The Relationship Between Lagrangian and Hamiltonian Mechanics: The Irregular Case

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Draft of April 28, 2025

Abstract

Lagrangian and Hamiltonian mechanics are widely held to be two distinct but equivalent ways of formulating classical theories. Barrett (2019) makes this intuition precise by showing that under a certain characterization of their structure, the two theories are categorically equivalent. However, Barrett only shows equivalence between "hyperregular" models of Lagrangian and Hamiltonian mechanics. While hyperregularity characterizes a large class of theories, it does not characterize the class of gauge theories. In this paper, I consider whether one can extend Barrett's results to show that Lagrangian and Hamiltonian formulations of gauge theories are equivalent. I argue that there is a precise sense in which one can, and I illustrate that exploring this question highlights several interesting questions about the way that one can construct models of Hamiltonian mechanics from models of Lagrangian mechanics and vice versa, about the role that constraints play, and the definition and interpretation of gauge transformations.

1 Introduction

Lagrangian and Hamiltonian mechanics are widely held to be two distinct but equivalent ways of formulating classical theories. While there have been challenges to this view in the philosophical literature by North (2009) and Curiel (2014), Barrett (2019) makes the intuition that these frameworks are equivalent precise by showing that under a certain natural characterization of the structure of Lagrangian and Hamiltonian mechanics, they are theoretically equivalent under the standard of theoretical equivalence given by categorical equivalence.

However, Barrett's equivalence result is restricted in an important way: he only shows equivalence between "hyperregular" models of Lagrangian and Hamiltonian mechanics. While hyperregularity characterizes a large class of theories, it does not characterize the class of *gauge theories*: theories that have local symmetries arising from Noether's Second Theorem. Instead, gauge theories fall under the class of "irregular" models. The question of whether Lagrangian and Hamiltonian gauge theories are (categorically) equivalent has not been discussed directly in the philosophical literature, despite the fact that it bears on other debates that are prominent in the literature. For one, there has been a recent debate about the correct characterization of the gauge transformations in the Hamiltonian formalism. Several authors have criticized the standard view on the basis that the resulting theory is inequivalent to the Lagrangian formalism (Pitts (2014*a*,*b*), Gracia & Pons (1988)). Second, an important question in modern physics is how to quantize a classical gauge system. If the Lagrangian and Hamiltonian characterizations of classical gauge systems are not equivalent, then one would also not expect the resulting quantized theories to be equivalent, which would have significant implications for evaluating different methods of quantization. Despite both of these important connections, one fails to find a clear exposition in the literature of which formulations of Lagrangian and Hamiltonian gauge theories are equivalent and in what sense.

In this paper, I aim to fill this gap. I demonstrate that the relationship between Lagrangian and Hamiltonian mechanics is made significantly more complicated when the assumption of hyperregularity is dropped, and I argue that the literature has so far failed to establish more than a notion of *dynamical equivalence* in the non-hyperregular context. However, I show that one can extend Barrett's result to prove an equivalence result in the irregular case by constructing hyperregular models of Lagrangian and Hamiltonian gauge theories through a process known as 'symplectic reduction'. In doing so, I argue that the claims in the literature that the standard approach to gauge transformations renders Hamiltonian mechanics inequivalent to Lagrangain mechanics are false: there is a natural formulation of Lagrangian mechanics in the irregular context that is equivalent to the formulation of Hamiltonian mechanics under the standard definition of gauge transformations.

While ultimately the paper supports the equivalence between Lagrangian and Hamiltonian mechanics in the context of classical gauge theories, exploring the question of whether the two frameworks are equivalent will highlight several interesting questions about the way that one can construct models of Lagrangian mechanics from models of Hamiltonian mechanics and vice versa, about the role that constraints play in relating the kinematics and dynamics of a theory, as well as the interpretation of gauge transformations.[1]

In section 2, I spell out the equivalence result in Barrett (2019), paying particular attention to the parts of the result that require the assumption of hyperregularity. In Section 3, I discuss how the situation changes when one considers gauge theories and present the constrained Hamiltonian formulation of gauge theories. In Section 4, I consider some arguments in the literature regarding equivalence between Lagrangian and Hamiltonian gauge theories, and I discuss why they fall short of providing an account of *theoretical* equivalence. In Sections 5 and 6, I show that one can reformulate Lagrangian mechanics as a constraint theory in a way that is analogous to the constrained Hamiltonian formulation, drawing from the work of Gotay & Nester (1979), and I show that the models of the reformulated Lagrangian gauge theory are related to the models of the Hamiltonian constraint theory in a natural way. In Section 7, I prove an equivalence result that extends the result in Barrett (2019) to the context of gauge theories. Finally, in Section 8 I discuss the upshots of this equivalence result and some possible responses.

2 The Regular Case

The standard geometric way of characterizing Lagrangian and Hamiltonian theories is as follows. Lagrangian mechanics has state space given by the tangent bundle of configuration space, T_*Q, whose points consist of the pair $(q_i, \dot{q}_i)$ encoding the generalized positions and velocities of the particles. The dynamics are given by specifying a Lagrangian function $L(q_i, \dot{q}_i)$, with equations of motion given by the Euler-Lagrange equations: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$. The fiber derivative of L is called the *Legendre transformation* and it is the map $FL : T_*Q \to T^*Q$ from the tangent bundle to the cotangent bundle that is defined as taking the point $(q_i, \dot{q}_i)$ to $(q_i, \frac{\partial L}{\partial \dot{q}_i})$. We say that the model (T_*Q, L) is *regular* if FL is a local diffeomorphism. When FL is a global diffeomorphism, we say that the model (T_*Q, L) is *hyperregular.*

Hamiltonian mechanics has as its state space the cotangent bundle of configuration space, T^*Q, whose points consist of the pair (q_i, p_i) encoding the positions and canonical momenta of the particles. The dynamics is given by specifying a Hamiltonian function $H(q_i, p_i)$, with dynamical equations given by Hamilton's equations: $\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$. The fiber derivative of H is the map $FH : T^*Q \to T_*Q$ from the cotangent bundle to the tangent bundle that is defined as taking the point (q_i, p_i) to $(q_i, \frac{\partial H}{\partial p_i})$. When FH is a (global) diffeomorphism, we say that the model (T^*Q, H) is (hyper)regular.

The cotangent bundle naturally comes equipped with a symplectic (closed, non-degenerate) two-form ω. We can write the equations of motion in terms of this two-form: $\omega(X_H, \cdot) = dH$ where X_H is the vector field associated with the Hamiltonian, which is unique by the non-degeneracy of the symplectic two-form. The integral curves of X_H correspond to solutions. We can also use this symplectic structure to define a two-form on the tangent bundle, $\Omega = FL^*(\omega)$. Ω is symplectic when FL is a (local or global) diffeomorphism. We can then show that the Euler-Lagrange equations are equivalent to $\Omega(X_E, \cdot) = dE$ where X_E is the vector field associated with the energy function $E = FL(\dot{q}_i)\dot{q}^i - L$. The integral curves of X_E correspond to solutions.

In discussions on the relationship between Lagrangian and Hamiltonian mechanics, the division between (hyper)regular and 'irregular' models is not often emphasized. For example, while North

(2009) defends the view that Hamiltonian mechanics has less structure than Lagrangian mechanics, and Curiel (2014) argues that Lagrangian mechanics better represents the structure of classical systems than Hamiltonian mechanics, neither of these arguments make explicit reference to the fact that the relationship between these two theories depends on which kind of system one is considering. The exception is Barrett (2019), who demonstrates that if one's criterion of theoretical equivalence is categorical equivalence, then Lagrangian and Hamiltonian mechanics can be shown to be equivalent as long as one is only concerned with the *hyperregular* models.

The question of interest in this paper is whether one can still show that Lagrangian and Hamiltonian mechanics are equivalent when one drops the assumption of hyperregularity to allow for gauge theories. In attempting to answer this question, we will inherit the view that if we can show that the corresponding theories are *categorically* equivalent, then we have shown that they are theoretically equivalent. We do not have space to defend this view here.[2] But the core idea behind categorical equivalence is that for two theories to be equivalent, it should not only be that one can map models of one theory to models of the other; there should also be agreement about which models are equivalent to each other i.e. there should be a natural relationship between the *isomorphisms* of both theories. Given that our focus is ultimately on Lagrangian and Hamiltonian gauge theories, it is clear why a categorical approach would be appropriate: part of what we want to capture is that the theories agree about the *gauge symmetries*, and the gauge symmetries capture a notion of equivalence between states/solutions of a theory. However, it will be helpful to first spell out the way that Barrett (2019) proves categorical equivalence in the hyperregular case in order to highlight the changes that occur when one allows for gauge theories.

