**Numbers as Ordered Pairs**

**J.-M. Kuczynski**

*Abstract: According to Frege, n=Kn, where n is any cardinal number and Kn is the class of all n-tuples. According to Von Neumann, n=Kpn, where Kpn is the class of all of n's predecessors. These analyses are prima facie incompatible with each other, given that Kn≠Kpn, for n>0. In the present paper it is shown that these analyses are in fact compatible with each other, for the reason that each analysis can and ultimately must be interpreted as being to the effect that n=Cn, where Cn is the class of all ordered pairs <Kn#,Rn#>, where Kn# is an arbitrary class and Rn# is an arbitrary relation such that a class k has n-many members exactly if k bears Rn# to Kn#.*

 According to Frege, the cardinal number n is Kn, where Kn is the class of all n-tuples. (Thus, Ø=0; K1=1; K2=2; and so on.) According to Von Neumann, the cardinal number n is Kpn, where Kpn is the class of all of n's predecessors. (Thus, Ø=0; {Ø}=1; {Ø,{Ø}}=2; and so on.)

 These analyses seem to be incompatible, given that Kn≠Kpn, for n>0. At the same time, each analysis is viable, given that the content of any statement of the form "…n…"is perspicuously represented---i.e. represented in such a way that its inferential properties can be read off of its syntax---by some statement of the form "… Kn…" and also by some statement of the form "…Kpn…".We thus have a paradox: Kn≠Kpn and yet n=Kn and n=Kpn.

 In an attempt to deal with this paradox, some philosophers have taken the desperate measure of saying that numbers, and therefore relations among numbers, do not exist mind-independently. Those authors who have rejected this position have failed to identify an entity with which a given cardinal number both can and must be identified.

 We will now identify just such an entity.

 Frege's analysis can be interpreted as being given by the proposition that, for any cardinal n, n=<Kn,∈>, where ∈ is the relation between class-member and class; and Von Neumann's analysis can be interpreted as being given by the proposition that n=<Kpn,≈>, where ≈ is the relation between set and equipollent set. Thus, if KJ is the class of John's cars, then Frege's analysis maps (i) *John has exactly one car* onto (ii) *KJ∈K1*, whereas Von Neumann's analysis maps (i) onto (iii) *KJ≈Kp1*.

 This suggests that, for any cardinal n, n can be identified with the class Cn of all ordered pairs <Kn#,Rn#>, where Kn# is any class and Rn# is any relation such that a class k has n-many members exactly if kRn#Kn# (read: *k bears Rn# to Kn#*). Thus, 0=C0={<K0#,R0#>│(k) (kR0#K0#↔k∈K0)}, where K0={k│(x) (x∈k↔x≈Ø)}; 1=C1= {<K1#,R1#>│(k) (kR1#K1#↔k∈K1)}, where K1={k│(x) (x∈k↔x≈{Ø})}; 2=C2= {<K2#,R2#>│(k) (kR2#K2#↔k∈K2)}, where K2={k│(x)(x∈k↔x≈{Ø, {Ø}})}; and so on.

 Given an ordered pair <K,R>, let us use the notation <K,R>n to indicate that <K,R> is an *n-pair*, meaning that a class k is an n-tuple iff kRK. For example, <K,R>2 ↔(k≈Kp2↔kRK). ("<K,R> is a 2-pair exactly if k's bearing R to K is necessary and sufficient for k's having two members.") For any cardinal n, Cn is the smallest class containing every n-pair. Each of <Kpn,≈> and <Kn,∈> is an n-pair, and there are obviously others. What each Cn-member <x,y>has in common with each of <Kn,∈> and <Kpn,≈> is that <x,y> represents n's structure and is thus, as we might put it, an *n-isomorph*. Since nothing that isn't in Cn is an n-isomorph, membership in Cn is both necessary and sufficient for possession of the structural properties, and therefore of all of the properties, that are individuative of n. Therefore, a given cardinal n not only *can* but *must* be identified with Cn. Therefore, n=Cn={<x,y>│<x,y>n}.

 *For any cardinal number n, n=Cn.* This analysis is consistent with the fact that any n-pair adequately represents n's structure and also with the fact that any adequate representation of n's structure is an n-pair. But unlike the analyses of Frege and Von Neumann, our analysis is consistent with the presumption that there is only one entity that can be identified with n.