*Article*

# Relating quasi-sets and rough sets: from quantum entities to AI

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**Abstract:** At present, there are at least two set theories motivated by quantum ontology: Décio Krause's quasi-set theory ($\mathfrak{Q}$) and Maria Dalla Chiara and Giuliano Toraldo di Francia's quasi-set theory (QST). Recent work [Jorge-Holik-Krause, 2023] has established certain links between QST and Pawlak's rough set theory (RST), showing that both are strong candidates for providing a non-deterministic semantics of N matrices that generalizes those based on ZF. In this work, we show that the new atomless quasi-set theory $\mathfrak{Q}^-$ , recently introduced to account for a quantum property ontology [Krause-Jorge, 2024], has strong structural similarities with QST and RST. We study the level of extensionality that each theory presents, its relation to the Leibniz principle and the rigidity property. We believe that developing common features among these three theories can motivate common fields of research. By revealing shared structures, the developments of each theory can have a positive impact on the others.

**Keywords:** quasi-sets; quasets; rough sets; quantum mechanics; identity; Leibniz's principle; extensionality
(List three to ten pertinent keywords specific to the article; yet reasonably common within the subject discipline.)

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## 1. Introduction

'Quantum set theories' exist from a long time. We can mention those of David Finkelstein (1982) and Gaisi Takeuti (1981).[1] But these theories, despite being motivated by quantum physics, do not intend to cope with a possible *ontology* of quantum entities. 'Set' theories trying to cope with a metaphysics of quantum entities were presented by Dalla Chiara and Toraldo di Francia in 1985 (see their 1993) and one of the present authors in 1990 (see [6]). The first one, termed *quaset theory* (QST) is a kind of fuzzy set theory,[2] where we have not only two alternatives for an element does belong or not to a set; it can be also 'more or less inside'. The trick, as we shall see, is to make flexible the notion of membership. Krause's theory, which has been improved since them, deals with entities that can be completely indistinguishable without collapsing in being just one entity as it would be implied if the standard theory of identity (STI) holds. Below we shall see more details.

Among others, mathematician and physicist Yuri Manin (1976) suggested that the usual set theory would not be adequate to treat collections of quantum entities; the same was said by Dalla Chiara and Toraldo di Francia (1986, 1993) in their analysis of the logical structure of quantum physics. As Manin stated,

> I would like to point out that (...) [set theory] is rather an extrapolation of common-place physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behaviour. Even 'sets' of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the 'set' of grains of sand. (...) We should consider the possibilities of developing a totally new language to speak about infinity [set theory is known as the theory of the infinite]. [8].

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[1] Takeuti's quantum set theory was improved in several aspects by M. Ozawa, see [3,4].
[2] In the sense that Weidner considers Fuzzy sets [7], which is not using relations defined in the axiomatics of ZF (as is normally done).

Dalla Chiara and Toraldo di Francia go in the same direction when they contest the reasonableness of applying a 'standard' set theory to the quantum realm:

> Consider the electrons of an atom. We generally know perfectly well how many electrons there are, but cannot tell which is which. It is customary to talk of the 'set' of the electrons of that atom. But do they really constitute a set? Certainly not in a classical sense. How can we verify that the collections of electrons that are found around the atoms satisfy, say, the Zermelo-Fraenkel axioms, without being able to distinguish one element from another? [5]

In the view of these authors, which is also ours, a *quantum set theory* should be able not only for expressing the fundamental mathematical notions necessary for the theory, but also for expressing an ontology, that is, a reasonable metaphysics about quantum entities. It is in this direction that we look at quantum sets.

The main goal of our work is to establish structural relations between three non-standard 'set' theories: the quaset theory QST [10,11], the quasi-set theory $\mathfrak{Q}$ [6,12] and the *rough set theory* RST of Zdzisław Pawlak [13,14]. One of the main trait of both QST and $\mathfrak{Q}$ is that both, to a greater or lesser degree, depart from extensionality. That is, they are theories without an extensionality axiom like ZF's or with a weakened extensionality axiom. We will see that the way each theory departs from extensionality allows us to establish certain patterns that link them.

On the one hand, $\mathfrak{Q}$ (quasi-sets) and QST (quasets) theories were motivated by the ontology of quantum entities; as said before, $\mathfrak{Q}$ intends to deal with collections of entities that may be genuinely indiscernible without being identical, and QST, to account for indeterminate properties of quantum entities. On the other hand, rough set theory can be considered a new mathematical approach to vagueness. At the basis of RST is the assumption that with every object in the universe of discourse we associate certain information (knowledge). Within the framework of this theory, entities characterized by the same information are indiscernible (similar) in view of the information available about them.

Our aim is to make explicit some existing links between the formalisms proposed to deal with concepts apparently as different as *non-identity entities, indefinite properties* and *vague concepts*. Establishing relations between these set theories may suggest new applications, both logical-mathematical and technological, as well as motivating new developments of each one, suggested by the similarity with the others. After the developments and the links being established, we can turn to the application to quantum theory, but we shall not do that in this paper.

## 2. The QST theory and its extensions

First, we will present an adapted version of the original work of Maria Luisa Dalla Chiara (1938-) and Giuliano Toraldo di Francia (1916-2011) given in 1985 (see [5]). We consider the theory presented in [15], which is already an extension of QST but add some further considerations, adapted to the discussions that follow. Anyway, to honor the original contribution, and since its spirit is not modified, we continue to report to the resulting theory as 'QST'.

Next, we will present two possible extensions, originally proposed in [16] within the framework of a non-deterministic semantics of Nmatrices. The latter will allow us to better explain the similarities we wish to highlight.

As said before, QST is inspired by the properties of quantum systems. In particular, by the fact that their collections of indistinguishable quanta do not seem to form sets as in usual set theories, where the elements are always distinguishable from each other. The intention of the authors was to capture this particularity of quantum collections through an axiomatic system and use the idea to characterize a formal semantics for certain quantum languages, something that, for them, should not be done in usual extensional set theories of *individuals*.[3] But, instead of pushing the discussion to the analysis of an ontology of indistinguishable but not identical things, they conduced their approach to *indeterminacy*, to the consideration of situations where extensional objects may do

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[3] Roughly speaking, an individual is an item that can be re-identified as such in different contexts; for instance, despite the tiny differences due to age and other things, we can identify Donald Trump as being *that* guy in different situation. This cannot be said of quantum entities (even traces in a bubble chamber need to be taken with a grain of salt since what we observe are the drops of water and not the particles themselves).

not exist, so they need to be dealt only *via* intensions, that is, the things being viewed as "intensional like entities" [15]. We preserve most of the original theory here but below we enlarge it with necessary axioms and notions to discuss our issues.

At the metatheoretical level, QST theory is grounded on a first-order logical language with equality (the use of equality makes a huge difference to $\mathfrak{Q}$). Its non-logical language includes the following specific primitive concepts:[4]

1. a monadic predicate: urelement (or ur-object) $(O)$.
2. three binary predicates: the positive membership relation $(\in)$, the negative membership relation $(\notin)$, and the inclusion relation $(\sqsubseteq)$.
   When we write '$x \in y$', we mean that $x$ *certainly belongs to* $y$ , and if $x \notin y$, than $x$ *certainly does not belong* to $y$. Thus, the negation $\neg(x \in y)$ says that it is false that $x$ certainly belongs to $y$, which (according to the axioms, for example by Axiom 2.2) is not equivalent to saying that $x$ certainly does not belong to $y$.
3. a unary functional symbol: the quasicardinal (*qcard*).
4. a binary functional symbol: the quaset-theoretical intersection $(\sqcap)$.

**Definition 2.1.** A quaset is something that is not an ur-object:

$$Q(x) := \neg O(x).$$

**Axiom 2.1.** Everything that has elements is a quaset:

$$\forall x \forall y \ \ (x \in y \to Q(y)).$$

**Axiom 2.2.** If we know for certain that something does not belong to a quaset, then it is not the case that it belongs with certainty to the given quaset. However, the converse is not true in general:

$$\forall x \forall y \ (x \notin y \to \neg(x \in y)).$$

The above axiom has the consequence that there exist instances of the principle $((x \in y) \vee (x \notin y))$ that are refutable, and therefore the possibility of indeterminate membership relations exists. On the other hand, as its authors suggest [5], the expression *knowing for certain* can be interpreted as *belonging for certain* if one does not want to suggest an epistemic reading.

For every formula $\phi$ of ZF (or ZFC), let $\phi^z$ be its corresponding formula in QST relativized to sets (quasets that coincide with its *qextension*).

**Axiom 2.3.** If $\phi$ is any instance of an axiom of ZF, then $\phi^z$ is an axiom of QST.

This ensures that QST can be considered a conservative extension of ZF.

**Axiom 2.4.** The inclusion relation between quasets $(\sqsubseteq)$ is a partial order (reflexive, antisymmetric and transitive).

The symbol '$\sqsubseteq$' has an intensional meaning, but not necessarily an extensional one. '$x \sqsubseteq y$' can be read as "the concept $x$ implies the concept $y$" [5].

**Axiom 2.5.** The inclusion of quasets implies 'extensional inclusion', that is, the inclusion of ZF, but not the other way around.

$$\forall x \forall y (x \sqsubseteq y \to \forall z \ ((z \in x \to z \in y) \wedge (z \notin y \to z \notin x)).$$

**Definition 2.2** (Weak or indeterminate membership)**.** Let $x$ be a quaset and $y$ be any object. We say that $y$ weakly belongs to $x$, and we denote it by $y \in^{-} x$, if it is neither the case that $x$ belongs with certainty to $y$ nor it is the case that $x$ does not belong with certainty to $y$:

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[4] Some of the terminology presented does not correspond to the original article by Dalla Chiara and Toraldo di Francia, but was presented in [15].

$$y \in^- x := \neg(y \in x) \wedge \neg(y \notin x). \quad (1)$$

From now on, when we say that something $x$ *is an element* of a quaset $y$ we mean $x \in y$ (alternatively, '$x$ *strongly belongs* to $y$), but not that $x \in^- y$. In this last case, we say that $x$ *weakly belongs* to $y$.

Intuitively, if $y$ is something that is *not completely inside* $x$, then neither there is absolute certainty about it, nor is it *completely outside* it. For instance, suppose you are looking to a table in front of you and that there is also a chair touching it. Surely there are electrons in both mobiles, but there are also electrons which cannot be said to belong neither to the chair nor to the table; they are in the 'intermediary' region. It is important to realize that in most cases the question of 'determining' to which mobile an electron belongs is not an epistemological fault, but a core result from quantum physics.

**Theorem 2.1.** The following holds in QST:

$$\forall x \forall_Q y((x \in y) \vee (x \in^- y) \vee (x \notin y)). \quad (2)$$

Proof: Immediate from the above definitions. ▮

The above axioms motivate the following definition, which introduces the notion of the *qextension* of a quaset $x$, denoted by *qext(x)* by means of a binary predicate $qext(x, y)$ meaning 'the 'quaset $x$ is the extension of the quaset $y$'.[5] Informally speaking, it is the unique quaset that contains with certainty all the elements of $x$ and with certainty does not contain all other entities, that is,

**Definition 2.3** (qextension or quasi-extension)**.** Let $x$ and $y$ be quasets. Then,

$$qext(y, x) := \forall z(z \in y \leftrightarrow z \in x) \wedge \forall z(z \notin y \leftrightarrow \neg(z \in x)).$$

Alternatively, we will write $y = qext(x)$, that is, $qext(x)$ stands for the quasi-extension of $x$. With this definition, we can recover Dalla Chiara and Toraldo di Francia's concept of the extension of a quaset by means of the following theorem, whose proof is immediate.

**Theorem 2.2** (Dalla Chiara and di Francia's definition of extension)**.** The extension of a quaset x is the unique quaset that certainly contains all the certain elements of x and certainly does not contain all the other elements:

$$\forall_Q x \forall_Q y(y = qext(x) \leftrightarrow \forall z(z \in x \leftrightarrow z \in y) \wedge \forall z(z \notin y \leftrightarrow \neg(z \in x))). \quad (3)$$

Proof: Immediate. The unicity follows from the fact stated below that the extension of a quaset is a set, hence obeying the ZF axioms. ▮

We can define a monadic predicate $Z$ to represent 'classical sets'(or simply 'sets') within QST. QST's internal copy of ZF is given by all quasets whose *qextension* coincides with the quaset itself. That is,

**Definition 2.4.** $Z(x) := x = qext(x)$.

**Theorem 2.3.** The quasi-extension of a quaset $x$ is a set.
Proof: Recall that something is a set iff it coincides with its own quasi-extension. Now it is trivial to see that using Axiom 2.2, any quaset $x$ satisfy definition 2.3. ▮

**Theorem 2.4.** Every quaset has a unique quasi-extension.
Proof: Since the quasi-extension of a quaset is a set and since sets obey the axioms of ZF, they are unique by extensionality. ▮

The expressions $\forall_Q x P(x)$ and $\exists_Q x R(x)$ must be interpreted as $\forall x(Q(x) \to P(x))$ and $\exists x(Q(x) \wedge R(x))$ respectively.

