelizable, it is also orientable, so we have a further reduction of the structure group of LM to $\mathrm{SO}(1, n-1, \mathbb{R})$, i.e. the bundle $\mathrm{SO}(1, n-1, \mathbb{R}) \to LM_{\mathrm{SO}} \overset{\pi_L}{\to} M$ of (positively) oriented orthonormal frames (the relevant embedding is again the subspace embedding).

Next, note that we can pull back the coframe to LM_{SO} via this embedding; the result is a smooth equivariant one-form which defines, at each $p \in LM_{\mathrm{SO}}$, a linear and orientation-preserving isometry between $T_{\pi_L(p)}M$ and V. In what follows, we will abuse notation by using e to denote both the coframe field on LM and its pullback to LM_{SO} (which space is under consideration will always be clear from the context). Note also that ρ and η_{AB} induce a representation of $\mathrm{SO}(1, n-1, \mathbb{R})$ on V, so that we can construct the associated bundle $LM_{\mathrm{SO}} \times_{\mathrm{SO}} V$. Similarly, we will abuse notation and elide the distinction between ρ and the induced representation of $\mathrm{SO}(1, n-1, \mathbb{R})$ going forward.

Given the structure defined here, there exists a unique (and generically non-flat) principal connection ω on LM_{SO} whose associated exterior covariant derivative operator $\overset{\omega}{D}$ is torsion-free ($\overset{\omega}{D}_\alpha e^A_a = 0$). The associated covariant derivative operator for this connection $\overset{\omega}{\nabla}$ on $LM_{\mathrm{SO}} \times_{\mathrm{SO}} V$, pulled back to TM via e, is just the usual Levi-Civita connection on TM with respect to the metric $\overset{e}{g}_{nm}$ defined in (1), recovering the structure of GR. In the gauge-theoretic approach to TPG, one drops the torsion-freedom condition on ω and instead imposes flatness.[25]

We now have all the structures in place to define models of the gauge-theoretic approach to TPG. These are structures $\langle M, LM_{\mathrm{SO}}, \pi_L, LM_{\mathrm{SO}} \times_{\mathrm{SO}} V, e, \omega, \eta_{AB}\rangle$, where LM_{SO}, V, e, and η_{AB} are as above, and ω is a flat principal connection on LM_{SO}. To connect this up with Palatini TPG, recall first that any coframe field e induces a metric on M via (1). Likewise, any connection ω on LM_{SO} induces a covariant exterior derivative $\overset{\omega}{D}$ on LM_{SO} and hence a covariant derivative operator

defines, at each $p \in M$, a linear bijection between T_pM and the fibre of $LM \times_{\mathrm{Gl}} V$ at p, though we won't discuss this representation of e in what follows.

[25] There are subtleties here about when such a flat connection ω exists. In the case of reductions of the structure group of LM induced by a Riemannian (rather than pseudo-Riemannian) metric on M, the existence of a global section of LM (which is equivalent to parallelizability) descends to LM_{O}, but this is not in general the case for reductions of the structure group induced by pseudo-Riemannian metrics on M (though it is when M is contractible). Consider, e.g. any Lorentzian metric on $M \simeq \mathbb{R} \times S^n$ which restricts to a Riemannian metric on S^n for $n \neq 1, 3, 7$. In any case, these same issues apply when considering the existence of compatible connections on M in the Palatini approach discussed in §2.1, so they won't concern us much going forward, aside from the fact that we will, in proposition 2, implicitly assume that the space of models of GR is restricted to those for which a compatible flat connection exists.

$\overset{\omega}{\nabla}$ on $LM_{\text{SO}} \times_{\text{SO}} V$. We can then pull this back via e to obtain a connection on TM. Explicitly, given the flat connection ω and coframe field e on LM_{SO}, the Weitzenböck connection $\overset{e,\omega}{\nabla}$ on TM is defined as follows:[26] if κ^a is any vector field on M (i.e. any section of TM), then for any $p \in LM$ and any $\xi^a \in T_{\pi_L(p)}M$ at $\pi_L(p)$,

$$e^A_m(\xi^n \overset{e,\omega}{\nabla}_n \kappa^m) = \xi^n \overset{\omega}{\nabla}_n (e^A_m \kappa^m). \tag{2}$$

2.3. **TPG and Cartan geometry.** TPG theorists such as Aldrovandi and Pereira (2013) and Pereira and Obukhov (2019) declare explicitly that they wish the theory to be understood as a 'gauge theory of translations'; to this end, they invoke (usually but not always implicitly) a principal translation bundle as the appropriate means of encoding such a gauge theory geometrically. As Le Delliou et al. (2020a,b) argue, however, the attempt to assimilate the coframe field e into a gauge field in the standard framework of gauge theory, namely viewing e as (part of) an 'ordinary' principal (i.e. Ehresmann) connection, is met with difficulties.[27] According to the authors, this is because the principal bundle of translations can only be trivial, i.e. isomorphic to the product space $M \times \mathbb{R}^n$, and the base manifold would only have the trivial frame bundle, which would be too restrictive as a framework for GR (or any geometric alternative to GR, including TPG). The details and the implications of this argument are contested by Pereira and Obukhov (2019), but in any case it's at least reasonable to agree with Le Delliou et al. (2020a,b) that the mathematical content of TPG as a gauge theory for translations is *prima facie* unclear. To remedy this problem, the authors instead propose defining e as (part of) a Cartan connection, which requires a turn to Cartan geometry.

Let's recall the basic picture of Cartan geometry. According to Cartan geometry, every space is characterized locally by a homogeneous space, of which Euclidean space, Minkowski space, de Sitter and anti de Sitter spaces are examples. The notion of a homogeneous space is in turn built upon Klein geometry, according to which every homogeneous space is characterized by a Lie group quotient over a closed normal subgroup. For example, if we consider a Lie group SO(3) and its subgroup SO(2) (together with an embedding), then the quotient group SO(3)/SO(2) is isomorphic to the two-sphere S^2.[28] As a more relevant example, Minkowski spacetime is characterized by the quotient of the Poincaré group by the Lorentz group. Letting $\mathcal{G}$ be the larger Lie group in a Klein geometry and $\mathcal{H}$ a closed normal subgroup of $\mathcal{G}$, for the case of Minkowski spacetime we have[29]

$$\mathcal{G} = \text{ISO}(1, n-1) = \text{SO}(1, n-1) \ltimes \mathbb{R}^n, \quad \mathcal{H} = \text{SO}(1, n-1).$$

More formally, let $\mathcal{G}$ be a Lie group and $\mathcal{H}$ a closed normal (Lie) subgroup of

[26] See (Baez and Wise, 2015, p. 168).

[27] For the definition of an Ehresmann connection, see e.g. (Wise, 2007, p. 135).

[28] The elements of SO(3) can be thought of rotations of orthonormal frames with three legs, and the equivalence relation can be thought of rotating the frames with one leg fixed. The resulting quotient is then a three-dimensional rotation of a unit vector (the fixed leg), namely S^2.

[29] Note that the fonts used for the Cartan groups are distinguished from those used in the cross-module definition of a 2-group, found below. In the Cartan context, the relevant $\mathcal{H}$ for teleparallel gravity is $\text{SO}(1, n-1)$, while in the cross-module definition for $\mathbf{Tel}(1, n-1)$ 2-group, H refers to $\mathbb{R}^{1,n-1}$, so these should be properly distinguished.

$\mathcal{G}$, and fix a smooth (right) action α of $\mathcal{H}$ on $\mathcal{G}$. The pair $(\mathcal{G}, \mathcal{H})$ denotes a Klein geometry, namely, the quotient space $\mathcal{G}/\mathcal{H}$ with respect to α. Now let $\mathcal{H} \to P \xrightarrow{\pi} M$ be a principal $\mathcal{H}$ bundle, and consider the associated bundle $P \times_{\mathcal{H}} \mathcal{G}$ under α. (This is, of course, simply an extension of the structure group of P from $\mathcal{H}$ to $\mathcal{G}$.) Then a ($\mathfrak{g}$-valued) Cartan connection ω_c on P consists of the following data:

- A principal connection ω on $P \times_{\mathcal{H}} \mathcal{G}$;
- A reduction of the structure group $f : P \to P \times_{\mathcal{H}} \mathcal{G}$;

such that the pullback $f^*\omega$ is, at each $p \in P$, a linear isomorphism between the tangent space T_pP and the Lie algebra $\mathfrak{g}$. In this case, we define $\omega_c := f^*\omega$. Finally, a Cartan geometry, modelled on a Klein geometry $(\mathcal{G}, \mathcal{H})$, consists of a principal $\mathcal{H}$ bundle P together with a $\mathfrak{g}$-valued Cartan connection ω_c on P.

This makes it clear that compared to an 'ordinary' principal connection, a Cartan connection has the following unique features along with usual properties such as being $\mathcal{H}$-equivariant:

1. It takes values in a larger algebra $\mathfrak{g} \supset \mathfrak{h}$ of $\mathcal{G} \supset \mathcal{H}$ than that of the gauge group of the bundle.
2. It is, at each $p \in P$, a linear isomorphism between the tangent space T_pP and the Lie algebra $\mathfrak{g}$.

Both properties call for some elaboration. To explain the geometric intuition underlying them, let us use Wise's analogy of a hamster in a ball in the simple case of SO(3)/SO(2). Consider a hamster in a ball atop a Riemann surface (see (Wise, 2010, p. 12), which also includes helpful illustrations). The hamster, while moving in the ball, is always at the point of tangency between the ball and the surface. The configuration of the hamster at a time can be specified by a triple of numbers: two specifying the point of tangency and the third being the hamster's orientation. The transformation group of the hamster at a point is thus SO(2). The motion of the ball is determined by the hamster, i.e., without slipping and twisting. The transformation group of the ball's configurations is SO(3). The quotient group SO(3)/SO(2) is exactly S^2, which is isomorphic to the ball. As can be expected, SO(3)/SO(2) describes a Klein geometry, and the Riemann surface together with (and surveyed by) the rolling ball is an analogy for a Cartan geometry.

There are important things to notice in this analogy which illuminate the unique properties of a Cartan connection. Although it might seem superficially that the hamster's possible motions are more restrictive than the ball's possible motions, the hamster's location at the surface together with its infinitesimal motion *determines* the infinitesimal motion of the ball, which is obvious based on our physical intuition about the scenario. This reflects the second property above, which says that the tangent space of a bundle with the smaller structure group is *isomorphic* to the Lie algebra of the larger group. Relatedly, we also see more intuitively how a Cartan connection can take value in the Lie algebra of a larger group than the gauge group of the principal bundle, just as the ball configurations at each point of any trajectory are completely determined by the trajectory together with the hamster configurations, constrained by the smaller gauge group of SO(2).

What does this have to do with TPG? Here is the idea. As mentioned above, a simple characterization of TPG consists of a flat principal connection ω, which can be represented by an $\mathfrak{so}(1, n-1)$-valued one-form $\omega^{\mathfrak{A}}{}_{\alpha}$ on P, together with a coframe field e^A_a, which can be represented as a $V \cong \mathbb{R}^n$-valued one-form on P. Since the Poincare group $\mathrm{ISO}(1, n-1)$ is a semidirect product of $\mathrm{SO}(1, n-1)$ and $\mathbb{R}^n$, and similarly the Lie algebra of the former is isomorphic to a semidirect sum of the those of the latter, we can combine the two fields into an $\mathfrak{iso}(1, n-1)$-valued connection, and thus make the coframe field part of a Cartan connection.

More formally, for any *reductive* Cartan geometry (of which $\mathcal{G} = \mathcal{H} \ltimes \mathcal{G}/\mathcal{H}$ is a special case), the Cartan connection ω_c can be decomposed uniquely into ω and θ, where ω is an ordinary principal connection on the principal H-bundle, and θ is a $\mathfrak{g}/\mathfrak{h}$-valued one-form on the H-bundle (Huguet et al., 2021a, p. 3).[30] θ is akin to e in being a 'translation-valued' one-form, but at this point they are technically still different. However, we now have everything we need to obtain the ordinary coframe field e; to see this, we appeal to the following result from Wise (2007, pp. 162–3):

Proposition 1. *The following two definitions of generalized coframe field e are equivalent, given a principal $\mathcal{H}$-bundle P and an Ehresmann connection on P:*

1. $e : TP \to \mathfrak{g}/\mathfrak{h}$.
2. $e : TM \to P \times_{\mathcal{H}} \mathfrak{g}/\mathfrak{h}$.