The element missing from our characterization of the theories above in order to define the relevant categories is the isomorphisms (the structure-preserving maps) between models of the theories. The structure-preserving maps of tangent space are given by point$_*$-transformations T_*f, defined as follows: given a diffeomorphism $f : M_1 \to M_2$, $T_*f : (q_i, \dot{q}_i) \to (f(q_i), f_*(\dot{q}_i))$. Similarly, the structure-preserving maps on cotangent space are given by point*-transformations: given a diffeomorphism $f : M_1 \to M_2$, $T^*f : (q_i, p_i) \to (f^{-1}(q_i), f^*(p_i))$. Therefore, we can define categories of hyperregular Lagrangian and Hamiltonian models, **Lag** and **Ham**, in the following way:

1. An object in the category **Lag** is a hyperregular model (T_*Q, L). An arrow $(T_*Q_1, L_1) \to (T_*Q_2, L_2)$ is a point$_*$-transformation $T_*f : T_*Q_1 \to T_*Q_2$ that preserves the Lagrangian in the sense that $L_2 \circ T_*f = L_1$.

2. An object in the category **Ham** is a hyperregular model (T^*Q, H). An arrow $(T^*Q_1, H_1) \to (T^*Q_2, H_2)$ is a point*-transformation $T^*f : T^*Q_1 \to T^*Q_2$ that preserves the Hamiltonian in the sense that $H_2 \circ T^*f = H_1$.

Next, define the functor F that takes a hyperregular model of Lagrangian mechanics to a hyperregular model of Hamiltonian mechanics via $F : (T_*Q, L) \to (T^*Q, E \circ FL^{-1})$ and that acts on arrows as $F : T_*f \to T^*(f^{-1})$.[3] Similarly, define the functor G that takes a hyperregular model of Hamiltonian mechanics to a hyperregular model of Lagrangian mechanics via $G : (T^*Q, H) \to (T_*Q, (\theta_a(X_H)^a - H) \circ FH^{-1})$ where θ_a is the canonical one-form such that $\omega_{ab} = -d_a\theta_b$, and that acts on arrows as $G : T^*f \to T_*(f^{-1})$. Then:

> **Theorem** (Barrett (2019)): F : **Lag** $\to$ **Ham** and G : **Ham** $\to$ **Lag** are equivalences that preserve solutions.[4]

The proof of this theorem relies on hyperregularity in several ways. First, notice that the functors F and G rely on the maps FL^{-1} and FH^{-1} to construct a Hamiltonian model in terms of a Lagrangian model and vice versa. These maps are only well-defined functions (globally) if FL and FH are (global) diffeomorphisms. Second, the proof works by showing that F and G are inverses in the sense that $GF(T_*Q, L) = (T_*Q, L)$, $FG(T^*Q, H) = (T^*Q, H)$, and similarly on arrows. This relies on the fact that $FL^{-1} = FH$ and $FH^{-1} = FL$, which is only true in the hyperregular context.

Given the importance of hyperregularity in showing that the categories of Lagrangian and Hamiltonian models are equivalent, one might conclude that the class of irregular Lagrangian and Hamiltonian theories cannot be categorically equivalent.[5] However, gauge theories do not fall under the class of hyperregular, or even regular, models, since the Legendre transformation defines a submanifold of T^*Q. It would be surprising, and significant, if the class of Lagrangian gauge theories and the class of Hamiltonian gauge theories were not equivalent. Therefore, our aim will be to consider whether there is a way to construct the models of Lagrangian and Hamiltonian gauge theories such that one can extend the above categorical equivalence result to this irregular context.

3 The Irregular Case

The standard approach to gauge theories begins in the Lagrangian context: gauge theories are theories that have local symmetries arising from Noether's Second Theorem. The existence of gauge symmetries in a theory implies that there is underdetermination in the evolution of the system; there are multiple possible solutions from some initial state. Given this underdetermination and the desire for unique evolution, one seeks to distinguish the variables whose evolution is underdetermined (the "gauge variables") from those whose evolution is not underdetermined (the "observables"). In the 1960s, work by Dirac (1964) and Bergmann (1961) demonstrated that there is a Hamiltonian formalism for describing gauge theories that provides a systematic method for isolating the gauge variables from the observables by connecting the presence of gauge variables to the presence of *constraints* on the Hamiltonian variables. This allows one to recover unique dynamics for a gauge system and provides the basis for the method of quantization called "canonical quantization", which relies on taking the observables to be the quantities that one uses to formulate a quantum theory. The standard modern treatment of the constrained Hamiltonian formalism can be found in Henneaux & Teitelboim (1994); we follow their treatment closely here.

We say that a Lagrangian is *irregular* when the Hessian $W_{ij} = \frac{\partial L}{\partial \dot{q}^i \dot{q}^j}$ is not invertible i.e. when it is singular. The gauge theories correspond to those irregular Lagrangian theories whose Legendre transformation defines a submanifold of T^*Q called the *primary constraint surface* Σ_p, defined by the satisfaction of a collection of (primary) constraints of the form $\phi_a(q_i, p_i) = 0$ where

$a = 1, ..., A$ is the number of primary constraints. It is therefore natural to formulate a Hamiltonian gauge theory on the primary constraint surface if we want to relate it to the Lagrangian theory.

If we start with a Hamiltonian theory on T^*Q, then one can specify the theory on the primary constraint surface in the following way. First, we can define an induced presymplectic (degenerate) two-form $\tilde{\omega} = i^*\omega$ where $i : \Sigma_p \to T^*Q$ is the inclusion map. The null vector fields of $\tilde{\omega}$ are the vector fields corresponding to the *primary first-class constraints*, which geometrically correspond to the primary constraints whose vector field is tangent to the constraint surface (while the *second-class* constraints are those constraints whose vector field is not tangent to the constraint surface).

Using this presymplectic two-form, the equations of motion on this submanifold can be written as $\tilde{\omega}(X_H, \cdot) = dH$ where H is the Hamiltonian on T^*Q restricted to the constraint surface (this is sometimes called the Hamilton-Dirac equation). Notice that since $\tilde{\omega}$ is degenerate, the solutions to this equation of motion are not unique; we can think of this fact as related to the gauge nature of the theory. In particular, $\tilde{\omega}(X_H, \cdot) = dH$ only defines X_H up to arbitrary combinations of the null vector fields.

This provides a well-defined theory on the primary constraint surface. However, there are inconsistencies that can arise with this theory: it may not be that the primary constraints hold at all points along a solution, which corresponds to the fact that the vector fields X_H that define the solutions to this equation may not be tangent to the constraint surface. In order for the solutions to be tangent to the constraint surface, it must be that $\tilde{\omega}(X_H, X_{\phi_a}) = dH(X_{\phi_a}) = 0$ for vector fields X_{ϕ_a} associated with the primary constraints. This may define a further collection of constraints called *secondary constraints*, and we can think of these additional constraints as leading to the specification of a further submanifold.

Continuing this process of requiring that the solutions to the equations of motion are tangent to the constraint surface terminates in a *final constraint surface*, $(\Sigma_f, \tilde{\omega}_f, H|_{\Sigma_f})$, defined by the satisfaction of the full collection of $M + S$ constraints, where the null vector fields of $\tilde{\omega}_f$ are those M vector fields associated with the M first-class constraints, and S is the number of second-class constraints. The integral curves of the null vector fields are called the *gauge orbits*. They are M-

dimensional surfaces on the constraint surface spanned by the null vector fields. In this way, on the final constraint surface, the gauge transformations are given by transformations along the integral curves of the null vector fields associated with the first-class constraints.

Following standard usage, let us define the 'Total Hamiltonian' as the equivalence class of Hamiltonians defined up to arbitrary combinations of *primary* (first-class) constraints i.e. the equivalence class of Hamiltonians on the primary constraint surface. Similarly, we define the 'Extended Hamiltonian' as the equivalence class of Hamiltonians defined up to arbitrary combinations of *primary and secondary* (first-class) constraints i.e. the equivalence class of Hamiltonians on the final constraint surface. Going forward, we will use the term 'Total Hamiltonian formalism' to refer to the formulation of irregular Hamiltonian mechanics on the primary constraint surface and 'Extended Hamiltonian formalism' to refer to the formulation of irregular Hamiltonian mechanics on the final constraint surface.

4 Inequivalence Argument

In the previous section, we showed that a Hamiltonian gauge theory is naturally formulated on the final constraint surface with the Extended Hamiltonian as the equivalence class of Hamiltonians. However, we also pointed out that if we start with a Lagrangian theory, the Legendre transformation defines the primary constraint surface, corresponding to the Total Hamiltonian being the right equivalence class of Hamiltonians (see Figure 1). This fact has led some authors to conclude that Extended Hamiltonian formalism is inequivalent to the Lagrangian formalism, and that this is reason to think that the Extended Hamiltonian formalism is mistaken.

For example, Gracia & Pons (1988) state:

> "No "extended hamiltonian" is needed, since it would introduce new solutions of the equations of motion that would break the equivalence between Lagrangian and Hamiltonian formalisms".

Similarly, Pitts (2014*b*) argues:

$$
\begin{tikzcd}
 & (T^*Q, H) \\
(T_*Q, L) \arrow[r, "FL"] & (\Sigma_p, H) \arrow[u] \\
 & (\Sigma_f, H) \arrow[u]
\end{tikzcd}
$$

Figure 1: The irregular case.

