Global Gauge Symmetry Breaking in the Abelian Higgs Mechanism

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Abstract

This paper aims to resolve the incompatibility between two extant gauge-invariant accounts of the Abelian Higgs mechanism: the first account uses global gauge symmetry breaking, and the second *eliminates* spontaneous symmetry breaking entirely. We resolve this incompatibility by using the constrained Hamiltonian formalism in symplectic geometry. First we argue that, unlike their local counterparts, global gauge symmetries are physical. The symmetries that are spontaneously broken by the Higgs mechanism are then the global ones. Second, we explain how the dressing field method singles out the Coulomb gauge as a preferred gauge for a gauge-invariant account of the Abelian Higgs mechanism. Based on the existence of this group of global gauge symmetries that are physical, we resolve the incompatibility between the two accounts by arguing that the correct way to carry out the second method is to eliminate only the redundant gauge symmetries, i.e. those local gauge symmetries which are not global. We extend our analysis to quantum field theory, where we show that the Abelian Higgs mechanism can be understood as spontaneous global $\mathrm{U}(1)$ symmetry breaking in the C^*-algebraic sense.[1]

[1]This article is based on the master thesis of one of the authors, which was supervised by the other author and by Hessel Posthuma. See (Borsboom 2024).

Contents

1 Introduction

In the original physics literature, the Higgs mechanism is presented as an instance of spontaneous local gauge symmetry breaking (Higgs 1964; Englert and Brout 1964; Guralnik, Hagen, and Kibble 1964), similar to symmetry breaking in the Ginzburg-Landau (GL) theory of superconductivity (Ginzburg and Landau 2009; Fraser and Koberinski 2016).

But over the past two decades, philosophers of physics have pointed out the incongruity of the notion of "local gauge symmetry breaking," and they have consequently pleaded for a gauge-invariant account of the Higgs mechanism. In this literature, the main conceptual problem that is levelled against the standard narrative seems to stem from the idea that "a genuine property like mass cannot be gained by eating descriptive fluff, which is just what gauge is" (Earman 2004). This problem is further exacerbated by Elitzur's theorem (Elitzur 1975; Smeenk 2006), which forbids local gauge symmetry breaking by showing that gauge-dependent quantities must have a vanishing vacuum expectation value (VEV). This applies in particular to the Higgs field, which supposedly acquires a nonzero VEV in the electroweak phase transition.

Two incompatible proposals. In response to the problems outlined above, philosophers have sought to reformulate the Higgs mechanism in a way that avoids the dubious concept of 'local gauge symmetry breaking'. For example, Lyre has argued that the Higgs mechanism does not exist at all, because it is a mere rewriting of Lagrangians using different choices of gauge (Lyre 2008). His argument comes down to the idea that, unlike symmetry transformations in other cases of spontaneous symmetry breaking (SSB), gauge transformations do not correspond to anything physical. Thus, according to Lyre, the Higgs mechanism is only a heuristic with no explanatory value.

In this paper, we will be concerned with two incompatible accounts of the Higgs mechanism. While the first account acknowledges SSB, the second denies that a gauge symmetry can be spontaneously broken (and in this particular aspect it agrees with Lyre), as follows:

(i) *The Higgs mechanism as the spontaneous breaking of a global gauge symmetry.* Struyve has proposed that the Abelian Higgs mechanism can be understood in a substantive gauge-invariant way, in terms of the spontaneous breaking of a *global* (also known as *rigid*), rather than a local, gauge symmetry (Struyve 2011). In this approach, one redefines the Hamiltonian of the Abelian Higgs model in terms of the transverse component of the electromagnetic potential, which is gauge-invariant, and a dressed version of the Higgs field, which is invariant under all gauge transformations except the global ones. This Hamiltonian contains a massless gauge field and exhibits a residual global $U(1)$ gauge symmetry. When this global symmetry is broken in the usual way, an expansion up to quadratic order around the chosen vacuum configuration yields a Hamiltonian with a massive gauge field. By thus breaking the *global* gauge symmetry, this account succeeds in producing massive gauge fields. This is also in line with the Josephson effect (Josephson 1962; Wezel and Brink 2008), in which a current flows between two superconductors depending on the relative *global* phase of the symmetry-breaking wavefunction of Cooper pairs.

However, in this account it is obscure why global and local gauge symmetries are being treated differently, so that the latter must be preserved while the former may be broken. Indeed, why should a reformulation of the Higgs mechanism in terms of fields that are invariant under local, but not global, gauge transformations be called gauge-invariant? Does this not suffer from the same conceptual issues? In fact, (Friederich 2013) discusses the spontaneous breaking of various remnant global gauge symmetries and points out that they do not match up with actual phase transitions, rendering it difficult to interpret the Higgs mechanism in that way. Thus it may be more promising to attempt to rid the Higgs mechanism of SSB altogether.

(ii) *The Higgs field sliding down a potential: no SSB.* This approach is advocated by (Berghofer et al. 2023).[2] They use the dressing field method (DFM) to rewrite the Abelian Higgs and electroweak Lagrangians in terms of *gauge-invariant dressed fields*. Since the dressed fields are gauge-invariant, there can be no gauge symmetry breaking, and the Higgs mechanism must be interpreted by way of the Higgs field sliding down its potential $V\colon \mathbb{R}^+ \to \mathbb{R}$, which in this approach only takes strictly positive values as its input (i.e. the Higgs field is not permitted to vanish anywhere: and recall that the Higgs potential only depends on the modulus of the Higgs field). These ideas can be generalized to perturbative quantum field theory (QFT) through the Fröhlich-Morchio-Strocchi (FMS) approach (Fröhlich, Morchio, and Strocchi 1980; Fröhlich, Morchio, and Strocchi 1981), in which n-point functions of gauge-invariant composite fields are expressed in terms of n-point functions of the elementary fields. It can then be shown that quantities like gauge boson masses agree on either side of this expression, i.e. for the gauge-invariant composite fields and gauge-dependent elementary fields (Maas 2019; Maas 2023).

These two gauge-invariant accounts of the Higgs mechanism are clearly incompatible: (i) Struyve reformulates the Abelian Higgs mechanism in the Coulomb gauge in terms of fields that are invariant under all the gauge symmetries except the global ones, which are *spontaneously broken*. (ii) Berghofer et al., and others, advocate the idea that, by using the DFM and FMS approach, SSB should be *eliminated* from the Higgs mechanism altogether.

The above suggests our central question: what does it mean to give a "gauge-invariant account" of the Higgs mechanism? Should such an account include, or remove, global gauge symmetry, i.e. should we dress our elementary fields in order to turn them into entirely gauge-invariant composite objects, or should we leave room for global symmetries?

In this paper, we will build upon recent results which show how global gauge symmetries arise as the physical symmetry group of Yang-Mills theory with asymptotic boundary conditions (Borsboom and Posthuma 2025). Global gauge symmetries respect asymptotic boundary conditions on the Yang-Mills fields, but are not 'trivial' or 'redundant' because they are not generated by the Hamiltonian constraints of the theory. In Yang-Mills-Higgs theory the same holds, but only in the *unbroken* phase. In the *broken* phase, on the other hand, the boundary-preserving and trivial symmetries overlap (up to topological terms). These results suggest that gauge symmetry breaking in the Higgs mechanism should be understood as global gauge symmetry breaking.

Because of the initial presence of a physical global gauge symmetry, the DFM cannot be used at the level of the theory's finite-dimensional principal fibre bundle to remove all gauge symmetry. It must instead be used at the level of the infinite-dimensional configuration space of fields, a method proposed in (Gomes and Riello 2018; Gomes 2019; Gomes and Riello 2021). In fact, this corresponds—at least in the Abelian case—to projecting down to the Coulomb gauge. We will argue that the dressing field method, thus corrected, vindicates Struyve's gauge-invariant account (i).

However, even if this incompatibility is resolved, the problem of the Higgs mechanism in QFT remains. Indeed, Stöltzner already expressed the worry that, even though Struyve provided a gauge-invariant account of the classical Higgs mechanism, the same remained to be done for the quantum case (Stöltzner 2012). This refers to Earman's original desire to relate SSB in the Higgs mechanism to the algebraic definition of SSB in quantum systems in terms of unitary inequivalence of GNS representations (Earman 2003a). Although the quantization of gauge theories is poorly understood, the desired link can be made by considering global gauge symmetry breaking. This was proved in (Morchio and Strocchi 2007), who showed that there

[2] See also (Masson and Wallet 2010; Fournel et al. 2014; François 2019; Attard et al. 2018; Zajac 2023).

can only be massive photons in the Abelian Higgs model if the global $U(1)$ symmetry of the theory is spontaneously broken in the C^*-algebraic sense. Conceptualizing this result is the second major aim of this article.

Thus the paper falls into two parts: first, Sections 2 to 4 discuss classical field theory. Second, Section 5 discusses QFT. In the first part, Section 2 introduces preliminary terminology on the physical status of gauge symmetries. After considering the details of the Hamiltonian formulation of electromagnetism in Section 3, we turn, in Section 4, to resolving the incompatibility between the two accounts, (i) and (ii). We explain how the Coulomb gauge is produced by the DFM, i.e. (ii), as precisely the gauge whose remnant subgroup is the group of global gauge transformations, and we discuss how this is an agreement with Struyve's account, i.e. (i). Finally, in Section 5, we recall the definition of SSB in algebraic quantum theory and show how the Abelian Higgs mechanism can be formulated by means of global gauge symmetry breaking in that language.

2 Global Gauge Symmetries as the Physical Gauge Group

The aim of this Section is to explain how global gauge symmetries arise as the group of physical symmetries in Yang-Mills theory with boundary conditions. To this end, we begin with an overview of the philosophical terminology surrounding the physical status of gauge symmetries in Section 2.1, focussing on the distinction between two conceptions of the term 'gauge', one mathematical and the other interpretative. In Section 2.2 we then introduce various subgroups of the gauge group and explain how a certain quotient of two of these subgroups should be viewed as the group of physical gauge symmetries.

2.1 Gauge fields, gauge transformations and redundancies

This Section does two things: First, it recalls the basic mathematical notions that we will use, and how they relate to physicists' way of thinking about gauge transformations. Second, it introduces the relevant interpretative notions from the literature, and especially that of a *redundant* gauge symmetry.

(1) *Formal description of a gauge theory.* In mathematical gauge theory, a *gauge field* is a connection on a principal G-bundle P over a differentiable manifold M which represents space(time). Such a bundle locally looks like a product space $U \times G$, where $U \subset M$, but its global structure can be more complicated. Here, G is called the *structure group*, and it is a compact matrix Lie group that acts on the bundle P. It is a global group, i.e. its generators do not vary over the manifold M. The bundle has a projection $\pi : P \to M$ onto the base manifold. In what follows, when working in the Hamiltonian formalism, we will work with a bundle that is defined only over space and not spacetime. In that case we denote the base space by Σ instead of M. Points on a spatial three-manifold Σ will be denoted by the boldface letter $\mathbf{x}$, while points on the spacetime four-manifold M will be denoted by x.

A *gauge transformation*, g, is a bundle automorphism, $g : P \to P$. The group $\mathcal{G}(P)$ (or simply $\mathcal{G}$), of such automorphisms under composition, is called the *gauge group*. One should clearly distinguish between the structure group G, which is global, and the gauge group $\mathcal{G}$, which is local, i.e. such that its generators vary over the points of the base manifold M.