For a proof, see (Wise, 2007, pp. 162-3). Here e in (1) is exactly θ. As a special case, P can be the bundle LM_{SO}, in which case (2) amounts to the ordinary definition of the coframe field. Since we are indeed given an Ehresmann connection on the $\mathcal{H}$-bundle, namely ω, we have recovered the ordinary coframe field e through this equivalence. That is, we have recovered the ordinary tetrad field e, and thus the requisite objects with which to construct a version of TPG, from the reductive Cartan connection $\omega_c = \omega + \theta$. In this way, the idea of making teleparallel gravity a gauge theory for translations has thus been made rigorous by Le Delliou et al. (2020a,b).

2.4. **TPG as a higher gauge theory.** Finally, let's introduce the resources needed to understand TPG as a higher gauge theory, which is the formulation of the theory proposed and preferred by Baez and Wise (2015). Just as ordinary Yang–Mills gauge theory involves gauge transformations construed as maps between connections on a principal bundle, higher gauge theory involves gauge transformations between higher connections on a higher principal bundle. The geometric picture behind these higher connections is that they encode facts about the parallel transport not just of vectors (as for standard connections) but also of higher-dimensional objects—for more on this geometric picture, see (Baez and Huerta, 2011).

Just as a standard principal bundle can be defined as a quadruple $\langle P, M, G, \pi\rangle$, where P is the total space, M is the base space, G is the structure Lie group, and the usual projection $\pi : P \to M$ is a smooth surjective map,[31] a *principal 2-bundle* can be likewise be defined as a quadruple $\langle \mathbf{P}, M, \mathbf{G}, \boldsymbol{\pi}\rangle$, save that now $\mathbf{G}$ is a *2-group*, $\mathbf{P}$ is a right $\mathbf{G}$ *2-space*, and the projection $\boldsymbol{\pi} : \mathbf{P} \to M$ is a smooth map.[32]

[30] A Cartan geometry is *reductive* just in case there is an $\mathrm{Ad}(\mathcal{H})$ invariant H-module decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ (Sharpe, 2000, p. 197).

[31] See (Weatherall, 2016b, p. 2414) for a philosophical primer.

[32] Really, it's a smooth functor, since any manifold can be understood as a Lie groupoid whose only morphisms are identity morphisms.

To understand a 2-bundle, then, evidently it is crucial to understand the notions of 2-groups and 2-spaces.

To understand 2-groups, we first need to understand 2-categories. We begin by recalling the definition of a category. In general, a category $\mathbf{C}$ consists in a collection of objects $\mathrm{ob}(\mathbf{C})$ and a collection of arrows (or morphisms) $\mathrm{mor}(\mathbf{C})$ between the objects of $\mathbf{C}$, satisfying several conditions. Here, it is useful to introduce the concept of *hom-sets*, which are the sets of arrows between two objects in a category, i.e. for $a, b \in \mathrm{ob}(\mathbf{C})$, one defines $\mathrm{hom}_{\mathbf{C}}(a, b) := \{f | f : a \to b, f \in \mathrm{mor}(\mathbf{C})\}$. Then a category $\mathbf{C}$ is:

- A collection of objects $\mathrm{ob}(\mathbf{C})$;
- For all $a, b \in \mathrm{ob}(\mathbf{C})$, a function $(a, b) \to \mathrm{hom}_{\mathbf{C}}(a, b)$;
- (Composition of arrows) For all $a, b, c \in \mathrm{ob}(\mathbf{C})$, a function $\circ : \mathrm{hom}_{\mathbf{C}}(b, c) \times \mathrm{hom}_{\mathbf{C}}(a, b) \to \mathrm{hom}_{\mathbf{C}}(a, c)$;
- (Identity arrow) For all $a \in \mathrm{ob}(\mathbf{C})$, an arrow $\mathrm{id}_a \in \mathrm{hom}_{\mathbf{C}}(a, a)$;

satisfying

1. (Associativity) For all $a, b, c, d \in \mathrm{ob}(\mathbf{C})$, all $h \in \mathrm{hom}_{\mathbf{C}}(c, d)$, all $g \in \mathrm{hom}_{\mathbf{C}}(b, c)$, all $f \in \mathrm{hom}_{\mathbf{C}}(a, b)$, $h \circ (g \circ f) = (h \circ g) \circ f$.
2. (Unity) For all $a, b, c \in \mathrm{ob}(\mathbf{C})$, all $g \in \mathrm{hom}_{\mathbf{C}}(b, c)$, and all $f \in \mathrm{hom}_{\mathbf{C}}(a, b)$, $\mathrm{id}_b \circ f = f$ and $g \circ \mathrm{id}_b = g$.
3. (Disjointness) If $(a, b) \neq (a', b')$ then $\mathrm{hom}_{\mathbf{C}}(a, b) \cap \mathrm{hom}_{\mathbf{C}}(a', b') = \varnothing$.

A *strict 2-category* is a category enriched in **Cat** (which is the category whose objects are categories and whose arrows are functors)—which means, very roughly, that instead of assigning a set of morphisms to any ordered pair of objects in the category, we now assign a category of morphisms to any ordered pair of objects in the category. That is, a strict 2-category $\mathbf{C}$ consists of the following data:

- A collection of objects $\mathrm{ob}(\mathbf{C})$;
- For all $a, b \in \mathrm{ob}(\mathbf{C})$, a *hom-category* $\mathbf{hom}_{\mathbf{C}}(a, b)$;
- (Horizontal composition) For all $a, b, c \in \mathrm{ob}(\mathbf{C})$, a (bi)functor $\circ : \mathbf{hom}_{\mathbf{C}}(b, c) \times \mathbf{hom}_{\mathbf{C}}(a, b) \to \mathbf{hom}_{\mathbf{C}}(a, c)$;
- (Identity arrow) For all $a \in \mathrm{ob}(\mathbf{C})$, a functor $\mathrm{id}_a : \mathbf{1} \to \mathbf{hom}_{\mathbf{C}}(a, a)$ (here, $\mathbf{1}$ is the category with one object and one arrow).

satisfying

i. (Associativity) For all $a, b, c, d \in \mathrm{ob}(\mathbf{C})$, $\circ_{(a,c,d)}(\mathrm{id}_{\mathbf{hom}_{\mathbf{C}}(c,d)} \times \circ_{(a,b,c)}) = \circ_{(a,b,d)}(\circ_{(b,c,d)} \times \mathrm{id}_{\mathbf{hom}_{\mathbf{C}}(a,b)}) : \mathbf{hom}_{\mathbf{C}}(c, d) \times \mathbf{hom}_{\mathbf{C}}(b, c) \times \mathbf{hom}_{\mathbf{C}}(a, b) \to \mathbf{hom}_{\mathbf{C}}(a, d)$.

ii. (Unity) For all $a, b \in \mathrm{ob}(\mathbf{C})$, $\circ_{(a,b,b)}(\mathrm{id}_b \times \mathrm{id}_{\mathbf{hom}_{\mathbf{C}}(a,b)}) = \mathrm{proj}_{\mathbf{hom}_{\mathbf{C}}(a,b)} : \mathbf{1} \times \mathbf{hom}_{\mathbf{C}}(a, b) \to \mathbf{hom}_{\mathbf{C}}(a, b)$, and likewise $\circ_{(a,a,b)}(\mathrm{id}_{\mathbf{hom}_{\mathbf{C}}(a,b)} \times \mathrm{id}_a) = \mathrm{proj}_{\mathbf{hom}_{\mathbf{C}}(a,b)} : \mathbf{hom}_{\mathbf{C}}(a, b) \times \mathbf{1} \to \mathbf{hom}_{\mathbf{C}}(a, b)$.

The objects of hom-categories are called *1-morphisms*; the arrows of hom-categories are called *2-morphisms*; and composition of arrows within a hom-category is called *vertical composition*. In a *weak 2-category*, the associativity and unity axioms are modified so that horizontal composition is only required to be associative and unital up to isomorphism; i.e. we have instead:

i′. For all $a, b, c, d \in \mathrm{ob}(\mathbf{C})$, a natural isomorphism between the functors $\circ_{(a,c,d)}(\mathrm{id}_{\mathbf{hom}_\mathbf{C}(c,d)} \times \circ_{(a,b,c)}) : \mathbf{hom}_\mathbf{C}(c, d) \times \mathbf{hom}_\mathbf{C}(b, c) \times \mathbf{hom}_\mathbf{C}(a, b) \to \mathbf{hom}_\mathbf{C}(a, d)$ and $\circ_{(a,b,d)}(\circ_{(b,c,d)} \times \mathrm{id}_{\mathbf{hom}_\mathbf{C}(a,b)}) : \mathbf{hom}_\mathbf{C}(c, d) \times \mathbf{hom}_\mathbf{C}(b, c) \times \mathbf{hom}_\mathbf{C}(a, b) \to \mathbf{hom}_\mathbf{C}(a, d)$.

ii′. For all $a, b \in \mathrm{ob}(\mathbf{C})$, a pair of natural isomorphisms, respectively between $\circ_{(a,b,b)}(\mathrm{id}_b \times \mathrm{id}_{\mathbf{hom}_\mathbf{C}(a,b)}) : \mathbf{1} \times \mathbf{hom}_\mathbf{C}(a, b) \to \mathbf{hom}_\mathbf{C}(a, b)$ and $\mathrm{proj}_{\mathbf{hom}_\mathbf{C}(a,b)} : \mathbf{1} \times \mathbf{hom}_\mathbf{C}(a, b) \to \mathbf{hom}_\mathbf{C}(a, b)$, and between $\circ_{(a,a,b)}(\mathrm{id}_{\mathbf{hom}_\mathbf{C}(a,b)} \times \mathrm{id}_a) : \mathbf{hom}_\mathbf{C}(a, b) \times \mathbf{1} \to \mathbf{hom}_\mathbf{C}(a, b)$ and $\mathrm{proj}_{\mathbf{hom}_\mathbf{C}(a,b)} : \mathbf{hom}_\mathbf{C}(a, b) \times \mathbf{1} \to \mathbf{hom}_\mathbf{C}(a, b)$.

Next, recall that a *groupoid* is a category in which every morphism has an inverse. Very roughly, then, a *Lie groupoid* is a groupoid in which both the collection of objects of the category, and the collection of arrows of the category, have the structure of smooth manifolds. Namely, a Lie groupoid is a groupoid $\mathbf{G}$ in which:

- The collection of objects $\mathrm{ob}(\mathbf{G})$ is a smooth manifold.
- The collection of morphisms $\mathrm{mor}(\mathbf{G})$ is a smooth manifold.
- The maps $s, t : \mathrm{mor}(\mathbf{G}) \to \mathrm{ob}(\mathbf{G})$ sending each morphism to, respectively, its source and target are smooth.
- The map $\mathrm{id} : \mathrm{ob}(\mathbf{G}) \to \mathrm{mor}(\mathbf{G})$ sending each $a \in \mathrm{ob}(\mathbf{G})$ to id_a is smooth.
- The map $\mathrm{inv} : \mathrm{mor}(\mathbf{G}) \to \mathrm{mor}(\mathbf{G})$ sending each morphism to its inverse is smooth.
- The collection of composable pairs of morphisms is a submanifold of $\mathrm{mor}(\mathbf{G}) \times \mathrm{mor}(\mathbf{G})$, and composition of arrows is a smooth map from this submanifold to $\mathrm{mor}(\mathbf{G})$.

To recover the usual notion of a (Lie) group, we can restrict attention to (Lie) groupoids with only a single object.

Combining these ideas, we can now define (Lie) 2-groups. First, we begin by defining 2-groupoids. A strict 2-groupoid is a 2-category in which

- Every 1-morphism has an inverse.
- Every 2-morphism has a (vertical) inverse.