> "The extended Hamiltonian breaks Hamiltonian-Lagrangian equivalence. Requiring Hamiltonian-Lagrangian equivalence fixes the supposed ambiguity permitting the extended Hamiltonian".

Such claims have been used to argue that the right definition of a gauge transformation in the Hamiltonian formalism is not given by a transformation relating solutions to the Extended Hamiltonian, but rather it is a transformation relating solutions to the Total Hamiltonian. And one can show that the transformations relating solutions to the Total Hamiltonian are not given by arbitrary combinations of first-class constraints but rather by a *particular* combination of first-class constraints, contrary to the standard definition.[6] Therefore, the claim that the Lagrangian formalism is equivalent only to the Total Hamiltonian formalism has significant implications not only for how one formulates Hamiltonian gauge theories but also for the characterization of the gauge transformations themselves.

However, to evaluate these claims, we ought to understand the sense of (in)equivalence that is at stake. This hasn't been discussed in detail in the literature; indeed, what one finds are references to certain results that are taken to show that the solutions to the Euler-Lagrange equations are equivalent to the solutions to the Hamilton-Dirac equations on the primary constraint surface. One particular result that is widely cited is found in Batlle et al. (1986), so let us spell out this result and consider the notion of equivalence that it supports.

> **Theorem (Batlle et al. (1986)):** If $(q_i(t), \dot{q}_i(t))$ satisfies the Euler-Lagrange equations, then $FL(q_i(t), \dot{q}_i(t))$ satisfies the Hamilton-Dirac equations on the primary con-

straint surface. Similarly, if $(q_i(t), p_i(t))$ satisfies the Hamilton-Dirac equations on the primary constraint surface, then $FL^{-1}(q_i(t), p_i(t))$ satisfies the Euler-Lagrange equations, where $FL^{-1}(q_i(t), p_i(t))$ is constructed via:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + v_a(q_i, \dot{q}_i)\frac{\partial \phi_a}{\partial p_i}$$

$$-\frac{\partial L}{\partial q^i} = \frac{\partial H}{\partial q^i} + v_a(q_i, \dot{q}_i)\frac{\partial \phi_a}{\partial q^i}$$

where ϕ_a are the primary constraints and $v_a(q_i, \dot{q}_i)$ are arbitrary.

The theorem shows that the solutions to the Euler-Lagrange equations map to the solutions to the Hamilton-Dirac equations on the primary constraint surface and vice versa.[7] But notice that the inverse Legendre transformation maps one point on the primary constraint surface to multiple points on the tangent space since it is defined in terms of arbitrary functions v_a. It therefore maps one solution on the primary constraint surface to multiple solutions on tangent space. If these solutions are not considered equivalent from the perspective of the Lagrangian formalism, then this result cannot establish that a Lagrangian gauge theory defined on tangent space and its corresponding Hamiltonian theory defined on the primary constraint surface have equivalent solutions.

Moreover, even if we do interpret these solutions as equivalent, it seems that the most that this theorem can tell us is that there is a *dynamical* equivalence between Lagrangian mechanics and Hamiltonian mechanics on the primary constraint surface: the two theories agree on the equivalence classes of solutions. One cannot use Barrett's result to establish categorical equivalence since we do not have a way of translating the models and symmetries of one theory to those of the other. For one, it was important for Barrett's result that $FL^{-1} = FH$, which follows from the fact that these maps are global diffeomorphisms. The maps between tangent space and the primary constraint surface do not satisfy this property. Moreover, we have not yet formally defined the symmetries – the isomorphisms – of the corresponding theories. The natural account of the isomorphisms of a Hamiltonian gauge theory is that they are the transformations that preserve the presymplectic structure of the constraint surface and preserve the Hamiltonian on the constraint surface. But it isn't clear how to relate such symmetries to transformations of the points of tangent space. Therefore, the theorem above is not sufficient to infer theoretical equivalence between Lagrangian gauge theories and Hamiltonian gauge theories defined on the primary constraint surface.

On the other hand, one can use the theorem to argue that there is a dynamical, and therefore theoretical, *inequivalence* between a Lagrangian gauge theory and the corresponding Extended Hamiltonian theory. Indeed, this seems to be the core of the argument that the Extended Hamiltonian gets the gauge transformations wrong, from the perspective of the Lagrangian formalism. The reasoning is as follows. There are distinct solutions in the Total Hamiltonian formalism that are equivalent in the Extended Hamiltonian formalism: the solutions that are related by vector fields associated with the *secondary* first-class constraints. Therefore, the Total Hamiltonian formalism and the Extended Hamiltonian formalism disagree about the equivalence classes of solutions. Since the Lagrangian formalism has the same equivalence classes of solutions as the Total Hamiltonian formalism (under an appropriate interpretation of the Lagrangian formalism), the Lagrangian formalism is dynamically inequivalent to the Extended Hamiltonian formalism.

However, there are some lingering puzzles. First, there is a sense in which the Total Hamiltonian formalism is empirically equivalent to the Extended Hamiltonian formalism: if we take secondary constraints to be a physical requirement in the Total Hamiltonian formalism, then the solutions must lie on Σ_f, and on Σ_f, the Hamiltonian in the Total Hamiltonian formalism is equivalent to the Hamiltonian in the Extended Hamiltonian formalism. Therefore, although the Total Hamiltonian theory distinguishes solutions that the Extended Hamiltonian theory does not, the differences between these solutions cannot be recognized by the structure of the final constraint surface on which these solutions lie. This suggests that the Total Hamiltonian theory, and correspondingly the Lagrangian theory, distinguishes more solutions than can be distinguished empirically.

Second, given that we have motivated two formulations of Hamiltonian mechanics in the presence of gauge symmetry—the Total Hamiltonian formalism and the Extended Hamiltonian formalism—it is natural to ask whether, in the context of gauge theories, one could also reformulate *Lagrangian* mechanics such that the equivalence classes of solutions match the Extended Hamiltonian formalism. If we could, then this would suggest that the inequivalence that we find between Lagrangian mechanics and the Extended Hamiltonian formalism is an accident of the way we set up the Lagrangian formalism in the first place.

These puzzles lead to the following questions: First, can one motivate an alternative formulation of Lagrangian mechanics that captures the same empirical content but is dynamically equivalent to the Extended Hamiltonian formalism? Second, can one provide a stronger account of theoretical equivalence between formulations of Lagrangian and Hamiltonian gauge theories?

In what follows, I will argue that the answer to both questions is yes: we can both reformulate Lagrangian mechanics in the presence of gauge symmetry such that the resulting theory is dynamically equivalent to the Extended Hamiltonian formalism, and we can set up an equivalence result between categories of Lagrangian and Hamiltonian models that naturally capture the content of this reformulated Lagrangian theory and the Extended Hamiltonian theory. This will refute the claim that from the perspective of (equivalence with) the Lagrangian formalism, the Total Hamiltonian formalism is motivated over the Extended Hamiltonian formalism.

More carefully, I first demonstrate, drawing from Gotay & Nester (1979), that one can formulate Lagrangian gauge theories on a constraint submanifold of tangent space, and that the relationship between the Lagrangian constraint surface and the Hamiltonian final constraint surface is the same as the relationship between tangent space and the Hamiltonian primary constraint surface. I will use this to show that the equivalence classes of solutions of the reformulated Lagrangian theory match the equivalence classes of solutions of the Extended Hamiltonian formalism.[8] Next, I argue that there is a way to redefine the models of these theories using a process known as *reduction* such that one can set up a categorical equivalence result between classes of models of the reduced theories. This will demonstrate that the sense in which Lagrangian and Hamiltonian gauge theories are equivalent is essentially that their *gauge-invariant* descriptions are equivalent.

5 Lagrangian Constraint Formalism

To see how we can think of constraints in the Lagrangian formalism, let us start by writing the Euler-Lagrange equations as:

$$W_{ij}\ddot{q}^j + K_i = 0 \tag{1}$$

where $W_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ is the Hessian and $K_i = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j - \frac{\partial L}{\partial q^i}$. The singular case is characterized by the vanishing of the determinant of W_{ij}. Let us say that the rank of W_{ij} is $n - m_1$ so that W_{ij} has m_1 null vectors, φ_μ, such that $W_{ij}\varphi^j_\mu = 0$. We call these "gauge identities" because they hold at all points in T_*Q.

Contracting the equations of motion with the null vectors, we get:

$$\chi_\mu = K_i \varphi^i_\mu = 0 \tag{2}$$

We call these the first m_1 "Lagrangian constraints". We now require for consistency that these constraints are preserved under time evolution i.e. $\frac{d}{dt}\chi_\mu = 0$. This gives rise to new Lagrangian constraints $\chi_{\mu'}$. We can continue this process until we are left with all of the Lagrangian constraints. As in the Hamiltonian case, there are certain constraints whose time evolution allows one to determine some of the undetermined accelerations; as we will see, these constraints correspond to the second-class constraints on the Hamiltonian side.