[5] The introduction of this predicate satisfy the conditions mentioned by Suppes in [17, p.156] and is in conformity with Definition 1 of [5].

**Axiom 2.6.** There exists a quaset that necessarily does not contain any element:

$$\exists_Q y \forall x (x \notin y).$$

**Theorem 2.5.** The empty quaset is a set and it is unique.
Proof: Trivial. ▮

If '∩' stands for the usual relation of intersection between sets, then the *weak conjunction* (or quaset-theoretical intersection) between quasets is a primitive concept denoted by ⊓ and we have that

**Axiom 2.7.** The weak conjunction coincides with the usual intersection for the particular case of sets:

$$\forall_Q x \forall_Q y ((x \sqcap y \subseteq x \wedge x \sqcap y \subseteq y) \wedge (Z(x) \wedge Z(y) \rightarrow x \sqcap y = x \cap y)).$$

This means that a *separation* procedure can be applied. Note that the axioms do not require the existence of *proper qua-classes* (other than sets, similar to 'proper classes' in standard set theory).

We will not explain all the axioms that the original version of QST incorporates for quasi-cardinals, since we will not need them for our comparison with the other axiomatic systems. The interested reader can consult them in [10,16]. We will only present one of them, which generalizes the original axiom of the existence of cardinals for quasets. In it, it is not required that every quaset admit a qcardinal. That is, we can have quasets associated with collections without a determined number of elements, as happens in Quantum Field Theory with some collections of photons.

**Axiom 2.8** (qcardinal)**.**

$$\forall_Q x \exists_Z y (y = qcard(x)) \longrightarrow \exists ! y (card(y) \wedge y = qcard(x) \wedge (Z(x) \longrightarrow y = card(x)))$$

If the quaset $x$ has a quasicardinal, then its (only) quasicardinal is a cardinal (defined in the *classical* part of the theory) and coincides with the cardinal of $x$ stricto sensu if $x$ is a set. As recalled above, this axiom does not guarantee that every quaset has a well-defined quasicardinal.

For what comes next, it is important to clarify the following: in the theories associated with quantum ontology that we will discuss below (QST, $\mathfrak{Q}$, $\mathfrak{Q}^-$) we cannot derive the propositions associated with the cardinals from the rest of their axioms, as is the case in ZF, where cardinals can be introduced from the axiomatics. For this reason, they must be added as extra axioms. This is because, at least in $\mathfrak{Q}$ and $\mathfrak{Q}^-$, the cardinals should not be defined as limit ordinals when the collections admit genuinely indiscernible entities. There are qsets that cannot be related bijectively to any ordinal.

A further explanation is this. Usually (for instance, in the von Neumann's stance), cardinals are particular ordinals. Hence, in attributing a cardinal to a set (in the presence of the Axiom of Choice), one is associating to it also an ordinal, and this entails an ordering of its elements, hence a distinction among them. The idea in both $\mathfrak{Q}$ and $\mathfrak{Q}^-$ is to have a cardinal, termed a *quasi-cardinal* (q-cardinal for short) which is not an ordinal. So, a collection of entities simulating a cluster of elementary quantum entities can have a cardinal but its elements continue to be not distinguished from one each other.

## 3. Further extending QST

We have extended the original QST already with new notions and postulates. Next we increase the theory with the introduction of other important notions.

The formalism of QST was not further developed by its authors or by other researchers from 1998 (when [15] was published) until 2023 (with [16]).[6] This is in sharp contrast to the case of $\mathfrak{Q}$, where its developments have continued steadily to date (see for instance [19]). Dalla Chiara and Toraldo di Francia intended to use their system as the basis for a semantics of certain languages dealing with quantum entities (see section 4); with a similar objective, the topic has been taken up again in 2023, but this time within the framework of Nmatrices

6 We recall that [18] is already an extension of the original theory of [5].

semantics [16]. This required expanding the original QST to the following ends: to guarantee the existence of non-classical quasets, so extending the expressive power of defining q-functions (non-deterministic valuations), allowing the existence of quasets without a well-defined cardinality (as happens in $\mathfrak{Q}$, see [20]), so as to formally characterize the *complement* of a quaset, etc. We will therefore present the minimal issues concerning two possible extensions. To the interested reader, we recommend [16, s.3.1-3.5].

The following axiom introduces a new binary connective for the union of quasets ($\sqcup$), which belongs to the extended non-logical language of QST, where '$\cup$' is the usual union of sets. This axiom can be considered the dual of the weak conjunction axiom of quasets (2.7).

**Axiom 3.1** (Union of quasets)**.**

$$\forall x \forall y((x \subseteq x \sqcup y \wedge y \subseteq x \sqcup y) \wedge (Z(x) \wedge Z(y) \rightarrow x \sqcup y = x \cup y)).$$

The following three axioms will be necessary to characterize quasets and establish relations with rough sets and quasi-sets. But before to state them we need to introduce the notion of the *closure* of a quaset. This is similar to the notion of the *cloud* of a quasi-set of $\mathfrak{Q}$ (7.3). Intuitively, the closure of a quaset $x$ is the quaset of the elements which *could be* elements of $x$ or which are *potential elements* of $x$, that is, the elements which it is false that they certainly do not belong to $x$.

**Axiom 3.2** (Schema of Separation)**.** Given a quaset $x$ that is a subquaset of some quaset $w$ and a formula $F(z)$, where $z$ is free, there exists a subquaset $y$ of $x$ whose elements that certainly do not belong to it also certainly do not belong to $x$ (but belong to $w$) and satisfy the property; in symbols,

$$\forall_Q w \forall_Q x(x \subseteq w \rightarrow \exists_Q y(y \subseteq w \wedge \forall z(z \in y \leftrightarrow z \in w \wedge F(z)))).$$

**Definition 3.1** (Closure of quasets)**.** Let $x$ and $w$ be as in the formulation of the Schema of Separation. Taking $F(z)$ as $\neg(z \notin x)$, the schema grants the existence of a quaset $y$ whose elements are those elements of $w$ to which it is false to say that they certainly do not belong to $x$, that is, those elements of $w$ that *could be* elements of $x$, or that weakly belong to it. We denote $\overline{x}$ such a quaset and call it *the clausure* of $x$.

This axiom has among its consequences that if $x \subseteq w$, then $\overline{x} \subseteq w$. In the weakest extension of QST (QST$^+$) we do not want this consequence; therefore, we will not have this axiom, and the closure of a quaset will be guaranteed by an axiom.[7]

**Axiom 3.3** (Closure)**.** Every quaset has a unique closure.

$$\forall_Q x \exists_Q ! y \forall z((z \notin y \leftrightarrow z \notin x) \wedge (z \in y \leftrightarrow \neg(z \notin x))).$$

As can be seen, this axiom is a dual of the axiom that allowed us to define the *qextension* of a quaset (2.3).

Several axiom, including those we will introduce below, do not determine the extension of a quaset, but its closure. Anyway, they enable that the extension can vary between the empty quaset and its closure. Thus, it results defined a family of quasets whose elements vary their qextension, but with a fixed closure. These quasets can be discerned by the elements that weakly belong to them and are generated by the axiom of anti-standardization. When the qextensions need to be restricted, some additional restrictions need to be introduced, such as the condition imposed to the qextension of the cartesian product.

Some remarks are important at this point to make things clear. First is that the inclusion relation entails (as it is easy to prove) that $x \subseteq \overline{x} \rightarrow qext(x) \subseteq \overline{x}$. But by the definition, it results that $qext(\overline{x}) = \overline{x}$. The remark is that $qext(x)$ and $\overline{x}$ are not the same notion, but dual of one another: every element which certainly weakly belongs to $x$ does not necessarily belongs to $qext(x)$, although surely it certainly belongs to $\overline{x}$. Notwithstanding, the converse, namely, $\overline{x} \subseteq qext(x)$ holds only for sets. Summing up, we have $qext(x) \subseteq \overline{x}$ for quasets and $qext(x) = \overline{x}$ for sets.

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7 Another option is to weaken the Schema of Separation a little; for example, by changing $y \subseteq w$ to $y \subseteq \overline{w}$ in the consequent of the conditional.

It is important to note that both the Schema of Separation and the concept of closure do not belong to the original formulation of QST.[8] In the original axiomatics of QST, the separation process could only be performed through the weak conjunction of quasets 2.7. We incorporate them in order to generate some sets that will be of interest in a possible nondeterministic semantics of Nmatrices based on QST or also on RST or $\mathfrak{Q}^{-}$.

**Theorem 3.1.** Every quaset $x$ is a subquaset of its closure.
Proof: By Axiom 2.2, $z \notin x \rightarrow \neg(z \in x)$. Hence $z \in x \rightarrow \neg(z \notin x)$, that is, $F(z)$. ▮

An immediate consequence of the definition is the following theorem, which is important for the sense of 'unique' to be introduced below.

**Theorem 3.2.** The closure of a quaset is a set.
Proof: By the above definitions, we conclude that for every $z$, either $z \in \overline{x}$ or $z \notin x$. ▮

**Axiom 3.4** (Partial standardization)**.**

$$\forall_Q x \forall z \exists_Q y (z \in^- x \longrightarrow (z \in y) \wedge (x \subseteq y) \wedge qext(y) = qext(x) \cup \{z\} \wedge \overline{x} = \overline{y})).$$

We can understand partial standardization as a process by which an element that has indeterminate membership in the quaset $x$ comes to belong with certainty to the quaset $y$, which is a partial standardization of $x$. This process can be repeated sequentially for each element that has indeterminate membership in the quaset $x$, culminating when the *closure* of the quaset is reached. In a way, this axiom allows us to generate all the quasets that are 'between' $qext(x)$ and $\overline{x}$. Therefore, having this axiom guarantees the existence of the closure of every quaset without the need for the axiom 3.3. The reason we require for the closure axiom is because there will be extensions of QST (see below), such as $QST^{+}$, that do not have the standardization or anti-standardization axioms. These axioms are incorporated into stronger extensions, such as $\overline{QST}^{+}$.

The reverse process, in which elements that belong with certainty to $x$ become indeterminately part of the new quaset $y$, is called *partial anti-standardization*. This axiom is the first to guarantee the existence of non-classical quasets, which in the axiomatics of QST are not required by its authors. The original axiomatics is compatible with the absolute absence of non-classical quasets. Given a classical set $x$, the following axiom guarantees the existence of a quaset $y$ whose qextension and closure do not coincide.

**Axiom 3.5** (Partial anti-standardization)**.**

$$\forall_Q x \forall z \exists_Q w (z \in x \longrightarrow z \in^- w \wedge w \subseteq x \wedge qext(x) = qext(w) \cup \{z\} \wedge \overline{x} = \overline{w}).$$

**Axiom 3.6** (General anti-standardization)**.**

$$\forall_Q x \forall_Q y \exists_Q \gamma \forall z (y \subseteq x \wedge z \in y \longrightarrow z \in^- \gamma \wedge \gamma \subseteq x \wedge qext(x) = qext(\gamma) \cup qext(y) \wedge \overline{x} = \overline{\gamma}).$$

This procedure allows all elements of any sub-quaset $y \subseteq x$ to go from belonging with certainty to belonging weakly. From this last axiom, the one of *partial anti-standardization* can be deduced by taking $y = \{z\}$.

**Definition 3.2.** The above axiom allows us to define a quaset ($\gamma$), which we will call anti-standardization of $x$ and we will denote it by $w^x_y$. Where it is made explicit that the subquaset $y$ was anti-standardized to the quaset $x$.

The general anti-standardization axiom proves the existence of non-empty quasets with empty qextension. In the notation presented above, these quasets are denoted as $w^x_x$. The anti-standardization process preserves the closure of the quaset, changing its qextension between the sets $\overline{x}$ and $\emptyset$. Therefore, we are guaranteed the existence of quasets whose elements all have weak membership. For every set $x$ in ZF (of the isomorphic copy containing QST), there exists a unique quaset $w^x_x$, such that

$$qext(x) = \overline{x} = \overline{w^x_x} \wedge qext(w^x_x) = \emptyset.$$

[8] The Schema of Separation was also not required this axiom in the formulations presented in [16].

It can be seen that, by definition, it is true that (for every quaset $x$) $w^x_x = w^x_{qext(x)}$.

The next axiom is for the power quaset. To physically justify the existence of subquasets of a given quaset. In [5], the authors express:

> *It is intriguing to note that there are even 'subsets' inside the 'set', each one with its own cardinality. For example, you can say that inside a sodium atom there are two electrons in the shell Is, two electrons in 2s, six electrons in 2p, one electron in 3s. Thus there is a sort of isolation procedure. You can state a property (e.g. having the [azimuthal] quantum number $\ell = 1$) and you can tell how many electrons form the 'subset' having that property (six), even if you cannot distinguish those electrons from all the others!*[9]

**Axiom 3.7** (Power quaset)**.**

$$\forall_Q x \exists_Q y \forall z((\neg(z \notin y) \longleftrightarrow (z \subseteq x)) \wedge (z \subseteq qext(x) \longrightarrow z \in y)).$$

We will denote the power quaset by $\mathcal{P}_q(x)$. When $x$ is equal to its qextension ($x = qext(x)$), this axiom prohibits the power quaset from admitting elements with indeterminate membership.