(In the weak case, these are replaced with almost-inverses, i.e. inverses up to natural isomorphism.) Equivalently, a strict 2-groupoid is a groupoid enriched in the category **Grpd** (whose objects are groupoids and whose arrows are functors). A (strict) Lie 2-groupoid then, very roughly, is a (strict) 2 groupoid in which the collections of objects, 1-morphisms, and 2-morphisms all form smooth manifolds.[33] Finally, a (Lie) 2-group is a (Lie) 2-groupoid with one object.

The foregoing makes it clear that a (Lie) 2-group $\mathbf{G}$ determines four pieces of data (making use of the fact that there is only one object, say a, in the category):

- A (Lie) group G, whose object is a, whose morphisms are $\mathrm{ob}(\mathbf{hom_G}(a,a))$, and with composition of morphisms as horizontal composition in $\mathbf{G}$ restricted to its action on 1-morphisms.
- A (Lie) group H, whose object is the image id of $\mathrm{ob}(\mathbf{1})$ under id_a, whose arrows are the set $\mathrm{hom}_{\mathbf{hom_G}(a,a)}(\mathrm{id},\cdot)$, and with composition of morphisms as vertical composition in $\mathbf{hom_G}(a,a)$.
- A (Lie) group homomorphism $t : H \to G$, defined via the condition that for each $g \in \mathrm{ob}(\mathbf{hom_G}(a,a))$ and each $h \in \mathrm{hom}_{\mathbf{hom_G}(a,a)}(\mathrm{id}, g)$, $t(h) = g$.
- An action α of G on H, defined through conjugation in $\mathrm{hom}_{\mathbf{hom_G}(a,a)}(\mathrm{id},\cdot)$ by the image of $\mathrm{mor}(\mathbf{1})$ under id_a for each $g \in \mathrm{hom}_{\mathbf{hom_G}(a,a)}(\mathrm{id},\cdot)$.

Together, the data (G, H, t, α) form a *(Lie) crossed module*. Conversely, given any (Lie) crossed module, one can define a corresponding (Lie) 2-group, and we will make use of this characterisation for ease of exposition later.

At this point, we can define two important 2-groups:

- The *Poincaré 2-group* ($\mathbf{Poinc}(1, n-1)$) consisting of a single object (say, a), and a hom-category $\mathbf{hom}_{\mathbf{Poinc}(1,n-1)}(a,a)$ defined as follows. $\mathrm{ob}(\mathbf{hom}_{\mathbf{Poinc}(1,n-1)}(a,a))$ is the Lorentz group $\mathrm{SO}(1, n-1)$, and $\mathrm{mor}(\mathbf{hom}_{\mathbf{Poinc}(1,n-1)}(a,a))$ is the Poincaré group $\mathrm{ISO}(1, n-1) = \mathrm{SO}(1, n-1) \ltimes \mathbb{R}^n$. Morphisms of $\mathbf{hom}_{\mathbf{Poinc}(1,n-1)}(a,a)$ are associated to the objects of $\mathbf{hom}_{\mathbf{Poinc}(1,n-1)}(a,a)$ so that the source and target of each morphism $(g,h) \in \mathrm{ISO}(1, n-1)$, $g \in \mathrm{SO}(1, n-1)$, $h \in \mathbb{R}^n$ are both g. Vertical composition is given by the map $(g,h) \circ (g,h') = (g, h+h')$ where $+$ denotes addition in $\mathbb{R}^n$, and horizontal composition is group composition in $\mathrm{ISO}(1, n-1)$, i.e. $(g,h) \circ (g',h') = (gg', h + gh')$.
- The *Teleparallel 2-group* ($\mathbf{Tel}(1, n-1)$) consisting of a single object (say, a), and a hom-category $\mathbf{hom}_{\mathbf{Tel}(1,n-1)}(a,a)$ defined as follows. $\mathrm{ob}(\mathbf{hom}_{\mathbf{Poinc}(1,n-1)}(a,a))$ is the Poincaré group, and $\mathrm{mor}(\mathbf{hom}_{\mathbf{Poinc}(1,n-1)}(a,a))$ is the group $\mathrm{Tel}(1, n-1) = \mathrm{ISO}(1, n-1) \ltimes \mathbb{R}^n$, where the semidirect product is defined via the group action $\alpha : \mathrm{ISO}(1, n-1) \times \mathbb{R}^n \to \mathbb{R}^n$, $\alpha((g,h), v) = gv$. Morphisms of $\mathbf{hom}_{\mathbf{Poinc}(1,n-1)}(a,a)$ are associated to the objects of $\mathbf{hom}_{\mathbf{Poinc}(1,n-1)}(a,a)$ so that the source of each morphism $((g,h),v) \in \mathrm{ISO}(1, n-1)$, $g \in \mathrm{SO}(1, n-1)$, $h, v \in \mathbb{R}^n$ is (g,h), and the target of each morphism $((g,h),v)$ is $(g, h+v)$. Vertical composition is given by the map $((g,h),v) \circ ((g',h'),v') = ((g',h'), v+v')$, and horizontal composition is group composition in $\mathrm{Tel}(1, n-1)$, i.e. $((g,h),v) \circ ((g',h'),v') = ((gg', h+gh'), v+gv')$.

[33] We will not repeat the full list of smoothness axioms for a Lie 2-groupoid here. The important point is that one takes the definition of a 2-groupoid $\mathbf{G}$ and requires that each hom-category $\mathbf{hom_G}(a,b)$ is a Lie groupoid, the space of objects is a smooth manifold, and that the collection of hom-categories is a smooth manifold over the space $\mathrm{ob}(\mathbf{G}) \times \mathrm{ob}(\mathbf{G})$, per the smoothness axioms for a Lie groupoid given above.

The Lie crossed module data for **Poinc**$(1, n-1)$ and **Tel**$(1, n-1)$ are then as follows:

- **Poinc**$(1, n-1)$: $G = \mathrm{SO}(1, n-1)$, $H = \mathbb{R}^n$, t is trivial, α is given by the usual group action of $\mathrm{SO}(1, n-1)$ on $\mathbb{R}^n$ determined by, e.g. the fundamental representation of $\mathrm{SO}(1, n-1)$.

- **Tel**$(1, n-1)$: $G = \mathrm{SO}(1, n-1)$, $H = \mathbb{R}^n$, t is the inclusion homomorphism, α is as described in the definition of **Tel**$(1, n-1)$ above.

Finally, we define **G** *2-spaces*. Let **G** be a strict Lie 2-group. A *strict right* ***G*** *2-space* is a Lie groupoid **X** equipped with a map $\alpha : \mathbf{X} \times \mathbf{G} \to \mathbf{X}$ satisfying the usual axioms for a right group action. (*Mutatis mutandis* for a strict left **G** 2-space; and for the case of a *weak* (right or left) **G** 2-space.)

We can now define two important principal 2-bundles, corresponding to the Lie 2-groups **Poinc**$(1, n-1)$ and **Tel**$(1, n-1)$. To prepare, first define the *fake tangent bundle* $\mathcal{T} \to M$ to be a vector bundle isomorphic (albeit not canonically) to the tangent bundle $TM \to M$. As before, a *coframe field* e is defined as an isomorphism from TM to $\mathcal{T}$. Let $\mathcal{T}$ be equipped with a metric of signature $(1, n-1)$ (in analogy with the metric η_{AB} introduced above). From the fake tangent bundle, we can build a principal $\mathrm{SO}(1, n-1)$ bundle $\mathcal{F} \to M$ called the *fake frame bundle*, which at each point $x \in M$ consists of linear orientation-preserving isometries $\mathbb{R}^{1,n-1} \to \mathcal{T}_x$.[34] Then we can introduce:

- The *Poincaré 2-bundle* $\mathbf{2}\mathcal{F} = \mathcal{F} \times_{\mathrm{SO}(1,n-1)} \mathbf{Poinc}(1, n-1)$. Here, $\times_{\mathrm{SO}(1,n-1)}$ determines an associated 2-bundle, defined in a precisely analogous way to the case of 1-bundles: if $\mathbf{P} \to M$ is a principal **G**-bundle, and **F** is a left **G** 2-space, then $(\mathbf{P} \times_{\mathbf{G}} \mathbf{F})_i$ $(i = 0, 1)$ is equal to $\mathbf{P}_i \times \mathbf{F}_i$ modulo the equivalence relation $(xg, f) \sim (x, gf)$, for $x \in \mathbf{P}_i, g \in \mathbf{G}_i, f \in \mathbf{F}_i$.

- The *Teleparallel 2-bundle* $\mathbf{Tel}(\mathcal{F}) = \mathcal{F} \times_{\mathrm{SO}(1,n-1)} \mathbf{Tel}(1, n-1)$.

Let's now turn to *2-connections* on 2-bundles. Detailed and rigorous treatments of these can be found in Bartels (2006); Baez and Schreiber (2004); in what follows, we will outline only the basic facts necessary to the development of TPG as a higher gauge theory.

Just as in ordinary (1-)gauge theory with group G, a connection can be seen locally as a $\mathfrak{g}$-valued 1-form A, in a 2-gauge theory based on a Lie 2-group with crossed module (G, H, t, α), a 2-connection can be seen locally as a $\mathfrak{g}$-valued 1-form A and an $\mathfrak{h}$-valued 2-form B on M which together are constrained to obey the 'fake flatness condition', which imposes that $\underline{t}(B)$ equal the curvature of A, where $\underline{t} : \mathfrak{h} \to \mathfrak{g}$ is the differential of the map t (Baez and Wise, 2015, p. 154).[35] The important point is that in the context of TPG, this allows us to encode the connection ω, which locally can be seen as a $\mathfrak{so}(1, n-1)$-valued 1-form, the coframe field e, which locally can be seen as an $\mathfrak{r}(n)$-valued 1-form (where $\mathfrak{r}(n)$ is the Lie

[34] One might question why one needs to work in this context with the fake frame bundle $\mathcal{F}$ rather than the 'real' frame bundle. As far as we can tell, the motivation stems from considerations to do with Cartan geometry—see (Gielen and Wise, 2013). But again, the details won't much matter going forward.

[35] The nomenclature 'fake flatness' derives from the theory of 'fake curvature', on which see (Breen and Messing, 2005).

algebra of the translation group), and its torsion $d_\omega e$, which locally can be seen as an $\mathfrak{t}(n)$-valued 2-form, into one unified geometric object (Baez and Wise, 2015, p. 155).[36] In particular, (Baez and Wise, 2015, theorem 32) prove that a 2-connection on $\mathbf{Tel}(1, n-1)$ 2-bundle consists of the following: (a) a flat connection ω on $\mathcal{F}$, (b) a $\mathcal{T}$-valued 1-form e, and (c) the $\mathcal{T}$-valued 2-form $d_\omega e$. These, of course, are precisely the objects which one needs in order to write down the dynamics of TPG; now, however, they are encoded in one specific geometric object. Going forward, we'll denote this Teleparallel 2-connection by $\bar{\omega}$.

3. General relativity and teleparallel gravity

With the relevant mathematics in hand, we can now consider GR and various distinct formulations of TPG, and their categorical equivalence (or otherwise). In this article, we'll work with a number of *prima facie* distinct theories:[37]

GR: Kinematical possibilities given by Lorentzian manifolds $\langle M, g_{ab}\rangle$; dynamical possibilities given by the Einstein equation.[38]

TPG_∇: Kinematical possibilities given by $\langle M, g_{ab}, \nabla\rangle$ for some torsionful connection ∇ compatible with the Lorentzian metric g_{ab}; dynamical possibilities given by the teleparallel equivalent of the Einstein equation.[39]

$\text{TPG}_{e,\omega}$: Kinematical possibilities given by $\langle M, LM_{\text{SO}}, \pi, LM_{\text{SO}} \times_{\text{SO}} V, e, \omega, \eta_{AB}\rangle$; dynamical possibilities given by the teleparallel equivalent of the Einstein equation written in terms of e and ω.