Note that this is *not* the definition of a gauge transformation that is commonly used by physicists, who usually think of gauge transformations as G-valued maps, i.e. as *maps from the* *manifold* M *into the structure group* G. However, it *is* possible to think of gauge transformations in this way, by using the following two isomorphisms:

(i) The group isomorphism[3] $\mathcal{G}(P) \cong C^\infty(P, G)^G$, where $C^\infty(P, G)^G$ denotes the group of smooth maps $f\colon P \to G$ satisfying $f(ph) = h^{-1}ph$ for all $p \in P, h \in G$ (for more details and proofs, see Chapter 5 of (Hamilton 2017)). By this isomorphism, gauge transformations can be viewed as G-*valued* maps on the principal bundle.

These are not quite G-valued maps *on the manifold* yet, i.e. we would like to work with $C^\infty(M, G)$ rather than $C^\infty(P, G)^G$, which contains the larger and more abstract space P. This will be possible, thanks to the second isomorphism, but only if we choose a (local) gauge. A (local) *gauge* is a (local) section of the bundle P, i.e. a smooth map $s : U \to P$ that satisfies $\pi \circ s = \mathrm{id}_U$, where $U \subset M$ is an open subset of the manifold. If this open subset is the whole manifold M, then the bundle is called *trivialisable*.[4]

(ii) It can be shown that a section of the bundle, s, defines an isomorphism[5] of groups $C^\infty(P_U, G)^G \cong C^\infty(U, G)$, where P_U is the restricted principal G-bundle over U. Thus, when we work with a trivialisable principal bundle, we can extend $C^\infty(U, G)$ to $C^\infty(M, G)$.

In other words: through these two isomorphisms, gauge transformations can be viewed as G-valued maps on the manifold, but *only in a specific gauge*. This allows us to define *global gauge transformations* as the constant G-valued maps on (a region of) the manifold. We note that, if there does not exist a global section of the principal bundle (i.e. if the bundle is not trivializable), then there is no well-defined notion of 'global gauge transformation' on the whole manifold, but only locally. However, a 'local global gauge transformation' seems like a *contradictio in terminis*, since we would like to think of global gauge transformations as extending out to infinity, since they do not depend on spacetime coordinates. For this reason, some authors prefer the word *rigid* over *global*. (Since for us there will be no confusion, we will continue to say 'global'.)

If one works on a *trivialisable* principal bundle over a manifold M, the gauge group $\mathcal{G}$ will always be isomorphic to $C^\infty(M, G)$. Thus, whenever we work with a trivialisable bundle in this article, we will not need to insist on the distinction between $\mathcal{G}$ and $C^\infty(M, G)$, and we will sometimes simply write: $\mathcal{G} = C^\infty(M, G)$.

Note that, on a trivialisable principal bundle, the *subgroup of* $\mathcal{G}$ *that consists only of global gauge transformations* can be identified with (i.e. is isomorphic to) the structure group G. It is only in the Abelian case that the action of the global gauge group can literally be viewed as the free and transitive right action of G on P.[6]

(2) *Interpretative aspects of gauge symmetries.* We will now clarify, in general terms, why global gauge transformations are physical. For they are a subgroup of the total gauge group $\mathcal{G}$, and so one might naively think that, if the transformations in $\mathcal{G}$ are unphysical, mathematical redundancies, then so are the global gauge transformations (and this point seems to be close to the view (ii) of the Higgs mechanism that we discussed in Section 1). However, it not true that the whole gauge group $\mathcal{G}$ is unphysical. For it has been shown that the subgroup of global gauge transformations *is* physical. This is the upshot of the constrained Hamiltonian analysis, which provides a formal correlate of the meaning of 'gauge' in the sense of 'redundant', which

[3]It is not too hard to check that the isomorphism is provided by sending $f \in \mathcal{G}(P)$ to $\sigma_f\colon P \to G$, defined via $f(p) = p\sigma_f(p)$ for any $p \in P$.

[4]For examples of non-trivialisable bundles, one looks for example at a manifold M that is topologically non-trivial (where, by 'topologically trivial' manifold, we mean one where all the homotopy groups are the identity). This is for example the case for BPST SU(2) instantons on S^3 (where 'BPST' stands for the names of the authors of (Belavin et al. 1975)), which are characterized by the second Chern class of the principal bundle.

[5]This isomorphism simply sends $\sigma \in C^\infty(P_U, G)^G$ to $\sigma \circ s\colon M \to G$.

[6]A gauge transformation is a bundle automorphism, but in the non-Abelian case the free and transitive group action of G on P need not actually yield bundle automorphisms. To see this, take $h_1, h_2 \in G$ that do not commute and consider the map $f\colon P \to P$ given by $p \mapsto ph_1$. Then we have $f(ph_2) = (ph_2)h_1 = p(h_2h_1) \neq p(h_1h_2) = f(p)h_2$.

constrasts with ‘formal’ as follows (see (Teh 2016)):

Redundant: A gauge transformation of the fields that has no physical effect, i.e. it does not lead to distinct empirical predictions.

Formal: A gauge transformation as defined in mathematical terms, i.e. an element of the gauge group $\mathcal{G} \cong C^\infty(\Sigma, G)$ (where Σ is space and G is a compact matrix Lie group) that acts on the fields.

As (Teh 2016) emphasizes, confusion ensues from mixing these two connotations of the word ‘gauge’. The statement that a gauge transformation is *redundant* is an interpretative statement, while *formal* is a mathematical i.e. non-interpretative statement. In this paper, we always use the word ‘gauge’ in the sense of *formal*, i.e. in the sense of an automorphism of a principal bundle P.

Furthermore, we will contrast the class of *redundant* gauge transformations with the class of gauge transformations that have:

Direct empirical significance (DES): A gauge transformation of the fields has DES if it has a physical effect, i.e. it leads to distinct empirical predictions.

In the recent philosophical literature, there has been substantial debate about whether, and if so how, gauge symmetries can exhibit DES in scenarios like Galileo’s ship.[7] The global gauge group has variously been put forward as the group of gauge symmetries carrying DES, and a number of authors have used sophisticated mathematical methods to approach this issue.[8] A notable result[9] that supports our analysis is Theorem 1 from (Gomes 2021), which can be rephrased as:

Theorem 2.1 (Rigid variety for $U(1)$)**.** For the Maxwell theory as coupled to a Klein-Gordon scalar field in a simply-connected universe: given the physical content of two regions, for matter vanishing at the boundary but not in the bulk of the regions, the universal state is undetermined, resulting in a residual variety parametrised by an element of $U(1)$. Here the particular action of $U(1)$ is that which leaves the gauge fields, but not the matter fields, invariant.

The results discussed in this literature thus suggest that DES is indeed exhibited by the *global gauge transformations*, which transform the system as a whole, and in particular on the edges.[10] The analogy is here with Galileo’s ship, where a transformation that is made locally on the system (i.e. the ship) does not change the physics inside the system, while a transformation that changes the boundary of the system does change the physical state. The difficulty with the Higgs mechanism, however, is that one does not think of the universe as a true subsystem. The boundaries of the universe are not “real” boundaries with another system outside, but asymptotic or conformal boundaries. Therefore, physical gauge symmetries in this context are *asymptotic symmetries*, which have been intensively studied in general relativity.[11] In (Borsboom and Posthuma 2025) the global gauge group is shown to be the asymptotic symmetry group of Yang-Mills theory on a Cauchy surface.

The philosophical literature has also considered the indirect empirical significance of a gauge transformation:[12]

[7]See (Kosso 2000; Brading and Brown 2004; Healey 2009; Greaves and Wallace 2014; Friederich 2015; Ramírez and Teh 2021; Wallace 2022a; Wallace 2022b).

[8]For a sample of the most recent literature, see (Gomes, Hopfmüller, and Riello 2019; Gomes and Riello 2021; Gomes 2021; Gomes 2022; Gomes 2025). For a more detailed analysis of these articles in relation to our ideas on the Higgs mechanism we refer the reader to Section 4.3.4 of (Borsboom 2024).

[9]However, this is a result on subsystems with a boundary in between, not on *asymptotic* boundary conditions.

[10]Hence the relation to edge modes, see e.g. (Donnelly and Freidel 2016; Carrozza and Höhn 2022; Riello 2021)

[11]See (Bondi, Burg, and Metzner 1962; Sachs 1962; Strominger 2014; Henneaux and Troessaert 2018; Henneaux and Troessaert 2020). For a philosophical discussion of this topic, see (De Haro 2017a).

[12]See e.g. (Brading and Brown 2004, p. 648)

Indirect empirical significance (IES): A gauge transformation of the fields has IES if it does not itself have a physical effect, i.e. it does not itself lead to distinct empirical predictions, but the properties of the fields and laws that are connected with the existence of this symmetry do have DES.

In other words, the physical significance of this gauge transformation is indirect, through its connection with fields and laws that do have DES. Although the symmetry transformation itself is not observable, its existence does have observable consequences. A well-known example is the local gauge symmetry of the Maxwell theory: although (as we will discuss) it has no DES, it does have IES in that it implies that the Maxwell theory can only be coupled to a current that is conserved. Thus it implies the physically significant fact of charge conservation. Therefore, gauge symmetries that are *redundant* can, and often do, have IES.

2.2 The group of physical symmetries

Section 2.1 argued, on general grounds, that the gauge transformations that are interpreted as having DES are the global ones. The aim of this Section is to explain precisely how this result is derived from taking the quotient of all *boundary-preserving* symmetries by all *redundant* symmetries.

Physical systems often come with boundary conditions. If a system exhibits a symmetry, then it is important that the symmetry group acts on that system in such a way that the boundary conditions are preserved to guarantee that our description of the system is well-defined. In particular, if we consider gauge theories with boundary conditions, then we must ensure that the action of the gauge group is boundary-preserving, i.e. respects the boundary conditions on the fields in the theory. In what follows, we denote the group of boundary-preserving gauge transformations by $\mathcal{G}^{\mathrm{I}}$, where the notation I stands for 'invariant', since these are the transformations that leave the boundary conditions invariant.

Thus, the group of physical symmetries equals all those elements of $\mathcal{G}^{\mathrm{I}}$ that are not *redundant*. But how can we characterize which gauge transformations are redundant and which ones are not? To this end, one can make use of the Hamiltonian theory of constraints, or the *constrainted Hamiltonian formalism*. This is a formalism that allows one to identify *redundant* gauge transformations by investigating which transformations are generated by the constraints of the theory. This generation occurs by means of taking Poisson brackets (see Section 3). For Yang-Mills theory the relevant constraint is the so-called *Gauss law constraint*, which is just the usual Gauss law $\nabla \cdot \mathbf{E} = 0$ in the special case of electromagnetism. We will explain precisely how constraints arise and generate gauge transformations in Section 3. The remainder of this Section is dedicated to the introduction of various subgroups of the formal gauge group $\mathcal{G}$ which will appear frequently throughout the article.

Gauge transformations that are generated by the Gauss constraint. To identify the gauge transformations that are generated by the Gauss law constraint, (Teh 2016) quotes a result in (Balachandran 1994) that states that the simplest asymptotic condition that secures that a gauge transformation can be generated by the Gauss constraint, is that the gauge transformation $g \in \mathcal{G}$ goes to unity at infinity at the appropriate rate:

$$g(\mathbf{x}) \to 1 \quad \text{as} \quad |\mathbf{x}| \to \infty\,, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^3$. This condition is also used e.g. in (Struyve 2011; Wallace 2022b). We denote the subgroup of maps that satisfy this condition by $\mathcal{G}^{\infty}$. It is important to note that, in order to even be able to speak of the identity at infinity, we need a section or frame, i.e. a trivialisation, of the bundle at infinity. If we work on a trivialisable bundle, we need not worry about the possibility of finding such a trivialisation.[13]

In addition, the transformations generated by the Gauss constraint are *small*, i.e. homotopic to the identity map, $x \mapsto 1 \in \mathcal{G}$. The gauge transformations generated by the Gauss constraint are small because the Gauss constraint works on elements of the infinite-dimensional Lie algebra $\mathrm{Lie}(\mathcal{G})$ of the gauge group $\mathcal{G}$, and these must be exponentiated to get an element of the Lie group $\mathcal{G}$. And it is a well-known fact that the image of the exponential map lies in the connected component of the identity of the Lie group, which explains why the Gauss constraint generates only small gauge transformations. For the empirical significance of *large* gauge transformations see (Gomes and Riello 2024).

