ON A PUZZLE ABOUT SUMS (Forthcoming in *Analysis*)

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ABSTRACT

While the problem of the philosophical significance of Riemann's theorem on conditionally convergent series has been discussed in detail for some time, specific versions of it have appeared in the literature very recently, over which there have been widespread disagreements. I argue that such discrepancies can be clarified by introducing a rather conventional type of composition rule for the treatment of some infinite systems (as well as supertasks) while analysing and clarifying the role of the concept of continuity by stripping it of the excesses that its application by the Leibnizian tradition has led to. The conclusion reached is that the indeterminacy associated with conditional convergence has a clear philosophical significance, but no fundamental ontological significance.

Keywords: Conditional Convergence; Continuity; Expansionist Analysis; Balance Principle; Ross Paradox.

1. A problematic example of conditional convergence?

Lee (forthcoming) studies a puzzle formulated by Linnebo (2023) in detail. Henceforth referred to as Linnebo's example:

Suppose you have an infinite number of iron balls and helium balloons. The balls have mass 1Kg, 1/3Kg, 1/5Kg, etc., while the balloons are capable of lifting 1/2Kg, 1/4Kg, 1/6Kg, etc. You also have a scale to which you are able to successively attach the balls and balloons in any chosen order – the whole infinite lot of them. Thus, you are able to weigh the balls and balloons, which make a positive and negative contribution to the reading of the scale, respectively. [...]. One option is to alternate between attaching one ball and one balloon, each in their standard order [...]. In this case, the scale will show (in kg):

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln(2) \approx 0.69 \quad (1)$$

Now comes the exciting bit. By adding the balls and balloons to your scale in a different order, while still using all of them, you can, I claim, produce any finite positive or negative reading of the scale whatsoever. (Linnebo 2023: 189).

This conclusion depends (as is widely known) on the conditionally convergent nature of the alternating harmonic series via Riemann's Rearrangement Theorem. Linnebo wonders whether such an outcome is metaphysically impossible, and even considers that its acceptance requires 'a dramatic departure from physics as we know it'. However, doubts about the metaphysical possibility and intuition of a 'dramatic departure' are dispelled if one conveniently distinguishes between synchronic composition rules and diachronic composition rules when dealing with the infinite system of balls, balloons and scales (a system which I shall refer to as X). Composition rule S enables the study of an infinite system's properties in the conventional way.

Composition rule S (synchronous): To study the properties of a system S comprising infinite components, the properties of a subsystem S_n of S with n components are considered, and then limit $n \to \infty$ is taken for all the magnitudes for which this limit exists. Whatever these n components are should be irrelevant, so long as a selection method is guaranteed to have been followed so that, when going to the limit, all components have been taken into consideration.

Assume that we are interested in value *M* of magnitude **M** in system S. In a subsystem of n components, value *M* (a function of n that can be expressed by M_n) is the sum of a series of n terms $a_1 + a_2 + a_3 + \cdots + a_n$; a sum reflecting the influence of each subsystem component on outcome M_n. Finally, $M = \lim_{n\to\infty} M_n$ is the sum of an infinite series. This series usually proves to be absolutely convergent, so the order followed in choosing an n-component subsystem (with the abovementioned caution) is irrelevant. However, if the series' convergence is conditional, the order is no longer irrelevant. The sum is dependent on the order of the summands and, consequently, the **M** value for the system depends on how the subsystems are chosen in order to take the limit. This is unacceptable. An objective magnitude's value, *M*, cannot be dependent on a purely formal choice. When this is applied to the weight of the infinite system of balls and balloons in system X, Linnebo's doubts arise with regard to its metaphysical possibility and its 'dramatic departure'. However, the question is ill-posed. Conditional convergence indicates that composition rule S does not apply in this case. Moreover, it fails to take time into account. On the contrary, in Linnebo's example, system X is subjected to external actions by an external manipulator operating on balls and balloons in one order or another. Composition rule D enables the study of a system's evolution under an infinite sequence of actions in the conventional way (characteristic, for example, of the literature on supertasks).

Composition rule D (diachronic): In order to study the evolution (in particular the properties) of a system S under the infinite sequence $A_1, A_2, A_3, \ldots A_n, A_{n+1}, \ldots$ of actions (executed at times $t_1 < t_2 < t_3 < \cdots < t_n < t_{n+1} < \cdots, with \lim_{n\to\infty} t_n = t^*$), the evolution of S under the finite sequence of actions $A_1, A_2, A_3, \ldots A_n$ is considered, and limit n→∞ is then taken for all the magnitudes which vary continuously at instant $t^* = \lim_{n\to\infty} t_n$.