$\text{TPG}_{[e,\omega]}$: Kinematical possibilities given by $\langle M, LM_{\text{SO}}, \pi, LM_{\text{SO}} \times_{\text{SO}} V, [e, \omega], \eta_{AB}\rangle$, where $[e, \omega]$ denotes an equivalence class elements of which are related by vertical principal bundle automorphisms $e \mapsto \Psi^* e$, $\omega \mapsto \Psi^*\omega$. Dynamical possibilities given by the teleparallel equivalent of the Einstein equation (for each pair in the equivalence class).[40]

TPG_{ω_c}: Kinematical possibilities given by $\langle M, LM_{\text{SO}}, \pi, LM_{\text{SO}} \times_{\text{SO}} V, \omega_c, \eta_{AB}\rangle$, where ω_c is a reductive Cartan connection on LM; dynamical possibilities given by the teleparallel equivalent of the Einstein equation written in terms of the constituent objects of $\omega_c = \omega + \theta$.

BW: Kinematical possibilities given by $\langle M, \mathbf{Tel}(\mathcal{F}), \bar{\omega}\rangle$, where $\mathbf{Tel}(\mathcal{F})$ is a $\mathbf{Tel}(1, n-1)$ 2-bundle and $\bar{\omega}$ is a $\mathbf{Tel}(1, n-1)$ 2-connection. Dynamical

[36] This geometric unification clearly also arises in the case of TPG as a Cartan gauge theory, introduced in the previous subsection. In the sections to follow, we'll compare these two approaches with respect to geometric unification.

[37] Just as in the case of electromagnetism (specifically EM2) considered above, understanding these theories as categories will give rise to yet further versions of TPG, as we'll discuss below.

[38] To repeat: in this article we drop reference to matter, just as earlier we worked with source-free electromagnetism.

[39] On which see e.g. (Aldrovandi and Pereira, 2013).

[40] For more on this 'equivalence class' formulation of TPG, see (Hohmann, 2022) or (Krššák et al., 2019, p. 20).

possibilities given by the TPG equations written in terms of the constituent objects of $\mathbf{Tel}(\mathcal{F})$.[41]

Here, 'BW' refers to Baez and Wise (2015), who developed the version of TPG whereby it is understood as a higher gauge theory; TPG_∇ is nothing other than the Palatini version of the theory introduced above; likewise, TPG_{ω_c} is nothing other than the verison of TPG understood as a Cartan gauge theory, also introduced above. In general, it is crucial to appreciate that different authors in the physics literature work with different versions of TPG. For example, authors who work on the 'geometric trinity' of gravity (on which see (Beltrán Jiménez et al., 2019)—the third node of said 'trinity' being 'symmetric teleparallel gravity', in which gravitational degrees of freedom are represent neither by curvature nor by torsion, but instead by non-metricity) typically work using TPG_∇ (the reason of course being that it is more natural to use a metric formalism to represent non-metricity!). On the other hand, those who prefer to think of teleparallel gravity as a 'gauge theory of translations', e.g. Aldrovandi and Pereira (2013), typically work with $\text{TPG}_{e,\omega}$ since it is easiest to understand such 'gauging' in terms of transformations enacted upon a coframe field e. (Whether this formalism is really most appropriate for understanding TPG as a 'gauge theory of translations' is of course another matter—this, indeed, is precise what motivates Le Delliou et al. (2020a,b) to move to the framework of Cartan connections.) Those who also work in this 'gauge theory' paradigm but who worry about redundancy in $\langle e, \omega \rangle$ might prefer to work with $\text{TPG}_{[e,\omega]}$—see e.g. (Hohmann, 2022; Krššák et al., 2019) for something like this approach. And then we have BW, the construction of which was motivated by attempts to find physical applications of higher gauge theory.[42]

We should be clear at this point that this list of formulations of TPG is by no means exhaustive: three other formulations of TPG which are of particular conceptual interest in their own right are (i) a 'gauge fixed' version of $\text{TPG}_{e,\omega}$, in which the components of the connection ω are set to vanish (this version of TPG, sometimes called 'pure tetrad TPG', was used widely in the earlier literature on the topic—see (Krššák et al., 2019) for some more recent discussion), (ii) a version of TPG in which one redefines $e \to h$ such that it is 'dressed' to be invariant under local translations and/or Lorentz transformations (see e.g. dressing via the introduction of the fields A and B by Aldrovandi and Pereira (2013))[43]—as we have already alluded to above and as we'll discuss in more detail below, TPG_{ω_c} and BW can be understood to be ways of making mathematically precise this desideratum of TPG being a 'gauge theory of the translations', and (iii) a version of TPG built in analogy with the programme of 'pre-metric electromagnetism' of Hehl and collaborators (see (Hehl and Obukhov, 2003) for a detailed exposition of this approach), in which the TPG field equations are formulated in analogy with the 'pre-metric Maxwell equations', and different 'constitutive relations' (used to fix the metric in the case of electromagnetism) yield a variety of distinct torsionful theories—see (Krššák et al.,

[41] See (Baez and Wise, 2015, p. 177). Ultimately, for reasons discussed below, it's in fact not clear to us that these dynamics are well-defined, but for the time being we'll set these concerns aside.

[42] Developing some elegant mathematics and then reverse-engineering some physical application for said mathematics might strike one as questionable methodology. In any case, we discuss in detail in §6 the virtues of formulating TPG as a higher gauge theory.

[43] For recent philosophical literature on dressing, see (François, 2019).

	Ob	Mor
GR	$\langle M, g_{ab} \rangle$	$g_{ab} \mapsto \chi_* g_{ab}$
$\mathbf{TPG}_{\nabla}$	$\langle M, g_{ab}, \nabla \rangle$	$g_{ab} \mapsto \chi_* g_{ab}$
$\mathbf{TPG}_{e,\omega}$	$\langle M, LM_{\mathrm{SO}}, \pi, LM_{\mathrm{SO}} \times_{\mathrm{SO}} V, e, \omega, \eta_{AB} \rangle$	$e \mapsto \tilde{\chi}_* e;\ \omega \mapsto \tilde{\chi}_* \omega$
$\overline{\mathbf{TPG}}_{e,\omega}$	$\langle M, LM_{\mathrm{SO}}, \pi, LM_{\mathrm{SO}} \times_{\mathrm{SO}} V, e, \omega, \eta_{AB} \rangle$	$e \mapsto \Psi_* e;\ \omega \mapsto \Psi_* \omega$
$\mathbf{TPG}_{[e,\omega]}$	$\langle M, LM_{\mathrm{SO}}, \pi, LM_{\mathrm{SO}} \times_{\mathrm{SO}} V, [e, \omega], \eta_{AB} \rangle$	$[e, \omega] \mapsto [\tilde{\chi}_* e, \tilde{\chi}_* \omega]$
$\mathbf{TPG}_{\omega_c}$	$\langle M, LM_{\mathrm{SO}}, \pi, LM_{\mathrm{SO}} \times_{\mathrm{SO}} V, \omega_c, \eta_{AB} \rangle$	$\omega_c \mapsto \Psi_* \omega_c$
BW	$\langle M, \mathbf{Tel}(\mathcal{F}), \bar{\omega} \rangle$	$\bar{\omega} \mapsto \bar{\Psi}_* \bar{\omega}$

Table 2: Objects and morphisms for GR and various formulations of TPG, understood categorically. Notation for morphisms and associated notion of preservation follow the shorthand introduced in Table 1.

2019, §9.4) or (Itin et al., 2017).[44] To render the narrative manageable, in this article we'll set aside all three of (i)–(iii).

In order make further progress in understanding the equivalence (or otherwise) of GR and the versions of TPG listed above, we can again avail ourselves of some resources from category theory. As in the case of electromagnetism, to each of these theories we can associate a category (potentially multiple categories, as we'll see), objects and morphisms of which are given in Table 2.[45] In that table, $\chi : M \to M'$ denotes a spacetime diffeomorphism, Ψ denotes a principal bundle morphism (Ψ, χ). $\Psi : LM_{\mathrm{SO}} \to LM'_{\mathrm{SO}}$, $\chi : M \to M'$, and $\tilde{\chi}$ denotes the lift of a spacetime diffeomorphism $\chi : M \to M'$ to LM_{SO}, defined as follows. First, recall that the coframe field e was originally defined on LM (see §2.2). Let $\chi : M \to M'$. Then we can define LM'_{SO} to be the bundle given by the reduction of the structure group induced by $\tilde{\chi}_* e$, where $\tilde{\chi}$ is the unique lift of χ to LM. Then the restriction of $\tilde{\chi}$ to LM_{SO} is a principal bundle isomorphism $LM_{\mathrm{SO}} \to LM'_{\mathrm{SO}}$. (Why? Because some $p \in LM_{\mathrm{SO}}$ will be an oriented orthonormal frame with respect to $\overset{e}{g}_{ab}$ iff $\tilde{\chi}(p)$ is an oriented orthonormal frame with respect to $\overset{\tilde{\chi}_* e}{g}{}_{ab}$.) By abuse of notation, we denote this map $\tilde{\chi}$, which we can then use to push forward e.g. ω from LM_{SO} to LM'_{SO}.

Evidently, there is a sense in which $\mathbf{TPG}_{e,\omega}$ is analogous to **EM2**, $\overline{\mathbf{TPG}}_{e,\omega}$ is analogous to $\overline{\mathbf{EM2}}$, and $\mathbf{TPG}_{[e,\omega]}$ is analogous to **EM2′**. However, it is important to note there are also disanalogies here: the morphisms of **EM2** exhaust the isomorphisms of its objects; not so for $\mathbf{TPG}_{e,\omega}$, as the isomorphisms of its objects are in fact the morphisms of $\overline{\mathbf{TPG}}_{e,\omega}$. By contrast, the morphisms of $\overline{\mathbf{EM2}}$ include transformations (the gauge transformations of the vector potential) which are *not* isomorphisms of its objects. Accordingly, $\mathbf{TPG}_{[e,\omega]}$ is somewhat less natural than **EM2′**, for why be motivated to take equivalence classes of geometric objects which are already understood as being isomorphic?

In any case, with these categories in hand, let's now consider whether or not

[44] This provides an interesting heuristic for the generalisation of TPG. For recent philosophical discussion of pre-metric electromagnetism, see (Chen and Read, 2023).

[45] As before (i.e. for the case of electromagnetism considered previously), dynamics are suppressed in this table.

they are equivalent.

4. Categorical equivalence

In this section, after recalling some background on categorical equivalence in the philosophy literature (§4.1—skippable for *cognoscenti*), we consider the categorical (in)equivalence of the versions of TPG presented (as categories) in the previous section (§4.2).

4.1. Background on categorical equivalence.

Weatherall (2016a) proposed a criterion of equivalence of physical theories, according to which two given theories are equivalent just in case (a) their associated categories of models are equivalent, and (b) the functors realising said equivalence preserve empirical content. The category of models associated with a theory $\mathcal{T}$ is a category the objects of which are models of $\mathcal{T}$, and the morphisms of which relate models regarded as having the 'same structure'.[46]

What is it for two categories to be equivalent? Two categories $\mathbf{A}$ and $\mathbf{B}$ are equivalent just in case there exist functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{A}$ such that $FG \cong 1_{\mathbf{B}}$, and $GF \cong 1_{\mathbf{A}}$.[47] Equivalently, the categorical equivalence of $\mathbf{A}$ and $\mathbf{B}$ amounts to the existence of a functor relating them which is:

Full: For all objects $a, b \in \mathbf{A}$, the map $(f : a \to b) \mapsto (F(f) : F(a) \to F(b))$ induced by F is surjective.

Faithful: For all objects $a, b \in \mathbf{A}$, the map $(f : a \to b) \mapsto (F(f) : F(a) \to F(b))$ induced by F is injective.

Essentially surjective: For every object $x \in \mathbf{B}$, there is some object $a \in \mathbf{A}$ and arrows $f : F(a) \to x$ and $f^{-1} : x \to F(a)$ such that $f \circ f^{-1} = 1_x$.