If $x = qext(x)$ and $y = \mathcal{P}_q(x)$, then (using the previous axiom)

$$z \in^- y \equiv \neg(z \in y) \wedge \neg(z \notin y) \Rightarrow \neg(z \notin y) \Rightarrow z \subseteq x \Rightarrow z \subseteq qext(x) \Rightarrow z \in y.$$

That is, there cannot exist elements with indeterminate membership in the quaset power of a classical set.

So far, the axiom is in agreement with what is classically expected. But let us now consider the following case: let $x, w$ be two quasets such that they do not coincide with their respective *qextensions*, but are so that $qext(x) = \overline{w}$. If we want the quaset power for the quaset $x$, by the axiom 3.7, it should be satisfied:

$$\forall z((\neg(z \notin y) \longleftrightarrow (z \subseteq x)) \wedge (z \subseteq \overline{w} \longrightarrow z \in y)).$$

Since (4) implies that $w \subseteq \overline{w}$, it follows that $w \in y$. That is, the candidate power quaset of $x$ includes $w$ (which is a non-classical quaset) within its *qextension* (since $w \in y \longleftrightarrow w \in qext(y)$). This departs from the classical case where $(qext(y) \neq \mathcal{P}(qext(x)))$. Notice that this axiom does not determine the power quaset univocally, as we had already anticipated.

The following axiom can be considered another alternative for the power quaset.

**Axiom 3.8.** $\forall_Q x \exists_Q y \big(\forall z(\neg(z \notin y) \longleftrightarrow (z \subseteq x)) \wedge (Z(x) \longrightarrow y = \mathcal{P}(x))\big).$

Where $\mathcal{P}$ denotes the power set in the sense of ZF. If $x$ is not a classical set, then the axiom 3.8 does not determine the *qextension* of the power set uniquely. This is because it is not specified which subquasets of $x$ belong with certainty to $y$. This means that, in the most general case, the power quaset is not uniquely defined. As far as our semantic goals are concerned, this will not cause any problems. Something similar happens in $\mathfrak{Q}$, where it is also not provable that this set is unique (see [21]).

To clarify this a little bit more, and to see what kind of indefiniteness the power quaset admits, let us analyze further the previous axiom. Since the first term of the conjunction is equivalent to $z \notin y \longleftrightarrow \neg(z \subseteq x)$, for each quaset $x$, it is absolutely defined which entities do not surely belong to the power quaset $y$. The non-uniqueness in the power quaset is due to the entities that have indeterminate membership in $y$, for which $qext(y)$ is not univocally defined. That is, if $y$ and $y'$ are two quasets that satisfy the conditions of the previous axiom (power quasets of $x$), then they only differ in that some elements that have indeterminate membership in one of them belong with certainty to the other (or vice versa). The extreme cases would be when all of them belong with certainty, for example, to $y$, and indeterminately to $y'$.

[9] The quantum numbers of the electrons in the 2p shell are $n = 2$, $\ell = 1$, $m_\ell \in \{-1, 0, 1\}$ and $m_s \in \{1/2. -1/2\}$. The quantum numbers of all other electrons involve $\ell = 0$. So, in saying that the property is 'having quantum number $\ell = 1$', they authors are referring to the electrons in that shell.

With what was said above, given a quaset $x$, the following is fulfilled:

$$qext(x) \subseteq x \subseteq \overline{x}. \tag{4}$$

It can be easily seen that if $A^q$ is a quaset identical to its *qextension*, then

$$qext(x) = x = \overline{x}.$$

The (4) equation has a very particular consequence. It states that classical quasets (sets) can have non-classical quasets as subqsets. This is a direct consequence of the inclusion of quasets and the definition of *closure*, which is always a classical set. This peculiarity of quasets has a direct impact on the different possibilities when defining the power quaset. It also reflects an important difference of QST with respect to $\mathfrak{Q}$, where classical qsets cannot include non-classical qsets.

For each quaset $X$, we can define a new quaset, $AE(X)$, consisting of all the *general anti-standardizations*, $w_y^x$. For this, we introduce the following axiom.

**Axiom 3.9.** $\exists_Q \Gamma(\forall_Q \gamma(\gamma \in \Gamma \leftrightarrow \gamma = w_y^x) \wedge \overline{\Gamma} = qext(\Gamma))$

Where $w_y^x$ satisfies the axiom 3.2.

**Definition 3.3** ($AE(x)$)**.** We call the set $\Gamma$ of the previous axiom the Anti-standardization set of x and denote it by $AE(x)$.

If $y = \emptyset \subseteq x$, then $w_\emptyset^x = x$, which means that *anti-standardization* is trivial. If $y = x \subseteq x$, then the *anti-standardization is total*. It follows from the above that $x \in AE(x)$. In the same way as we did with *anti-standardization*, we can generalize the axiom of *partial standardization*.

To characterize the complement of quasets, we present the following axiom:

**Definition 3.4** (complementary quasets)**.** Let $A$ and $x$ be quasets such that $x \subseteq A$. The complement of $x$ relative to $A^q$, denoted by $x_A^c$ (we will dispense with the subscript $A$ when the context is clear), is characterized by:

$$x \sqcup x^c = \overline{A} \quad \wedge \quad x \sqcap x^c = \emptyset \quad \wedge \quad (x^c)^c = x \tag{5}$$

$$(qext(x))^c = qext(x^c) \sqcup \{y \in \overline{A} : y \in^- x\}. \tag{6}$$

Where the primitives $\sqcap$ and $\sqcup$ are those introduced in 2.7 and 3.1.

**Axiom 3.10** (Unordered pair quaset)**.** $\forall x \forall y \exists_Q z\, (\neg(x \notin z) \wedge \neg(y \notin z))$

This axiom says that given two entities $x$ and $y$, whether they are quasets or urelements, there exists a quaset $z$ for which each either belongs with certainty to the quaset $z$ or has indeterminate membership. It does not require that such a quaset be unique, nor does it explain anything about its *qextension*. If both $x$ and $y$ belonged with certainty to the quaset, one would expect their quasi-cardinality to be greater than or equal to 2. But both could have indeterminate membership in $z$ (along with other elements). We will denote by $\{x, y\}_A$ a quaset that satisfies the previous axiom and whose *qextension* has at most $x, y$. This does not imply $x \in \{x, y\}_A$, nor $y \in \{x, y\}_A$, since both could have indeterminate membership. But it does imply that if $x' \neq x$ and $x' \neq y$, then $\neg(x' \in \{x, y\}_A)$. The subscript $A$ is due to the fact that we are going to relativize the unordered pair to a classical set $A$, such that both the elements that belong with certainty to the pair and those that belong indeterminately to it are elements of $A$ (if $A$ were not classical, we could work with its *closure*). Therefore, we will have $\{x, y\}_A \subseteq A$ and $qext(A) = A$. We do this in order to have a little more control over the elements that can belong indeterminately. On the other hand, the set $A$ can be directly related to what in section 3 of [22] (and in our next section) is called the *cloud* of the qset. This concept of $\mathfrak{Q}$ can be key in establishing an important link between QST and $\mathfrak{Q}$. In view of what was mentioned above about the closure of a quaset, we can affirm:

$$qext(\{x, y\}_A) \subseteq \{x, y\}_A \subseteq \overline{\{x, y\}_A} \subseteq A \tag{7}$$

**Definition 3.5** (Orderer pair)**.** .

$$(x,y)_A := \{\{x\}_A, \{x,y\}_A\}_{\mathcal{P}(\mathcal{P}(A))}.$$

Where the power quaset can be taken as any of those given in 3.7 or 3.8.

The definitions of Cartesian product, q-relation and q-function can be found in [16].

### *3.1. Possible QST extensions*

This section presents some possible extensions of QST that will be useful for our purposes. We will define two new quaset theories, $QST^+$ and $\overline{QST}^+$, as alternative extensions of the original QST axiomatics. One goal is to use these formalisms in the metalanguage of nondeterministic Nmatrices semantics.

**QST**$^+$

The language of $QST^+$ is the original language of QST expanded with a primitive union of quasets symbol ($\sqcup$). All the axioms of QST are part of the axioms of $QST^+$, but this one has extra axioms such as the axioms of generalized cardinal 2.8, union of quasets (3.1), unordered pair (3.10), power quaset (3.7 or 3.8), *closure* (3.1). Furthermore, conditions such as the one relative to the complement (6) and the one we include below, which determines the *qextension* of the Cartesian product (8), are considered to be part of the theory.

$$qext(A^q \times B^q) = qext(A^q) \times qext(B^q) \tag{8}$$

The consequences of having imposed this condition can be seen in [16] (section 3.4).

Therefore, we have that:

$$\begin{aligned} QST^+ \text{ Language} &= QST \text{ Language} + \{\sqcup\} \\ QST^+ \text{ Axioms} &= \text{QST Axioms} + \{\text{Generalized Card (2.8), Union (9), Pairing (3.10),} \\ &\qquad \text{Power quaset ((3.7) o (3.8)), Closure (3.3)}\} \end{aligned}$$

QST$^+$ does not have the Schema of Separation (3.2). Instead, it has the axioms (2.7) and (3.3).

The additional axioms (3.4) (partial standardization) and (3.5) (partial anti-standardization) allow access to more quasets, which could be useful at some point for purposes beyond semantics. For this reason, we'll include them as an alternative in what we'll call $\overline{QST}^+$.

$\overline{\textbf{QST}}^+$

$$\overline{QST}^+ \text{ Language} = QST^+ \text{ Language}$$

$$\begin{aligned} \overline{QST}^+ \text{ Axioms} = QST^+ \text{ Axioms} + \{&\text{Schema of Separation (3.2), Partial standardization (3.4),} \\ &\text{Partial anti-standardization (3.5)}\} \end{aligned}$$

If necessary, the general versions of the last two axioms can be used. Versions where the Schema of Separation is weakened can also be considered.

## 4. A glimpse on the application of QST to quantum semantics

The main purpose of the quantum set theories mentioned in this paper (QST, $\mathfrak{Q}$, $\mathfrak{Q}^-$) is the discussion of possible semantics for some quantum languages.The possibility of the application of QST to these problems was explored in [10,23]. Due to problems of space, we present only a short resume of the main ideas to show how QST and its extensions. We shall refer to a system with just two indistinguishable components, but the results can be extended to $N$ with no difficulty.

Suppose we have a quantum system involving two 'identical' systems whose wave functions are $\psi_1(x)$ and $\psi_2(x)$. Calling the systems $p_1$ and $p_2$, it is clear that $\{p_1, p_2\}$ cannot be viewed as a set in the standard sense since, by hypothesis, the systems are indiscernible. The whole system is represented by the wave function (representing a pure state of the system) $\psi = c\psi_1 + d\psi_2$ in the product space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of the Hilbert spaces of the compound systems (which as usual we call 'particles'), being $c$ and $d$ complex numbers such that $c^2 + d^2 = 1$. The Schrödinger equation governs the time evolution of $\psi$. Let us call $\mathcal{L}$ a language for speaking of all of this. The language would encompass monadic predicates $Q_i$ to represent the *meaningful properties* of the single systems. The first problem is that $\mathcal{L}$ cannot incorporate individual constants $a_1$ and $a_2$ to *name* the particles since they are indiscernible and, as said in [5], "microphysics is the land of anonymity". Proper names do not exist in this realm, and the labels we attach to the particles do not act as rigid designators, which means that they would name the *same* particle in different contexts (or possible worlds). Even if we mockingly name the particles, say by $a_1$ and $a_2$, these 'names' do not identify them as individuals but are just linguistic devices we use to *refer* to the fact that we have two particles. In $\mathfrak{Q}$ and $\mathfrak{Q}^-$, this is expressed by positing that we have a quasi-set whose elements are indistinguishable quantum systems of a kind and that the quasi-cardinality of the whole collection is two; there is no identification. This reflects the idea that when we have a molecule such as $H_2SO_4$, we have only the kinds of things being involved (hydrogen, oxygen and sulfur atoms) and their quantities, without particularisation of the atoms (something emphasised by John Dalton in 1808).

In QST, identity holds for all entities and hence we would be able, at least in principle, of distinguishing (and name) among the atoms of the same kind. But, even so, we do not wish to compromise the theory with particular (proper) names. As recalled by Dalla Chiara, the problem is not that "whether or not *we are allowed* to introduce names (. . . ) but rather whether or not *we are able* to determine a reasonable denotation-function for such names." [23]. To be faithful to quantum physics, we should not be able to think of such names.