When this is applied to the weight of the infinite system of balls and balloons in system X, any doubts about its metaphysical possibility and 'dramatic departure' are dispelled. When the manipulative actions on the balls and balloons are carried out in one order or another, the final weights given by the scale prove to be different. The weight value is not therefore dependent on a purely formal choice (as it was under rule S), but on the real order in which these operations (actions) are carried out. Once the metaphysical doubt is dispelled, there is no dramatic departure either. Even in ordinary experience, it is common for the outcome of a sequence of actions to depend on the order of actions. Consider the case of the composition of two rotations, one followed by the other. As is widely known, such a composition can be represented by a matrix product. However, the matrix product is not commutative. The result of two rotations is therefore dependent (except in special cases) on the order in which they are performed. This is an elementary standard result. In the case of Linnebo's example there is non-commutativity of infinite sums (Knopp 1954: 138) instead of non-commutativity of matrix products.

2. The role of continuity.

Hoek (2023) finds Linnebo's example problematic. He attributes to this author the implicit use of what he calls the Continuity Principle. In his words:

Continuity Principle: If a certain natural quantity converges to a particular limit value *l* over the course of a certain time interval [t_0, t_1), and nothing further happens to affect its value at t_1, then the quantity in question attains the limit value *l* at t_1. (Hoek 2023: 1793)

This principle is clearly untrue and Linnebo would be mistaken should he attempt to justify his example on its basis. Justification of Linnebo's example does not require the continuity principle, rather composition rule D. This rule enables limit n→∞ to be taken for all magnitudes which vary continuously at instant $t^* = \lim_{n\to\infty} t_n$. However, this does not mean that it allows limit n→∞ to be taken for all magnitudes which tend towards a definite limit when n→∞ (i.e. at $t = t^*$). The reason for this is that the existence of such a limit means nothing if the magnitude in question does not vary continuously at $t = t^*$. The question now is: which magnitudes vary continuously? I doubt that a blanket answer can be given to this whatever kind of infinite system is under consideration. However, if it is a physical system (as in Linnebo's example and the like) then there is a simple criterion: a magnitude varies continuously at a certain instant t if, and only if, this is compatible with the underlying physical theory. To illustrate this criterion, let us consider the case of the well-known Ross paradox. We have two empty urns A and B and an infinite denumerable set IN of numbered balls. At instant $t_n = \frac{n}{n+1}$ (with positive integer n) the balls numbered $10n - 9$ to 10n are placed in urn A while the ball numbered n is placed in urn B. What is the state of the urns at $t = 1 = \lim_{n\to\infty} \frac{n}{n+1}$? If $N_A(t)$ is the number of balls in urn A at instant t and $N_B(t)$ is the number of balls in urn B at instant t, it is therefore clear that

$$\lim_{t\to 1} N_A(t) = \lim_{t\to 1} N_B(t) = \infty$$

However, $N_A(t)$ and $N_B(t)$ do not both vary continuously at t = 1 (i.e. at limit $n\to\infty$ [1]). The relevant physical theory to see why simply needs to make use of two basic and intuitive principles (made explicitly clear by Earman long ago, Earman 1986: 38), namely, the fact that the world lines of material bodies are continuous functions of time, and that these same world lines have no starting or end points (i.e. there are no points at which particles are created and no points at which particles disappear). Since any one of the IN balls is in urn B at time instants prior to $t = 1$ that are sufficiently close to $t = 1$, the continuity of their world lines implies (by definition) that any one of them will be in urn B at $t = 1$. Moreover, if the IN balls are the only balls present prior to $t = 1$, no new balls can emerge from nothing at $t = 1$ to occupy urn A because their world lines would

[1] Obviously $N_A(t)$ and $N_B(t)$ experience discontinuities at all instants $t_n = \frac{n}{n+1}$ but, as we shall see, $N_B(t)$ (and not $N_A(t)$) is continuous at limit t = 1.

have starting points. In conclusion, $N_B(1) = \infty$ but $N_A(1) = 0$. All the balls are at $t = 1$ in urn B. At that instant urn A is empty. Only $N_B(t)$ is a continuous function of time at t = 1 (i.e. at limit n→∞ [2]).