We denote by $\mathcal{G}_0^\infty \subset \mathcal{G}$ the subgroup of gauge transformations that satisfy the above two conditions, i.e. they:

(i) are small, i.e. $g \in \mathcal{G}_0$,
(ii') satisfy the asymptotic condition Eq. (1).

Thus Teh takes the gauge transformations $\mathcal{G}_0^\infty$ to be the *redundant* ones, i.e. the ones that are not physical. (Anticipating that we will replace the condition (ii') by the more general (ii) below, we gave this label a tilde.)

Smallness, i.e. condition (i), is automatically satisfied for the $U(1)$ structure group of electromagnetism in three spatial dimensions, but it is not automatically satisfied for the non-Abelian structure group $SU(2)$: and so, in general, (i) and (ii') are independent conditions.

Gauge transformations that are not generated by the Gauss constraint. To find the set of all admissible gauge transformations, i.e. not only those that are redundant but also those that have DES, we should drop the smallness condition (i). Also, we should replace (ii') by a more general asymptotic condition that is not specific to those gauge transformations that are generated by constraints, but which makes sure that gauge transformations are boundary-preserving. Thus we consider the following gauge transformations, which, as shown in (Borsboom and Posthuma 2025), leave invariant whatever asymptotic boundary conditions are imposed on the fields:

(ii) the subgroup of gauge transformations that are constant at infinity, i.e. $\exists g_0 \in G: \quad g(\mathbf{x}) \to g_0$ as $|\mathbf{x}| \to \infty$.

Clearly, (ii') is a special case of (ii) with $g_0 = 1$. We denote by $\mathcal{G}^I$ the subgroup of transformations that satisfy condition (ii), but not necessarily condition (i). Because these gauge transformations preserve the boundary conditions, we say that they are 'boundary-preserving'.

The reason we want to impose condition (ii) on gauge transformations is that boundary conditions are an integral part of how we represent a physical system: they restrict the types of configurations that are allowed. A transformation that does not respect the boundary conditions can therefore not be viewed as a symmetry of a physical system (in either the senses of *redundant*, or of DES that are of our interest here), since it maps the fields describing one physical system into those of a different physical system, and in some cases even into a set of fields that do not correspond to a physical system at all (e.g. because they are inconsistent with the theory's dynamics). Specifically, in field theory the boundary conditions are chosen such that physical quantities like the energy are finite, and we will assume that the conditions (ii) are of this type.

[13] One may worry, however, that choosing a trivialisation at infinity amounts to breaking gauge-invariance there. For details on how to treat asymptotic conditions on gauge transformations in a gauge-invariant manner we refer to (Borsboom and Posthuma 2025).

To sum up: $\mathcal{G}_0^\infty$ is the subgroup of gauge transformations that satisfy (i) and (ii') (and therefore also (i) and (ii)), $\mathcal{G}^\infty$ satisfy (ii') (and therefore also (ii)), and $\mathcal{G}^{\mathrm{I}}$ satisfy (ii). Thus we have the following hierarchy of subgroups (Teh 2016):

$$\mathcal{G}_0^\infty \subset \mathcal{G}^\infty \subset \mathcal{G}^{\mathrm{I}} \subset \mathcal{G}\,.$$

We now follow (Teh 2016) in defining the symmetries that exhibit DES to be those that are boundary-preserving but non-redundant,[14] i.e. not generated by the Gauss law constraint. Thus we obtain:

$$\mathcal{G}_{\mathrm{DES}} = \mathcal{G}^{\mathrm{I}}/\mathcal{G}_0^\infty\,. \tag{2}$$

However, this construction was only recently made precise and conceptually clear in (Borsboom and Posthuma 2025) for pure Yang-Mills theory on Euclidean space with asymptotic boundary conditions. The main point made in that article is that the requirement that boundary-preserving gauge transformations $g \in \mathcal{G}^{\mathrm{I}}$ become constant at asymptotic infinity follows not directly from the boundary conditions on the fields, but rather from the additional requirement that the *Lagrangian of Yang-Mills theory be defined on a tangent bundle to configuration space*. Extending this requirement to Yang-Mills-Higgs theory, it can be shown that $\mathcal{G}^{\mathrm{I}}$ is different in the unbroken and broken phases. In the unbroken phase $\mathcal{G}^{\mathrm{I}}$ consists of all asymptotically constant gauge symmetries, like for pure Yang-Mills theory. But in the broken phase, i.e. when we take a minimum of the Higgs potential $V(\varphi)$ to have zero energy, rather than $\varphi = 0$, the boundary-preserving gauge symmetries $\mathcal{G}^{\mathrm{I}}$ are all transformations that become the identity at infinity. Thus, we find that $\mathcal{G}_{\mathrm{DES}}$ consists of a copy of the global gauge group for every homotopy class only in the unbroken phase. It is trivial (or discrete) in the broken phase.

We will use these results throughout this article, while referring to (Borsboom and Posthuma 2025) for the details. However, we do need to understand in detail how the group of redundant gauge transformations $\mathcal{G}_0^\infty$ can be found by studying which transformations are generated by the Gauss law constraint. The next Section will therefore be dedicated to this. Doing so will provide us with the necessary tools to give our harmonized account of gauge symmetry breaking in the Higgs mechanism in Section 4.

3 The Gauss Law Constraint in Electromagnetism

Now that we understand what the group $\mathcal{G}^{\mathrm{I}}$ looks like in the unbroken and broken phases of the Higgs model, we turn to the group $\mathcal{G}_0^\infty$ of redundant gauge transformations. The aim of this Section is to show that this group consists of gauge transformations that become the identity at asymptotic infinity. We first explain how gauge transformations relate to constraints in Section 3.1. We then specialize to electromagnetism in Section 3.2. Finally we show how the Coulomb gauge arises as the natural gauge for studying global gauge symmetry breaking in Section 3.3.

3.1 Redundant gauge symmetries are generated by constraints

Constraints in the Hamiltonian formalism generate redundant gauge transformations because we require determinism. While constraints can arise in the Hamiltonian formalism for a number of reasons, for us, since we are interested in gauge theories, the appearance of constraints will signal the presence of a redundant symmetry: in other words, it will signal the failure of the configuration

[14] This is similar to and inspired by (Greaves and Wallace 2014), in which symmetries exhibiting DES are defined as those that are boundary-preserving but *non-interior*.

space to be in a one-to-one correspondence with the space of physical states. And, as we discuss below, we will define the latter in terms of determinism under time evolution.

In the Lagrangian formalism, if the Hessian matrix of the Lagrangian has a zero eigenvalue, so that its determinant is zero,[15] the acceleration is not uniquely determined by the positions and velocities, and the solutions contain arbitrary functions of time (Henneaux and Teitelboim 1992). This implies that, in such a situation, the time evolution is in general not deterministic. In a gauge theory, this lack of determinism, i.e. the vanishing of the determinant of the Hessian, is the consequence of a gauge symmetry.

From this perspective, the constrained Hamiltonian formalism is a way to construct a deterministic theory with a unique time evolution, by identifying points on the phase space that are related by a redundant symmetry. The submanifold of points on phase space that are related by a redundant symmetry, and that thus correspond to the same physical state, is generated by a set of constraints. Hence the slogan 'constraints generate redundant gauge transformations', which we will take as our criterion for *redundant* gauge transformations.

Note that this criterion has, as it of course should, a crucial interpretative aspect. This is because our consideration of constraints as generating redundancies is directly motivated by the requirement of having a *deterministic theory with a unique time evolution*. Indeed, if one drops this latter requirement, then the gauge transformations that are generated by constraints are *not* in general redundant. Alternatively, if one uses a different dynamical principle to define one's set of physical states, one will in general have a different criterion for a gauge transformation being *redundant* (see (Henneaux and Teitelboim 1992), (Earman 2003b, p. 149)).[16]

The constraints give the set of points in the cotangent bundle where the Lagrangian transformation is invertible. Mathematically speaking, constraints arise because the Legendre transformation is not a diffeomorphism. More specifically, constraints relate to the fact the velocities in the Lagrangian formalism cannot be obtained from the canonical momenta in the Hamiltonian formalism. In such a case, the canonical momenta are not independent, but satisfy constraint relations. We now fill in some of the mathematical details.

Recall that, in the Lagrangian formalism, we work with positions and velocities,[17] i.e. pairs $(q^i, \dot{q}_i) \in TQ$, which take values in the tangent bundle, TQ, to the configuration space, with $q^i \in Q, \dot{q}^i \in T_qQ$. By contrast, in the Hamiltonian formalism, we work with positions and momenta, i.e. pairs $(q^i, p_i) \in T^*Q$, which take values in the cotangent bundle, T^*Q.

Thus the Legendre transformation is a map, $TQ \to T^*Q$, from the tangent bundle to the cotangent bundle. In simple cases, this map is invertible, i.e. *hyperregular*, so that it is in fact a diffeomorphism of the tangent bundle. However, in cases where the Lagrangian has e.g. internal symmetries, this map need not be hyperregular (Marsden and Ratiu 1999, p. 186).

If the Legendre transformation is not surjective, then there are relations $c_m(q^i, p_j) = 0$ between the momenta p_j and generalised positions q^i, called *constraints*. A constraint expresses an equivalence relation of a set of points in the cotangent bundle that, under the Legendre transformation, do not have a unique inverse in the tangent bundle. Since these are precisely the points where determinism fails due to a gauge symmetry, by interpreting them as corresponding to the same physical state, we restore determinism. As a consequence, the corresponding gauge transformation is *redundant*. Thus, in the Hamiltonian formulation, we may wish to quotient our gauge transformations by the *redundant* ones.[18]. The best known example of a gauge

[15]The Hessian matrix of a Lagrangian L is the matrix $\partial^2 L/\partial\dot{q}^i\partial\dot{q}^j$. For details, see (Henneaux and Teitelboim 1992, pp. 4-5).

[16]For an example of a treatment of a gauge theory where one gives up the requirement of determinism, see (Belot 1995, pp. 88-90).

[17]In field theories these are field configurations and their time derivatives.

[18]A precise treatment of this reduction procedure, even in the presence of boundaries, is given in e.g. (Binz, Śniaiycki,

constraint is probably Gauss's law from electromagnetism, which we will discuss in Section 3.2.