In this sense, the extension of a predicate $Q_i$ should not determine a well-defined subset $Ext(Q_i)$ of the domain of entities; when we speak of the property 'to be an electron', there are several *electron quasets* that could be taken as its extension. All we have are the *intensions*, that is, the *descriptions* of the involved entities, and it is assumed that these intensions give rise to quasets as their extensions. The problem is that if $Q_i(a_1)$ is intended to express that 'particle 1 has the property described by the predicate $Q_i$' (something that can be described in the formalism), the denotation function (let us call it '$\varrho$'), in associating a quaset $\varrho(Q_i)$ to the predicate, does not enable us to *know with certainty* if the element $\varrho(a_1)$ certainly belongs or not to the quaset. By the way, $\varrho(a_1)$ does not identify a very specific element of the domain as the referent of the mock name $a_1$. As Dalla Chiara reminds us,

> "According to a main trend in traditional semantics (from Stuart Mill to Kripke) the use of proper names seems to be essentially connected with an *ostensive* function. The behaviour of identical particles in QM shows clearly that 'at this level of reality' proper names cannot be used in an ostensive way. (. . . ) One might go further and ask: to what extent is our way of arguing about the external world essentially founded on *names* and *predicates*? Are these traditional syntactic categories really essential for an adequate logical description of the microworld?" [23].

The semantics Dalla Chiara constructs is a kind of Kripke semantics and interesting results are advanced.

The same remarks can be put concerning the applications of $\mathfrak{Q}$ and of $\mathfrak{Q}^-$ to the same problem. But in this last case, the very notion of identity does not hold for all objects as it happens in QST, so it seems that these theories are more adequate to express a semantic of non-individuals and not only of blurred names and properties. We proceed to present the theories of qsets $\mathfrak{Q}$ y $\mathfrak{Q}^-$.

Another possible application enters the difficult issue of the interpretation of quantum superpositions. In our account, our idea is still just a guess, so we will only sketch it and leave the development to another paper.

In quantum physics, an observable is associated to a self-adjoint operator. The theory also says that a photon can behave either like a particle or like a wave, and can exist in a superposed state of both states, let us call them $|P\rangle$ and $|W\rangle$ for simplicity. Then the superposed state is (in a simplified way) $|\phi\rangle = |P\rangle + |W\rangle$. As Abner Shimony says,

> " If a particle-like property is measured, the photon behaves like a particle, and if a wave-like property is measured, the photon behaves like a wave. Whether the photon is wave– or particle like *is indefinite* until the experimental arrangement is specified." (our emphasis) [24]

That is, the state $|\phi\rangle$ will collapse into either $|P\rangle$ or $|W\rangle$, but before the measurement, the observable not have a definite value, and that it not simply unknown, as says Shimony, "by the scientist who seeks to describe the system". We can understand this as follows. Let us associate a quaset $x$ to the observable $P$ (particle) and another $y$ to $W$ (wave). The elements of these quasets are the possible values that the observable can assume with certainty when a series of measurements of the observable are made (in a probabilistic sense). The closures of these quasets represent the *indefinite* values of each observable. This makes more sense if we consider the union of the closure of the quasets, $\overline{x}$ and $\overline{y}$, that is, $\overline{x} \sqcup \overline{y}$. It seems clear that *it is indefinite*, and not merely unknown, if the elements of this union belong to either $\overline{x}$ or $\overline{y}$, so we may say that this is the quaset associated to the superposition. This way, we can say that we have a *quaset theoretical* view of a superposition.

But notice that this is not so in classical logic, that is, in a set theory such as ZF. In this case, there are no *indefinite* values; either a quantity has a certain value in a quantum state or it does not; the only accepted ignorance is epistemological. There is no alternative (the Excluded Middle Law holds).[10]

## 5. The $\mathfrak{Q}$ and the $\mathfrak{Q}^-$ theories

Quasi-set theory $\mathfrak{Q}$ was proposed to deal with collections of completely indiscernible things. A quasi-set can have as elements entities that cannot be discerned in any way. Other qsets can be formed by discernible entities, and standard sets can be viewed as particular cases of qsets. Thus, in that we call standard sets or just 'sets' to abbreviate, the elements are always distinct from each other. The meaning of "distinct" will become clear later when we discuss rigid structures.

$\mathfrak{Q}$ is a semi-extensional theory. The expression 'semi-extensional' reports to the fact that the theory of quasi-sets do not encompass an Axiom of Extensionality in the usual sense, but a Weak Extensionality Axiom which informally says that qsets comporting 'the same quantity' (in terms of quasi-cardinals, see below) of elements of the same kind (indiscernible among them) are indiscernible, as exemplified above (for instance, when we say that two sulfuric acid molecules $H_2SO_4$ are indiscernible).

The first theory of quasi-sets was a theory with atoms, entities that can be elements of the qsets but which (in principle) do not have elements. This is in accordance with the set theories with atoms, such as the ZFA set theory (Zermelo-Fraenkel with atoms; see [25,26]). In the theory of quasi-sets, it was supposed to be the possibility of existence (the theory does not postulate their existence) of two kinds of atom, the m-atoms, which are supposed to be entities to which the standard theory of identity (STI) does not apply, and the M-atoms, which satisfy the axioms of ZFA, so are entities endowed with identity conditions.[11] Collections of these entities, perhaps also involving other qsets, are called *quasi-sets*. *Sets* are those qsets whose transitive closure does not contain m-atoms; so, they are those qsets constructed in *classical part* of the theory. When we restrict the theory to this 'classical part', it becomes equivalent to ZFA, and to ZFC if we drop also the M-atoms.

A qset may have a *quasi-cardinal*, something that is intended to express the quantity of elements it has, so that it resembles the cardinals of sets. If $x$ is a qset, then $qc(x)$ expresses its quasi-cardinal, or simply 'q-cardinal' for short.

The main motivation for the theory is the interpretation that ascribes to quantum entities (electrons, protons, photons, atoms, etc.) an absence of identity. In some situations, m-atoms cannot be distinguished from other m-atoms in any way, and even so they can be counted as numerically distinct. This fact is contrary to what STI says: any mathematical theory encompassing STI and which does not consider any kind of substratum is such that given two entities whatever, they are *distinct* (different) and this entails that there exists (even if only in principle) a property satisfied by just one of them. But in the interpretation assumed here, quantum entities can sometimes be completely indiscernible, such as bosons in a bosonic condensate (a BEC, see [27,28]). Fermions cannot be in the same quantum state due to obedience to Pauli's Exclusion Principle, but even if they present a difference, the situation can be one in which we cannot identify which item is which. For instance, if we consider the two electrons of a neutral helium atom, the total pure state is a superposition of states, say, of one electron

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10 When we say ""Excluded Middle Law" we mean having "$\in$" and "$\notin$" as the only alternatives. What is said in this section can be related to what is stated in "[16]" about complementary quasets (see section 3.3).

11 STI is formulated standardly as in ZFC: we have a primitive binary predicate '=' subjected to Reflexivity, Substitutivity and Extensionality; see [12].

having spin UP in some direction and the other having spin DOWN in the same direction. But ‘to have spin UP’ (or DOWN) cannot be considered as a property of the electrons. In the superposed state, we can say that they do not have particular properties; only the entire system has, say, total spin zero. Only after a measurement of the spin in the chosen direction can we say that *one of them* has spin UP and the other has spin DOWN, but there is no sense to say something like ‘electron Peter has spin UP’ as if ‘Peter’ were a proper name. The quantum realm is a world of anonymity; proper names do not act as rigid designators [5]. Due to this fact, there is a sense in saying that even fermions can be considered as indiscernible.

Even if a qset has only m-atoms as elements, this does not imply that they are all completely indiscernible. m-atoms can differ by some properties, as electrons differ from protons and neutrons, and even quantum things of the same kind, say photons, can differ in polarization among other things. We assume that in quantum theory, two things are of fundamental importance: *kinds* (of things or items) and *quantities*. For instance, a proton is considered as being formed by two quarks up and one quark down. No differences between the two quarks up are given. What matters is that they are quarks up and that they are two. No identity is involved (in the sense of STI, which would imply that they present some difference). Kinds and quantities; this is what the theory of quasi-sets enables us to consider.

### $\mathfrak{Q}^-$: A quantum set theory without atoms

The main difference between the theory presented here and previous proposals is that it does not commit itself to atoms (urelements): it is a ‘pure’ theory [29]. Thus, ‘all we have’ are quasi-sets, which can be regarded as extensions of predicates (in a different sense than in ZFC). Furthermore, the new theory does not resort to cardinals (limit ordinals of ZF) to assign numbers of elements to its collections, thereby avoiding discerning elements due to ordering.

Like the rest of the theories that we will discuss in this work, $\mathfrak{Q}^-$is a formal theory.[12] So, despite that we refer to qsets, sets, and other entities, the theory is in principle not compromised with any intuitive sense of these words. The intended interpretation is this: *sets* are copies of sets in ZFC, the Zermelo-Fraenkel set theory with the Axiom of Choice. Quasi-sets (qsets) are items to which STI does not apply. So, it makes no sense to give an interpretation to expressions such as $x = y$ or $\neg(x = y)$ when at least one of these variables represents a qset.

This does not mean that we cannot define the identity for qsets; the problem is that we avoid doing that precisely to be able to express the indiscernibility of qsets, which will be taken as the semi-extensions of formulas with one free variable, which we call *properties*. So, properties can be indiscernible and this does not mean (as would be implied by STI) that they are *the same* property.

A possible interpretation of indiscernible properties could be this: indiscernible properties would be properties that we can measure repeatedly, say when a physicist reports that she is measuring *the same property* more than once when, in reality, she is measuring an *indiscernible* property got by preparing the system in a similar (*indiscernible*) way.

We begin by supposing the existence of an intended universe $\mathfrak{V}$ (*Bereich*, as called by Zermelo [30]) whose elements are of three kinds: the quasi-sets (qsets), the sets, and the finite quasi-cardinals (finite q-cardinals). The language $\mathcal{L}$ of the theory encompasses the following category of primitive symbols:

(i) Standard logical symbols (let us assume negation, ‘$\neg$’, implication ‘$\rightarrow$’, and the universal quantifier ‘$\forall$’ as primitive). Improper symbols such as parentheses and commas are also supposed. The other standard propositional connectives (‘$\wedge$’, ‘$\vee$’, and ‘$\leftrightarrow$’), as well as the existential quantifier (‘$\exists$’) are defined as usual.

(ii) Individual variables of two kinds: $x, y, z, \ldots$ are metavariables for qsets and sets, and $m, n, p, \ldots$ are metavariables for finite q-cardinals.

It is remarkable that despite the variables $x, y, z, \ldots$ being called *individual variables*, they may range over a domain whose elements are non-individuals, that is, entities devoid of identity conditions, which form our ‘phenomenology’. Furthermore, for the application we have in mind, we should recall what we have said in [20]: “An ontological domain populated exclusively by properties and non-individual bundles

[12] The minus sign above the letter $\mathfrak{Q}$ indicates that the atoms are being ruled out.

of properties cannot be adequately apprehended by any language that includes individual constants and variables." Anyway, there is no easy sense of starting with a language not comprising individual variables.

(iii) An unary predicate symbol $S$ to designate sets.

(iv) The equality symbol, '$=$'.

(v) The indiscernibility symbol, '$\equiv$'.

(vi) The membership relation '$\in$'.

(vii) A binary predicate symbol $K$.

(viii) Three specific primitive symbols to be used with finite q-cardinals: an individual constant $\overline{0}$, a unary functional symbol $s$ and two binary functional symbols, $\otimes$ and $\oplus$.

The *terms* of $\mathcal{L}$ are the individual variables, the individual constants $\overline{0}$, the expressions of the form $s(m)$ (which we abbreviate writing simply '$sm$'), $m \oplus n$ and $m \otimes n$.

**Definition 5.1** (Order for q-cardinals)**.**

$$n \leq m := \exists p(m = n \oplus p)$$

The formulas are defined recursively as usual, with the proviso that the predicate $K$ is used this way: only expressions of the form $K(x, m)$ are formulas. If $S(x)$, then we say that $x$ is a set. We make use of restricted quantifiers; therefore, $\forall_S x\varphi$ stands for $\forall x(S(x) \rightarrow \varphi)$, while $\exists_S x\varphi$ abbreviates $\exists x(S(x) \wedge \varphi)$, where $\varphi$ is a formula.

The logical axioms, that is, the postulates (axiom schema and inference rules) of the underlying logic are those below; we insist in showing them despite being 'classical' because of the care we need to enlighten due to the presence of q-cardinals. So, we have the following axiom schemata, where $\alpha, \beta$, etc. are formulas and $x$ is a variable:

(1) $\alpha \rightarrow (\beta \rightarrow \alpha)$

(2) $(\neg\alpha \rightarrow \neg\beta) \rightarrow ((\neg\alpha \rightarrow \beta) \rightarrow \alpha)$

(3) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$

(4) $\forall u(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \forall u\beta)$, where $u$ is either a variable for quasets or for q-cardinals and does not appear free in $\alpha$.