In light of the above, analysis of Linnebo's example in greater detail is of some use. In order to simplify the exposition, the balls will be considered positive weights and the balloons negative weights. Suppose that (with positive integer i) the weights $p_i = \frac{(-1)^{i+1}}{i}$ (corresponding to the terms of the alternating harmonic series) are placed on the scale at instants $t_i = \frac{i}{i+1}$. According to composition rule D, the total weight on the scale at $t = 1$ is $\sum_{i=1}^{\infty} p_i = \ln 2$, as Linnebo says. Indeed, the continuity of the weight at t = 1 is compatible with the underlying physical theory, which says nothing about the total weight of the mereological sum of the balls and balloons. Nevertheless, this total weight can change if the temporal order in which the weights are placed on the scale is altered (i.e. the order of the actions A_i

[2] Function $f(x)$ is continuous at point x = x* if and only if $\lim_{x \to x^*} f(x) = fx*$ (intuitively, $f(x)$ experiences no jump at point x = x*). Since $\lim_{t \to 1} N_A(t) = \infty \neq 0 = N_A(1)$, $N_A(t)$ is not continuous at $t = 1$. However, $\lim_{t \to 1} N_B(t) = \infty = N_B(1)$. So, in an extended sense of the concept of function that admits ∞ in its range of values, $N_B(t)$ is a continuous function of time at t = 1.

is altered in composition rule D). Consider $\sum_{i=1}^{\infty} p_{F(i)}$ as a rearrangement of series $\sum_{i=1}^{\infty} p_i$ so that $\sum_{i=1}^{\infty} p_{F(i)} = H$ (where H is any given number $\neq \ln 2$). It is clear that F is a certain bijective function of the positive integers over the positive integers. Now weights $p_{F(i)} = \frac{(-1)^{F(i)+1}}{F(i)}$ are placed on the scale at instants $t_i = \frac{i}{i+1}$. The bijective nature of F evidently implies that at $t = 1$, the same positive or negative masses will be on the scale as previously (when, at instants $t_i = \frac{i}{i+1}$, weights $p_i = \frac{(-1)^{i+1}}{i}$ were placed on the scale). Yet now, as t = 1 is approached, the total mass of the placed weights moves increasingly closer to H. So, the natural answer (appealing to composition rule D in the same way as earlier, given the continuity of the weight at t = 1) is that, if weights $p_{F(i)} = \frac{(-1)^{F(i)+1}}{F(i)}$ are placed at instants $t_i = \frac{i}{i+1}$, the total weight at $t = 1$ is H, and not ln2. In order to arrive at this outcome, it is essential that the bijection F leads to the altering of the temporal order of placement on the scale of an infinite number of weights. If only the order of placement of a finite number of weights is altered, it can immediately be seen that the total weight would continue to be ln2. Therefore, in general, the total weight shown on the scale is dependent on the temporal order in which the weights have been placed. In order to determine the total weight of a configuration of infinite weights at a given instant, one must first know how it was arrived at. This is not as strange as it may seem. In order to determine the velocity of a freely moving material body occupying a certain point P in space at a given instant t, one must first know how it arrived at P at instants preceding t. This is because the velocity depends on the world line followed. The same thing applies to the weight when the series is conditionally convergent: the weight of configuration C at an instant t depends on how it arrived at this configuration at instants preceding t. Just as the velocity of P at t can be altered by manipulating its world line at instants prior to t, so too can the weight of configuration C at t be altered by manipulating the procedure to arrive at such a configuration at instants preceding t.

The above discussion allows one to see the fallacy hidden in an interesting argument made by Hoek (involving the idea of continuity) in relation to Linnebo's example. I shall analyse it more precisely than Hoek does, modifying the presentation somewhat for the sake of clarity. Consider two separate scales: A and B. Suppose that at instants $t_i = \frac{i}{i+1}$, weights $p_i = \frac{(-1)^{i+1}}{i}$ have been placed on A. As seen above, the total weight on A at $t = 1$ is $\ln 2 \approx 0.69$. Suppose also that after $t = 1$ the weights on A are manipulated and repositioned on scale B in the following order: at instant $t_i = 1 + \frac{i}{i+1}$ weight $p_{F(i)} = \frac{(-1)^{F(i)+1}}{F(i)}$ is placed on B. F is a bijection of the positive integers Z^+ over the positive integers Z^+ such that, at successive instants $t_i = 1 + \frac{i}{i+1}$, the sequence of weights that are successively placed on B is $1, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{8}, \frac{1}{5}, \ldots$ [3] What is the total weight at $t = 2$? Hoek believes he sees an absurdity here. Prior to $t = 2$ only a finite number of the weights that were on A have been relocated on B. Therefore, the total weight (adding the weights on A to those on B) continues to be ln2. However, at instant $t = 2$ all the weights are relocated on B and, given the order in which this relocation has taken place, their total weight is:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n} \right) = \frac{\ln 2}{2} \approx \frac{0.69}{2} \approx 0.35 \qquad (2)$$