It will be helpful to consider the picture in geometric terms. We can define, as in the regular case, the Lagrangian state space to be endowed with a two form $\Omega = FL^*\omega$ that is given in coordinate form by $\Omega = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} dq^i \wedge d\dot{q}^j$. When the Hessian $W_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ is non-invertible, Ω is degenerate and so it is a pre-symplectic two-form.

The geometric equations of motion can be written as before:

$$\Omega(X_E, \cdot) = dE \tag{3}$$

Because Ω is not symplectic in the irregular case, there will not be a unique solution to the equations of motion; indeed there may not be any solution at some points. However, the null vector fields of Ω allow us to define a submanifold where one can solve the equations at every point, in the following way. The null vector fields Z of Ω are such that $\Omega(Z, \cdot) = 0$. So, for the equations of motion to hold and be tangent to T_*Q, we must have that $dE(Z) = 0$. This motivates restricting to the submanifold P_1 defined by $dE(Z) = 0$ for null vector fields Z. We can therefore think of $dE(Z)$ as *constraints*.

Next, we require that the solutions to the equations of motion everywhere lie tangent to P_1 i.e. that the constraints hold at all points along a solution. But this is just to require that $dE(Y) = 0$ where Y is a null vector field of Ω restricted to P_1, which we can write as Ω_1. So we should restrict to a submanifold where in addition $dE(Y) = 0$. Therefore, we can think of $dE(Y)$ as further constraints.

Reiterating this process, we find a constraint surface P_k for K constraints where the solutions of the equations of motion $\Omega_k(X_E, \cdot) = dE$ are tangent to the constraint surface (where E is the energy function on T_*Q restricted to the points of the constraint surface P_k). The null vector fields of Ω_k correspond to the null vector fields of Ω and the vector fields associated with the constraints. Therefore, we can think of this formalism as providing a way on the Lagrangian side to associate constraints with gauge transformations: the vector fields associated with the constraints generate (a subset of) the gauge transformations, understood as transformations along the integral curves of the null vector fields.

However, there are some constraints $K_i \varphi^i_\mu = 0$ that are not accounted for by this geometric procedure. These are the constraints that do not correspond to null vector fields of the (induced) presymplectic two-forms. As Gotay & Nester (1980) show, these constraints are determined by requiring that the equation of motion is second-order, which corresponds to requiring that a solution to the equation of motion, written in coordinate-dependent form as $X = \alpha^i \frac{\partial}{\partial q^i} + \beta^i \frac{\partial}{\partial \dot{q}^i}$, is such that

$\alpha^i = \dot{q}^i$ (this follows from the two-form written in coordinate form above). If constraints of this kind arise, we can find their time derivative and thereby determine potentially new constraints. So take the final constraint surface to be given by $(P_f, \Omega_f, L|_{P_f})$ where P_f is the sub-manifold defined by the satisfaction of $K + J$ constraints where J is the number of constraints arising from the second-order condition.

6 Relationship between Final Constraint Surfaces

We have seen that we can construct submanifolds of tangent space in a similar way to the construction of submanifolds in the Hamiltonian formalism through constraints, and that we can write the equations of motion intrinsically on these submanifolds. So the natural question is whether the theory defined on the final constraint submanifold on the Lagrangian side is equivalent to the theory defined on the final Hamiltonian constraint manifold. To present an equivalence result of this kind, we will start by using the results in Gotay & Nester (1979) to show that the relationship between the models on the final constraint manifolds is the same as the relationship one finds between the original Lagrangian model and the model on the primary constraint surface.[9]

We will restrict ourselves, following Gotay & Nester (1979), to *almost regular* Lagrangian models. An almost regular Lagrangian model is associated with two assumptions. First, FL is a submersion onto its image i.e. its differential is surjective. Second, the fibers $FL^{-1}(FL(q, \dot{q}))$ are connected submanifolds of T_*Q. These two assumptions guarantee that $FL^*(H) = E$ defines a single-valued Hamiltonian, since they imply that the energy function E is constant along the fibers $FL^{-1}(FL(q, \dot{q}))$.[10] We can think of the almost regular Lagrangian models as characterizing the Lagrangian gauge theories: they are the models of Lagrangian mechanics for which there is a well-defined corresponding Hamiltonian theory on the primary constraint surface, with the Hamiltonian related to the energy function via $FL^*H = E$.

We also assume that we have no ineffective constraints[11], which means that there is a clear separation between first-class and second-class constraints i.e. a first-class constraint does not become second-class when considering its evolution and vice versa. To start, we will assume that we just have first-class constraints on the Hamiltonian side and constraints that correspond to null vector fields on the Lagrangian side.

Let us first consider the relationship between T_*Q and the primary Hamiltonian surface Σ_p. Take i_p to be the inclusion map $i_p : \Sigma_p \rightarrow T^*Q$. Then we can define the Legendre transformation between T_*Q and Σ_p, $FL_p : T_*Q \rightarrow \Sigma_p$ via $i_p \circ FL_p = FL$. Since FL is assumed to be a submersion onto its image and its image is precisely Σ_p, FL_p is also a submersion and is surjective (but not injective nor an immersion). Moreover, take FL_{p*} is the pushforward map associated with Fl_p. The kernel of FL_{p*} (or $Ker(FL_{p*})$) is the collection of vector fields Z on T_*Q such that $FL_{p*}(Z)$ is the zero vector everywhere.

> **Proposition 1:** The dimension of the space of null vector fields on T_*Q is equal to the dimension of the space of null vector fields on Σ_p plus the dimension of $Ker(FL_{p*})$.[12]

Proposition 1 tells us that for every null vector field on tangent space there is a corresponding null vector field on the primary Hamiltonian constraint surface, and that every null vector field on the primary Hamiltonian constraint surface corresponds to a null vector field on tangent space, but that the relationship is many-to-one from tangent space to the Hamiltonian primary constraint surface. The reason for this many-to-one relationship is that for any null vector field on tangent space, adding a vector field in the kernel of FL_{p*} (of dimension equal to the number of primary first-class constraints) gives rise to a distinct null vector field on tangent space that corresponds to the same null vector field on the Hamiltonian primary constraint surface.

It turns out that the same relationship holds between the final constraint surfaces P_f and Σ_f. Define the induced Legendre transformation between these spaces as follows. Define $i_L : P_f \rightarrow T_*Q$ as the inclusion map from the final Lagrangian constraint surface to the tangent space and $i_H : \Sigma_f \rightarrow T^*Q$ as the inclusion map from the final Hamiltonian constraint surface to the cotangent space. Then $FL_f : P_f \rightarrow \Sigma_f$ is given implicitly by $i_H \circ FL_f = FL \circ i_L$ (see Figure 2).

$$
\begin{tikzcd}
(T_*Q, L) \arrow[dr, "FL"] & (T^*Q, H) \\
(P_f, L) \arrow[u] \arrow[dr, "FL_f"] & (\Sigma_p, H) \arrow[u] \\
& (\Sigma_f, H) \arrow[u]
\end{tikzcd}
$$

Figure 2: Relationship between final constraint surfaces.

> **Proposition 2:** The dimension of the space of null vector fields on P_f is equal to the dimension of the space of null vector fields on Σ_f plus the dimension of $Ker(FL_{f*})$.[13]

Proposition 2 tells us that the relationship between null vector fields on the final constraint surfaces is also many-to-one from the Lagrangian to the Hamiltonian constraint surface, where the difference in dimension is given by the dimension of $Ker(FL_{f*})$. One can show that $Ker(FL_{f*}) = Ker(FL_{p*})$, and so the difference in dimension of null vector fields on the final constraint surface is given by the number of primary first-class constraints.

We can also show how the solutions to the equations of motion on the final constraint surfaces are related, using the fact that $FL_f^*(H) = E$ on the final constraint surfaces:

> **Proposition 3:** The dimension of the space of solutions to $\Omega_f(X_E, \cdot) = dE$ on P_f is equal to the dimension of the space of solutions to $\tilde{\omega}_f(X_H, \cdot) = dH$ on Σ_f plus the dimension of $Ker(FL_{f*})$.[14]

Proposition 3 tells us that the relationship between solutions is many-to-one in the sense that there are distinct solutions on the Lagrangian final constraint surface – related by the addition of vector fields in $Ker(FL_{f*})$ – that correspond to the same solution on the Hamiltonian final constraint surface. This provides the analogue to the theorem from Batlle et al. (1986) that we discussed in Section 4 for the final constraint surfaces. We can therefore give a partial response to the claim that the Extended Hamiltonian formalism is inequivalent to the Lagrangian formalism: there is an alternative formulation of Lagrangian gauge theories whose relationship to the

Extended Hamiltonian formalism is the same as the relationship between the original formulation of Lagrangian mechanics and the Total Hamiltonian formalism.

Propositions 1 through 3 also clarify the sense in which certain formulations of Lagrangian and Hamiltonian gauge theories are dynamically equivalent: they agree on solutions *up to* null vector fields. And it is clear why one would want to treat solutions that differ by null vector fields as being equivalent; such solutions cannot be distinguished by the presymplectic structure of the state space. More generally, the null vector fields are naturally thought to generate the (gauge) symmetries of the theories precisely because transformations along the null vector fields leave intact the relevant structure. This suggests that if we want to set up a *categorical* equivalence result, we need a way of characterizing the structure of the theories that includes the redundancy associated with the null vector fields.