(5) $\forall u\alpha \rightarrow \alpha(t)$, where $t$ is a term free for $u$ in $\alpha$ and of the same kind of $u$.[13] We remark that if $\alpha$ is $K(x, m)$, then the term must be in accordance with the kind of entity it is representing, that is, qsets or sets for $x$ and q-cardinals for $m$.

(6) The inference rules are Modus Ponens and Generalization, that is,

$$\frac{\alpha, \alpha \rightarrow \beta}{\beta}, \qquad \frac{\alpha}{\forall u\alpha},$$

where $v$ is a variable for qsets or for q-cardinals, the standard conditions being observed.

(7) $\forall_S x(x = x)$

(8) $\forall_S x\forall_S y(x = y \rightarrow (\alpha(x) \rightarrow \alpha(y)))$

(9) $\forall m(m = m)$

(10) $\forall m\forall n(m = n \rightarrow (\alpha(m) \rightarrow \alpha(n)))$

Standard conditions are imposed for (8) and (10), given the above restrictions mentioned in the case of $K(x, m)$.

[13] As usual, $\alpha(u_1, \ldots, u_n)$ means that the variables $u_i$ are among the free variables of $\alpha$; therefore, $\alpha(t_1, \ldots, t_n)$ is obtained by the substitution of $u_i$ by $t_i$.

The postulates say that for sets and q-cardinals, the standard postulates of the first-order calculus with equality hold, but for qsets the same calculus *without* identity holds. Since the notion of equality (or identity since we are not making a difference between these two notions) does not hold for qsets, the Principle of Identity in the form $\forall x(x = x)$ does not apply to either of them. Since this is also called the reflexive rule of identity, this logic enters in the realm of *non-reflexive logics* ([12])

For the purposes of this article, we do not consider it necessary to explain the axioms corresponding to the q-cardinals. For the same reasons, we will present only the strictly necessary axioms about qsets. Such axioms and their respective reflections can be found in [29].

The basic idea is to suppose a domain comprising *quasi-sets*, some of them being classified as *sets*, which will obey the postulates of ZFC. For other qsets, termed 'pure' in the next definition, the axioms are as follows.

**Definition 5.2** (Pure qsets)**.** A 'pure' qset is defined as follows using the primitive predicate $S$:

$$P(x) := \neg S(x).$$

In other words, a pure qset is an item that is not a set. This makes them do not obey *all* the ZFC axioms, especially STI.

Thus, some specific axioms of $\mathfrak{Q}^-$ are:

(1) The ZFC axioms for sets, that is, for items that satisfy the predicate $S$ and the the postulates of first-order Peano arithmetic for finite q-cardinals.

(2) $\forall_S x \forall_S y(x \equiv y \to x = y)$

(3) $\exists_P x(x \equiv x) \wedge \forall x(x \equiv x)$

(4) $\forall x \forall y(x \equiv y \to y \equiv x)$

(5) $\forall x \forall y \forall z(x \equiv y \wedge y \equiv z \to x \equiv z)$

(6) $\forall y(S(y) \to \forall x(x \in y \to S(x)))$.

The last axiom says that the members of a set are also sets. In saying that the q-cardinals obey the axioms of Peano's arithmetic, we are identifying then with either $\overline{0}$, $s\overline{0}$, $ss\overline{0}$, etc. Then we define the q-cardinals (really, 'q-numerals') $\overline{1} := \overline{0}, \overline{2} := \overline{1}$, etc. Notice that these q-cardinals *are not* ordinals since we are not identifying then with the elements of one of models, namely, with $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, etc. Of course these ordinals also model the q-cardinals, but *they are not to be identified* with them.

(7) [Separation Schema] Let $\alpha(z)$ be a formula with the variable $z$ free:

$$\forall x \forall m(K(x,m) \to \exists y \forall z(z \in y \leftrightarrow \exists w \forall n(w \in x \wedge z \equiv w \wedge K(y,n) \wedge n \leq m) \wedge \alpha(z)))$$

That q-set $y$ will be denoted by $[z : \alpha(z)]_x$, noting that its q-cardinality must not exceed the q-cardinality of $x$. Given the qsets $x$, $z \in x$ and let $\alpha$ be the formula defined by $\alpha(w) \leftrightarrow w \equiv z$, then we can infer (from axiom (7)) the existence of a qset $[w : z \in x \wedge w \equiv z]_x$, which is an *equivalence class* of indistinguishables from $z$ that are not necessarily in $x$ (although they can be).[14] Using the following union axiom, we can gather those classes so that we form the quotient set $x/_{\equiv}$, which will be used later.

(8) The q-cardinality of any empty set is zero:

$$\forall x(K(x, \overline{0}) \leftrightarrow \neg \exists y(y \in x)).$$

The union axiom of $\mathfrak{Q}^-$can be formulated to obtain a qset $\bigcup x$ from a qset $x$, whose elements are nonempty qsets. We admit this generalization, but we will formulate it only for qsets. Thus, given $x$ and $y$, the axiom says that there exists a qset, denoted by $x \cup y$, whose elements are indistinguishable from the elements of $x$ or $y$, and whose q-cardinality is no greater than the sum of the q-cardinalities of $x$ and $y$.

[14] We recall that the q-cardinal of that class is no larger than the q-cardinal of $x$.

(10) [Union]

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow \exists w' \exists w"((w' \in x \wedge w \equiv w') \vee (w" \in y \wedge w \equiv w")) \wedge \\ \forall m \forall n (K(x,m) \wedge K(y,n) \rightarrow K(z,p) \wedge p \leq m \oplus n)). \tag{9}$$

The elements of the q-union set $x \cup y$ need not be elements of any of those sets; it is enough that they are indistinguishable from elements of them.

**Definition 5.3** (Weak singleton)**.** Let $w$ be a qset and $x$ be an element of $w$. By means of the condition $\alpha(t) \leftrightarrow t \in w \wedge t \equiv x$, we get by Separation the qsets of the indiscernible from $x$ which belong to $w$, which we denote by '$[x]_w$'.

Then the Axiom of Pairing can be formulated like this:

(11)[Pairing] $\forall_Q w \forall x \forall y (x \in w \wedge y \in w \rightarrow \exists z \forall t (t \in z \leftrightarrow (t \in [x]_w \vee t \in [y]_w)))$

We denote this qset by

$$[x,y]_w.$$

In principle, nothing tells us about the q-cardinality of that qset, except that it cannot supersede the q-cardinality of $w$. If $x \equiv y$, it could be $\overline{1}$. The axiom could be much more general and say that we form $[x,y]_w$, not by taking elements that are in $w$, but that are indistinguishable from elements of $w$ (as long as we limit their q-cardinality so as not to form proper q-classes). Note that using the above axiom, we can form the elements of the q-set $t/_\equiv$, namely $[x]_t$, $[y]_t$, etc., for $x, y, \ldots \in t$.

(13) [Weak Extensionality Axiom, WEA]

$$\forall x \forall y \forall n \Big( (\forall z \in x/_\equiv)(\exists w \in y/_\equiv)(K(z,n) \rightarrow K(w,n)) \wedge \\ \forall u \forall v (u \in z \wedge v \in w \rightarrow u \equiv v)) \\ \wedge (\forall w \in y/_\equiv)(\exists z \in x/_\equiv)(K(w,n) \rightarrow K(z,n) \\ \wedge \forall u \forall v (u \in w \wedge v \in z \rightarrow u \equiv v))) \rightarrow x \equiv y \Big) \tag{WEA}$$

If we use the identity relation ($=$) instead of the indiscernibility relation, then the equivalence classes of $x/_\equiv$ will be unitary (same with $y/_\equiv$) and the axiom reduces to the Axiom of Extensionality of ZFC; in that case, $x \equiv y$ is nothing more than $x = y$.

What does this axiom say? A careful reading will show that it tells us that if qsets $x$ and $y$ have 'the same number' (given by q-cardinals) of indiscernible elements, then $x$ and $y$ are indiscernible. The axiom WEA allows us to prove what we said above, that all qsets with no elements are indiscernible. The proof is immediate, recalling that the q-cardinal of such qsets is $\overline{0}$.

## 6. The RST theory

As we said, Pawlak's Rough Sets theory can be considered as a mathematical approximation to *vagueness*. The theory considers that each entity in its domain is characterized by some (usually incomplete) information and when this information coincides for two different entities, they are indiscernible. The indiscernibility relation generated in this way is the mathematical basis of rough set theory. This understanding of indiscernibility is related to Gottfried Wilhelm von Leibniz's idea that objects are indiscernible if and only if all available properties take identical values. However, in the rough set approach, indiscernibility is defined relative to a given (possibly incomplete) set of functionals (properties).

At the metatheoretical level, RST has a first-order logical language with identity and its object language is that of ZF (although it can be replaced by another base theory).[15] The novelty of the theory comes through the structure and conclusions that can be drawn from its "coarse-grained" description. Because of the granularity of

15 There are several links between RST and Fuzzy Sets, which will not be discussed here. For the interested reader, we recommend [31,32].

knowledge, some objects of interest cannot be discerned and appear the same (similar). As a consequence, vague concepts, in contrast to precise concepts, cannot be characterized in terms of information about their elements. Any set of all indiscernible (similar) objects is called an elementary set, and forms a basic granule (atom) of knowledge about the universe. Any union of some elementary sets is referred to as crisp (precise) set– other wise the set is rough (imprecise, vague) (see [14]).

Thus, in the proposed approach, we assume that any vague concept is replaced by a pair of precise concepts, called the *lower approximation* and the *upper approximation* of the vague concept. The lower approximation consists of all objects that surely belong to the concept, and the upper approximation contains all objects that possibly belong to the concept. The difference between the upper and the lower approximation constitutes the boundary region of the vague concept. Depending on the presentation considered and the formalism used, rough set theory can express vagueness in different ways: by employing a boundary region for each set [14], through uncertainty functions $I$ [33], or also by using certain "membership primitives" [13]. There are many different presentations, as this theory has gained a lot of interest over the years and has achieved applications in multiple areas.[16]

The following presentation follows [14].

Let $\mathcal{A} = (U, A)$ (in ZF) be a pair called *information system*. Where $U$ and $A$ is a set whose elements are nonempty finite sets. The set $U$ is the domain of objects and $A$ is a set of functions (properties or attributes) whose domain is $U$. That is, if $a \in A$, then $a : U \to V_a$, where $V_a$ is the set of values corresponding to the attribute $a$. Every subset $B \subseteq A$ determines a dyadic relation on $U$ called the indiscernibility relation, defined by

$$x\, I(B)\, y := \forall a(a \in B \to a(x) = a(y)) \tag{10}$$

where $a(x)$ denotes the value that the attribute $a$ takes on the object $x$.

Hence, $I(B)$ is an equivalence relation for any choice of $B$ (which we assume is nonempty). Being an equivalence relation, it determines a partition of the domain $U$. We denote the set of all equivalence classes of $I(B)$ by $U/B$ and the particular class containing the element $x$ by $B(x)$ or $[x]_B$. Therefore, considering the data, we generally cannot observe individual objects and must instead reason based on the accessible fragments of knowledge.

If $(x, y) \in I(B)$ will say that $x$ and $y$ are *B-indiscernibles*. Equivalence classes of the relation $I(B)$ (or blocks of the partition $U/B$) are referred to as $B$-elementary sets or $B$-elementary granules. The elementary sets are the basic building blocks of our knowledge. The unions of $B$-elementary sets are called $B$-definable sets.

We now define two operations on any subset $X$ of $U$.

$$B_*(X) := \{x \in U : B(x) \subseteq X\} \tag{11}$$

$$B^*(X) := \{x \in U : B(x) \cap X \neq \emptyset\} \tag{12}$$

These two sets assigned to each subset $X$ are called $B$-lower and the $B$-upper approximation of $X$, respectively. We call the difference between these sets $B$-boundary region of $X$.

$$BN_B(X) := B^*(X) \setminus B_*(X) \tag{13}$$

$B$-lower approximation (of $X$) can be interpreted as the set of all objects that are certain to belong to $X$ in view of $B$. $B$-upper approximation can be interpreted as the set of all objects that possibly belong to $X$ in view of $B$. The boundary region of a set $X$ with respect to $B$ is the set of all objects (in view of $B$) that cannot be classified as either belonging to $X$ with certainty or belonging to its complement with certainty (fig. 1).