However, if a weight of (ln2/2) kg on A has been removed, there must still be another (ln2/2) kg left on A from the initial ln2 kg. This is absurd, says Hoek, because there can be no weight left on A since all the balls and balloons initially on it have been placed on B at $t = 2$. In order to undo this apparent contradiction (showing that there is no absurdity) note that, as

[3] It is easy to see that the bijection F leading to this sequence of weights is as follows: for each $n \in Z^+$, we have $F(3n-2) = 2n-1$; $F(3n-1) = 4n-2$; $F(3n) = 4n$. However, this detail is not necessary to understand the paper.

$t = 2$ is approached, the weight on B gets progressively closer to (ln2/2), and the weight on A also gets progressively closer to (ln2/2). The reason is that prior to $t = 2$ only a finite number of the weights initially placed on A have been relocated, so the total weight of what is on A plus what is on B is ln2. However, at $t = 2$ weight (of magnitude ln2/2) ONLY remains on B because when all the weights move from A to B, in total they weigh half of what they weighed on A (in effect, when all the weights were on A their total weight was $\ln 2$, whereas when they are all moved on B, in the above manner, their total weight is $\frac{\ln 2}{2}$). Hoek's error lies in failing to see this, and in considering (erroneously) that the weight on A must vary continuously at t = 2, [4] as does the weight on B. The continuity of the weight on A at $t = 2$ contradicts the underlying physical theory (because there is no weight on A at $t = 2$); the continuity of the weight on B, however, does not. The situation is therefore no more than a variant of the Ross paradox. In my analysis of the Ross paradox, urn A is left empty at

[4] A similar digression to note (1) is made here. Obviously, the weight on A as a function of time, $W_A(t)$, and the weight on B as a function of time, $W_B(t)$, experience discontinuities at all instants $t_i = 1 + \frac{i}{i+1}$ where there is a transfer of weights from scale A to scale B. However, as we can see, $W_B(t)$ (and not $W_A(t)$) is continuous at limit t = 2.

$t = 1$, thus function $N_A(t)$ exhibits an infinite discontinuity at that instant. For the same reasons as before, there is no weight left on scale A at $t = 2$, so the function providing the weight on A at instant t exhibits a jump discontinuity at $t = 2$ (the amount of jump at t = 2 is ln2/2). In the case of Hoek's argument, the underlying Ross paradox is somewhat masked by the 'surprise' (considered in §1) that the total weight of a conditionally convergent series of weights indicated by a scale is dependent on the temporal order in which they are placed on it.

3. The irrelevance of the spatial order...

What is the weight of a conditionally convergent series of weights NOT placed on a scale? Without further information the weight is indeterminate. However, it is not a fundamental ontological indeterminacy: [5] one value or another will be adopted depending, for example, on the temporal order in which they are placed on a scale (or, somewhat more generally, in a given configuration). Lee argues that, in the example given by Linnebo and others, 'it's spatial distribution, rather than temporal order, that matters'