The constraint relations define the *primary constraint set* $C \subset T^*Q$, which is the image of the Legendre transformation (provided the Lagrangian $\mathcal{L}$ is *regular* (Marsden and Ratiu 1999)). Primary constraints do not rely on the equations of motion (Henneaux and Teitelboim 1992), while secondary constraints arise as the requirement that the primary constraints be preserved in time as the system evolves. This can be iterated to obtain tertiary constraints etc. (Healey 2007). The space that is defined by requiring all constraints to be satisfied is called the *constraint surface*, and it is assumed to be a submanifold smoothly embedded in phase space.[19]

If a function F on phase space vanishes on the constraint surface, it is said to vanish *weakly*, written $F \approx 0$. Constraints whose Poisson brackets with all other constraints vanish weakly are called *first class*. A familiar slogan is that "first class constraints generate gauge transformations" (where 'generating' means taking Poisson brackets), although there are many subtleties about this statement that we do not consider here, see e.g. (Earman 2003b).[20]

To understand whence comes this notion of *gauge* as 'generated by the first class constraints', it is useful to consider the symplectic formalism underlying constrained Hamiltonian analysis.[21] The phase space cotangent bundle T^*Q is a symplectic manifold, i.e. a manifold equipped with a closed non-degenerate 2-form $\omega \in \Omega^2(T^*Q)$. It is written locally as

$$\omega = \sum_i dq^i \wedge dp_i \,. \tag{3}$$

On a symplectic manifold $(\mathcal{M}, \omega)$, every smooth function $H \in C^\infty(\mathcal{M})$ defines a Hamiltonian vector field $X_H \in \mathfrak{X}(\mathcal{M})$, where $\mathfrak{X}(\mathcal{M})$ denotes the set (more precisely: the C^∞-module) of vector fields on $\mathcal{M}$, through

$$dH = \omega(X_H, \cdot) \,.$$

Here the $\cdot$ denotes the vacant spot in which an arbitrary vector field from $\mathfrak{X}(\mathcal{M})$ can be inserted. Note that dH also eats such a vector field by definition of the differential operator $d : C^\infty(\mathcal{M}) \to \Omega^1(\mathcal{M})$ which sends a smooth function to a 1-form. The Poisson bracket between $f, g \in C^\infty(\mathcal{M})$ is then defined as

$$\{f, g\} = \omega(X_f, X_g) \,. \tag{4}$$

It is not difficult to see that for the canonical 2-form on the cotangent bundle this locally gives the familiar Poisson bracket:

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \,. \tag{5}$$

The idea behind Hamiltonian vector fields is that their integral curves are precisely the trajectories through phase space satisfying Hamilton's equations. Similarly, we can consider the Hamiltonian vector fields X_{c_m} associated with the constraints $c_m \colon \mathcal{M} \to \mathbb{R}$. If all constraints are first class, then these vector fields flow tangentially to the constraint surface. If not all constraints are first class, then one can use the Dirac algorithm to guarantee so. We will not be concerned with that possibility since in our case of interest all constraints are first class.

and Fischer 1988; Riello and Schiavina 2024; Riello and Schiavina 2025)

[19] In Yang-Mills theory, the constraint surface is given by $\mathcal{C} = J_{\mathfrak{h}}^{-1}(0)$, where $J_{\mathfrak{h}}$ is the momentum map of the group of infinitesimal localisable gauge symmetries, whose Lie algebra is $\mathfrak{h}$ (Binz, Śniaiycki, and Fischer 1988).

[20] Pitts has objected to this slogan (Pitts 2014), but this objection has again been objected to (Pooley and Wallace 2022).

[21] For a conceptual overview see (Gomes and Butterfield 2024).

If all constraints are first class then their associated vector fields are null directions of the symplectic form *on the constraint surface*, i.e. $\omega(X_{c_m}, X_{c_n}) = \{c_m, c_n\} \approx 0$. The existence of such null directions is possible because ω can be degenerate on the constraint surface, whereas it is by definition non-degenerate on the entire phase space. It is for this reason that we call these directions *gauge* and their integral curves *gauge orbits*. Points within one orbit are interpreted as physically equivalent, so any physical quantity must be gauge-invariant in the sense of being constant on every gauge orbit.

In addition to calculating the Poisson bracket, Eq. (5), between the constraints themselves, we can also use it to calculate the Poisson bracket of a constraint with a field F. Schematically, we write:

$$\{F, c_m\} = \delta_m F\,, \tag{6}$$

where F is any field of the theory, and c_m is any first-class constraint. The result is then that $\delta_m F$ is the corresponding *redundant* gauge transformation of F (Henneaux and Teitelboim 1992, p. 17). Thus if F is a gauge-invariant field, the right-hand side of this equation is zero. And if F is a non-gauge-invariant field, the right-hand side is its infinitesimal gauge variation. Thus by taking Poisson brackets with constraints, we move along the orbit of a *redundant* gauge symmetry in the phase space. And by exponentiating such infinitesimal variations, we can generate finite gauge orbits. In the next Section, we will illustrate this in the example of the Maxwell theory.

3.2 Electromagnetism

In this and the next Section we show explicitly how the Gauss law constraint generates redundant gauge transformations in electromagnetism (i.e. $G = U(1)$), and how this leads to the Coulomb gauge as the appropriate gauge to handle the physical gauge group derived in the previous Section.[22] We will need these results and explicit expressions in Section 4 for our harmonisation of the two conflicting gauge-invariant accounts of the Higgs mechanism anticipated in Section 1. The Coulomb gauge precisely removes the redundant gauge transformations generated by the Gauss law constraint, but it leaves the physical gauge group of global gauge symmetries. We will therefore use it to study global gauge symmetry breaking in the Abelian Higgs mechanism, combining the ideas from (Struyve 2011) and the DFM (Berghofer et al. 2023).

From now on we work on the trivial bundle $P = \Sigma \times U(1)$. The first thing to do is to find the constraints of electromagnetism. The covariant Lagrangian of Maxwell theory is:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}\,.$$

Here and from now on we use the Einstein summation convention for pairs of indices of which one is up and one is down. The Legendre transformation gives the following canonical momenta:

$$\Pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -\frac{1}{2} F^{\mu\nu} \frac{\partial}{\partial \dot{A}_i} (\partial_\mu A_\nu - \partial_\nu A_\mu) = -\frac{1}{2} F^{\mu\nu} (\delta^0_\mu \delta^i_\nu - \delta^0_\nu \delta^i_\mu) \tag{7}$$

$$= -\frac{1}{2}(F^{0i} - F^{i0}) = F^{i0} = -E^i\,. \tag{8}$$

In the Maxwell theory, one constraint takes the familiar form of Gauss's law in differential form, i.e. $\rho = J^0 = \nabla \cdot \mathbf{E}$, where ρ is the charge density and $\mathbf{E}$ the electric field. The other constraint

[22] In this respect we disagree with (Wallace 2024), where it is argued that unitary gauge is appropriate for electrodynamics with a Higgs field.

is $\Pi^0 = 0$, which indicates that A_0 is a Lagrange multiplier. This enables us to use a spacetime split $\Sigma \times \mathbb{R}$, where Σ is a Cauchy surface which in Minkowski spacetime can just be taken to be $\mathbb{R}^3$. The gauge fields are then functions $A_i \in C^\infty_I(\Sigma)$ which approach zero sufficiently quickly.[23] Similarly $E^i \in C^\infty_I(\Sigma)$, and phase space consists of such pairs $(\mathbf{A}, \mathbf{E})$. We can then use a smooth function $\lambda\colon \Sigma \to \mathbb{R}$, called a *Lagrange multiplier*, to obtain the *smeared Gauss constraint*

$$G_\lambda = \int_\Sigma d^3x\, \lambda\,(\nabla \cdot \mathbf{E} - \rho)\,, \tag{9}$$

which is a function on the phase space. Here the gauge transformation parameter λ is an element of the infinite-dimensional Lie algebra Lie$(\mathcal{G}^I)$ of the boundary-preserving gauge group.

We are interested in calculating how the smeared Gauss constraint acts on the fields, i.e. in its Poisson brackets with A_i and E^i. The symplectic form is the canonical one:

$$\Omega = \int_\Sigma d^3x\, \mathbb{d}A_i \wedge \mathbb{d}E^i\,, \tag{10}$$

We follow (Gomes and Butterfield 2024; Gomes and Riello 2021) in using the double struck $\mathbb{d}$ to indicate that this is the differential operator on the infinite-dimensional phase space coordinatized by the fields A_i and E^i, and not on the three-dimensional space Σ, for which we use the usual d. From the expression for the symplectic form Ω, it follows that the Poisson bracket of two functionals F and G of the fields is given by

$$\{F, G\} = \int_\Sigma d^3x \left(\frac{\delta F}{\delta A_i(\mathbf{x})} \frac{\delta G}{\delta E^i(\mathbf{x})} - \frac{\delta F}{\delta E^i(\mathbf{x})} \frac{\delta G}{\delta A_i(\mathbf{x})} \right), \tag{11}$$

where $\delta/\delta A_i(\mathbf{x})$ and $\delta/\delta E^i(\mathbf{x})$ denote the functional derivatives of phase space functions with respect to the fields $A_i(\mathbf{x}), E^i(\mathbf{x})$, which are themselves functions on Σ. The poisson bracket of the smeared Gauss constraint with the gauge potential thus becomes

$$\begin{aligned}
\{G_\lambda, A_i(\mathbf{x})\} &= -\frac{\delta G_\lambda}{\delta E^i(\mathbf{x})} = -\frac{\delta}{\delta E^i(\mathbf{x})} \int_\Sigma d^3y\, \lambda(\mathbf{y})(\nabla \cdot \mathbf{E}(\mathbf{y}) - \rho) = -\frac{\delta}{\delta E^i(\mathbf{x})} \int_\Sigma d^3y\, \lambda(\mathbf{y}) \nabla \cdot \mathbf{E}(\mathbf{y}) \\
&= -\frac{\delta}{\delta E^i(\mathbf{x})} \left(\int_{\partial\Sigma} d^2y\, \lambda(\mathbf{y}) \mathbf{E}(\mathbf{y}) \cdot \mathbf{n} - \int_\Sigma d^3y\, \nabla\lambda(\mathbf{y}) \cdot \mathbf{E}(\mathbf{y}) \right) \\
&= \partial_i \lambda(\mathbf{x}) - \frac{\delta}{\delta E^i(\mathbf{x})} \int_{\partial\Sigma} d^2y\, \lambda(\mathbf{y}) \mathbf{E}(\mathbf{y}) \cdot \mathbf{n}\,.
\end{aligned}$$

Here, we have performed partial integration and denoted the unit normal vector pointing out of the boundary $\partial\Sigma$ by $\mathbf{n}$. This boundary term should be viewed as an integral over a sphere of radius r, taking $r \to \infty$, since it is an asymptotic boundary. But from this derivation it becomes clear that, if the Poisson bracket $\{G_\lambda, A_i(\mathbf{x})\}$ really is to generate gauge transformations, then the boundary term must vanish. In order to guarantee this we require that $\lambda \to 0$ asymptotically, which is how we obtain the result that redundant gauge transformations should become trivial at infinity. For a more mathematical treatment and for detailed considerations on the rate at which (gauge) fields and gauge parameters should vanish asymptotically, we refer to (Borsboom and Posthuma 2025).

[23] Here we again use the notation I to indicate behavior relating to asymptotic boundary conditions. In this case it means that we consider fields which vanish at infinity, whereas before it signified gauge transformations that respect asymptotic boundary-conditions.

Furthermore, it is not difficult to see that the following Poisson bracket holds between the smeared Gauss constraint and the electric field:

$$\{G_\lambda, E^i(\mathbf{x})\} = \frac{\delta}{\delta A_i(\mathbf{x})} \int_\Sigma d^3y\, \lambda(\mathbf{y})(\nabla \cdot \mathbf{E} - \rho) = 0\,.$$

Thus, the smeared Gauss constraint G_λ does not modify the electric field: which is observable, and therefore gauge-invariant.

3.3 Deriving Coulomb gauge

As we just discussed, the Gauss law in Eq. (9) states that the divergence of the electric field equals the distribution of charges in space. We will now discuss how the general solution of this constraint equation gives, in a natural way, what is usually called the 'Coulomb gauge'. To this end, we will first solve the Gauss constraint for the electric field, **E**. Since, as we discussed in Eq. (7), the electric field is the canonical momentum associated to the gauge potential **A**, the solution that we write down has to be compatible with the canonical Poisson bracket. This requirement will naturally give us the Coulomb gauge.