We can define two membership relationships as follows [13]:

$$x \underline{\in}_B X := x \in B_*(X) \tag{14}$$

[16] We recommend the official Rough Sets website: https://www.roughsets.org/

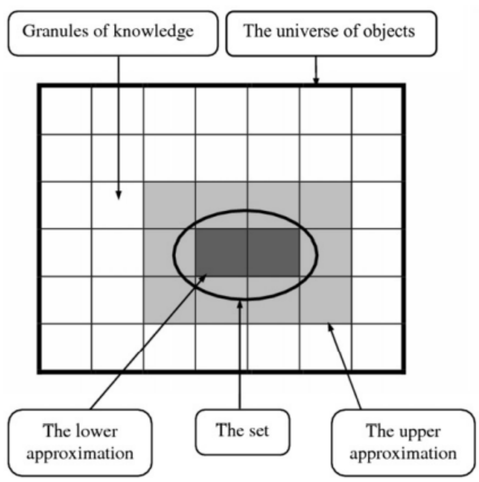


**Figure 1.** The intuitive idea of Rough Sets, a figure inspired by [34].

$$x \overline{\in}_B X := x \in B^*(X) \tag{15}$$

In the first case, we say that $X$ surely belongs to $X$ in $B$, while in the second case we say that X possibly belongs to $X$ in $B$. In this way, it is naturally fulfilled that

$$B_*(X) = \{x \in U : x \underline{\in}_B X\} \quad ; \quad B^*(X) = \{x \in U : x \overline{\in}_B X\}$$

This last form will be used when comparing this formalism with that of QST. Some of the properties of these approximations are (see more properties in [14]).

$$\begin{array}{rc} i) & B_*(X) \subseteq X \subseteq B^*(X) \\ ii) & B_*(\emptyset) = B^*(\emptyset) = \emptyset \quad ; \quad B_*(U) = B^*(U) = U \\ iii) & B^*(X \cup Y) = B^*(X) \cup B^*(Y) \\ iv) & B_*(X \cap Y) = B_*(X) \cap B_*(Y) \\ v) & B_*(X^c) = (B^*(X))^c \\ vi) & B^*(X^c) = (B_*(X))^c \\ vii) & B_*(B_*(X)) = B^*(B_*(X)) = B_*(X) \end{array} \tag{16}$$

# 7. Establishing relationships

## *7.1. Relating QST to RST*

We can establish several structural relationships between QST and RST through their respective membership relations. If we translate the primitives '$\in$', '$\notin$' of QST by their corresponding membership relations of RST, '$\underline{\in}_B$', '$\overline{\notin}_B$' [(14), (15)], then $qext(X)$ and $\overline{X}$ [(2.3), (3.3)] translate to $B_*(X)$ and $B^*(X)$ [(11), (12)] respectively. For details of this and other translations between QST and RST, see section 3.7 of [16].

We can see the above as follows. Given fixed $B$ and $X$ (not empty), the partition of $U$ generated by $I(B)$ can be separated into three disjoint and exhaustive regions (a new partition): $B_*(X)$, $BN_B(X)$ y $U \setminus B^*(X)$ (o $(B^*(X))^c$). It is proven by definition that, for all $B$ and $X$:

$$B_*(X) \cap (B^*(X) \setminus B_*(X)) \cap (B^*(X))^c = \emptyset$$

$$B_*(X) \cup (B^*(X) \setminus B_*(X)) \cup (B^*(X))^c = U$$

In terms of belongings, it can be expressed as:

$$\forall X \forall x \Big( x \underline{\in}_B X \vee x \overline{\notin}_B X \vee \big( \neg (x \underline{\in}_B X) \wedge \neg (x \overline{\notin}_B X) \big) \Big) \tag{17}$$

with $B \subseteq A$, where it is verified

$$\forall X \forall x ( x \underline{\in}_B X \rightarrow \neg ( x \overline{\notin}_B X ) ) \tag{18}$$

, but it is false that the inverse implication holds. That is,

$$\neg ( \forall X \forall x ( \neg ( x \overline{\notin}_B X ) \rightarrow x \underline{\in}_B X ) ) \tag{19}$$

This is due to the existence of the $BN_B(X)$ zone.

These same relationships are expressed in QST as:

$$\forall X \forall x \Big( x \in X \vee x \notin X \vee \big( \neg ( x \in X ) \wedge \neg ( x \notin X ) \big) \Big) \tag{20}$$

Where '$\in$' and '$\notin$' are primitives of QST, unlike RST, where they are defined by the equivalence relation $I(B)$. In their intended interpretation, these memberships are read as "$x$ belongs with certainty to the quaset $X$ or $x$ does not belong with certainty to the quaset $X$ or $x$ belongs indeterminately to such a quaset".

We define weak or indeterminate membership $\in^-$ (in QST) through the following expression:

$$y \in^- x := \neg ( y \in x ) \wedge \neg ( y \notin x ). \tag{21}$$

Then, the equation (20) can be expressed as

$$\forall X \forall x ( x \in X \vee x \in^- X \vee x \notin X ) \tag{22}$$

Which reflects the three disjoint regions of the domain, just as in RST. The three regions in QST are characterized by $qext(X)$ (quasi-extension of $X$ (2.3)), $qext((\overline{X})^c)$[17] (elements that certainly do not belong to the complement of the closure of $X$ (3.1)) and the zone of indeterminacy, which is the analogue of $BN_B(X)$ in QST.

Before establishing the analogous relations for the case of $\mathfrak{Q}^-$, we might ask the following: if there is a strong relation between the regions that separate its domains, characterized both by coarse grain and by their membership relations, what is the equivalence relation analogous to $I(B)$ that exists in QST? The equivalence relation for the case of QST is the indiscernibility relation, defined from the intensional operation "$\subseteq$" (2.4): $x\mathcal{R}y := x \equiv y \leftrightarrow x \subseteq y \wedge y \subseteq x$.[18]

Another (not necessarily equivalent) way to obtain an indiscernibility relation is:

$$x \equiv y \leftrightarrow qext(x) = qext(y) \wedge \overline{x} = \overline{y}.$$

To prove equivalence between these two proposals, some conditions must be imposed that we have not asked for the moment, but that could be considered if necessary. Our intention is to show the possibility of generating the graining in QST from one of its indiscernibility relations. The crucial difference between the proposal in QST and that of RST is that, in the latter theory, the coarse grain can be controlled through $B$, while in QST it is inherent to its formalism. We could propose QST as a limiting case of RST when the graining cannot be made finer, which would be the same as generating RST, not with a basis in ZF, but in QST. Defining the two approximations of each rough set through quasets, instead of ZF sets.[19] In quantum mechanics, this coarse grain limit would be determined by $\hbar$.

We will now present the corresponding treatment for $\mathfrak{Q}^-$.

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17 The operation $qext$ is applied to $(\overline{X})^c$ because the complement is defined in QST from its primitives $\sqcup$ and $\sqcap$ (3.4), producing that there exist entities that have weak membership both to the quaset $X$ and to its complement (see [16, 3.3]). This is the formal correlate of entities whose state is described by a quantum superposition.

18 Recall that, since the inclusion of quasets is an intentional operation, double inclusion should be interpreted as a double implication, not as the identity of quasets.

19 This was already developed for the case of fuzzy sets, where fuzzy upper and lower approximations are proposed for each rough set (see [35,36].

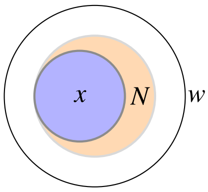


**Figure 2.** A qset $x$ and its cloud $N$ relative to the q-set $w$.

### *7.2. Relating QST to $\mathfrak{Q}^-$*

A comparison between early versions of $\mathfrak{Q}$ and QST can be found in [11]. To proceed with our goal, we need to define weak membership ($\in^*$) in $\mathfrak{Q}^-$(following [29]).

**Definition 7.1** (Weak membership)**.** $x \in^* y := \exists w(w \in y \land w \equiv x)$.

If $x$ or $y$ are classical sets, then $x \in y \leftrightarrow x \in^* y$.

With that notion, we can talk about *potential elements* of a qset in the following way. Suppose we have a qset $w$ and let $x$ be one of its sub-q-sets. We then define the *Cloud* of $x$ relative to $w$ as (intuitively speaking) the qset formed by the elements of $w$ that have indiscernibles in $x$, that is, that 'could' be in $x$. Intuitively, if we have a sample of an element that can exchange components with other elements or with the environment, then we can talk about entities that *could* belong to the sample due to the possibility of exchange between entities of the same nature (e.g. electrons for electrons).[20]
We can now define a new concept, weak inclusion ($\subseteq^*$).

**Definition 7.2** (Sub-q-sets)**.** We define sub-q-sets in two ways:

1. $x \subseteq y := \forall z(z \in x \to z \in y)$.
2. $x \subseteq^* y := \forall z(z \in x \to \exists w \forall m \forall n(w \in y \land z \equiv w \land K(x,m) \land K(y,n) \to m \leq n))$

Some consequences of the preceding definition are the following:

**a)** $x \subseteq y \to x \subseteq^* y$

**b)** $x \subseteq^* y \land y \subseteq^* x \leftrightarrow x \equiv y$

It is immediate to verify that if $x$ is a set, then $x \subseteq y \leftrightarrow x \subseteq^* y$.

As always, examples taken from situations in the particular sciences are important. Suppose we have a lithium atom, Li, whose electron decay is $1s^2\ 2s^1$. We can think of a qset with three elements that simulates the electrons of that atom, and also of sub-q-sets of it, for example one containing as an element only the electron that is in the second energy level. In that case, we are thinking of an electron *that is* in the atom and so we would use the symbol $\subseteq$. But we can also think of the ionization of the atom that would eliminate the outermost electron, giving us a $Li^+$ cation. We can also do the reverse operation, making the cation capture an electron so as to obtain a neutral atom again. In that case, we cannot say either that the captured electron is the same as the one that was eliminated or that the 'new' neutral atom is the same as the first. They are indistinguishable. But now, dealing with the first atom and the captured electron, it would be more appropriate to say that the qset formed by the captured electron is a sub-q-set of the first atom in the sense of $\subseteq^*$.

**Definition 7.3** (Cloud of a qset)**.** Let $w$ be a qset and let $x \subseteq w$. The cloud of $x$ relative to $w$ is the qset $N(x,w)$ defined as follows:

$$[y \in w : \exists z \in x \land z \equiv y]$$

It is desirable that the q-cardinality of $x$ be maintained. Thus, its cloud would be something like its extension to $w$.

If we consider all clouds of elements of a q-set $w$, we can define operations between them, such as intersection, complement and union of clouds. What we obtain is a *algebra of clouds* whose structure is not boolean, as would be the case for subsets of a classical set. The structure obtained is a different lattice, which

[20] The notion of a cloud of a qset was introduced in [37].

resembles an orthomodular lattice, as is the case for the algebra of subspaces in quantum physics; the details are in [38]. With clouds, we can also approximate the idea of *quasets* of Dalla Chiara and Toraldo di Francia [5], as indicated in [16].

Using the definition 7.1 of weak membership, we can rewrite the definition 7.2 of sub-q-set as follows:

**Definition 7.4** (Sub-q-sets)**.**

$$x \subseteq^* y := \forall z(z \in x \to z \in^* y) \wedge \forall m \forall n(K(x,m) \wedge K(y,n) \to m \leq n) \tag{23}$$

We could also express that the cloud of a qset $x$, relative to any qset $\omega$, is included (in the sense of $\subseteq$) in the qset $y$ in the following way. If $x \subseteq y \subseteq \omega$,

$$N(x,\omega) \subseteq y \leftrightarrow \forall z(z \in^* x \leftrightarrow z \in y)$$

As we have remarked above, $x \subseteq^* y \wedge y \subseteq^* x$ implies indiscernibility between $x$ and $y$. In [39], a quantum mereology is presented where the notion of "being a physical part of", denoted by $\sqsubset$, which is a primitive of its formalism, can be used in a similar way to obtain an indiscernibility relation between physical systems. In our case, the fundamental primitive concept is the indiscernibility relation (and the membership relation of ZF), with which we define weak membership and weak inclusions between qsets. However, if our primitive concept were to be both memberships (or could also be inclusions), the definition 7.1 could be interpreted as an implicit definition of the concept of indiscernibility. We could also define indiscernibility through the double inclusion of qsets $\subseteq^*$, which in turn rests on the concepts of weak and standard membership.

The notion of weak membership allows us to partition the qset universe as follows. Given qsets $x$ and $y$, we have the following exclusive and exhaustive options:

$$(x \in y) \vee (x \notin y \wedge x \in^* y) \vee (x \notin^* y) \tag{24}$$

Comparing the above equation with (22), we might suggest:

**i)** The membership of $\mathfrak{Q}^-(\in)$ is associated with the corresponding membership of QST. That is, one could associate the *qextension* of a quaset with the qsets of $\mathfrak{Q}^-$; The elements that belong with certainty to the quaset are associated with those that belong with certainty to the qset. The role that $B_*(X)$ plays in RST would be played by $qext(X)$ in QST, and $X$ in $\mathfrak{Q}^-$. We could express the idea (conceptually) as: $\in_{QST} \leftrightsquigarrow \in_{\mathfrak{Q}^-} \leftrightsquigarrow \underline{\in}_B$

**ii)** Equivalently, we associate $\notin^*$ $(\mathfrak{Q}^-)$ with $\notin$ (QST) and $\overline{\notin}_B$ (RST). In this way, the Cloud of qsets (relative to a given set) would be associated with the *closure* of quasets (also relative to a certain set) and with the $B$-upper-approximation. Entities that do not belong with certainty to a quaset are associated with objects that do not belong weakly to a qset and with those that do not belong to the $B$-upper approximation.