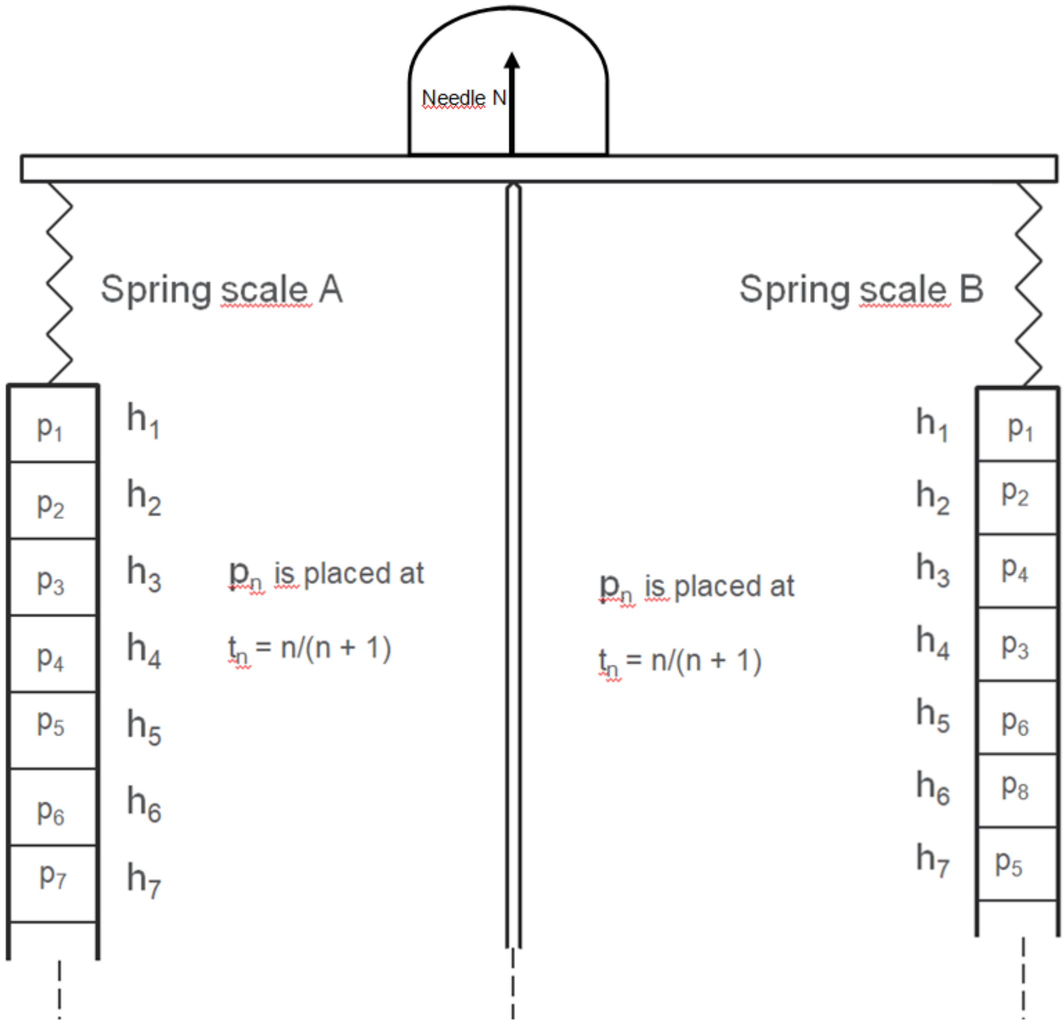
[5] This does not mean that examples such as Linnebo's are of 'no real philosophical significance' (MacKenzie, forthcoming: 1), which this paper has sought to prove.

(forthcoming: 22). Lee's theory, which he calls 'The Expansionist Analysis', holds that

> the weight of a collection A is x iff for every spatial point, if we consider an ever-expanding sequence of balls centred on that point, then the weights of the finite subcollections of A contained with those balls will approach x. (Lee forthcoming: 16).

Consequently, 'it's convergence over regions of space, rather than intervals of time, that matters for wieght' (forthcoming: 22). Lee frames his argument by restricting himself to infinitary regions of space and by assuming that any finite region of space only contains a finite number of weights. Under these conditions he considers it a virtue of expansionist analysis that, for any region of space, there is some determinate answer as to the amount of weight contained within that region, no matter how we individuate the collection of items within that region. However, this is not true. Consider the following distribution: for each positive integer i, one particle of weight -1 at $x = -i$ and one of weight +1 at $x = +i$. If I start by constructing balls centred at point $x = 0$, the total weight is then 0. However, if they are centred at $x = +i$, the weight is +2i (and -2i if they are centred at $x = -i$). This unacceptable indeterminacy (dependent on a purely formal choice) is circumvented by composition rule D: the weight (when determined) depends on the temporal filling order of coordinate points $x = \pm i$.

The most clear-cut argument against expansionist analysis is the following. Consider two spring scales A and B. Each one ends in an infinite vertical cylinder (of finite weight z) containing an infinite number of identical hollow compartments $h_1, h_2, h_3, \dots h_n, h_{n+1}, \dots$ separated by a fixed inner wall. The desired weight can be placed in each compartment. Compartment h_{i+1} is immediately below compartment h_i. See the figure. At instant $t = 0$ both scales show the same weight z, corresponding to the identical empty cylinders hanging below them. Suppose that, following the alternating harmonic series pattern, at $t_n = \frac{n}{n+1}$ we place a weight $p_n = \frac{(-1)^{n+1}}{n}$ in one of the hollows in the spring scale A cylinder as well as a weight $p_n = \frac{(-1)^{n+1}}{n}$ in one of the hollows in the spring scale B cylinder (but each weight always in a different hollow). Clearly, the two spring scales will show the same weight at any instant prior to $t = 1$.