A functor 'forgets structure' just in case it is not full; 'forgets stuff' just in case it is not faithful, and 'forgets properties' just in case it is not essentially surjective.[48]

4.2. Categorical equivalence of TPG formulations.

Let us now consider the categorical (in)equivalence of the theories (i.e., GR and various versions of TPG) presented as categories in §3.

Begin with $\mathbf{GR}$ and $\mathbf{TPG}_\nabla$, and consider a functor $F_1 : \mathbf{TPG}_\nabla \to \mathbf{GR}$ which takes each object $\langle M, g_{ab}, \nabla \rangle$ to $\langle M, g_{ab} \rangle$ and each arrow to an arrow generated by the same diffeomorphism. We have then the following proposition:

Proposition 2. *F_1 forgets (only) structure.*

[46] This will be an interpretative matter—see e.g. (March, 2024a). For relevant background on category theory, see (Mac Lane, 1998). Recently, a number of authors have proposed refinements of the categorical equivalence programme—see e.g. March (2024c); Hudetz (2019)—in order to accommodate concerns raised by *inter alia* Hudetz (2019); Weatherall (2018b). We will set aside these issues in what follows.

[47] Two categories $\mathbf{A}$ and $\mathbf{B}$ are equivalent just in case their *skeletons* sk($\mathbf{A}$) and sk($\mathbf{B}$) are isomorphic, where isomorphic objects in $\mathbf{A}$ are equal in sk($\mathbf{A}$), *mutatis mutandis* $\mathbf{B}$.

[48] For more detail on the interpretation of 'structure', 'stuff', and 'properties', see (Baez et al., 2004).

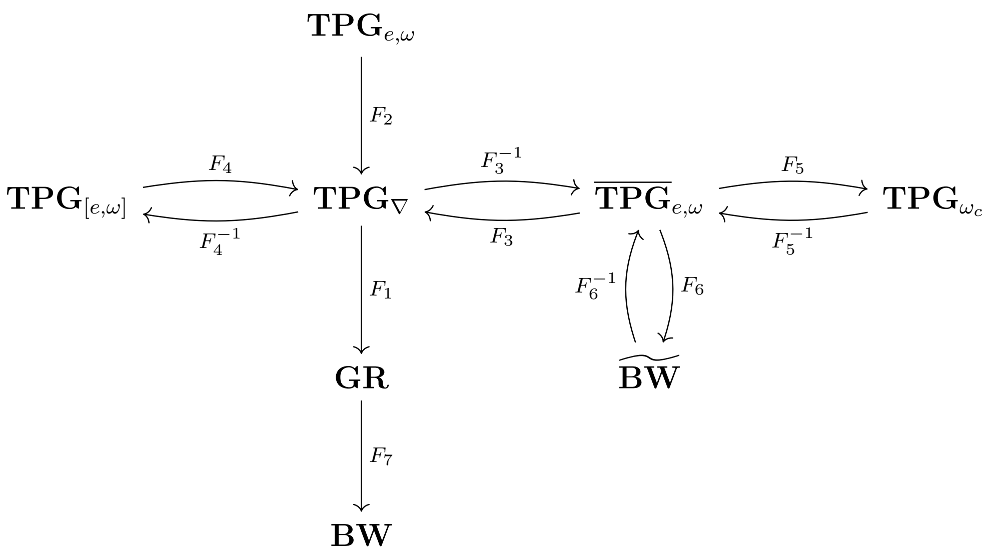

Figure 1: Functorial relationships between categories used in this article (modelled on Nguyen et al. (2018, fig. 2)).

Proof. See (Weatherall and Meskhidze, 2024, p. 16). □

From this, it follows that **GR** is categorically inequivalent to $\mathbf{TPG}_\nabla$. This is the main result of Weatherall and Meskhidze (2024). But what of our other formulations of TPG?

Before we get to the issue of categorical equivalence with respect to these other formulations of TPG, we note first that the relationship between models of TPG_∇ and $\text{TPG}_{e,\omega}$ is given by the following pair of propositions:

Proposition 3. *Let $\langle M, LM_{\text{SO}}, \pi, LM_{\text{SO}} \times_{\text{SO}} V, e, \omega, \eta_{AB}\rangle$ be a model of $TPG_{e,\omega}$. Then there exists a unique metric $\overset{e}{g}_{ab}$ and connection $\overset{e,\omega}{\nabla}$, as defined in equations* (1) *and* (2)*, such that $\langle M, \overset{e}{g}_{ab}, \overset{e,\omega}{\nabla}\rangle$ is a model of TPG_∇.*

Proposition 4. *Let $\langle M, g_{ab}, \nabla\rangle$ be a model of TPG_∇, and fix a vector space V of dimension n, a representation ρ of $\text{Gl}(n, \mathbb{R})$ and a (flat) Lorentzian metric η_{AB} on V. Then there exist a coframe e and connection ω on LM_{SO} such that $g_{ab} = \overset{e}{g}_{ab}$, $\nabla = \overset{e,\omega}{\nabla}$, where $\overset{e}{g}_{ab}$ and $\overset{e,\omega}{\nabla}$ are as defined in equations* (1) *and* (2)*, and $\langle M, LM_{\text{SO}}, \pi, LM_{\text{SO}} \times_{\text{SO}} V, e, \omega, \eta_{AB}\rangle$ is a model of $TPG_{e,\omega}$. Moreover, the pair $\langle e, \omega\rangle$ is not unique. If $\langle e, \omega\rangle$ is any such pair, then so is $\langle e', \omega'\rangle$ iff $\langle e', \omega'\rangle = \langle \varphi^* e, \varphi^* \omega\rangle$ for some vertical principal bundle automorphism $\varphi : LM_{\text{SO}} \to LM_{\text{SO}}$.*

For proofs, see Appendix A. Note that vertical principal bundle automorphisms of LM_{SO} correspond to local Lorentz transformations in the TPG literature; the usual expressions for the behaviour of the coframe and connection under such a transformation can, as ever, be recovered by fixing a (local) trivialisation of LM_{SO} and computing expressions for the (local) representatives of e and ω. Note also that we have explicitly set aside non-uniqueness of the pair $\langle V, \eta_{AB}\rangle$ in proposition 4.

We also make use of the following result:

Proposition 5. *Let $\mathfrak{M}_\nabla = \langle M, g_{ab}, \nabla\rangle$ be a model of TPG_∇ and let $\chi : M \to M'$ be a diffeomorphism. Let $\mathfrak{M}_{e,\omega} = \langle M, LM_{\mathrm{SO}}, \pi_L, LM_{\mathrm{SO}} \times_{\mathrm{SO}} V, e, \omega, \eta_{AB}\rangle$, $\mathfrak{M}'_{e,\omega} = \langle M', LM'_{\mathrm{SO}}, \pi'_L, LM'_{\mathrm{SO}} \times_{\mathrm{SO}} V, e', \omega', \eta_{AB}\rangle$ be any two models of $TPG_{e,\omega}$ corresponding to $\mathfrak{M}_\nabla$, $\chi_*\mathfrak{M}_\nabla$ respectively in the sense of proposition 4. Then there exists a unique bundle morphism (Ψ, χ) such that $\langle e', \omega'\rangle = \langle \Psi_* e, \Psi_*\omega\rangle$.*

Again, see Appendix A for proofs. With these propositions in hand, consider then $\mathbf{TPG}_{e,\omega}$, and now consider a functor $F_2 : \mathbf{TPG}_{e,\omega} \to \mathbf{TPG}_\nabla$ which takes each object of $\mathbf{TPG}_{e,\omega}$ to the corresponding object of $\mathbf{TPG}_\nabla$ as given in proposition 3, and each arrow $(\tilde{\chi}, \chi) \mapsto \chi$. Then we have:

Proposition 6. *F_2 forgets (only) structure.*

Proof. F_2 is essentially surjective by proposition 3 and faithful by the proof of proposition 5. But it is not full. To see this, consider an object $\mathfrak{M}_\nabla = \langle M, g, \nabla\rangle$ of $\mathbf{TPG}_\nabla$, and let $\mathfrak{M}_{e,\omega}$, $\mathfrak{M}'_{e,\omega}$ be any two distinct objects in $\mathbf{TPG}_{e,\omega}$ which correspond to $\mathfrak{M}_\nabla$ in the sense of proposition 3 (such exist, by proposition 4). Then the arrow $\mathrm{id}_M \in \hom_{\mathbf{TPG}_\nabla}(\mathfrak{M}_\nabla, \mathfrak{M}_\nabla)$ is not the image of any arrow in $\hom_{\mathbf{TPG}_{e,\omega}}(\mathfrak{M}_{e,\omega}, \mathfrak{M}'_{e,\omega})$ under F_2. □

So, $\mathbf{TPG}_{e,\omega}$ has more structure than $\mathbf{TPG}_\nabla$. What about $\overline{\mathbf{TPG}}_{e,\omega}$? Consider the functor $F_3 : \overline{\mathbf{TPG}}_{e,\omega} \to \mathbf{TPG}_\nabla$ which sends objects of $\overline{\mathbf{TPG}}_{e,\omega}$ to their corresponding objects of $\mathbf{TPG}_\nabla$ given in proposition 3, and takes each arrow $(\tilde{\chi}^*\varphi, \chi)$ to χ. Then we have:

Proposition 7. *F_3 forgets nothing.*

Proof. F_3 is essentially surjective by proposition 3 and full and faithful by proposition 5. □

The next category to consider is $\mathbf{TPG}_{[e,\omega]}$. Consider a functor $F_4 : \mathbf{TPG}_{[e,\omega]} \to \mathbf{TPG}_\nabla$ which takes morphisms of the former category (i.e., diffeomorphisms χ such that $[e', \omega'] = [\tilde{\chi}_* e, \tilde{\chi}_*\omega]$ to χ.

Proposition 8. *F_4 forgets nothing.*

Proof. F_4 is essentially surjective by proposition 3 and full and faithful by the proof of proposition 5. □

Note that this matches the fact that $\mathbf{EM2}'$ and $\overline{\mathbf{EM2}}$ are categorically equivalent, on which see (Weatherall, 2016a, p. 1084).[49]

The next version of TPG to which we turn is $\mathbf{TPG}_{\omega_c}$. Again, we begin with a pair of propositions:

Proposition 9. *Let $\langle M, LM_{\mathrm{SO}}, \pi, LM_{\mathrm{SO}} \times_{\mathrm{SO}} V, e, \omega, \eta_{AB}\rangle$ be a model of $TPG_{e,\omega}$. Then there exists a unique reductive Cartan connection $\omega_c = \omega + e$ such that $\langle M, LM_{\mathrm{SO}}, \pi, LM_{\mathrm{SO}} \times_{\mathrm{SO}} V, \omega_c, \eta_{AB}\rangle$ is a model of TPG_{ω_c}.*

Proof. See theorem 2 of (Kobayashi, 1956). □

[49] Thinking about morphisms for categories the objects of which include equivalence classes is a little delicate, as noted in the case of electromagnetism by Nguyen et al. (2018). We won't discuss this issue further in this article.

Proposition 10. *Let* $\langle M, LM_{\text{SO}}, \pi, LM_{\text{SO}} \times_{\text{SO}} V, \omega_c, \eta_{AB}\rangle$ *be a model of* TPG_{ω_c}*. Then there exists a unique coframe field-connection pair* (e, ω) *such that* $\omega_c = \omega + e$ *and* $\langle M, LM_{\text{SO}}, \pi, LM_{\text{SO}} \times_{\text{SO}} V, e, \omega, \eta_{AB}\rangle$ *is a model of* $TPG_{e,\omega}$*.*

Proof. See theorem 2 of (Kobayashi, 1956). □

Next, note that since the decomposition of the reductive Cartan connection $\omega_c = \omega + e$ is $\text{Ad}(\text{SO}(1,3))$-invariant, bundle isomorphisms $\Psi : LM_{\text{SO}} \to LM'_{\text{SO}}$ have the following action on e and ω:[50]

$$\begin{aligned} e &\mapsto e' = \Psi^* e, \\ \omega &\mapsto \omega' = \Psi^* \omega. \end{aligned} \tag{3}$$

Consider, then, a functor $F_5 : \overline{\mathbf{TPG}}_{e,\omega} \to \mathbf{TPG}_{\omega_c}$ which takes each object in $\mathbf{TPG}_{e,\omega}$ to its corresponding object of $\mathbf{TPG}_{\omega_c}$ as given in proposition 9, and arrows in $\mathbf{TPG}_{e,\omega}$ to an arrow generated by the same principal bundle diffeomorphism.