Before turning to this task, let us consider how the situation changes when we also have second-class constraints on the Hamiltonian side. Since we assumed that there are no ineffective constraints, this means that we only need to consider the case where we have primary second-class constraints, since the time derivative of these constraints will generate any additional second-class constraints.

We have shown that we can relate the first-class constraints to null vector fields on the Lagrangian side. But since second-class constraints do not correspond to null vector fields, we cannot relate them to a Lagrangian constraint in the same way. However, it turns out that for every (distinct) primary second-class Hamiltonian constraint, there is a corresponding (distinct) Lagrangian constraint whose associated vector field is not null. In particular, the additional Lagrangian constraints are the pullback under the (induced) Legendre transformation of the time derivative of a second-class Hamiltonian constraint (Batlle et al. (1986), Pons (1988)). Generalizing, the final Lagrangian constraint surface will be reduced in dimension by the number of second-class constraints on the Hamiltonian side. Therefore, the difference in dimension of the final Lagrangian constraint surface and the final Hamiltonian constraint surface is given by the number of primary first-class constraints.

7 Reduction and Equivalence

Although we now have a picture under which both the Lagrangian formalism and the Hamiltonian formalism can be written intrinsically on constraint manifolds that are systemically related, we do not yet have a theoretical equivalence result. The barrier is that we do not have a way to define a translation from Lagrangian to Hamiltonian models and vice versa via the relationship between FL, FL^{-1}, FH and FH^{-1} since the final constraint submanifolds are not of the same dimension. However, we have seen that there are indications that we should be able to set up an equivalence result: while the dimensions of the final constraint surfaces are different, the difference seems to be due to arbitrariness in the Lagrangian formalism coming from the null vector fields in the kernel of FL_*. More generally, if we take null vector fields to generate symmetries, then it seems that the two formalisms agree on all symmetry-invariant content.

One way of characterizing the idea that theories agree on all symmetry-invariant content is to consider whether one can formulate the theories directly in terms of the equivalence classes under such symmetries. Indeed, there is a well-known construction for specifying a Hamiltonian theory in terms of the equivalence class of states along the integral curves of the null vector fields called *reduction*: the process of reduction defines a manifold that "quotients out" the gauge transformations.[15] This is not a construction that one often finds discussed for a Lagrangian theory.[16] However, we have shown that we can think of a Lagrangian gauge theory in an analogous way to the Hamiltonian formalism as defined on a presymplectic manifold. This suggests that we should be able to equally construct a reduced space for the final Lagrangian constraint surface. The question then becomes: are the *reduced* versions of Lagrangian and Hamiltonian gauge theories categorically equivalent?

The reason that reduction will help us to set up a categorical equivalence result is that one can show that reduction induces a symplectic two-form on the reduced space. Recall that being symplectic means that the Lagrangian/Hamiltonian models are *regular*: the two-form is non-degenerate and so we can, at least locally, define the inverse of the fiber derivatives. Therefore, if we can show that the Legendre transformation of a reduced Lagrangian model gives rise to a reduced Hamiltonian model and vice versa, then this suggests that we can set up an equivalence result in an exactly analogous way to Barrett (2019), if we restrict to the hyperregular reduced models.

Consider first a presymplectic Hamiltonian manifold $(\Sigma, \tilde{\omega}, H)$ that is foliated by the gauge orbits at each point. We can define a smooth, differentiable manifold $\bar{\Sigma}$ by taking the quotient of Σ by the kernel of $\tilde{\omega}$ i.e. the null vector fields of $\tilde{\omega}$. Recall that the integral curves of the null vector fields define the gauge orbits, and so the points of the quotient manifold are just the equivalence classes of points along the gauge orbits. This is well-defined since the gauge orbits foliate the constraint surface in such a way that one can define a transverse manifold that meets each leaf of the foliation in at most one point i.e. through each point there is only one gauge orbit.[17] Recall that on the final constraint surface, the dimension of the gauge orbits is the number of first-class constraints M and the dimension of Σ_f is $2N - M - S$ where N is the dimension of configuration space and S is the number of second-class constraints. So the quotient manifold of the final Hamiltonian constraint surface $\bar{\Sigma}$ has dimension $2N - 2M - S$.

Define an open, surjective projection map $\pi : \Sigma_f \to \bar{\Sigma}$ such that we define the reduced two-form $\bar{\omega}$ via $\tilde{\omega}_f = \pi^*(\bar{\omega})$, which acts according to $\bar{\omega}(\bar{X}, \bar{Y}) = \tilde{\omega}_f(X, Y)$ where $\bar{X} = \pi_*(X)$. One can show that $\bar{\omega}$ is well-defined and is symplectic.[18] We can also define a reduced Hamiltonian $\bar{H}$ as the value of H on the equivalence class of points along the gauge orbits i.e. $H = \pi^*(\bar{H})$. This is well-defined because H is constant along the gauge orbits on the final constraint surface (since the solutions to the equations of motion are tangent to the final constraint surface). We can therefore write the equations of motion on the reduced space in terms of the reduced Hamiltonian $\bar{H}$ as $\bar{\omega}(\bar{X}_{\bar{H}}, \cdot) = d\bar{H}$, and the solutions are just the projection of the solutions to the equations of motion on Σ_f to $\bar{\Sigma}$; they are the solutions defined for the gauge-invariant quantities.

Therefore, there is a well-defined Hamiltonian theory on the reduced space of the final Hamiltonian constraint surface in terms of a symplectic two-form and a reduced Hamiltonian function. However, this only required that we had a presymplectic manifold with a foliation induced by the null vector fields of the associated two-form and that the Hamiltonian function was constant along the gauge orbits. Given that the same is true for the Lagrangian final constraint surface, we can do the same reduction procedure on the Lagrangian side to produce a reduced Lagrangian space $\bar{P}$ with an associated symplectic two-form $\bar{\Omega}$. This space will have dimension $2N - 2K - J$ where K is the number of Lagrangian constraints associated with null vector fields and J is the number of additional Lagrangian constraints. As in the Hamiltonian case, the equations of motion $\bar{\Omega}(\bar{X}_{\bar{E}}, \cdot) = d\bar{E}$ are well-defined because the energy function E is constant along gauge orbits on P_f, and so the reduced Lagrangian function $\bar{L}$ – and consequently the reduced energy function $\bar{E}$ – are also well-defined.

Let us now turn to the relationship between models of the reduced theory. First, let us consider the relationship between the dimensions of the reduced spaces corresponding to models on the final constraint surfaces P_f, Σ_f that are related via FL_f. Recall that the dimension of the Lagrangian final constraint surface P_f is equal to the dimension of the Hamiltonian final constraint surface Σ_f plus the number of primary first-class constraints. But by Proposition 2, the difference in dimension of the null vector fields is also given by the number of primary first-class constraints. Therefore, the dimension of the reduced Lagrangian space $\bar{P}$ is equal to the dimension of the reduced Hamiltonian space $\bar{\Sigma}$.

Next, define an induced transformation $\bar{FL} : \bar{P} \rightarrow \bar{\Sigma}$ that satisfies $\pi_H \circ FL_f = \bar{FL} \circ \pi_L$ where $\pi_H : \Sigma_f \rightarrow \bar{\Sigma}$ and $\pi_L : P_f \rightarrow \bar{P}$ are the projection maps. This provides a way to map from the reduced Lagrangian space to the corresponding reduced Hamiltonian space. Since $\bar{L}$ is regular (since the induced two-form is symplectic), the Legendre transformation on $\bar{P}$, $F\bar{L}$, will be a (local) diffeomorphism. Therefore, since $\bar{P}$ and $\bar{\Sigma}$ have the same dimension, the induced transformation $\bar{FL}$ is precisely the Legendre transformation on $\bar{P}$, $F\bar{L}$.[19] Similarly, since $\bar{H}$ is regular, the fiber derivative of $\bar{H}$, $F\bar{H}$, will be a (local) diffeomorphism and it will map $\bar{\Sigma}$ to $\bar{P}$. Using the reduced Legendre transformation, one can also show that the reduced symplectic two-forms are related via $F\bar{L}^*(\bar{\omega}) = \bar{\Omega}$ and the reduced Hamiltonian and energy function are related via $F\bar{L}^*(\bar{H}) = \bar{E}$.[20]

Finally, since (P_f, L_f) is, by assumption, an almost regular system, $(\bar{P}, \bar{L})$ will also be almost regular. This implies that $F\bar{L}$ is injective.[21] Moreover, the image of $F\bar{L}$ is $\bar{\Sigma}$ by construction of the induced transformation so $F\bar{L}$ is surjective. But this means that $F\bar{L}$ is a global diffeomorphism, and so $(\bar{P}, \bar{L})$ is in fact a *hyperregular* system. Therefore, we can define the inverse $F\bar{L}^{-1} : \bar{\Sigma} \to \bar{P}$. This allows us to define $\bar{H} = \bar{E} \circ F\bar{L}^{-1}$.