**iii)** Finally, the entities that in QST belong weakly to a given quaset would be associated with those that belong to $BN_B(X)$ and with those that in $\mathfrak{Q}^-$belong weakly to a qset, but without belonging (they do not belong to the qset, but some entity that is indiscernible does belong to it).

This suggested relationship between the "regions" of the three formalisms can be adopted, but not before noting the following difference at the logical level. Let us return to the focus on (22) and (24). In both expressions, the corresponding weak membership associated with its formalism appears (2.2 and 7.1 respectively), but while in QST weak membership satisfies that

$$\forall x \forall y(x \in y \to \neg(x \notin y))$$

and consequently

$$\forall x \forall y(x \notin y \to \neg(x \in y)).$$

In $\mathfrak{Q}^-$, it is verified that

$$\forall x \forall y (x \in y \rightarrow x \in^* y)$$

, and consequently

$$\forall x \forall y (x \notin^* y \rightarrow x \notin y).$$

That is, the logical consequences that follow from weak membership (or from which they follow) are different in each case. In QST, membership with certainty implies not membership weakly (as in RST).

We could emphasize their similarities if, within the framework of $\mathfrak{Q}^-$, we proposed from the beginning two primitives of membership, $\in$ and $\notin^*$, and from them we defined the relations of indiscernibility and **subq**$^*$.[21] Thus, to highlight the similarity with the QST case, we could express the three possibilities as:

$$\forall x \forall y (x \in y \vee (\neg(x \in y) \wedge \neg(x \notin^* y)) \vee (x \notin^* y)).$$

This suggests the following definition:

$$x \in_* y := \neg(x \in y) \wedge \neg(x \notin^* y).^{22}$$

Thus, we can express the above as: $(x \in y) \vee (x \in_* y) \vee (x \notin^* y)$.

And in this way, we have to

$$\forall x \forall y (x \in y \rightarrow \neg(x \notin y)) \quad \wedge \quad \forall x \forall y (x \in y \rightarrow \neg(x \in^- y)) \quad [\text{QST}]^{23}$$

$$\forall x \forall y (x \in y \rightarrow \neg(x \notin^* y)) \quad \wedge \quad \forall x \forall y (x \in y \rightarrow \neg(x \in_* y)) \quad [\mathfrak{Q}^-]$$

$$\forall x \forall_Q y ((x \in y) \vee (x \in^- y) \vee (x \notin y)) \quad [\text{QST}]$$

$$\forall x \forall y ((x \in y) \vee (x \in_* y) \vee (x \notin^* y)) \quad [\mathfrak{Q}^-]$$

With these membership symbols, the similarity between QST and $\mathfrak{Q}^-$is best expressed. In terms of this new membership ($\in_*$), which we could call *extra weak*, the possibilities of the universe of qsets can now be expressed in a way equivalent to that of QST. This may have interesting consequences for the semantics of quantum logic. In [16], a generalization of the nondeterministic semantics of Nmatrices for the quantum projector lattice in the framework of QST is presented (see also [40–42]). Based on the structural-logical similarities just shown, such results could be extended to $\mathfrak{Q}^-$and have interesting applications for the Kochen-Specker theorem. A quantum Nmatrix based on $\mathfrak{Q}^-$could be used to give a semantics that avoids the Kochen-Specker contradiction, due to considering that the projectors associated with the same observable are identical (instead of indiscernible) in different incompatible contexts as shown in [43].

To conclude this section, we will say a few words about the conditions analogous to (16) in QST.

To see if the corresponding relationships hold in QST, we need to translate:

- $B_*(X)$ to $qext(X)$,
- $B^*(X)$ to $\overline{X}$,
- ZF's $\cup, \cap$ operations by QST's $\sqcup, \sqcap$ operations,
- the inclusion of ZF by the respective operation between quasets
- and finally, equality between sets by double inclusion of quasets.

The proof that such conditions hold (asking for some extra minimal conditions on $\sqcup$ and $\sqcap$ regarding quasi-extension and closure) can be seen in section 3.6 [p.26] and the appendix of [16].

21 Expressing the axioms of $\mathfrak{Q}^-$formally through two primitives of membership is proposed to make their axiomatic presentations more similar, but it is not necessary if we only want to highlight the separation of domains presented.

22 En el marco de $\mathfrak{Q}^-$, $\neg(x \in y)$ es equivalente a $x \notin y$ y $\neg(x \notin^* y)$ es equivalente a $x \in^* y$.

23 Remember that inverse implications do not apply in QST.

The analogous study for the $\mathfrak{Q}^-$case is not yet fully done. Conditions on *clouds* of qsets (analogous to $B^*(X)$) have already been given in [22], but issues regarding weak inclusion ($\subseteq^*$) (7.2) remain to be resolved (see a more detailed treatment in [29]). Weak inclusion of qsets implies standard inclusion of qsets, but it remains to be established under what conditions in (16) one should change "$\subseteq$" to "$\subseteq^*$". These issues will be further developed in future work.

| QST and extensions | $\mathfrak{Q}^-$ | RST |
|---|---|---|
| $\forall x \forall y_Q((x \in y) \vee (x \in^- y) \vee (x \notin y))$ | $\forall x \forall y((x \in y) \vee (x \in_* y) \vee (x \notin^* y))$ | $\forall X \forall x(x \underline{\in}_B X \vee x \overline{\notin}_B X \vee x \in BN_B(X))$ |
| $x \in^- y := \neg(x \in y) \wedge \neg(x \notin y)$ | $x \in^* y := \exists z(z \in y \wedge z \equiv x)$ ; $x \in_* y := (x \notin y) \wedge (x \in^* y)$ | $x \underline{\in}_B X := x \in B_*(X)$; $x \overline{\in}_B X := x \in B^*(X)$ |
| $\forall x \forall y(x \in y \to \neg(x \notin y)) \forall x \forall y(x \in y \to \neg(x \in^- y))$ | $\forall x \forall y(x \in y \to \neg(x \notin^* y)) \forall x \forall y(x \in y \to \neg(x \in_* y))$ | |
| $qext(x)$ | $x$ | $B_*(X)$ |
| $\overline{x}$ | $N(x, \omega)$ | $B^*(X)$ |

**Table 1.** Summary of the main results obtained by relating these theories

## 8. Extensionality

In this section we will analyze the extensionality of each of the theories involved. This property, in ZF, is stated as an axiom, directly linking the membership primitive (the only non-logical primitive of ZF) with the logical identity of the metatheory. According to this principle, two sets are identical when they share their extensions. Only their extensions come into play to decide whether two sets are identical or not. That is, to decide whether they count as one or not. Therefore, the issue of identity, together with its relation to extensionality, directly affects the cardinality of the entities involved.

In certain situations, even in the context of ZF, the meaning of extensionality is not sufficiently clear. According to what has been said above, one could interpret the extensional nature of sets as consisting in the fact that their membership structure uniquely determines their identity. We think that this is not precise enough. To make the idea more precise, it is necessary to distinguish between: *criteria of identity* and *criteria of individuation* [44]. In our view, identity is an *absolute* concept: there is no half-identity; either a thing has identity or it does not, and the latter case, for us, occurs with quantum entities. Individuation is committed to some metaphysical principle that establishes an ontological relation between entities; as J. Lowe says, *what individualizes an object is whatever makes the object the singular object that it is*, that is, *whatever makes it an object, distinct from all others, and the same object that it is in opposition to anything else* [44].[24]

It might be thought that the membership structure of a set is what individuates the set, what makes it the set that it is as opposed to any other set. However, this is not exactly what the axiom of extensionality says, since individuation brings with it peculiar extra consequences: if the membership structure is what makes a set the set that it is, then it is reasonable to demand that whenever two sets share the same membership structure they are, in fact, the same set. This implication is not guaranteed by the axiom of extensionality alone (see [45]).

Let us consider the following example in ZF without the Regularity axiom: Let the sets $A = \{B, C\}$, where $B = \{B\}$ and $C = \{C\}$. Whether we wish to make a claim about the identity or individuality of $A$ or to pronounce on its cardinality, we must decide whether $B = C$ or not. The extensionality axiom is of no use here, since it simply tells us that $B = C$ if and only if $B = C$. It will not violate extensionality, or any of the other axioms of ZFC (ZFC without the Regularity axiom), whether $B$ and $C$ are unequal or not. Nothing in the formalism allows us to decide. Therefore, assuming that the membership structure is what individualizes the sets and, consequently, that whenever two sets share the same membership structure they are the same set, is not enough to decide in cases like the one presented. When it comes to the set $A$ (Boffa set), it turns out that $B$ and $C$ have the same membership structure (there is a non-trivial automorphism that exchanges the two nodes) even though they are not identical. Something else is needed to know whether $B$ and $C$ are or are not the same set.

[24] Lowe distinguishes between individuation in two senses; one metaphysical, which is what we are adopting, and one epistemological, saying that this one presupposes the other.

The authors of [45] argue against the membership structure being responsible for the individuation of sets. Their position is supported by the fact that there is at least one other plausible candidate in the vicinity:

> [...] perhaps, for some sets, what makes a set be the very set that it is is its membership structure together with the fact that the set is identical to itself and distinct from any other, with this last fact obtaining in virtue of nothing more fundamental; which is to say, in some cases, membership structure together with primitive identity facts are responsible for set individuation. [45, p.4]

The identity of these entities can be determined either by a primitive identity, external to the formalism, or by ontological issues, for example, through a theory of *substratum*. The moral is that, even in ZF (without regularity) extensionality is not a sufficient criterion of individuation if one wishes that every time two entities satisfy the same criterion of individuation, they are the same entity. The identity must be given (or not) externally. Of course, in the context of ZF whit Regularity, the axiom of Extensionality serves to guarantee that there is just one empty set at the base of the hierarchy of sets and that any identity question concerning sets can ultimately be settled by reference to the empty set through repeated applications of the axiom of Extensionality. For the moment, we will leave the extensionality of ZF and delve into that of QST.

QST was presented as an intensional set theory. That is, an axiomatic set theory where *neither intensions uniquely determine an extension, nor extensions uniquely determine their intensions*. Its motivation is due to the behavior of quantum entities, which in certain circumstances, seem to be represented by intensional sets. This is closely linked to the problem of the identity of quantum entities. Since quantum entities can be considered entities without identity, they defy Leibniz's principle, and the claim to label them or give them proper names (rigid designators) seems not to be well founded. According to Dalla Chiara and Toraldo Di Francia, since quantum entities are mainly nomological entities, they are better captured by intensional descriptions [5]:

> The lack of proper names is due fundamentally to the fact that the objects of microphysics are nomological. All their characteristics are fixed by physical law and are identical for objects of one and the same kind. As far as we know today, one electron is identical to another electron, one proton to another proton, and so on.

As the authors say, the intension of a term always determines, at least in principle, an extension for the term, but unlike what happens in standard semantics, this extension will not necessarily be unique. For example, once the intension of the term *electron* is stipulated, we have the possibility of recognizing, by theoretical or experimental means, whether a given physical system is a collection of electrons or not; if so, we can also enumerate all the quantum states available in it. In fact, we can do this in several different ways. To illustrate, the authors take the case of spin. If we consider electrons defined by $s = 1/2$, $m = 9.1\ 10^{-28}g$, $q = 4.8\ 10^{-10}e.s.u.$,

> we can choose a z-axis and state how many electrons have $s_z = +1/2$ and how many have $s_z = -1/2$. But we could instead refer to the x-axis, or the y-axis, or any other direction, obtaining different sets of quantum states, all having the same cardinality. We thus arrive at a situation, which is usually believed to be impossible in classical semantics: different extensions can correspond to one and the same intension. Of course, the reverse situation of one and the same extension corresponding to different intensions is trivially possible, as in classical semantics (for instance, instead of giving the mass of a particle, one could give its rest energy).

This peculiarity of collections of quantum entities to admit *different extensions compatible with a given intension is one of the main motivations of the theory*. Systems of microobjects exhibit irreducibly intensional behavior: they generally do not determine precise extensions and are not determined by them. Consequently, a basic feature of QST will be a strong violation of the extensionality principle.

Neither the original axiomatization of QST (see [10]) nor its possible extensions (see [16]) have an axiom of extensionality for non-classical quasets. For quasets not belonging to the domain isomorphic to ZF containing QST, we have no principle of extensionality. By what was said in the introduction, the primitive "⊆" of QST must be interpreted intensionally. $x \subseteq y$ must be interpreted as "the concept $x$ implies the concept $y$". Therefore, double inclusion must be interpreted as a double implication between concepts, indicating that such concepts cannot be discerned within the framework of this axiomatic.[25]

RST has in its basic structure the axioms of ZF (in their standard formulations), however after generating the partition associated with the amount of available information, one forgets the extensionality axiom of ZF and goes on to treat as indiscernible certain sets that, after granulation, are shared by both approximations. Even though in ZF they are different sets, they are treated as indiscernible for certain partitions. In the limit, when the available information is the maximum possible and the area of the granulation tends to zero, this criterion of indiscernibility between rough sets is transformed into the extensionality axiom of ZF. Both approximations coincide in this limit, causing the rough set to become a well-defined set.