The most general solution of the Gauss constraint, which is a first-order linear differential equation, is given, in the familiar way, by a linear combination of: (i) the *general solution of the homogeneous equation*, i.e. $\nabla \cdot \mathbf{E} = 0$ (i.e. the *transverse* part of the electric field, whose divergence is zero but whose curl is non-zero), and (ii) a *particular solution of the inhomogeneous equation* (i.e. the *Coulombic* part of the electric field). In other words, we seek the *Helmholtz decomposition* $E^i = E^i_L + E^i_T$ of the electric field into longitudinal (irrotational, i.e. curl-free) and transverse (solenoidal, i.e. divergence-free) components. Since the longitudinal part E^i_L is curl-free, it can be written as the gradient of a scalar function (Poincaré lemma), i.e. $E^i_L = \partial^i \epsilon$ with $\epsilon \in C^\infty_I(\Sigma)$.[24] We then generate the Coulombic electric field coordinates in phase space by the vector field

$$\mathbb{E}^L = \int_\Sigma d^3x\, E^L_i \frac{\delta}{\delta E_i(\mathbf{x})} = \int_\Sigma d^3x\, \partial_i \epsilon \frac{\delta}{\delta E_i(\mathbf{x})}\,,$$

where, following (Gomes and Riello 2021; Gomes and Butterfield 2024), we have again used the double struck notation to stress that this vector field lives on infinite-dimensional phase space. The analogy here is with a vector v in usual finite-dimensional tangent space, which can be written in a coordinate basis as $v = v^i \partial/\partial x^i$. Similarly $\delta/\delta E_i(\mathbf{x})$ is the basis vector field in infinite-dimensional tangent space. Additionally we have to integrate over Σ, since $\delta/\delta E_i(\mathbf{x})$ is a functional derivative (it derives with respect to a field which itself is a function on Σ). A vector field on the infinite-dimensional phase space should be thought of as being tangent to a curve through the space of fields $(\mathbf{A}, \mathbf{E})$.

The last step is to require that the canonical momentum (i.e. **E**, see Eq. (7)) has zero Poisson bracket with the gauge potential, i.e. **A**. To require this, we first extend the Coulombic-radiative split to the vector potential **A**, i.e. we look for the component $\mathbf{A}^T$ of **A** that is symplectically orthogonal to $\mathbf{E}^L$. To find this radiative component, we define another vector field in phase space (orthogonal to the previous one):

$$\mathbb{A}^T = \int_\Sigma d^3x\, A^T_i \frac{\delta}{\delta A_i(\mathbf{x})}\,,$$

[24] We recall that the subscript I in $C^\infty_I(\Sigma)$ signifies appropriate asymptotic fall-off behaviour, a notation inherited from the group $\mathcal{G}^I$ of asymptotically constant gauge transformations. In this case we need the functions $\epsilon \in C^\infty_I(\Sigma)$ to be such that their derivatives $\partial^i \epsilon$ exhibit the same asymptotic behaviour as the longitudinal electric fields E^i_L.

and we require that the two are orthogonal with respect to the symplectic structure on the phase space (see Eq. (10)):

$$0 = \Omega(\mathbb{A}^T, \mathbb{E}^L) = \int_\Sigma d^3x \, \mathbb{d}A_i \wedge \mathbb{d}E^i(\mathbb{A}^T, \mathbb{E}^L) = \int_\Sigma d^3x \left(\mathbb{A}^T(A_i)\mathbb{E}^L(E^i) - \mathbb{E}^L(A_i)\mathbb{A}^T(E^i)\right) \tag{12}$$

$$= \int_\Sigma d^3x \, \mathbb{A}^T(A_i)\mathbb{E}^L(E^i) = \int_\Sigma d^3x \, A_i^T (E^L)^i = \int_\Sigma d^3x \, A_i^T \partial^i \epsilon = -\int_\Sigma d^3x \, \epsilon \partial^i A_i^T \tag{13}$$

for all $\epsilon \in C_I^\infty(\Sigma)$, again using partial integration and assuming the boundary term to vanish due to sufficiently rapid asymptotic fall-off behaviour of ϵ and A_i. This equation must hold for any $\epsilon \in C_I^\infty(\Sigma)$, so we find $\partial^i A_i^T = 0$, which, as we announced, is the Coulomb gauge condition. Thus the Coulomb gauge is singled out by the Hamiltonian formalism as the part of the gauge potential that is transverse, i.e. orthogonal to the longitudinal part of the electric field.

We can project the gauge field $\mathbf{A}$ onto the component satisfying this condition. This is called the *radiative projection*, and it is given by (Gomes and Butterfield 2024):

$$A_i^T(\mathbf{A}) = A_i - \partial_i(\Delta^{-1}\partial^j A_j)\,, \tag{14}$$

where $\Delta^{-1} = \nabla^{-2}$ is the inverse of the Laplacian with Green's function $-\frac{1}{4\pi r}$. The radiatively projected vector potential is gauge-invariant: under a gauge transformation $A_i \to A_i + \partial_i\lambda$, we have

$$\begin{aligned} A_i^T &\to A_i + \partial_i\lambda - \partial_i(\Delta^{-1}(\partial^j A_j + \partial^j\partial_j\lambda)) = A_i + \partial_i\lambda - \partial_i(\Delta^{-1}\partial^j A_j) - \partial_i\Delta^{-1}\Delta\lambda \\ &= A_i + \partial_i\lambda - \partial_i(\Delta^{-1}\partial^j A_j) - \partial_i\lambda = A_i - \partial_i(\Delta^{-1}\partial^j A_j) = A_i^T\,. \end{aligned}$$

This was certainly to be expected, since we removed precisely the part of the gauge field that is symplectically orthogonal to the part of the electric field that satisfies the constraint (the Gauss law). In other words, we removed the part of the gauge field forming a null direction of the symplectic form on the constraint surface, i.e. the *pure gauge* part in the sense of the constrained Hamiltonian formalism.

In summary: the Coulomb gauge rolls out of the Hamiltonian formalism as the condition that singles out the gauge-invariant, transverse component of the gauge field.

4 Resolving the Incompatibility

In the previous Sections, we used constrained Hamiltonian analysis to illustrate how, in electromagnetism, the Coulomb gauge allows one to remove the gauge symmetries generated by the Gauss law constraint, i.e. the redundant gauge symmetries. This has set us up to address the incompatibility, introduced in Section 1, between:

(i) Struyve's account of the Abelian Higgs mechanism, and

(ii) the dressing field method (DFM).

The main point we endeavor to explain is that, to resolve this incompatibility, the use of the DFM to study the Higgs mechanism, i.e. (ii), that has so far been made in the philosophical literature, needs to be corrected. For one cannot cover the full configuration space of classical fields by considering gauge-invariant composite objects like in these applications of the DFM. Indeed, one should not try to give a gauge-invariant account of the Higgs mechanism by eliminating the whole gauge group. Rather, one must apply the DFM at the higher level of infinite-dimensional phase space[25] in order to create fields in the Coulomb gauge that are nearly gauge-invariant,

[25]The details of this approach can be found in (Gomes 2019; Gomes, Hopfmüller, and Riello 2019; Gomes and Riello 2021; Riello and Schiavina 2024).

i.e. invariant under all but the global gauge transformations. This highlights how one element from Section 3 is crucial for the correct application of the DFM: namely, breaking the global gauge symmetry in the Coulomb gauge.

The structure of this Section therefore mirrors our two points, (i) and (ii), in the Introduction: Section 4.1 first reviews Struyve's gauge-invariant account of the Higgs mechanism. Then, Section 4.2 gives the correct version of the DFM gauge invariant account. With this correction, (i) and (ii) are rendered compatible.

4.1 Struyve's gauge-invariant account of the Higgs mechanism

As we discussed in Section 3, the Coulomb gauge condition $\partial^i A_i = 0$, which is enforced by the radiative projection

$$A_i^T(\mathbf{A}) = A_i - \partial_i(\Delta^{-1}\partial^j A_j)\,, \tag{15}$$

renders the gauge field invariant under the total gauge group $\mathcal{G}$, and in particular under the subgroup $\mathcal{G}^I$ of gauge transformations that are constant asymptotically, so that they preserve $\mathbf{A}$'s boundary conditions. This seems to suggest that the entire discussion above was unnecessary: the Coulomb gauge simply removes the entire gauge group, as any proper gauge should. Why then the hassle of distinguishing between physical (global) and unphysical gauge transformations?

Crucially, however, we have not said anything about matter fields. In the Abelian Higgs model, we consider a complex scalar field φ, understood to be the local expression of a section of an associated vector bundle. Under a gauge transformation with parameter λ, this field transforms as:

$$\varphi \to e^{ie\lambda}\varphi\,. \tag{16}$$

There is something noteworthy about this transformation: the field clearly also changes under a global gauge transformation $g = e^{i\lambda_0}$ with $\lambda_0 \in \mathbb{R}$ constant. This is not the case for the gauge field, because of the derivative in $A_i \to A_i + \partial_i\lambda$. Thus, although the global gauge group can be abstractly understood as the group of gauge transformations that have DES, its action can only be seen in the presence of matter fields. Indeed, it is the fact that the global gauge group acts trivially on gauge fields but non-trivially on matter fields that is used in the construction of Galileo's ship scenarios for $U(1)$ gauge theories in thought experiments of the "'t Hooft beam splitter" sort (Greaves and Wallace 2014). This fact is also what makes the issue of *gluing* interesting (and leads to Theorem 2.1), see (Gomes and Riello 2021; Gomes 2021).

Thus we need to examine how, in the Abelian Higgs model, the matter field can be made gauge-invariant too, in an appropriate sense. To this end, we use a redefinition similar to the radiative projection. Namely, we will dress the Higgs field with a Coulomb tail, as follows:

$$\varphi \mapsto \varphi' = e^{-ie\Delta^{-1}\partial^i A_i}\varphi\,, \tag{17}$$

and we will always work in terms of φ', which as we will now argue has the right properties under gauge transformations. Indeed, the thus dressed Higgs field, Eq. (17), is invariant under all gauge transformations *except the global ones*, precisely because the global gauge transformations do not change A_i. For consider a *redundant*, i.e. local but not global, gauge transformation. Then we see that the transformation of the original Higgs field φ, in Eq. (16), is precisely cancelled by the transformation coming from the Coulomb dressing by the gauge field. By contrast, if the transformation is global (rigid), then the dressing does not transform, and the dressed Higgs field φ', transforms just like φ. Thus φ' is invariant under all gauge transformations, except the global ones.

In effect, the Coulomb dressed field φ' in Eq. (17) defines equivalence classes on the space of fields: namely, the orbits of the Higgs field under *redundant* gauge transformations. Thus by working only with the Coulomb dressed field φ', we in effect *reduce* the space of fields.

By using the radiatively projected A_i^T and the redefined Higgs field φ', Struyve is able to reformulate the Abelian Higgs mechanism gauge-invariantly. Since the only symmetry that is left when using these fields is the global gauge group, the only possible notion of SSB is global gauge symmetry breaking. Struyve then works out that all the usual consequences of the classical Abelian Higgs mechanism can indeed be derived from *global gauge symmetry breaking*.