Therefore, for an almost regular Lagrangian model defined on the final constraint surface, we can construct a reduced model such that this model is hyperregular and its Legendre transformation is precisely the (hyperregular) reduced model of the corresponding Hamiltonian final constraint surface. This implies that as long as we are concerned with almost regular Lagrangian models and their corresponding Hamiltonian models, the reduced formulations of these theories bear exactly the same relationship as hyperregular models of Lagrangian and Hamiltonian mechanics (see Figure 3).

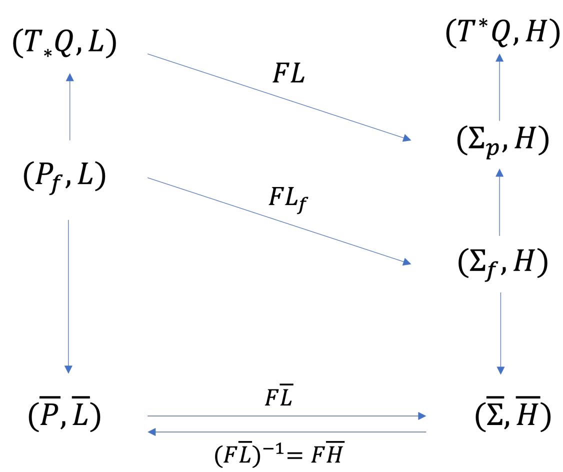

Figure 3: Relationship between reduced spaces.

We are now at the point where we can set up a categorical equivalence result. Recall that to do so, we need to define the models and symmetries of the associated theories. In the hyperregular case discussed in Section 2, the symmetries were taken to be the point-transformations that preserved the Lagrangian/Hamiltonian. However, in order for the point-transformations to be well-defined for the reduced theories, we need that the reduced state space has the structure of a (co)tangent bundle. This is not guaranteed by the above.[22] On the other hand, we do have that the reduced spaces are symplectic manifolds. Therefore, it seems that the natural notion of symmetry is rather given by *symplectomorphisms:* diffeomorphisms that preserve the symplectic two-form on the reduced space (and preserve the Lagrangian/Hamiltonian).

In light of this, define the category **LagR** as having objects $(\bar{P}, \bar{\Omega}, \bar{L})$ and arrows between objects $(\bar{P}_1, \bar{\Omega}_1, \bar{L}_1)$ and $(\bar{P}_2, \bar{\Omega}_2, \bar{L}_2)$ given by symplectomorphisms that preserve the Lagrangian i.e. diffeomorphisms $f : \bar{P}_1 \to \bar{P}_2$ such that $f^*(\bar{\Omega}_2) = \bar{\Omega}_1$ and $f^*(\bar{L}_2) = \bar{L}_1$.

Similarly, define the category **HamR** as having objects $(\bar{\Sigma}, \bar{\omega}, \bar{H})$ and arrows between objects $(\bar{\Sigma}_1, \bar{\omega}_1, \bar{H}_1)$ and $(\bar{\Sigma}_2, \bar{\omega}_2, \bar{H}_2)$ given by symplectomorphisms that preserve the Hamiltonian i.e. diffeomorphisms $g : \bar{\Sigma}_1 \to \bar{\Sigma}_2$ such that $g^*(\bar{\omega}_2) = \bar{\omega}_1$ and $g^*(\bar{H}_2) = \bar{H}_1$.

Define the functor J as taking the object $(\bar{P}, \bar{\Omega}, \bar{L})$ of **LagR** to $(\bar{\Sigma}, \bar{\Omega} \circ F\bar{L}^{-1}, \bar{E} \circ F\bar{L}^{-1})$ where $\bar{\Sigma}$ is the image of $\bar{P}$ under $F\bar{L}$, and that takes arrows $f : \bar{P}_1 \to \bar{P}_2$ to $F\bar{L}_2 \circ f \circ F\bar{L}_1^{-1}$. Similarly, define the functor K as taking objects $(\bar{\Sigma}, \bar{\omega}, \bar{H})$ of **HamR** to $(\bar{P}, \bar{\omega} \circ F\bar{H}^{-1}, (\bar{\theta}_a(X_{\bar{H}})^a - \bar{H}) \circ F\bar{H}^{-1})$ where $\bar{P}$ is the inverse image of $\bar{\Sigma}$ and $\bar{\theta}$ is the reduced one form, and takes arrows $g : \bar{\Sigma}_1 \to \bar{\Sigma}_2$ to $F\bar{H}_2 \circ g \circ F\bar{H}_1^{-1}$. Then we can show that:

> **Proposition 4:** J : **LagR** $\to$ **HamR** and K : **HamR** $\to$ **LagR** are equivalences that preserve solutions.[23]

8 Upshots

Proposition 4 provides a sense in which irregular Lagrangian mechanics and irregular Hamiltonian mechanics are equivalent: one can formulate these theories geometrically on a presymplectic final constraint manifold such that the hyperregular class of reduced models are categorically equivalent. This provides an extension to the result in Barrett (2019) that hyperregular Lagrangian and Hamiltonian theories are categorically equivalent. Moreover, it has several interesting consequences.

First, we discussed in Section 4 the view that the correct Hamiltonian formulation is the Total Hamiltonian formalism on the basis that it is dynamically equivalent to the Lagrangian formalism in the context of gauge theories. But our arguments have suggested that the Extended Hamiltonian formalism can be motivated in a similar, and even stronger, way: there are reasons to move to the final Lagrangian constraint surface from the perspective of the Lagrangian formalism, and not only are the models formulated on the Lagrangian final constraint surface dynamically equivalent to models of the Extended Hamiltonian formalism, one can also prove a theoretical equivalence result between the reduced version of such models.

To deny that Proposition 4 supports the Extended Hamiltonian formalism, one would have to maintain that there is something mistaken about the Lagrangian constraint formalism we presented. One avenue might be to argue that we shouldn't think of Lagrangian constraints as restricting the state space of Lagrangian mechanics: they should be thought of as *dynamical* constraints and not *kinematical* constraints, and therefore they should not restrict the kinematically possible models that we use to define the theory. According to this view, the correct formulation of the Lagrangian models is the usual tangent bundle formulation. Given this, one might argue that we should interpret Proposition 4 as instead providing stronger support for the claim that the Lagrangian formalism is equivalent to the Total Hamiltonian formalism: we can consider the result in the case where the reduced models are formulated by taking the reduction of the tangent bundle/primary constraint surface models.

Setting aside the subtleties of taking reduction to happen at the level of the tangent bundle/primary constraint surface[24], I think that the discussion in this paper shows that there are good reasons to take the formulation on the final Lagrangian constraint surface to be well-motivated. First, the Lagrangian constraints are motivated in an identical way to the secondary Hamiltonian constraints: they are required for the equations of motion to be well-defined everywhere. Therefore, although one could take the points off the final Lagrangian constraint surface to be "kinematically possible", there is a sense in which they play no role in the empirical content of the theory. Second, given that the Lagrangian constraints give rise to null vector fields in the same way as Hamiltonian constraints do, they are important for capturing the redundancy that a Lagrangian theory has. In other words, the reason that one would want to formulate the Lagrangian theory on the final constraint surface is for the same reason one would want to formulate the Hamiltonian theory on the final constraint surface: it provides an intrinsic characterization of the dynamics and the symmetries of a theory.

Another reason that Proposition 4 is significant is that it goes against some commonly found remarks in the literature. For example, Earman (2002) says: "Is there then some non-question begging and systematic way to identify gauge freedom and to characterize the observables? The answer is yes, but specifying the details involves a switch from the Lagrangian to the constrained Hamiltonian formalism." The geometric formulation shows that one can provide the same characterization from the Lagrangian side: the observables are the functions that are constant along the null vector fields of the associated two-form, and these are equivalent to the Hamiltonian observables. Moreover, on the usual understanding of gauge theories, one starts with a Lagrangian gauge theory and uses it to define the Hamiltonian one. Proposition 4 suggests that one could equally start with a Hamiltonian theory with constraints, reduce the final constraint surface, and use this to define the corresponding (reduced) Lagrangian theory.

However, there are several subtleties with the equivalence result given by Proposition 4. For one, we restricted to a subset of the irregular Lagrangian models, the 'almost regular' ones, and then considered the corresponding Hamiltonian models defined via the Legendre transformation. While we argued that the almost regular Lagrangian models and the corresponding Hamiltonian models have hyperregular reduced models, we did not show that this exhausts the class of hyperregular reduced models. It would therefore be interesting to consider whether there are hyperregular reduced models that cannot be thought of as coming from a 'gauge theory' in the sense of being an almost regular Lagrangian model or its corresponding Hamiltonian model. Moreover, 'almost regularity' referred to the Lagrangian model, but there doesn't seem to be a clear Hamiltonian analogue: the fiber derivative of the Hamiltonian on the primary/final constraint surface does not construct an almost regular Lagrangian model. Therefore, it seems that we need some alternative way to characterize the relevant class of gauge theories in Hamiltonian terms.[25]

Second, we have been assuming that it is adequate to think of Lagrangian gauge theories as having (pre)symplectic structure. But we defined the associated two-form by pulling back the (pre)symplectic two-form on the Hamiltonian state space along the Legendre transformation; tangent space does not come naturally equipped with a (pre)symplectic two-form.[26] Therefore, one might argue that we are enforcing *Hamiltonian* structure on a Lagrangian theory that it shouldn't be taken to have. Whether this is right depends on what one takes to constitute a Lagrangian vs. Hamiltonian gauge theory, which itself is related to the background debate between North (2009) and Curiel (2014).