One can even imagine a mechanism that moves a needle N to the left or right of the equilibrium position E depending on whether the weight shown by A is heavier or lighter than the weight shown by B. It is clear that the position of N will continue to be E at $t = 1$. This is further corroborated by the intuitive principle previously mentioned: the world lines of material bodies are continuous functions of time. Applied to the needle N (and/or its constituent parts,) it is clear that it will remain at position E at $t = 1$ (and there is no causal mechanism to remove it from this position at a later point in time). However, suppose that the hollows $h_1, h_2, h_3, \dots h_n, h_{n+1}, \dots$ in the cylinder under spring A were filled with the respective weights $1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \frac{1}{5}, \dots$ while the hollows $h_1, h_2, h_3, \dots h_n, h_{n+1}, \dots$ in the cylinder under spring B were filled with the respective weights $1,\ -\frac{1}{2},\ -\frac{1}{4},\ \frac{1}{3},\ -\frac{1}{6},\ -\frac{1}{8},\ \frac{1}{5}, \dots$, as shown in the figure. According to Lee's expansionist analysis, at $t = 1$ the weight under spring scale A ($z + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = z + \ln 2$) will suddenly be different to the weight under spring scale B ($z + 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} \dots = z + \frac{\ln 2}{2}$). This is certainly not empirically refutable, but it is, nonetheless, metaphysically absurd: there is no causal mechanism to justify this sudden difference in weights at $t = 1$ because there was no such difference before $t = 1$ (the same weights $p_n = \frac{(-1)^{n+1}}{n}$ were placed under each of the scales at the same instants in time $t_n = \frac{n}{n+1}$; their distribution in the respective hollows was the only difference).

4. ... and the falsity of the Infinitary Balance Principle.

What I call the Infinitary Balance Principle (IBP) reads as follows: if an infinite weight and an infinite counterweight lie on a scale, then the scale is in the same state of equilibrium as when it holds no weights. Both Lee and Hoek accept this explicitly (though not by this name). Its falsity follows directly from my earlier argument, now trivially modified, against expansionist analysis. Suppose that, at instant $t_n = \frac{n}{n+1}$, we place a weight $p = \frac{2}{n}$ in spring scale A compartment h_n and a weight $p = \frac{1}{n}$ in spring scale B compartment h_n. Clearly, the difference between the weight that A shows and the weight that B shows will increase as $t \to 1$. Therefore, needle N will increasingly tilt to the left for $t \to 1$. Since the world lines of material bodies are continuous, N cannot be in equilibrium position E at $t = 1$ (and there is no causal mechanism to take it to this position at a later point in time).

5. Recapitulation on the continuity criterion.

Note the importance in my discussion of the continuity criterion ('a magnitude varies continuously at a certain instant t if, and only if, this is compatible with the underlying physical theory') introduced on p. 6. It was seen in the Ross-Littlewood paradox that the continuity of $N_A(t)$ at t = 1 is incompatible with the underlying physical theory (unless ex nihilo creation is admitted), while the discontinuity is not. For the same reason, $W_A(t)$ (the weight on scale A in the discussion in §2) cannot be continuous at t = 2. Alternatively, in the cases of spring scales A and B discussed in §3 and §4, it was seen that the continuity of the weight difference at t = 1 is compatible with the underlying physical theory, while the discontinuity is not (unless, as hinted at in my analysis, exotic causal mechanisms are introduced [6]).

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REFERENCES

Earman, J. 1986. *A primer on determinism*. Dordrecht: Reidel.

Hoek, D. 2023. Oystein vs Archimedes: A Note on Linnebo's Infinite Balance. *Erkenntnis* **88**, 1791-96.

Knopp, K. 1954. *Theory and Application of Infinite Series*. London and Glasgow: Blackie & Son Limited.

Lee, A. Y. Forthcoming. A Puzzle about Sums. In *Oxford Studies in Metaphysics*, eds. K. Bennett and D. W. Zimmerman. Oxford: Oxford University Press. <philarchive.org/rec/LEEAPA-15>.

Linnebo, Ø. 2023. Riemann's Scale: A Puzzle About Infinity. *Erkenntnis* **88**, 189-91.

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McKenzie Alexander, J. Forthcoming. On the Incompleteness of Classical Mechanics. *British Journal for the Philosophy of Science*. https://doi.org/10.1086/731828.

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