Proposition 11. F_5 *forgets nothing.*

Proof. F_5 is essentially surjective by proposition 9, and full and faithful by construction given equation 3. □

This just leaves **BW**. First, we note the following pair of propositions:

Proposition 12. *Let* $\langle M, \textbf{\textit{Tel}}(\mathcal{F}), \bar{\omega}\rangle$ *be a model of* BW*. Then there exists a unique pair* (e, ω) *such that* $\bar{\omega} = (\omega, e, T)$ *and* $\langle M, LM_{\text{SO}}, \pi, LM_{\text{SO}} \times_{\text{SO}} V, e, \omega, \eta_{AB}\rangle$ *is a model of* $TPG_{e,\omega}$*.*

Proof. See proposition 27 of (Baez and Wise, 2015). □

Proposition 13. *Let* $\langle M, LM_{\text{SO}}, \pi, LM_{\text{SO}} \times_{\text{SO}} V, e, \omega, \eta_{AB}\rangle$ *be a model of* $TPG_{e,\omega}$*. Then there exists a unique* $\bar{\omega} = (\omega, e, T)$ *such that* $\langle M, \textbf{\textit{Tel}}(\mathcal{F}), \bar{\omega}\rangle$ *is a model of* BW*.*

Proof. See proposition 27 of (Baez and Wise, 2015). □

Next note that weak vertical bundle automorphisms of $\mathbf{Tel}(\mathcal{F})$ can be shown to be equivalent to the maps (Baez and Wise, 2015, eq. 6):

$$\begin{aligned} e &\mapsto e' = \Lambda e + d_{\omega'} v + a, \\ \omega &\mapsto \omega' = \Lambda \omega \Lambda^{-1} + \Lambda d \Lambda^{-1}, \\ T &\mapsto T' = d_{\omega'} e'. \end{aligned} \tag{4}$$

Strict vertical $\mathbf{Tel}(\mathcal{F})$ bundle automorphisms correspond to the case $a = 0$. As Baez and Wise (2015, p. 178) also note, neither strict gauge transformations with $d_{\omega'} v \neq 0$ nor weak gauge transformations preserve the TPG action written in terms of the Teleparallel 2-connection; hence, they do not preserve dynamical possibilities. We'll return later to the philosophical significance of this fact; for the time being, we note that this motivates the introduction of a new category $\widetilde{\mathbf{BW}}$, in which morphisms of $\mathbf{BW}$ associated with transformations for which $d_{\omega'} v \neq 0$ and $a \neq 0$ are excluded.

[50]For discussion related to this, see (Wise, 2010, §3.4).

That this can be done consistently is a consequence of the fact that the $d_{\omega'}v = 0$ and $a = 0$ bundle automorphisms are precisely those bundle automorphisms of $\mathbf{Tel}(\mathcal{F})$ induced by strict bundle automorphisms of $\mathbf{2}\mathcal{F} = \mathcal{F} \times_{\text{SO}} \mathbf{Poin}(1, n-1)$; since $\mathbf{Tel}(\mathcal{F}) = \mathbf{2}\mathcal{F} \times_{\mathbf{Poin}(1,n-1)} \mathbf{Tel}(1, n-1)$, we can take the arrows of $\widetilde{\mathbf{BW}}$ to be principal 2-bundle isomorphisms $\mathbf{Tel}(\mathcal{F}) \to \mathbf{Tel}(\mathcal{F}')$ induced by strict principal 2-bundle isomorphisms $\mathbf{2}\mathcal{F} \to \mathbf{2}\mathcal{F}'$. Such bundle automorphisms are in one-to-one correspondence with bundle automorphisms of LM_{SO}, since $\mathbf{SO}(1, n-1)$ is the Lie groupoid of objects of $\mathbf{Poin}(1, n-1)$ (Baez and Wise, 2015, §2.7).

Consider a functor $F_6 : \overline{\mathbf{TPG}}_{e,\omega} \to \widetilde{\mathbf{BW}}$ which each object in $\overline{\mathbf{TPG}}_{e,\omega}$ to its corresponding object of $\widetilde{\mathbf{BW}}$ as given in proposition 13, and takes bundle automorphisms of LM_{SO} to restricted bundle automorphisms of $\mathbf{Tel}(\mathcal{F})$ in the sense discussed above. In this case, we have:

Proposition 14. *F_6 forgets nothing.*

Proof. It is clear from the above discussion that there is a one-to-one correspondence between the objects and morphisms between the two categories. □

How does **BW** compare with the other formulations of TPG given in this article in respect of its amount of structure? This, indeed, is our final question to address when it comes to the categorical equivalence (or otherwise) of **GR** and the various versions of TPG considered in this article. In order to answer this question, note first that the weak/strict gauge transformations with (respectively) $a \neq 0$ or $d_{\omega'}v \neq 0$ transcend the symmetries of the formulations of TPG considered previously—moreover, these transformations do not preserve the spacetime metric, as illustrated straightforwardly for the case of strict gauge transformations with $d_{\omega'}v \neq 0$:

$$\begin{aligned} g_{ab} &= \eta_{MN} e^M{}_a e^N{}_b \\ &\mapsto \eta_{MN}(e^M{}_a + (d_{\omega'}v)^M{}_a)(e^N{}_b + (d_{\omega'}v)^N{}_b) \\ &= \eta_{MN}(e^M{}_a + d_a v^M + \omega'_a{}^M{}_O v^O)(e^N{}_b + d_b v^N + \omega'_b{}^N{}_P v^P) \\ &= \eta_{MN} e^M{}_a e^N{}_b + \cdots \end{aligned} \tag{5}$$

(The same is true for weak gauge transformations with $a \neq 0$.) The key point is that both of these gauge transformations are not metric-preserving. So now consider a functor $F_7 : \mathbf{GR} \to \mathbf{BW}$ which takes objects of **GR** to their corresponding objects of **BW** via propositions 4 and 13, and isometries of g_{ab} to bundle automorphisms of $\mathbf{Tel}(\mathcal{F})$:

Proposition 15. *F_7 forgets (only) structure.*

Proof. It is clear that the distinct isometries of g_{ab} in **GR** are mapped to distinct bundle automorphisms of $\mathbf{Tel}(\mathcal{F})$ in **BW**—to see this, note that $\chi_* g_{ab} = \eta_{MN} \chi_* e^M{}_a \chi_* e^N{}_b$; $e^M{}_a$ and $\chi_* e^M{}_a$ are distinct but will be associated to a bundle automorphism of $\mathbf{Tel}(\mathcal{F})$, and any distinct such transformation of the coframes will correspond to a distinct bundle automorphism. It is also clear that all objects in **BW**, which are determined by some ω and e, are mapped onto by F_7. So, the functor is faithful and essentially surjective. But for $d_{\omega'}v \neq 0$ and $a \neq 0$, the morphisms between $\langle M, \mathbf{Tel}(\mathcal{F}), \bar{\omega}\rangle$ and $\langle M, \mathbf{Tel}(\mathcal{F}), \overset{v,a}{\bar{\Psi}}_* \bar{\omega}\rangle$ are not mapped onto (where $\overset{v,a}{\bar{\Psi}}$ is the bundle automorphism associated with v, a), for we have seen above that such transformations are not metrically equivalent (even up to isometry). So, the functor F_7 is not full. □

Let's take stock. Our conclusions from this section regarding the categorical equivalence (or otherwise) of GR and the various formulations of TPG considered in this article (all understood categorically) are as follows:

1. Of all the theories considered in this article, $\mathbf{TPG}_{e,\omega}$ has the most structure.
2. $\mathbf{TPG}_{\nabla}$, $\overline{\mathbf{TPG}}_{e,\omega}$, $\mathbf{TPG}_{[e,\omega]}$, $\mathbf{TPG}_{\omega_c}$, and $\widetilde{\mathbf{BW}}$ all have equal amounts of structure, and are all categorically equivalent.
3. **GR** has less structure than the theories in (2).
4. **BW** has less structure than **GR**; it has the least amount of structure of any of the theories considered in this article.

So much for categorical equivalence. Let's now consider whether these theories are reduced or sophisticated versions of each other.

5. Reduction and sophistication

How does the pattern of reduced/sophisticated theories carry over to the case of teleparallel gravity from the case of electromagnetism? There are several points to make here:

A. We have already seen that one can map many models of $\mathbf{TPG}_{\nabla}$ to the same **GR** model—namely, that with the same metric g. As such, one sees that **GR** is a reduced theory associated with $\mathbf{TPG}_{\nabla}$.

B. $\mathbf{TPG}_{\nabla}$ is categorically equivalent to $\overline{\mathbf{TPG}}_{e,\omega}$; moreover, both of these theories have as their morphisms all the automorphisms of their objects. As such, these equivalent theories are both internally sophisticated.

C. As already discussed, $\mathbf{TPG}_{e,\omega}$ is a somewhat unnatural theory as its morphisms do not exhaust the automorphisms of its objects. Without the morphisms associated with local Lorentz transformations, one regards those transformations as relating distinct (but empirically equivalent) states of affairs. In this case, one can just insert more morphisms in the category—those associated with the local Lorentz transformations—in order to arrive at a new category, $\overline{\mathbf{TPG}}_{e,\omega}$. While this might look like a case of external sophistication, it is in fact *also* a case of internal sophistication, for the morphism-related models are *already isomorphic*, without the need for any mathematical reformulation.

D. $\mathbf{TPG}_{[e,\omega]}$ is also *already* a sophisticated theory; however, as already mentioned above, it is also somewhat unnatural, as typically one does not take equivalence classes of objects which are already isomorphic.

E. $\mathbf{TPG}_{\omega_c}$ and **BW** are also already internally sophisticated versions of $\mathbf{TPG}_{e,\omega}$, because every morphism in the category is an isomorphism of the objects in the category.

F. Despite $\widetilde{\mathbf{BW}}$ being equivalent to $\mathbf{TPG}_{\nabla}$, $\overline{\mathbf{TPG}}_{e,\omega}$, $\mathbf{TPG}_{[e,\omega]}$, and $\mathbf{TPG}_{\omega_c}$, it is a theory which is sophisticated with respect to the bundle automorphisms corresponding to local Lorentz transformations, but not with respect to bundle automorphisms corresponding to strict gauge transformations with $d_{\omega'}v \neq 0$ or to weak gauge transformations. As before, one could insert further arrows here, thereby both externally *and* internally sophisticating, and thereby arrive at **BW**.

Let's compare again with the case of electromagnetism. There, one could reduce **EM2** in order to arrive at **EM1**; alternatively, one could take equivalence classes of vector potentials in order to arrive at **EM2′**—also a reduced theory, albeit not one formulated 'intrinsically' (cf. (March, 2024d)). One could also externally sophisticate in order to arrive at $\overline{\mathbf{EM2}}$, or internally sophisticate in order to arrive at **EM3**.

The situation is somewhat similar in TPG, but there are important differences. Beginning with $\mathbf{TPG}_{e,\omega}$, one can sophisticate this theory to arrive at $\overline{\mathbf{TPG}}_{e,\omega}$, but as stressed above this case of external sophistication is *also* a case of internal sophistication, for the additional morphisms are isomorphisms of the objects in the category anyway! One can take equivalence classes *per* $\mathbf{TPG}_{[e,\omega]}$, but unlike moving from **EM2** to **EM2′** this does not *yield* an internally sophisticated (but not intrinsically formulated) theory, because the theory was internally sophisticated to begin with! And in the case of TPG, we see with $\mathbf{TPG}_{\omega_c}$ and **BW** that there are multiple different ways to sophisticate the theory, which have varying amounts of structure.