Finally, while symplectic reduction is well-founded, it runs into problems in certain applications; notably, in the context of time-reparameterization-invariant theories such as General Relativity where the Hamiltonian function is itself a first-class constraint, symplectic reduction, inasmuch as it is well-defined[27], leads to (a version of) the "Problem of Time": one ends up with a theory without a meaningful notion of evolution. This puzzle has led to views that either reject formulating a gauge theory on the reduced space or extend the formalism in some way.[28] On these views, the symplectic reduced space is not (in all cases) the relevant structure to consider when asking whether Lagrangian and Hamiltonian gauge theories are equivalent.

I take the claim that the reduced space does not always correctly characterize the content of a gauge theory to be an important limitation of the argument that Lagrangian and Hamiltonian gauge theories are equivalent on the basis that their reduced theories are equivalent. However, I see the results presented here as providing the basis for considering this issue from a new perspective. For example, one could ask whether the category of models on the final constraint surface (for either the Lagrangian or Hamiltonian theory) is equivalent to the category of reduced models given by **LagR** and **HamR**. If the answer is yes, then this would suggest that the corresponding Lagrangian and Hamiltonian theories on the final constraint surface are equivalent. If the answer is no, then the question about the equivalence between Lagrangian and Hamiltonian gauge theories formulated on the final constraint surface is left open.

To end, I think the lesson of this paper is that even if cases where one has an intuition that two theories are equivalent, proving categorical equivalence can be hard, and considering what assumptions go into proving categorical equivalence can shed light on what one means by, for example, a "Lagrangian gauge theory" and a "Hamiltonian gauge theory", as well as how one should interpret such theories if one wants to maintain that they are (or are not) equivalent. Therefore, rather than this paper providing a definitive answer to whether Lagrangian and Hamiltonian gauge theories are equivalent, I think one should view it as providing a first step towards using categorical relationships to shed light on the structure of, and relationship between, Lagrangian and Hamiltonian gauge theories.

A Appendix

A.1 Proposition 1

First, we show that every null vector field on Σ_p gets mapped to by a null vector field on T_*Q via FL_{p*}. Then, we show that every null vector field on T_*Q maps to a null vector field on Σ_p via FL_{p*}. Finally, we show that the vector fields in $Ker(FL_{p*})$ are null vector fields on T_*Q, which implies that for any null vector field Z on T_*Q, there is another vector field Y on T_*Q such that $FL_{p*}(Y) = FL_{p*}(Z)$ where the difference between Y, Z lies in the kernel of FL_{p*}. By the linearity of FL_{p*}, this is the only way that one could have distinct null vector fields on T_*Q that map to the same null vector field on Σ_p. Therefore, this suffices to show that the dimension of the space of null vector fields on T_*Q is equal to the dimension of the space of null vector fields on Σ_p plus the dimension of $Ker(FL_{p*})$.

For the first, suppose that $\tilde{\omega}_p(X, \cdot) = 0$ i.e. X is a null vector field on Σ_p. Since FL_{p*} is a submersion, every vector field X can be written as $FL_{p*}(Z)$ for some vector field Z on T_*Q. Therefore, we can write the supposition as $\tilde{\omega}_p(FL_{p*}(Z), \cdot) = 0$. This is equivalent to $(FL_p^*\tilde{\omega}_p)(Z, \cdot) = 0$. Since $FL_p^*\tilde{\omega}_p = \Omega$, this implies that Z is a null vector field of Ω.

For the second, suppose that $\Omega(Z, \cdot) = 0$. By the definition of Ω, this means that at all points $x \in T_*Q$, $(FL_p^*\tilde{\omega}_p)(Z, \cdot) = 0$. This is equivalent to $\tilde{\omega}_p(FL_{p*}(Z), \cdot) = 0$ at the point $FL_p(x)$. Since FL_p is a submersion, this means that at all points $y \in \Sigma_p$, $\tilde{\omega}_p(FL_{p*}(Z), \cdot) = 0$ where $FL_{p*}(Z)$ is defined in terms of some point $x \in T_*Q$ such that $y = FL_p(x)$. Therefore, although $FL_{p*}(Z)$ is not guaranteed to be a well-defined vector field, one can construct a null vector field on Σ_p via

$FL_{p*}(Z)$ in a point-wise sense.

Finally, to show that $Ker(FL_{p*}) \subseteq Ker(\Omega)$, suppose that $Y \in Ker(FL_{p*})$. Then, from the above, $\Omega(Y, \cdot) = (FL_p^* \tilde{\omega}_p)(Y, \cdot) = \tilde{\omega}_p(FL_{p*}(Y), \cdot)$. But $FL_{p*}(Y)$ is the zero vector at every point and so $\Omega(Y, \cdot) = 0$. Therefore, every vector field in $Ker(FL_{p*})$ is a null vector field on T_*Q.

A.2 Proposition 2

We can use a similar proof to that of Proposition 1 if FL_f is a (surjective) submersion. To show that FL_f is a submersion, it suffices to show that the number of Lagrangian constraints is equal to the number of secondary (first-class) Hamiltonian constraints, since each constraint reduces the dimension of the state space by one.

To see why this is the case, notice that Proposition 1 implies that if $dH(X)$ is a Hamiltonian constraint where X is a null vector field on Σ_p, then $dE(Z)$ is a Lagrangian constraint where $X = FL_{p*}(Z)$ and $E = FL_p^*(H)$. Similarly, if $dE(Z)$ is a Lagrangian constraint where Z is a null vector field on T_*Q, then one can construct a Hamiltonian constraint $dH(FL_{p*}(Z))$ via point-wise application of $FL_{p*}(Z)$. To show that there is only one such Hamiltonian constraint for each Lagrangian constraint, recall that by the assumption of almost regularity, E is constant along the fibers $FL^{-1}(FL(q, \dot{q}))$. This implies that $dE(Y) = 0$ for every $Y \in Ker(FL_{p*})$. Therefore, there are no Lagrangian constraints of the form $dE(Y)$ where $Y \in Ker(FL_{p*})$. Since the difference in dimension of the space of null vectors is equal to $Ker(FL_{p*})$ by Proposition 1, this shows that there is a one-to-one correspondence between Lagrangian constraints of the form $dE(Z)$ for null vector fields Z on T_*Q and the first generation of secondary, first-class Hamiltonian constraints defined on Σ_p. Reiterating this reasoning, the same will be true of all further constraint submanifolds. Therefore, since each constraint reduces the dimension of the state space by one, the relationship between P_f and Σ_f will be the same relationship as between T_*Q and Σ_p: the induced Legendre transformation FL_f will be a surjective submersion, where $Ker(FL_{f*}) = Ker(FL_{p*})$.

Therefore, we can use similar reasoning as the proof for Proposition 1 to show that the dimension of the space of null vector fields on P_f is equal to the dimension of the space of null vector fields on Σ_f plus the dimension of $Ker(FL_{f*})$, where $Ker(FL_{f*}) = Ker(FL_{p*})$ is the number of primary first-class constraints.

A.3 Proposition 3

Similar to the proof of Proposition 1, we first show that every solution on Σ_f gets mapped to by a solution on P_f via FL_{p*}. Then, we show that every solution on P_f maps to a solution on Σ_f. Since $Ker(FL_{f*}) \subseteq Ker(\Omega_f)$, if X_E is a solution to $\Omega_f(X_E, \cdot) = dE$, then so is $X_E + \alpha^i Y_i$ where $Y_i \in Ker(FL_{f*})$ and α^i is an arbitrary function on Ω_f. Therefore, since $FL_{f*}(X_E + \alpha^i Y_i) = FL_{f*}(X_E)$, this suffices to show that the dimension of the space of solutions on P_f is equal to the dimension of the space of solutions on Σ_f plus the dimension of $Ker(FL_{f*})$.

For the first, suppose that $\tilde{\omega}_f(X_H, \cdot) = dH$. Since FL_f is a submersion, every X_H can be written as $FL_{f*}(X_E)$ for some vector field X_E on P_f. Therefore, we can write the supposition as $\tilde{\omega}_f(FL_{f*}(X_E), \cdot) = dH$, which is equivalent to $(FL_f^*\tilde{\omega}_f)(X_E, \cdot) = FL_f^*(dH)$. Since $FL_f^*\tilde{\omega}_f = \Omega_f$ and $FL_f^*(dH) = d(FL_f^*H) = dE$, this implies that X_E is a solution to $\Omega_f(X_E, \cdot) = dE$, which is the equations of motion on P_f.