Interpretation of SSB in the gauge-invariant Higgs mechanism. In view of our discussion on global gauge symmetries as the physical asymptotic gauge group, we agree with Struyve that the above is the correct way to understand the Higgs mechanism. For it is a *gauge-invariant account* of the Higgs mechanism, in the correct sense: namely, it is invariant under *redundant* gauge symmetries, viz. the ones that are local but not global. And it is not, and it should not, be invariant under those symmetries that have DES. For, as we have discussed, those symmetries are physical in the sense that they correspond to different configurations of the physical system (here, the Higgs field). Thus spontaneously breaking the global symmetry means choosing a specific state of the Higgs field, out of a number of possible states. Namely, *the symmetry breaking in the Higgs mechanism is the specification of the physical state of the Higgs field*, and this state is invariant under the redundant gauge symmetries. One way to think of this state is as a specification of an asymptotic condition for the field, which determines the global phase value that the Higgs field must approach asymptotically (Of course, we have not yet said anything about the other aspect of the Higgs mechanism, beyond symmetry breaking: namely, the generation of mass. We will return to this in Section 5.)

4.2 The correct gauge-invariant DFM

In this light, we can finally resolve the tension between Struyve's account and the dressing field method as presented in (Masson and Wallet 2010; Fournel et al. 2014; François 2019; Berghofer et al. 2023; Attard et al. 2018; Zajac 2023; Berghofer and François 2024). Let us briefly review the main idea of the DFM: considering a principal G-bundle $\pi\colon P \to \Sigma$, and letting $H \subset G$ denote a closed subgroup, the fundamental object of the DFM is:

Definition 4.1. A smooth map $u\colon P \to H$ satisfying $u(ph) = h^{-1}u(p)$ for all $h \in H$ is called an H-*dressing field*.

We can think of dressing fields as trivializations in the direction of the subgroup H, as formalised in the following result (Fournel et al. 2014).

Proposition 4.2. A dressing field $u\colon P \to H$ exists if and only if there is an isomorphism of H-spaces $P \cong P/H \times H$, where the action of H on P/H is trivial.

Thus, a dressing field partially trivalizes the principal bundle. It can then be used to dress gauge and matter fields in order to make them invariant under H-valued gauge transformations.

Definition 4.3. Let u be an H-dressing field and $A \in \Omega^1(P, \mathfrak{g})$ a connection 1-form with curvature F. Let $\rho\colon G \to V$ be a representation giving an associated bundle $E = P \times_\rho V$ and $\varphi\colon P \to V$

a G-equivariant map (equivalently a section of E). Then we define the *dressed fields*

$$\begin{aligned} A^u &= u^{-1}Au + u^{-1}du, \\ \varphi^u &= \rho(u^{-1})\varphi, \\ F^u &= u^{-1}Fu = dA^u + \frac{1}{2}[A^u, A^u], \\ D^u\varphi^u &= \rho(u^{-1})D\varphi = d\varphi^u + \rho_*(A^u)\varphi^u. \end{aligned}$$

We note that A^u is not itself a connection 1-form, which should not be expected anyway since u is not a gauge transformation.

Critique of the standard use of DFM. In (Berghofer et al. 2023) the DFM is used to distinguish between *artificial* and *substantial* gauge symmetries, i.e. those for which a local dressing field can be found and those for which only a non-local dressing field can be obtained. This way, the well-known trade-off between locality and gauge-invariance, which is famously showcased by the Aharonov-Bohm effect (Healey 2007), is formalised. When applied to the Abelian Higgs model, the DFM then supposedly shows that the $U(1)$ gauge symmetry is artifical, since one can define a dressing field u through the polar decomposition $\varphi = u\sqrt{\varphi^*\varphi}$ of the Higgs field (Berghofer et al. 2023), in analogy to unitary gauge. Something similar is done to remove the $SU(2)$ symmetry of the electroweak $U(1) \times SU(2)$ theory. It is then concluded that, since there is no gauge symmetry left in the theory, there can be no SSB in the Higgs mechanism. This leads to the idea that the DFM provides "an alternative interpretation of the BEHGHK mechanism that is more in line with the conclusions of the community of philosophers of physics" (Attard et al. 2018, p. 4). In particular "the DFM approach to the electroweak model is consistent with Elitzur's theorem stating that in lattice gauge theory a gauge symmetry cannot be spontaneously broken" (Berghofer et al. 2023, p. 66). François' article even contains a section titled "there is no SSB in the electroweak model and we long suspected it" (François 2019).

However, as noted also in (Berghofer et al. 2023), there is something awkward about the application of the DFM to the Abelian Higgs mechanism or electroweak theory: the polar decomposition of the Higgs field is not defined when the Higgs field vanishes. Berghofer et al. suggest simply ignoring the vanishing field configurations, but we are now in a position to understand its deeper sense. The gauge group $\mathcal{G}^I$ does not act freely on the space of $U(1)$ gauge fields, because of the global gauge transformations. This means that one cannot take a smooth symplectic quotient[26] - one cannot properly parametrize the entire field space in terms of gauge-invariant variables, as is attempted in (Berghofer et al. 2023). Yet, when the Higgs field is added, the issue is apparently resolved, since the gauge group does act freely on the space of Higgs fields. After all, global gauge transformations do change the Higgs field by a phase factor. Naively, then, we can use the phase of the Higgs field as a reference frame. But unfortunately, the problem remains, since the configurations $(\varphi = 0, A_i)$ of vanishing Higgs field and arbitrary gauge field are still invariant under the action of the global gauge group. Thus, even on the enlarged phase space of gauge and Higgs fields one cannot perform smooth symplectic reduction. It should not come as a surprise that, if one ignores vanishing Higgs field configurations, such a symplectic quotient seems possible. After all, it is precisely these configurations that are problematic. But there is no legitimate reason to remove vanishing Higgs field configurations. Indeed, in doing so one removes the "original" symmetric state of the Higgs field "before" symmetry breaking, so it is not surprising that one does not find any SSB in this approach.

What, then, is left of the DFM? Should we dismiss it entirely? Not at all, but we must apply it at a different level, in the spirit of (Gomes and Riello 2018; Gomes, Hopfmüller, and Riello 2019).

[26] Instead one obtains a stratified space with strata labelled by the electric flux (Riello and Schiavina 2024).

We should not attempt to use it to remove, for instance, the $SU(2)$ part of the structure group of the electroweak theory. Instead, we should implement it to remove the entire *unphysical* subgroup $\mathcal{G}_0^\infty$ (which equals $\mathcal{G}^\infty$ for $G = U(1)$). In other words, we should seek a dressing field *at the level of infinite-dimensional field space*, i.e. an appropriately transforming $\mathcal{G}_0^\infty$-valued map on field space. But this is precisely provided by the object $\exp(-i\Delta^{-1}\partial^i A_i)$, viewed as a map on field space! Indeed, it is not hard to see that for any gauge transformation $e^{i\lambda} \in \mathcal{G}^\infty$ we have

$$e^{-i\Delta^{-1}\partial^i A_i} \to e^{-i\Delta^{-1}\partial^i(A_i+\partial_i\lambda)} = e^{-i\lambda}e^{-ie\Delta^{-1}\partial^i A_i},$$

as required in the definition of a dressing field. In addition, if we indeed take the dressing field $u = \exp(-i\Delta^{-1}\partial^i A_i)$ on field phase space, the "pure gauge term" $u^{-1}\mathbb{d}u$ reproduces the radiative projection. Thus, we see that at the higer level of infinite-dimensional field space, the DFM and Struyve's account actually harmonize perfectly. The Coulomb gauge is reproduced by the dressing field $\exp(-i\Delta^{-1}\partial^i A_i)$, and the two approaches conspire to reveal the true sense of a "gauge-invariant account" of the Higgs mechanism as one in which the unphysical gauge group $\mathcal{G}^\infty$ is removed, so that the focus lies entirely on the breaking of the physical global gauge group $\mathcal{G}^I/\mathcal{G}_0^\infty \cong U(1)$.

5 Abelian Higgs Mechanism in QED

Pleasing as the above solution of our first research question may be, the relation to quantum theory remains entirely unilluminated. The goal of this Section is to show that the classical ideas presented so far extend to QFT. Indeed, Morchio and Strocchi have shown that mass generation in the Abelian Higgs mechanism can be understood through SSB of global $U(1)$ gauge symmetry *in the algebraic sense*, answering (though probably unknowingly) the original call to do so in (Earman 2003a) (which was repeated and refined in (Stöltzner 2012)).

We begin by recalling the definition of SSB in algebraic quantum theory. We then present the main theorem of (Morchio and Strocchi 2007), which uses this algebraic language to show that there can only be massless photons in scalar QED if the global $U(1)$ gauge symmetry is *unbroken*, in which case the electric charge is *superselected*. If the $U(1)$ symmetry is *broken*, we do not have superselection but instead we have *charge screening*, and the photon must be massive. The result (Theorem 5.5) is fairly complicated and technical, so we spend some time preparing its exposition and explaining its various parts. We note that all relevant proofs are neatly collected in Chapters 5 and 6 of (Borsboom 2024).

5.1 SSB in algebraic quantum theory

Let us recall one of the two equivalent definitions of SSB in the algebraic language of C^*-algebras (Landsman 2017, p. 345).

Definition 5.1. Let $\mathcal{A}$ be a C^*-algebra and $\beta\colon \mathcal{A} \to \mathcal{A}$ a $*$-automorphism. Then a state $\omega\colon \mathcal{A} \to \mathbb{C}$ is said to spontaneously break the symmetry β if the GNS presentations $\pi_\omega\colon \mathcal{A} \to B(H_\omega)$ and $\pi_{\beta^*\omega}\colon \mathcal{A} \to B(H_{\beta^*\omega})$ are not unitarily equivalent, i.e. if there is no unitary operator $U\colon H_\omega \to H_{\beta^*\omega}$ such that

$$\pi_{\beta^*\omega}(a) = U\pi_\omega(a)U^*, \qquad a \in \mathcal{A}.$$

Here the state $\beta^*\omega$ is defined by $\beta^*\omega(a) = \omega(\beta^{-1}(a))$.

Now, this definition of SSB is simple and general, but not so easy to check (Strocchi 2008, p. 120). We should like to make use of an *order parameter*, i.e. an element of the algebra whose ground state expectation value is not invariant under the symmetry in question precisely when that symmetry is broken. This is how we will study the Higgs mechanism in Section 5.2, and to that end we have the following result (cf. Proposition II.8.2 in (Strocchi 2008)). For the proof the reader can use Strocchi's original work, but a more pedagogical treatment is provided under Proposition 5.26 in (Borsboom 2024).

Proposition 5.2. (Symmetry breaking). Let $\mathcal{A}$ be a C*-algebra with vacuum state $\omega_0 \in S(\mathcal{A})$ in whose GNS representation $(\pi_{\omega_0}, H_{\omega_0}, \Omega_{\omega_0})$ spacetime translations β_x are implementable through a strongly continuous family of unitary operators $U(x)$, such that Ω_0 is the unique translationally invariant state in H_{ω_0}. Then an *internal* symmetry (one that commutes with spacetime translations) $\beta \in \mathrm{Aut}(\mathcal{A})$ is unbroken in ω_0, in the sense of Definition 5.1, if and only if all ground state correlation functions are invariant under β, i.e. for all $a \in \mathcal{A}$,

$$\omega_0(\beta(a)) = \omega_0(a)\,. \tag{18}$$

In other words: SSB of an internal symmetry by the ground state can be detected by means of an order parameter, i.e. an element $a \in \mathcal{A}$ whose ground state expectation value is not invariant under the symmetry transformation β. The condition in Eq. (18) can also be made infinitesimal:

$$\omega_0(\delta a) = 0 \qquad \text{for all } a \in \mathrm{Dom}(\delta)\,. \tag{19}$$

Here δ is the derivation on $\mathcal{A}$ induced by the 1-parameter group of automorphisms β_λ, i.e.

$$\delta(a) = \left.\frac{d\beta_\lambda(a)}{d\lambda}\right|_{\lambda=0},$$

where $\beta_\lambda = \beta(\lambda)$ and $\mathrm{Dom}(\delta)$ consists of all elements of $\mathcal{A}$ for which this limit exists (cf. Proposition 9.19 in (Landsman 2017)). We note that the infinitesimal version of the symmetry implementability condition, i.e. Eq. (19), is also used in the algebraic version of Goldstone's theorem (Kastler, Robinson, and Swieca 1966; Buchholz et al. 1992; Stöltzner 2012) (cf. Theorem III.3.27 in (Haag 1996)).