Here are two upshots from these observations. First: inserting arrows between objects in a category when those objects are not isomorphic counts as a case of (merely) external sophistication (again, cf. (March, 2024d)); however, if a category *lacks* arrows between objects which are isomorphic, then adding those arrows counts as a case of external sophistication which is also a case of internal sophistication. And second—to repeat—sophistication (or reduction!) needn't be unique: we see this in the case of the transition from (e.g.) $\mathbf{TPG}_{e,\omega}$ to (e.g.) $\mathbf{TPG}_{\omega_c}$ or **BW**, where one can 'forget' structure' such that symmetries are isomorphisms, but one can in addition forget about varying degrees of *further* structure to yield distinct (i.e., categorically inequivalent) resulting theories.[51]

6. Assessing TPG as a Cartan or higher gauge theory

With the above, we take ourselves to have given some decisive and fair exhaustive verdicts on equivalence, reduction, and sophistication in teleparallel gravity. Let's now step back, and interrogate why one might be interested in formulating TPG as a Cartan or higher gauge theory to begin within. In particular, in this section we'll consider four different answers to this question which one might offer: (i) those to do with the Yang–Mills analogy (§6.1), (ii) those pushing the virtues of sophisticated theories (§6.2), (iii) those to do with theoretical unification (§6.3), and (iv) those to do with symmetry principles in physics (§6.4).

[51]An anonymous referee has suggested to us that making this claim in the context of **BW** is tendentious given the problematic features of **BW** to be discussed in the next section. Fair enough in this particular case; however, it remains the case that there is no *a priori* reason to think that sophistication (or reduction) need be unique.

6.1. **The Yang–Mills analogy.** Given that TPG can be formulated as a theory about a connection (or connection and coframe field) on a principal bundle, one might ask to what extent TPG can be thought of as a 'gauge theory' which is analogous to Yang–Mills theories (on which, for recent philosophical discussion, see (Wallace, 2015)), and if so, as a gauge theory for which group. Here, it is helpful to recall several features of 'standard' Yang–Mills type theories (for some gauge group $\mathcal{G}$):

1. The gauge fields are pullbacks (along sections) of a (principal) connection $\mathcal{G}$ connection ω on a principal $\mathcal{G}$ bundle to the tangent space.
2. The field strengths are the pullbacks (along sections) of the curvature two-form associated with the principal $\mathcal{G}$ connection ω.
3. The action is $\mathcal{G}$-invariant.

Now, if one takes $\mathcal{G} = \mathrm{SO}(1, n-1)$, i.e. the Lorentz group, then the standard gauge-theoretic approach to TPG satisfies (3) but not (1) or (2). This is because this formulation of TPG also includes as a gauge field the coframe e, and the torsion as a field strength. Meanwhile, none of the other formulations of TPG we have discussed here satisfy any of (1)–(3) for this choice of G.

Alternatively, if one takes $\mathcal{G} = \mathrm{SO}(1, n-1) \rtimes \mathbb{R}^n$ (motivated by the desire to understand TPG as a gauge theory of the translation group), then the Cartan approach to TPG satisfies (2) but not (3), and it is unclear how to assess (1). This is because the main principal bundle of interest in the Cartan approach is still LM_{SO} (and the Cartan connection is not a principal connection on LM_{SO}), but one does have the bundle $LM_{\mathrm{SO}} \times_{\mathrm{SO}} (\mathrm{SO}(1, n-1) \rtimes \mathbb{R}^n)$ 'in the background', so to speak, on which the Cartan connection *is* a principal connection.[52]

Finally, if one takes $\mathcal{G} = \mathbf{Tel}(1, n-1)$, then the approach of Baez and Wise satisfies (1) but not (2) (since the 2-curvature of the teleparallel 2-connection vanishes identically, or alternatively, since the relevant field strength, i.e. the torsion is now in fact part of the connection), nor does it satisfy (3). For this reason, on no (existing, to our knowledge) approach to TPG can the theory be understood as a 'standard' gauge theory.

6.2. **Virtues of sophisticated theories.** One of the by-now well-known advantages of internally sophisticating in the case of electromagnetism is that certain equations of motion become mathematical identities—for example, the Maxwell equation $d_a F_{bc} = 0$ in **EM1** becomes the Bianchi identity $d_a d_b A_c = 0$ in $\overline{\mathbf{EM2}}$.[53] This, the thought goes, leaves less to be explained in the resulting theory. This raises the question of whether there could be any advantage to the use of $\overline{\mathbf{TPG}}_{e,\omega}$ or $\mathbf{TPG}_{\omega_c}$ or **BW** on similar grounds. This question warrants careful assessment, for it could provide reasons which militate in favour of the use of TPG on conceptual grounds (obviously, while accepting the trade-off that some of these theories have more structure than **GR**). One possible illustration of a result like this is that the 'fake flatness condition' in **BW** constrains at the level of kinematics the spacetime

[52] Note also that the TPG action in the Cartan formalism is an instance of the Yang–Mills action $\Omega \wedge \star\Omega$, providing one uses a suitably-defined 'internal' Hodge dual—see Lucas and Pereira (2008).

[53] Also **EM2**, but that isn't a sophisticated theory.

connection to be flat—one does not seem to get this constraint 'for free' in anything but this sophisticated approach to the theory.

6.3. **Unification.** One might claim that there is some advantage to working with $\mathbf{TPG}_{\omega_c}$ or **BW** due to the fact that e and ω are unified into *one* object—either ω_c or $\bar{\omega}$. But to what extent is this a genuine virtue of these approaches to the theory? One worry on this front is that there is no physical correlation between the objects being unified, in the sense that e.g. there is no further non-trivial coupling between said objects in some mutual dynamics. According to Maudlin (1996), this kind of physical correlation (which he calls "nomic correlation") is one of the key criteria required in order to regard a theory as being unificatory. So, following Maudlin's lead, we would suggest that there is no unification in either of these formulations of TPG in a 'true' physical sense.[54]

6.4. **Earman's principles.** Following Jacobs (2021), define 'value space symmetries' as automorphisms of the value space (i.e., the spaces in which the physical fields take their values—for us, this will include the relevant bundles), and 'internal symmetries' as solutionhood-preserving transformations of a theory's models induced by bijections of the value space.[55] Then here, again following Jacobs (2021, §7.4), are the 'value space' versions of Earman's famous 'symmetry principles', SP1 and SP2:[56]

SP1: If φ induces an internal symmetry, then it is a value space symmetry;

SP2: If φ is a value space symmetry, then it induces an internal symmetry.

Now, given these principles, one might worry about their status in TPG. In particular, as Baez and Wise (2015, p. 178) acknowledge, strict gauge transformations with $d_{\omega'}v \neq 0$ and weak gauge transformations are not symmetries of the TPG action! Therefore, there seem to be value space symmetries which are not internal symmetries, violating SP2 in what Belot (2000) would (we expect) disparage as a case of "arrant knavery".

To get a handle on the consequences of this, it is instructive to ask: what kinds of models of BW (and what corresponding models of GR) are related by (strict or weak) gauge transformations? A partial answer to this question is given by Baez and Wise (2015, Theorem 36), who show that for any simply-connected manifold, *all* teleparallel 2-connections are gauge-equivalent under weak gauge transformations. In particular, and recalling that the teleparallel 2-connection of BW encodes the coframe field e of $\mathrm{TPG}_{e,\omega}$ and hence the metric of GR, this entails that for any simply connected manifold, all models of GR on that manifold are (weakly) gauge equivalent according to BW. This is, of course, one manifestation of the fact that **BW** has less structure than **GR**, as shown above. Nevertheless, put this way, it is striking, since it appears to show that weak teleparallel 2-connections lack local

[54]This isn't to deny that (e.g.) higher gauge theory might possibly be useful in shedding light on unification in the standard model, possibly in dialogue by recent work by Gomes (2024). This, however, is evidently a topic for another day.

[55]Arguably, 'dynamical symmetry' would be more perspicuous terminology here, but in what follows we'll continue to use Jacobs' nomenclature.

[56]Cf. (Earman, 1989, §3.4).

gauge-invariant degrees of freedom (consider an embedding of e.g. $\mathbb{R}^n$ into arbitrary M). This raises some immediate questions:

1. Is it acceptable to identify these models of GR *per* BW?
2. In what sense does BW have the resources to represent the full range of relativistic phenomena?
3. How does a model of BW acquire physical significance? How does it make predictions?

We'll take these in reverse order. *Ad* (3): by identifying distinct GR models (in fact, all possible GR models for a given manifold topology, as we've seen), it would seem that a given model of BW radically underdetermines empirical phenomena (due to its lack of local gauge-invariant degrees of freedom). As such, one's ability to use a specific BW model to make specific predictions about the evolution of some spacetime would seem to be severely undermined. *Ad* (2): while in a certain sense it's true that, due to BW models having less structure than GR models, they are able to represent more phenomena (cf. Bradley and Weatherall (2020); Wolf and Read (2023)), there is also a clear sense—exhibited in the above-discussed failure of Earman's principles—in which this representational flexibility of BW leaves the physical phenomena which the theory purports to describe problematically unconstrained (which, of course, is just to repeat our points regarding (3)). And *ad* (1): given that identifying GR models *per* BW leads to these difficulties of representation, in our view it is only reasonable to conclude that it is *not* acceptable to make this identification. Thus, mathematical elegance aside, there are good grounds on which to maintain that BW is *not* an acceptable alternative spacetime theory to GR.

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A. Proofs of various propositions

Proposition 3. *Let $\langle M, LM_{\mathrm{SO}}, \pi, LM_{\mathrm{SO}} \times_{\mathrm{SO}} V, e, \omega, \eta_{AB}\rangle$ be a model of TPG$_{e,\omega}$. Then there exists a unique metric $\overset{e}{g}_{ab}$ and connection $\overset{e,\omega}{\nabla}$, as defined in equations* (1) *and* (2)*, such that $\langle M, \overset{e}{g}_{ab}, \overset{e,\omega}{\nabla}\rangle$ is a model of TPG$_\nabla$.*

Proof. For this, note that (1) and (2) determine the action of $\overset{e}{g}_{ab}$ and $\overset{e,\omega}{\nabla}$ uniquely. Flatness of $\overset{e,\omega}{\nabla}$ follows from the fact that ω is flat. And to show that $\overset{e,\omega}{\nabla}$ is compatible with $\overset{e}{g}_{ab}$, let η^a be any vector field on M. We have

$$\begin{aligned}
\eta^n \overset{e,\omega}{\nabla}_n \overset{e}{g}_{ab} &= \eta^n \overset{e,\omega}{\nabla}_n \eta_{NM} e^N_a e^M_b \\
&= \eta_{NM}\eta^n (e^N_a \overset{e,\omega}{\nabla}_n e^M_b + e^M_b \overset{e,\omega}{\nabla}_n e^N_a) \\
&= \eta_{NM}\eta^n (e^B_b e^N_a \overset{\omega}{\nabla}_n e^m_B e^M_m + e^A_a e^M_b \overset{\omega}{\nabla}_n e^m_A e^N_m) \\
&= \eta_{NM}\eta^n (e^B_b e^N_a \overset{\omega}{\nabla}_n \delta^M_B + e^A_a e^M_b \overset{\omega}{\nabla}_n \delta^N_A) \\
&= 0
\end{aligned}$$

□

Proposition 4. *Let $\langle M, g_{ab}, \nabla\rangle$ be a model of TPG$_\nabla$, and fix a vector space V of dimension n, a representation ρ of $\mathrm{Gl}(n,\mathbb{R})$ and a (flat) Lorentzian metric η_{AB} on V. Then there exist a coframe e and connection ω on LM_{SO} such that $g_{ab} = \overset{e}{g}_{ab}$, $\nabla = \overset{e,\omega}{\nabla}$, where $\overset{e}{g}_{ab}$ and $\overset{e,\omega}{\nabla}$ are as defined in equations (1) and (2), and $\langle M, LM_{\mathrm{SO}}, \pi, LM_{\mathrm{SO}} \times_{\mathrm{SO}} V, e, \omega, \eta_{AB}\rangle$ is a model of TPG$_{e,\omega}$. Moreover, the pair $\langle e, \omega\rangle$ is not unique. If $\langle e, \omega\rangle$ is any such pair, then so is $\langle e', \omega'\rangle$ iff $\langle e', \omega'\rangle = \langle \varphi^* e, \varphi^*\omega\rangle$ for some vertical principal bundle automorphism $\varphi : LM_{\mathrm{SO}} \to LM_{\mathrm{SO}}$.*

Proof. First, note that given any metric g_{ab} we can always find a coframe field e such that (1) holds.[57] Then given e, we can (uniquely) define a flat connection $\overset{\omega}{\nabla}$ on the associated bundle $LM_{\mathrm{SO}} \times_{\mathrm{SO}} V$ via (2). Since any flat connection on the associated bundle is (uniquely) determined by some flat principal connection ω,[58] this establishes existence.