For the second, suppose that $\Omega_f(X_E, \cdot) = dE$. By the definition of Ω_f and E, this means that at all points $x \in P_f$, $(FL_f^*\tilde{\omega}_f)(X_E, \cdot) = FL_f^*(dH)$. This is equivalent to $\tilde{\omega}_f(FL_{f*}(X_E), \cdot) = dH$ at the point $FL_p(x)$. Since FL_f is a submersion, this means that at all points $y \in \Sigma_f$, $\tilde{\omega}_f(FL_{f*}(X_E), \cdot) = dH$ where $FL_{f*}(X_E)$ is defined in terms of some point $x \in P_f$ such that $y = FL_f(x)$. This means that $FL_{f*}(X_E)$ is a solution to the equations of motion on Σ_f at every point. Therefore, even though $FL_{f*}(X_E)$ is not guaranteed to be a well-defined vector field, one can construct a solution on Σ_f via $FL_{f*}(X_E)$ in a point-wise sense.

A.4 Proposition 4

To show that J is a functor, we need to show that J takes objects of **LagR** to objects of **HamR** and arrows to arrows. The first is trivial. To show the second, take an arrow f between objects $(\bar{P}_1, \bar{\Omega}_1, \bar{L}_1)$ and $(\bar{P}_2, \bar{\Omega}_2, \bar{L}_2)$. Since f is a symplectomorphism, $f^*\bar{\Omega}_2 = \bar{\Omega}_1$. Since $\bar{\Omega} = F\bar{L}^*\bar{\omega}$

by construction, this means that $f^*(F\bar{L_2}^*\bar{\omega}_2) = F\bar{L_1}^*\bar{\omega}_1$. We want to show that $F\bar{L}_2 \circ f \circ F\bar{L}_1^{-1}$ is an arrow in **HamR**. That is, we want to show that $(F\bar{L}_2 \circ f \circ F\bar{L}_1^{-1})^*\bar{\omega}_2 = \bar{\omega}_1$ and $(F\bar{L}_2 \circ f \circ F\bar{L}_1^{-1})^*(\bar{E}_2 \circ F\bar{L_2}^{-1}) = \bar{E}_1 \circ F\bar{L}_1^{-1}$. The first follows from the fact that $f^*(F\bar{L_2}^*\bar{\omega}_2) = F\bar{L_1}^*\bar{\omega}_1$. The second follows from the fact that $f^*\bar{E}_2 = \bar{E}_1$ since $f^*\bar{L}_2 = \bar{L}_1$. Similar reasoning can be used to show that K is a functor.

Since $F\bar{L}$ and $F\bar{H}$ are global diffeomorphisms, it follows from Abraham & Marsden (1987, Theorems 3.6.7, 3.6.8) that $F\bar{L}^{-1} = F\bar{H}$ and $F\bar{H}^{-1} = F\bar{L}$. This implies that the functors J and K are inverses on objects and on arrows, which suffices to show that J and K are equivalences.

Finally, that J and K preserve solutions follows from Proposition 3, which shows that FL_f preserves solutions up to gauge-equivalence, in combination with the fact that the solutions that are equivocated through reduction are the gauge-related solutions.

Notes

1. There is a particular philosophical puzzle concerning the Hamiltonian formulation of General Relativity and its gauge transformations known as the 'Problem of Time' (Anderson (2012)) We will not consider this problem directly in this paper; indeed, given that symplectic reduction is particularly problematic for General Relativity (Belot (2007), Thébault (2012)), the equivalence result will be less straightforward in this case. However, we will comment on it briefly in Section 8.

2. Categorical equivalence as an account of theoretical equivalence has been developed and defended in several places, including Halvorson (2012, 2016), Halvorson & Tsementzis (2017), Weatherall (2016, 2017), Barrett (2015*a*,*b*, 2019).

3. A functor is a structure-preserving map between categories that takes objects to objects and arrows to arrows.

4. Functors realize an equivalence between categories when they are *full, faithful,* and *essentially surjective*. For details, see, for example, Weatherall (2016). The functors F, G preserve solutions because they preserve the integral curves of X_E and X_H (Abraham & Marsden (1987, Theorem 3.6.2)).

5. Indeed, in a footnote (16), Barrett (2019) says: "One can, of course, consider the more general case, but I conjecture that there the theories will be inequivalent according to any reasonable standard of equivalence."

6. For more discussion on this debate, see Pitts (2014*b*), Pons (2005), Pooley & Wallace (2022).

7. One can therefore see this theorem as the analogue of Theorem 3.6.2 in Abraham & Marsden (1987), which Barrett (2019) uses as part of the proof of equivalence in the hyperregular case (see footnote 3), in the context of gauge theories.

8. In Gryb & Thébault (2023, ch.8) it is argued that the symmetries of the Extended Hamiltonian can be motivated from the Lagrangian perspective through careful consideration of Noether's Second Theorem. I take this to be complementary to the argument presented here. The reason for using the formalism in Gotay & Nester (1979) is that it directly allows us to compare the geometric structure of the two theories. However, it would be interesting to explore the extent to which the results here agree with the analysis in Gryb & Thébault (2023).

9. Although the core content of the results in this section can be found in Gotay & Nester (1979), they do not discuss in detail the kind of equivalence that these results imply, nor do they draw the implications that we do here for the debate about the Total vs. Extended Hamiltonian.

10. The proof can be found in Gotay & Nester (1979).

11. An ineffective constraint is one whose gradient vanishes weakly. For discussion, see Gotay & Nester (1984).

12. See A.1 for proof.

13. See A.2 for proof.

14. See A.3 for proof.

15. The procedure for reduction that we consider here is drawn from Henneaux & Teitelboim (1994), Souriau (1997). However, reduction often refers to a (more general) procedure sometimes called "Marsden-Weinstein reduction" due to its development by Marsden & Weinstein (1974). While these procedures are related, I leave to future work the question of how the arguments here can be cast in terms of Marsden-Weinstein reduction. For further discussion on Marsden-Weinstein reduction and its relation to Lagrangian and Hamiltonian approaches to gauge theories, see Butterfield (2006), Belot (2007).

16. An exception is Pons et al. (1999).

17. See Souriau (1997) §5 and §9 for details.

18. It is well-defined since the value of $\tilde{\omega}_f$ doesn't depend on which point along the gauge orbit one considers. It is closed since $\tilde{\omega}_f$ is closed and π is a surjective submersion, and it is non-degenerate since $Ker(\bar{\omega}) = Ker(\tilde{\omega}_f)/Ker(\tilde{\omega}_f) = 0$.

19. For further discussion of the properties that the Lagrangian theory must satisfy in order for the reduced Legendre transformation to be well-defined and correspond to the induced transformation between reduced spaces, see Cantrijn et al. (1986).

20. To see this, notice that $\pi_L^*(F\bar{L}^*\bar{\omega}) = FL_f^*(\pi_H^*\bar{\omega}) = FL_f^*\tilde{\omega}_f = \Omega_f$. Since π_L is a surjective submersion, this implies that $F\bar{L}^*(\bar{\omega}) = \bar{\Omega}$. The second follows by similar reasoning.

21. The reason is that for an almost regular system, the image of the Legendre transformation is the leaf space of the foliation generated by the kernel of the pushforward of the Legendre transformation. When a system is regular, this kernel is zero, and so it must be injective.

22. Moreover, even if one could think of the reduced state space as having the structure of a (co)tangent space, it isn't clear that one would want the symmetries to be given by point-transformations. In particular, Barrett (2015*b*) shows that there are point$_*$-transformations that don't preserve any symplectic two-form on T_*Q. Given that the symplectic two-form is an integral part of the construction of these reduced models, one might conclude that point$_*$-transformations are not the relevant symmetries to consider for the reduced Lagrangian models.

23. See A.4 for proof.

24. In particular, the solutions to the equations of motion are not tangent to the constraint surface in the case where there are Lagrangian/secondary constraints that are not represented in the structure of the state space, and so the reduced equations of motion are not well-defined. See Pons et al. (1999) for further discussion.

25. For example, is it the case that any regular Hamiltonian theory with the addition of constraints gives rise to a constraint surface model that is the Legendre transformation of some almost regular Lagrangian model?

26. For details, see Barrett (2015*a*).

27. In the case of General Relativity, there are barriers to applying reduction because the transformations generated by the constraints do not form a Lie group. For further discussion, see Thébault (2012), Gryb & Thébault (2016*a*).

28. For different views of this kind, see Rovelli (2004, 2002), Barbour & Foster (2008), Gryb & Thébault (2012, 2016*b*). For further discussion, see Thébault (2012).

Acknowledgements

I am especially grateful to Jim Weatherall for discussions about the proofs of the propositions in the paper and for comments on a previous draft. I am also grateful to the audience at New Directions 2024 for feedback, and to Karim Thébault and Sean Gryb for helpful discussion.

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