5.2 The Higgs mechanism as global gauge symmetry breaking

Now that we have a concrete test of SSB of an internal symmetry by the ground state, we can apply this to the Abelian Higgs mechanism. Indeed, gauge symmetries are by definition internal, so we can detect global $U(1)$ gauge symmetry breaking in the vacuum state by means of an order parameter. When doing so, we do not have to worry about Elitzur's theorem, since that theorem applies only to *local* gauge symmetries and not to global ones (Strocchi 2013).

The non-perturbative results we discuss in this section are based mostly on (Morchio and Strocchi 2007; De Palma and Strocchi 2013; Strocchi 2013; Strocchi 2019) and are formulated in the language of the Wightman axioms of QFT, in which fields $\varphi_i(x)$ are modelled as operator-valued tempered distributions.[27] Such tempered distributions "eat" test functions $f \in \mathcal{S}(\mathbb{R}^4)$ in Schwarz space to produce operators on a common dense domain $D \subset H$, containing the vacuum state Ψ_0, of a Hilbert space H. To a region $\mathcal{O} \subset M$ of Minkowski spacetime, we then associate the algebra generated by all polynomials of smeared fields $\varphi_i(f)$ with $\mathrm{supp}(f) \subset \mathcal{O}$. This will not necessarily yield a C*-algebra, and indeed the relation between the Haag-Kastler

[27] For details on these axioms, we refer the reader to (Wightman and Garding 1965; Streater and Wightman 2016).

and Wightman axioms is very complicated (Haag 1996). We assume, however, that the detection of SSB via an order parameter as in Proposition 5.2 is still possible for the Coulomb field algebra of QED considered in this section.

Now, the main result, Theorem 5.5, is centered around the usual Gauss law $\rho = j^0 = \nabla \cdot \mathbf{E} = \partial_i F^{i0}$, viewed as a consequence of Noether's second theorem. We will therefore also refer to it as the *Noether relation*, following Strocchi's terminology. Through the Noether relation the electric charge is expressed as a *Gauss charge*, i.e. in terms of the antisymmetric field strength tensor $F_{\mu\nu}$. In Theorem 5.5, it will be this *very relation which is at stake*, in the sense that it holds in QED only if the global $U(1)$ symmetry is unbroken. Its failure to hold when the $U(1)$ symmetry is broken by the vacuum state can be understood to be a consequence of current charge screening: if the electromagnetic force is carried by massive photons, then it is short-ranged, so electric charges cannot be detected infinitely far away by means of Gauss's theorem.

Thus, we first consider the quantum theoretical version of the Noether relation between the electric charge and the electromagnetic field strength, which is in fact not so straightforward to carry over from classical field theory to QFT. In particular, we explain how there is a conflict between the Gauss law and locality, which forces us to allow for non-local fields in the Coulomb gauge, thereby enabling us to recover the Noether relation. Continuing in the Coulomb gauge, we then present the main theorem and explain the meaning and relevance of superselection and current charge screening.

5.2.1 The Noether relation and locality

We recall that the Gauss law arises through Noether's second theorem as the constraint for Yang-Mills theory, such that the Gauss charge, i.e. the electric charge $Q^e = \int d^3x j_0(\mathbf{x}, 0)$ for electromagnetism (here $j_0(\mathbf{x}, 0)$ denotes the time-component at time $t = 0$ of the electric current $j_\mu = \partial^\nu F_{\nu\mu}$), can be expressed in terms of the antisymmetric field strength tensor $F_{\mu\nu}$. By Noether's first theorem this electric charge generates global $U(1)$ gauge transformations:

$$\{Q^e, \varphi\} = \delta^e \varphi = ie\varphi\,, \qquad \{Q^e, A_\mu\} = \delta^e A_\mu = 0\,.$$

In other words: by Noether's theorems, we know that the integral $\int d^3x\, j_0(\mathbf{x}, 0) = \int d^3x\, \partial^i F_{i0}(\mathbf{x}, 0)$ generates global $U(1)$ transformations. If, however, we attempt to extend this result to QFT by replacing Poisson brackets with commutators, we run into problems, as we will now see.

Let us attempt to extend the generation of global gauge transformations by the electric charge to quantum fields $\varphi(x), A_\mu(x)$ assumed to satisfy the Wightman axioms. We first need to regularise the charge by smearing it out in space and time:

$$Q_R = \int d^3x dt f_R(\mathbf{x})\alpha(t) j_0(\mathbf{x}, t) \equiv j_0\,(f_R \alpha)\,, \tag{20}$$

where $f_R(\mathbf{x}) = f(\mathbf{x}/R) \in \mathcal{D}(\mathbb{R}^3)$ with $f(x) = 1$ for $|\mathbf{x}| < 1$ and $f(\mathbf{x}) = 0$ for $|\,\mathbf{x}| < 1 + \delta$ with $\delta > 0$, and where $\mathrm{supp}(\alpha) \subset [-\gamma, \gamma]$ with $\int \alpha(t)dt = 1$ (Strocchi 2013, p. 147). We have to perform this regularisation because the limit Q_R for $R \to \infty$ does not actually exist. By regularising we can avoid this problem, since we only need the limits of the commutator of Q_R with the matter and gauge fields. These limits do exist and are actually independent of the choice of smearing function α (Strocchi 2013). It it then not too hard to show that, for a theory with a scalar field $\varphi(\mathbf{x}, t)$ and current $j_\mu(\mathbf{x}, t)$ that are local with respect to each other (i.e. that commute at spacelike separated points), we have (Strocchi 2013):

$$i \lim_{R\to\infty} [Q_R, \varphi(g, t)] = i \lim_{R\to\infty} [j_0\,(f_R, t)\,, \varphi(g, t)] = \delta^e \varphi(g, t), \qquad g \in \mathcal{D}(\mathbb{R}^3)\,,$$

where δ^e denotes the derivation corresponding to the global $U(1)$ gauge symmetry. This is indeed the desired quantum version of the classical Noether relation between the charge (the integral over the current density) and the global gauge symmetry transformation. This result, however, is spoiled in the presence of the Gauss law (Proposition 2.1 in (Strocchi 2013)).

Proposition 5.3. If a field φ is local with respect to the electric field $\mathbf{E}$ (i.e. if these fields commute at spacelike separated points), then the Gauss law gives $\lim_{R\to\infty}[j_0(f_R\alpha), \varphi] = 0$, implying that Q_R cannot generate the global $U(1)$ transformations.

Conversely, this proposition tells us that if a field φ is charged in the sense that $\delta^e\varphi \neq 0$, then it cannot be local w.r.t. the electric field. Thus, if we denote by $\mathcal{F}$ the local field algebra (generated by polynomials of the fields), then charged states cannot be in the closure of $\mathcal{F}\Psi_0$ (where Ψ_0 is the vacuum state), but must instead be generated from the vacuum by non-local charged fields (Strocchi 2013). In algebraic language, if $\mathcal{O}$ is a bounded region of spacetime with causal complement $\mathcal{O}'$, then for sufficiently large R we have $Q_R \in \mathcal{A}(\mathcal{O}')$. For any algebraic state ω which is localised in $\mathcal{O}$ we then get $\omega_0(Q_R) = \langle\Psi_0, Q_R\Psi_0\rangle = 0$, showing that ω is uncharged. In other words: localised states must be uncharged because of the Gauss law.

This conflict between locality and the Gauss law[28] forces us to accept that the field algebra becomes non-local, if we want to maintain the Gauss law as an operator equation (Strocchi 2013). This is what is done in the Coulomb gauge, which can be obtained from the local field algebra by the same field transformation as in Section 3.3. That is: if A_μ and φ are the local but gauge-dependent fields in the Feynman quantisation of QED, then the transformation to the non-local Coulomb gauge is the radiative projection introduced in Eq. (14) and Eq. (17):

$$\begin{aligned} \varphi_C &= e^{-ie\Delta^{-1}\partial^i A_i}\varphi, \\ A^C_\mu &= A_\mu - \partial_\mu\Delta^{-1}\partial^i A_i\,. \end{aligned}$$

We write A^C_μ instead of A^T_μ as before in order to stress that we are now considering a *quantum* field that generates the Coulomb field algebra. The field φ_C is still charged since it transforms under the global charge group $U(1)$, but it is gauge-invariant in the redundant sense. The cost is that the fields are non-local because of the non-local nature of the operator $\Delta^{-1}\partial^i$, which involves an integral over all of space.

5.2.2 Global gauge symmetry breaking in Coulomb gauge

We will now explain in what sense the Noether relation between the electric charge and the electromagnetic current density, which, as we saw in Proposition 5.3, is spoiled for local gauge quantisation, can be restored in the non-local Coulomb gauge whenever the global $U(1)$ gauge symmetry is *unbroken*. We then present and subsequently dissect the main result of (Morchio and Strocchi 2007).

Let $\mathcal{F}_C$ denote the field algebra generated by the Coulomb fields φ_C and A^C_i. We assume its vacuum correlations to be well-defined (see footnote 49 in chapter 7 of (Strocchi 2013)). We call the generator of the global $U(1)$ symmetry the electric charge Q^e, defined by

$$[Q^e, \varphi_C] = e\varphi_C, \quad Q^e\Psi_0 = 0\,.$$

The question now is whether this electric charge can indeed be written like Q_R in Eq. (20), i.e. as an integral over the current density j_0. The following result (Proposition 4.1 in (Morchio and Strocchi 2007), Proposition 5.3 in (Strocchi 2013)) shows that the correspondence between the generator of the global $U(1)$ symmetry and the current density still fails.

[28]This conflict can be seen as yet another incarnation of the familiar trade-off between locality and gauge-invariance.

Proposition 5.4. In the Coulomb gauge of QED, with field algebra $\mathcal{F}_C$ and vacuum vector Ψ_0, we have that for all $\Psi, \psi \in \mathcal{F}_C\Psi_0$ the limit

$$\lim_{R\to\infty} (\Psi, [j_0^C(f_R\alpha), \varphi_C]\psi)$$

exists. However, the limit is generally dependent on α, meaning that the time-independent global $U(1)$ symmetry cannot be generated by such integrals of the charge density.

This failure of the integral of the charge density to generate the time-independent global $U(1)$ transformations can be remedied by time-averaging the integral of j_0 with an improved smearing: we define $Q_{\delta R} = j_0(f_R\alpha_{\delta R})$, where $\alpha_{\delta R} = \alpha(x_0/\delta R)/\delta R$, with $\alpha \in \mathcal{D}(\mathbb{R}), 0 < \delta < 1$ and $\text{supp}(\alpha) \subset [-\epsilon, \epsilon]$ with $\epsilon \ll 1$ (Morchio and Strocchi 2007). However, this only works in the case of *unbroken* $U(1)$ symmetry, and this insight is at the heart of Morchio and Strocchi's main result Theorem 5.5. (Theorem 7.6.2 in (Strocchi 2013), Theorem 2.8.3 in (Strocchi 2019)).