We now move on to establish non-uniqueness. For the 'if' direction, let $\langle M, LM_{\mathrm{O}}, \pi_L, LM_{\mathrm{SO}} \times_{\mathrm{SO}} V, e, \omega, \eta_{AB}\rangle$ be a model of TPG, and let $\varphi : LM_{\mathrm{SO}} \to LM_{\mathrm{SO}}$ be a vertical principal bundle automorphism. First, we show that $\overset{\varphi^* e,\varphi^*\omega}{\nabla} = \overset{e,\omega}{\nabla}$. So let $\kappa^a : M \to TM$ be any vector field, and consider any $p \in M$ and any $\xi^a \in T_pM$. We know that for any section $\tau^A : M \to LM_{\mathrm{SO}} \times_{\mathrm{SO}} V$,

$$\xi^n \overset{\varphi^*\omega}{\nabla}_n \tau^A = \varphi^*(\xi^n \overset{\omega}{\nabla}_n \varphi_* \tau^A).$$

Here, $\varphi^* v^A(x) = v^A(\varphi(x))$ for all points $v^A : \pi_L^{-1}(p) \to V$ in $LM_{\mathrm{SO}} \times_{\mathrm{SO}} V$, and $\varphi_*\tau^A(p)(\varphi(x)) = \tau^A(p)(x)$, where $p \in M$ and $x \in \pi_L^{-1}(p)$. The remainder is just some computation:

$$\begin{aligned}
\varphi^* e^A_m (\xi^n \overset{\varphi^* e,\varphi^*\omega}{\nabla}_n \kappa^m) &= \xi^n \overset{\varphi^*\omega}{\nabla}_n \varphi^* e^A_m \kappa^m \\
&= \varphi^*(\xi^n \overset{\omega}{\nabla}_n \varphi_*(\varphi^* e^A_m \kappa^m)) \\
&= \varphi^*(\xi^n \overset{\omega}{\nabla}_n e^A_m \kappa^m) \\
&= \varphi^*(e^A_m(\xi^n \overset{e,\omega}{\nabla}_n \kappa^m)) \\
&= \varphi^* e^A_m(\xi^n \overset{e,\omega}{\nabla}_n \kappa^m).
\end{aligned}$$

[57] See e.g. (Tecchiolli, 2019).

[58] See e.g. (Michor, 2008).

Since $\varphi^* e^A_m$ is invertible, we can conclude that $\xi^n \overset{\varphi^* e,\varphi^*\omega}{\nabla}{}_n \kappa^m = \xi^n \overset{e,\omega}{\nabla}_n \kappa^m$ and hence that $\overset{\varphi^* e,\varphi^*\omega}{\nabla} = \overset{e,\omega}{\nabla}$. It remains to show that $\overset{\varphi^* e}{g}{}_{nm} = \overset{e}{g}_{nm}$. Let ξ^a, $\kappa^a \in T_p M$ and let $x \in \pi^{-1}(p)$. We have

$$\begin{aligned}
\overset{\varphi^* e}{g}{}_{nm} \xi^n \kappa^m &= \eta_{NM} (\varphi^* e^N_n)_x (\varphi^* e^M_m)_x \xi^n \kappa^m \\
&= \eta_{NM} (e^N_n)_{\varphi(x)} (e^M_m)_{\varphi(x)} \xi^n \kappa^m \\
&= \eta_{NM} (e^N_n)_{xg(x)} (e^M_m)_{xg(x)} \xi^n \kappa^m \\
&= \eta_{NM} (\rho(g^{-1}(x)))^N_R (\rho(g^{-1}(x)))^M_S (e^R_n)_x (e^S_m)_x \xi^n \kappa^m \\
&= \eta_{RS} (e^R_n)_x (e^S_m)_x \xi^n \kappa^m \\
&= \overset{e}{g}_{nm} \xi^n \kappa^m .
\end{aligned}$$

For the 'only if' direction, suppose that $\langle e', \omega' \rangle$ is another pair satisfying the stated conditions. It follows that $\eta_{NM} e^N_a e^M_b = \eta_{NM} e'^N_a e'^M_b$ so that $\eta_{NM} e'^N_n e'^M_m e^n_A e^m_B = \eta_{AB}$ and hence that for each $x \in LM_{\text{SO}}$, $(e'^A_n)_x (e^n_B)_x = (\rho(g(x)))^A_B$ for some (smooth) assignment $g(x) \in O(1, n-1, \mathbb{R})$. Thus $(e'^A_a)_x = (\rho(g(x)))^A_N (e^N_a)_x = (e^A_a)_{xg(x)}$, so defining $\varphi(x) = xg(x)$ (which is a vertical principal bundle automorphism) we have that $(e'^A_a)_x = (e^A_a)_{\varphi(x)} = (\varphi^* e^A_a)_x$. Finally, we know that $e^a_N \xi^n \overset{\omega}{\nabla}_n e^N_m \kappa^m = e'^a_N \xi^n \overset{\omega'}{\nabla}_n e'^N_m \kappa^m = \varphi^* e^a_N \xi^n \overset{\omega'}{\nabla}_n \varphi^* e^N_m \kappa^m$ for all ξ^a, κ^a. But we already know that $e^a_N \xi^n \overset{\omega}{\nabla}_n e^N_m \kappa^m = \varphi^* e^a_N \xi^n \overset{\varphi^*\omega}{\nabla}{}_n \varphi^* e^N_m \kappa^m$ for any vertical principal bundle automorphism, so we have $\varphi^* e^a_N \xi^n \overset{\omega'}{\nabla}_n \varphi^* e^N_m \kappa^m = \varphi^* e^a_N \xi^n \overset{\varphi^*\omega}{\nabla}{}_n \varphi^* e^N_m \kappa^m$ and hence $\overset{\omega'}{\nabla} = \overset{\varphi^*\omega}{\nabla}$. Since any connection on the associated bundle is (uniquely) the lift of some principal connection, we have $\omega' = \varphi^* \omega$. □

For proposition 5, we first prove a lemma:

Lemma 1. *Let M be a differentiable manifold (assumed connected, paracompact, and Hausdorff), and let $LM \xrightarrow{\pi} M$ be the frame bundle over M. Let (Ψ, χ) be a principal bundle morphism, where $\Psi : LM \to LM'$ and $\chi : M \to M'$ are diffeomorphisms. Then there exists a unique vertical principal bundle automorphism $\varphi : LM' \to LM'$ such that $\Psi = \tilde{\chi}^* \varphi$, where $\tilde{\chi} : LM \to LM'$ denotes the (unique) lift of χ to LM.*

Proof. Let Diff(M) denote the diffeomorphism group of M, Aut(LM) the group of principal bundle automorphisms of LM, and Ver(LM) the group of vertical principal bundle automorphisms of LM. Note that Ver(LM) is a normal subgroup of Aut(LM). Then we have the following split exact sequence:

$$1 \longrightarrow \mathrm{Ver}(LM) \xrightarrow{\ i\ } \mathrm{Aut}(LM) \underset{l}{\overset{j}{\rightleftarrows}} \mathrm{Diff}(M) \longrightarrow 1$$

Here, $i : \mathrm{Ver}(LM) \to \mathrm{Aut}(LM)$ is the group homomorphism taking each element of Ver(LM) to itself, $j : \mathrm{Aut}(LM) \to \mathrm{Diff}(M)$ is the group homomorphism taking each pair in Aut(LM) $(\Psi, \chi) \mapsto \chi$, and $l : \mathrm{Diff}(M) \to \mathrm{Aut}(LM)$ is the group homomorphism taking each diffeomorphism $\chi \mapsto (\tilde{\chi}, \chi)$, where $\tilde{\chi}$ is the unique lift of χ to LM. It follows that $\mathrm{Aut}(LM) \cong \mathrm{Diff}(M) \ltimes \mathrm{Ver}(LM)$, from which the result follows. $\square$

Proposition 5. *Let $\mathfrak{M}_\nabla = \langle M, g_{ab}, \nabla \rangle$ be a model of TPG$_\nabla$ and let $\chi : M \to M'$ be a diffeomorphism. Let $\mathfrak{M}_{e,\omega} = \langle M, LM_{\mathrm{SO}}, \pi_L, LM_{\mathrm{SO}} \times_{\mathrm{SO}} V, e, \omega, \eta_{AB} \rangle$, $\mathfrak{M}'_{e,\omega} = \langle M', LM'_{\mathrm{SO}}, \pi'_L, LM'_{\mathrm{SO}} \times_{\mathrm{SO}} V, e', \omega', \eta_{AB} \rangle$ be any two models of TPG$_{e,\omega}$ corresponding to $\mathfrak{M}_\nabla$, $\chi_* \mathfrak{M}_\nabla$ respectively in the sense of proposition 4. Then there exists a unique bundle morphism (Ψ, χ) such that $\langle e', \omega' \rangle = \langle \Psi_* e, \Psi_* \omega \rangle$.*

Proof. First, consider the bundle morphism $(\tilde{\chi}, \chi)$, where $\tilde{\chi} : LM_{\mathrm{SO}} \to LM'_{\mathrm{SO}}$ is the lift of χ to LM_{SO}, and let $\chi_* \mathfrak{M}_{e,\omega}$ denote the lift of $\mathfrak{M}_{e,\omega}$ via $(\tilde{\chi}, \chi)$. By construction, $\chi_* \mathfrak{M}_{e,\omega}$ is a model of TPG$_{e,\omega}$ corresponding to $\chi_* \mathfrak{M}_\nabla$ respectively in the sense of proposition 4. It follows from proposition (4) that there exists some vertical principal bundle automorphism $\varphi : LM'_{\mathrm{SO}} \to LM'_{\mathrm{SO}}$ such that $\varphi_*(\chi_* \mathfrak{M}_{e,\omega}) = \mathfrak{M}'_{e,\omega}$, and hence that $(\tilde{\chi}^* \varphi, \chi)$ is a principal bundle morphism satisfying the conditions of the proposition.

For uniqueness, suppose that $(\tilde{\chi}^* \varphi', \chi)$ is another diffeomorphism satisfying the stated conditions (that we can restrict attention to diffeomorphisms of this form is a consequence of lemma 1). Then $\tilde{\chi}^* e^A_a = (\varphi' \circ \varphi^{-1})^* \tilde{\chi}^* e^A_a$ i.e. $(\tilde{\chi}^* e^A_n)_x v^n = (\tilde{\chi}^* e^A_n)_{\varphi' \circ \varphi^{-1}(x)} v^n = (\tilde{\chi}^* e^A_n)_{x g^{-1}(x) g'(x)} v^n = (\rho(g^{-1}(x)))^A_N (\rho(g'(x)))^N_M (\tilde{\chi}^* e^M_n)_x v^n$ for all $x \in LM'_{\mathrm{SO}}$ and all $v^a \in T_{\pi'_L(x)} M'$. It follows that $(\rho(g^{-1}(x)))^A_N (\rho(g'(x)))^N_B = \delta^A_B$ and hence that $(\rho(g'(x)))^A_B = (\rho(g(x)))^A_B$ and $g'(x) = g(x)$. Since this holds for all $x \in LM'_{\mathrm{SO}}$, we have $\varphi' = \varphi$. $\square$

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