The theorem distinguishes between two situations: situation **(A)**, where there are *massless photons*, and the global $U(1)$ gauge symmetry is unbroken; and situation **(B)**, where the $U(1)$ symmetry is broken by the presence of an order parameter, and all bosons must be massive. As follows:

Theorem 5.5. Let $\mathcal{F}_C$ denote the Coulomb field algebra with vacuum state $\Psi_0 \in H$, generated by the gauge field A_i^C and the complex scalar Higgs field φ_C. Let β_λ denote the 1-parameter family of $*$-automorphisms of $\mathcal{F}_C$ corresponding to the continuous global $U(1)$ symmetry with generator Q^e (the electric charge), and let j_0 be the associated conserved Noether charge current. Then the following results hold:

(A) If the spectral measure $d\rho(m^2)$ of the electromagnetic field $F_{\mu\nu}$ contains a $\delta(m^2)$ contribution, i.e. if there are massless vector bosons, then:

(i) the global $U(1)$ gauge transformations are generated by the improved smeared current density, i.e. for any $F \in \mathcal{F}_C$

$$[Q^e, F] = \delta^e F = \left.\frac{d\beta_\lambda(F)}{d\lambda}\right|_{\lambda=0} = i \lim_{\delta\to 0} \lim_{R\to\infty} [j_0(f_R\alpha_{\delta R}), F]\,; \tag{21}$$

(ii) we have $\lim_{R\to\infty} j_0(f_R\alpha_{\delta R})\Psi_0 = 0$, so that $\langle\delta^e F\rangle_0 = 0$ for any $F \in \mathcal{F}_C$, meaning that the global $U(1)$ symmetry is unbroken (by Proposition 5.2);

(iii) the electric charge Q^e can be expressed in terms of j_0 not only in the commutators with fields $F \in \mathcal{F}_C$, but also in the matrix elements of the Coulomb charged states $\psi, \Psi \in \mathcal{F}_C\Psi_0$:

$$\langle\psi, Q^e\Psi\rangle = \lim_{\delta\to 0} \lim_{R\to\infty} \langle\psi, j_0(f_R\alpha_{\delta R})\Psi\rangle\,,$$

implying that Q^e is a superselected charge (see below for an explanation).

(B) If the global $U(1)$ symmetry is broken by some $F_{SSB} \in \mathcal{F}_C$ such that $\langle\delta^e F_{SSB}\rangle_0 \neq 0$, then

(i) the spectral measure $d\rho(m^2)$ of $F_{\mu\nu}$ cannot contain a $\delta(m^2)$ contribution, i.e. there are no massless vector bosons;

(ii) the global $U(1)$ gauge transformations are not generated by the current density, in fact

$$\lim_{\delta\to 0} \lim_{R\to\infty} [j_0(f_R\alpha_{\delta R}), F] = 0 \tag{22}$$

for any $F \in \mathcal{F}_C$, and $j_0(f_R \alpha_{\delta R})$ annihilates the vacuum, so that for any $\Psi \in \mathcal{F}_C \Psi_0$:

$$\lim_{\delta \to 0} \lim_{R \to \infty} j_0(f_R \alpha_{\delta R}) \Psi = 0 \,, \tag{23}$$

i.e. we have *current charge screening*;

(iii) the two-point function $\langle j_\mu(x) F_{SSB} \rangle_0$ does not vanish, and its Fourier spectrum coincides with the energy-momentum spectrum of $F_{\mu\nu}$, so that the absence of massless vector bosons coincides with the absence of massless Goldstone modes.

Let us unpack this result. As we mentioned above, it distinguishes between two situations: **(A)**, in which there are *massless photons*, in the sense of zero-mass contribution $\delta\rho(m^2)$ to the Källén-Lehmann spectral representation of the electromagnetic field $F_{\mu\nu}$, and in which the global $U(1)$ gauge symmetry is unbroken; and situation **(B)**, in which the $U(1)$ symmetry is broken in the sense of Proposition 5.2 by some element $F_{SSB} \in \mathcal{F}_C$. In the latter situation it is derived that the spectral measure of the electromagnetic field *cannot* have a zero-mass contribution, i.e. all bosons must be massive. In fact, there are not even Goldstone bosons, since their spectrum coincides with that of $F_{\mu\nu}$.

In situation **(A)**, the Noether relation between the electric charge Q^e and the improved smeared current density holds in commutators. That is: the action of the derivation δ^e, corresponding to the infinitesimal global $U(1)$ gauge transformation generated by Q^e, on an arbitrary field $F \in \mathcal{F}_C$, can indeed be written as an appropriate limit of a commutator $[j_0(f_R \alpha_{\delta R}), F]$. Furthermore, in this situation the vacuum is annihilated by the smeared current density:

$$\lim_{R \to \infty} j_0(f_R \alpha_{\delta R}) \Psi_0 = 0 \,,$$

from which it follows by the Noether relation (Eq. (21)) that the vacuum expectation values $\langle \delta^e F \rangle_0$ vanish[29] for any $F \in \mathcal{F}_C$, which is equivalent to the global $U(1)$ gauge symmetry being unbroken by Proposition 5.2. Thirdly, the Noether relation between the electric charge and the improved smeared current holds not only in commutators but also in matrix elements between states generated from the vacuum Ψ_0 by the action of the field algebra $\mathcal{F}_C$. From this, it can be deduced that for any observable $A \in \mathcal{F}_C$, which must be gauge-invariant, the commutator $[Q^e, A]$ vanishes on the states. This phenomenon is known as superselection. It provides a *superselection rule* for states carrying different charges: these cannot form a coherent superposition.[30] There is, in fact, a much more general and profound relation between global gauge symmetries and superselection, known as *DHR superselection theory* (Doplicher and Roberts 1990; Haag 1996). Indeed, representations of field algebras with gauge symmetry form different *superselection sectors* labelled by the gauge charges. We see the same phenomenon here, but only when the global $U(1)$ gauge symmetry remains intact. An important and interesting question for future work is precisely how the superselection seen here relates to the *flux superselection* sectors studied in (Gomes and Riello 2021; Riello and Schiavina 2024). Since flux superselection sectors relate to the presence of global gauge symmetries in classical field theories, it seems plausible that, through quantisation, they give rise to DHR superselection sectors. This relation is studied from the perturbative point of view in (Rejzner and Schiavina 2021).

In situation **(B)**, the global $U(1)$ symmetry is assumed to be broken by an order parameter, labelled F_{SSB}, in the sense of Proposition 5.2. In that case, everything turns out quite the opposite way. There cannot be massless contributions to the electromagnetic spectrum (as these

[29]After all, in the vacuum expectation values the object $\delta^e F$ is sandwiched between two copies of the vacuum state Ψ_0. Since $[\delta^e F]$ can be written as a limit of the commutator $[j_0(f_R \alpha_{\delta R}), F]$, the object $\lim_{R\to\infty} j_0(f_R \alpha_{\delta R})$ will always hit Ψ_0 on the left or right, giving zero.

[30]For a philosophical introduction to this topic, we refer the reader to (Earman 2008).

would restore the $U(1)$ symmetry), so all bosons must be massive. This is of course the desired and usual consequence of the Higgs mechanism, and it is now formulated entirely in terms of the quantum-theoretical definition of SSB. In addition, instead of superselection we have current charge screening: the Noether relation fails and the improved smeared current $j_0(f_R \alpha_{\delta R})$ annihilates any Coulomb state $\Psi \in \mathcal{F}_C \Psi_0$ in the appropriate limit. As we mentioned before, we can think of this as being due to the impossibility of detecting charges from infinitely far away, because of the short range of the screened electromagnetic field. Lastly, the usual avoidance of massless Goldstone modes is also guaranteed.

Now, the above result is powerful, but of course we must ask ourselves to what extent it can be generalized to the non-Abelian case. Whereas there does not seem to be any direct generalisation, De Palma and Strocchi have in fact proven the avoidance of Goldstone's theorem for the general case of global gauge symmetry breaking in Yang-Mills theory (De Palma and Strocchi 2013), as follows:

Theorem 5.6. In the BRST gauge of Yang-Mills theory with structure group G, if the global gauge group G is broken by the vacuum expectation value of an element F of the field algebra $\mathcal{F}$, i.e. $\langle \delta F \rangle_0 \neq 0$, then the Fourier transform of the two-point function $\langle J^a_\mu(x) F \rangle_0$, where J^a_μ are the conserved Noether currents defined by

$$J^a_\mu = \partial^\nu F_{\nu\mu} + \partial_\mu B^a + f^a_{bc} A^b_\mu B^c - i f^a_{bc} \bar{c}^b (D_\mu c)^c,$$

contains a $\delta(k^2)$, i.e. there are massless Goldstone modes. However, these modes do not belong to the physical spectrum.

Thus, also in the non-Abelian case, global gauge symmetry breaking in QFT agrees with one of the main ideas behind the Higgs mechanism: the "eating" of Goldstone modes. Yet, as far as the authors are aware, the actual existence of massive gauge bosons in the case of broken global non-Abelian gauge symmetry has not been proved anywhere.

6 Conclusion

We have discussed how the global gauge group is singled out as the subgroup of transformations that are boundary-preserving but not generated by the Gauss constraint, and therefore are not *redundant* but have direct physical significance, because they change the physical state at infinity.[31] As shown in the work of Struyve, here discussed in detail, the Abelian Higgs mechanism can be reformulated as the breaking of only the global gauge symmetry, thus assuaging the worries that various philosophers have expressed about the standard narrative of the Higgs mechanism. We have also demonstrated how this viewpoint harmonizes with the dressing field method advocated by Berghofer et al., but applied, along the lines of the work of Gomes and Riello, to the *redundant* gauge group $\mathcal{G}_0^\infty$ of small, local gauge transformations rather than the structure group $U(1)$ or $SU(2)$, as has been done in the earlier literature. We have also seen how the Abelian Higgs mechanism has been proved by Morchio and Strocchi to be an instance of SSB of global $U(1)$ symmetry in QFT.

These ideas are supported by the physics of superconductors, which also exhibit global and not local gauge symmetry breaking (Wezel and Brink 2008). Thus, our results suggest that the analogy between the Higgs mechanism and superconducitvity is not merely formal, as argued in (Fraser and Koberinski 2016), but physical, since global gauge symmetries are physical.

[31] There is here an analogy with diffeomorphisms in general relativity, where diffeomorphisms that generate symmetries at infinity can have a real physical effect. For a detailed discussion, see (De Haro 2017b); see also (De Haro, Teh, and Butterfield 2017; Belot 2018).

Still, we lack any causal, dynamical account of the Higgs mechanism.[32] However, this is the case for all SSB in quantum systems, although there have been various attempts at an account of dynamical symmetry breaking (Landsman and Reuvers 2013; Landsman 2017; Ven 2022; Wezel 2022). Thus, we have reduced the problem of gauge symmetry breaking to the general problem of quantum SSB, at least in the Abelian case. The non-Abelian case remains to be better understood, though significant steps have been made in e.g. (Lusanna and Valtancoli 1997a; Lusanna and Valtancoli 1997b; Lusanna and Valtancoli 1998; Gomes, Hopfmüller, and Riello 2019; Gomes and Riello 2021; Riello and Schiavina 2024). Another possibility for future research is to deepen the link of our work to edge modes and (quantum) reference frames, along the lines of, e.g. (Donnelly and Freidel 2016; Carrozza and Höhn 2022; Fewster et al. 2024).

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[32]Such a dynamical account should detail precisely how global gauge symmetries are broken in some kind of early universe phase transition or crossover.

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