What can we say about the extensionality of $\mathfrak{Q}$ and $\mathfrak{Q}^{-}$? Both are considered semi-extensional theories, which is reflected in the weak extensionality axiom of each. The weak extensionality axiom of $\mathfrak{Q}^{-}$(WEA) says that if we have a qset $A$ and we exchange some of its elements for another indiscernible one belonging to its cloud, the new qset is indiscernible from the first. Since these objects have no identity, we cannot say whether or not they are the same. They can be counted as two (or more) genuinely indiscernible entities. Entities that are *different only in number*. It is in this sense that qset theories are considered semi-extensional. They are invariant under permutation of indiscernibles. Although the corresponding axiom of $\mathfrak{Q}$ has not been presented, it can be interpreted in the same way for the purposes of this section (see, for example, [6,12,46]).

### *8.1. The relation of indiscernibility*

The indiscernibility relation is directly linked to the concept of extensionality. In both $\mathfrak{Q}$ and $\mathfrak{Q}^{-}$, this relation is a non-logical primitive that plays a prominent role in the corresponding weak extensionality axioms. In the $\mathfrak{Q}$framework, it is an equivalence relation, which does not collapse into congruence due to the existence of m-atoms. That is, m-atoms can be genuinely indiscernible without being identical. The (apparently) only situation where the substitution by indiscernibles doesn't hold is with respect to the membership relation. In the atomless $\mathfrak{Q}^{-}$theory, legitimate qsets are responsible for ensuring that such a relation does not collapse into the identity.

Despite not having the logical identity in its metatheoretical language, in $\mathfrak{Q}$ one can *define* an "equality" (different from the one we use in axioms (7)-(10)) called *extensional* (see [11, p.144] and, for the case of $\mathfrak{Q}^{-}$, discussion in [29, s.8]). However, this defined equality does not satisfy the requirements required for identity, such as *replacement of identicals*. We will take an example discussed in [47]. Analyzing the nature of strong singletons, it is stated: *even if* $x' \equiv x$, *the theory doesn't grant that any indistinguishable from x will belong to any strong singleton of x; these strong singletons are indiscernible, not identical, that is,* $x \equiv x'$ *, but not* $x = x'$. Immediately afterward, he discusses a possible objection to such a conclusion by suggesting that we could define an identity for m-atoms as follows: let $x$ and $y$ be indiscernible m-atoms and let $[\![x]\!]$ and $[\![y]\!]$ be strong singletons of these elements.[26] Then we can establish:

$$x =^{*} y := x =_{E} y$$

, where '$=_E$' denotes the extensional equality of $\mathfrak{Q}$, which for the example we can take as that of ZF.

Since both the q-cardinals $x$ and $y$ are equal to 1, we can conclude that their elements are the same, and hence the defined identity would have the properties of the standard identity. But this is a mistake. In the object language of $\mathfrak{Q}$, we cannot say that the elements of these strong singletons are 'the same', since this requires identity, and then we would be clearly begging the question by assuming that we want to define. Formally, if $x =^{*} y$, we can say that they are $*-$ *identical*, but never that they are 'the same'. Moreover, $=^{*}$ does not have all the properties of the standard identity; in particular, substitution fails (for details, see [47]).

[25] Although the original axiomatics of QST might suggest that $x \subseteq y \wedge y \subseteq x$ implies $x = y$, the treatment given in [16] shows that two quasets sharing both their *qextension* and their *closure* can be considered indiscernible. Within the framework of the extensions of QST presented in [16], the axiom 2.5 and the definition 2.4 should be weakened to allow double inclusion to be equivalent to indiscernibility (and not to identity).

[26] For the example, we ignore the qsets to which the singletons are relative so as not to run the risk of having their own classes.

What interests us in this example is the following: in the framework of $\mathfrak{Q}$, even with extensional equality, which respects an extensionality axiom, and the regularity axiom, we do not have enough to have a general individuation criterion. This is due to the lack of logical identity for m-atoms.[27] Something very similar happened when analyzing the example of the Boffa set in ZF without Regularity. Even though in this context we had identity, the extensionality axiom was not enough to give a good individuation criterion either. That is, we arrive at similar results; on the one hand, with the extensionality axiom, the identity and without regularity and, on the other hand, with a defined extensional identity, which enables an extensionality axiom, and the Regularity axiom, but without logical identity.

In $\mathfrak{Q}$ theory, indiscernibility is interpreted ontologically, that is, as an inherent characteristic of quantum entities (ontological correlate of m-atoms). This could be seen as a direct consequence of interpreting Heisenberg's uncertainty principle ontologically, rather than epistemically.[28] We say this because it contrasts with the epistemic character that indiscernibility acquires in both RST and QST. In RST, it is a direct consequence of the lack of information or "vagueness" about our system, in addition to being a *definite* (non-primitive) relation. We already said that two rough sets that share their two approximations are considered indiscernible. We also said that there are many alternative presentations of RST. A particular presentation can be seen in [48]. In this work, indiscernibility is characterized by certain sets called *the minimal description of the object*. Two objects are indiscernible, when they share their *minimal descriptions*, $Md(x) = Md(y)$. This represents another way of using the available information about the system. Its authors express (p.2): *we adopt the view that there is a mutual correspondence between intensions, i.e. properties (or characteristic features) of objects, and extensions, i.e. sets of objects possessing these properties*. This contrasts with what has been said regarding extensionality in the case of QST.

The target language of QST does not have a symbol for this relation, although we said that, due to the intensional character of the theory, it could be defined, for example, through the double inclusion of quasets. Whatever the way of defining this equivalence relation, it most likely inherits the epistemic character that its two membership primitives ($\in, \notin$) have in the intended interpretation. Therefore, neither in QST nor in RST can we speak of genuine indiscernibility. This is directly related to what is called *rigid structures* (see below).

$\mathfrak{Q}^{-}$is a special case in this sense, since it allows its indiscernibility to be interpreted both ontologically and epistemically. On the one hand, since it inherits many concepts and structures from its predecessor with atoms $\mathfrak{Q}$ we could interpret the indiscernibility ontologically. On the other hand, since we could consider expressing the formalism using two primitives of membership ($\in, \notin^{*}$), as in the case of QST, we could interpret the indiscernibility (in this case, no longer primitive, but defined) epistemically.

Let us take the following example in $\mathfrak{Q}^{-}$. The Separation Schema (axiom 7) allows us to show the existence of q-sets with no elements, which we will call *empty*, but without the identity we cannot prove their uniqueness. However, due to the Axiom of Weak Extensionality (WEA), we can prove that all such empties are indiscernible. The way to infer the existence of a q-set with no elements is to adopt a contradictory formula, like $x \not\equiv x$, together with axiom (3), in the Separation Schema. We will denote the empty q-sets regardless by '$\emptyset$'. Since the identity cannot be used for empty q-sets, and since the postulates of ZFC that hold for sets allow us to derive the existence of an empty *set*, the only relation we have between empty q-sets and the empty set is indiscernibility, and this can be proved from the definitions and from the axiom of weak extensionality. Does this indiscernibility between the empty set and empty qsets have ontological or epistemic status?

### *8.2. Equivalence relations*

Let us analyze the following relationships established by definition in RST.

$$x \underline{\in}_B X \leftrightarrow x \in B_*(X) \leftrightarrow [x]_B \subseteq X \leftrightarrow \forall y (y I(B) x \rightarrow y \in X) \tag{25}$$

[27] It is important to notice that *we can* define an identity for m-atoms in several distinct ways, but this would be contrary to the idea of the metaphysics of non-individuals.

[28] Recall that in Bohmian Mechanics, this principle is considered a simple epistemic limitation. It has no ontological character, since Bohmian particles have, at each instant, well-defined position and velocity.

This formalism was developed to deal with indiscernibility functions that depend on partial knowledge about the properties of entities. That is, to deal with the situation in which we do not have access to the value that all the properties take on the objects of the domain. When the indiscernibility is over all possible properties, as we said before, Leibniz's principle is fulfilled and the indiscernibility collapses into the identity. Therefore, we will analyze what happens when in the previous expression we replace $I(B)$ by '$\equiv$' of $\mathfrak{Q}^-$.

For this, remembering that quantification in RST runs through the finite (bounded) domain $U$, we will take a fixed qset $D$ (which can be a set) that fulfills the same function in $\mathfrak{Q}^-$for the purposes of quantification.

Therefore, the last expression of the above formula is expressed by:

$$\forall y \in D(y \equiv x \to y \in X).$$

Where we are considering that $X \subseteq D$ (with the inclusion is that of $\mathfrak{Q}^-$).

The above expression is equivalent to (by 5.3)

$$[x]_D \subseteq X.$$

Where it is clear, seeing (25), that the singleton of $x$ (relative to $D$) fulfills the function that, in RST, has the equivalence class of $x$ associated with $B$ ($[x]_B$ or $B_*(x)$). By the definition of *Cloud* 7.3, the above can be written as

$$N([x]_D, D) \subseteq X$$

So far we have the following:

$$N([x]_D, D) \subseteq X \leftrightarrow [x]_D \subseteq X \leftrightarrow \forall y \in D(y \equiv x \to y \in X) \tag{26}$$

If we compare this with (25), and we observe the terms that we have not yet related ($x \underline{\in}_B X, x \in B_*(X)$), we can see that we are still associated with the membership $\underline{\in}_B$ with the concept of *Cloud* in $\mathfrak{Q}^-$. In principle, this could surprise us, since we would expect the membership associated with the *Cloud* to be $\overline{\in}_B$. The complete expression, if we express the first two formulas of the first biconditional still in RST

$$x \underline{\in}_B X \leftrightarrow x \in B_*(X) \leftrightarrow \overset{N([x]_D,D)\subseteq X}{[x]_D \subseteq \quad X} \leftrightarrow \forall y(y \equiv x \to y \in X) \tag{27}$$

Let's focus on the color expression. If we follow the chain of biconditionals coming from the right, the color expression should be translated in $\mathfrak{Q}^-$as $x \in \overline{X}$ (closure of $X$). However, if we take into account the biconditional on the left and replace '$\underline{\in}_B$' by '$\in$' , the color expression would be rendered in $\mathfrak{Q}^-$as simply $x \in X$. Therefore, the only way for these replacements (translations into $\mathfrak{Q}^-$) to be consistent is if, for the indiscernibility relation '$\equiv$', the lower and upper approximations agree.

The above could be interpreted as follows. If within the framework of RST, we want, on the one hand, to maintain the interpretations '$\underline{\in}_B$' and '$\overline{\in}_B$' as *surely belongs* and *possibly belongs* and, on the other hand, to admit a genuine indistinguishability relation (with respect to all attributes) as '$\equiv$', there is no other option than to collapse both approximations (into the identity). Which leads us to conclude that RST satisfies Leibniz's principle when all the properties for the indistinguishability relation are taken into account. We will now proceed to analyze this principle.

## 9. Leibniz's Law

According to Quine, the so-called *Substitution Principle* or, as Quine calls it, the *Principle of Indiscernibility of Identicals* must always apply: identical objects can be substituted for each other in any context 'salva veritate'. In this framework, what is meant by identity is a subtle point. We often associate an object with individuality as if this indicated its identity, and we do so through the twin concept of discernibility. Objects are individuals, entities with identity, when we can discern them from others, even if they are similar. But what gives an object its individuality? Can there be two objects that are exactly the same, differing only in that one is one and the other is the other, or, as it is said, that they differ only in number? Leibniz answered this question absolutely negatively. For him, if two objects are two, there must be a quality, an attribute or property that distinguishes them. He crystallized this in his famous *Principle of Identity of Indiscernibles*: *It is not true that two substances can be completely alike and differ only in number* (see [49]).

The validity of this principle has been much debated, especially since the advent of quantum mechanics [12]. This principle is built into classical logic and mathematics, meaning that in any theory based on them there can be no absolutely indiscernible entities. What is known as Standard Identity Theory (STI) is the conjunction of these two principles: Leibniz's principle and the principle of substitution of identicals. Any mathematical theory encompassing STI and which does not consider any kind of substratum is such that given two entities whatever, they are *distinct* (different) and this entails that there exist (even if only in principle) a property satisfied by just one of them. RST satisfies TSI. This is because it is built on the basis of ZF, which satisfies STI by having (in its standard version) first-order logic with identity in its metalanguage. As its author says in [14, p.6]:

> This understanding of indiscernibility is related to the idea of Gottfried Wilhelm Leibniz that objects are indiscernible if and only if all available functionals take on them identical values (Leibniz's Law of Indiscernibility: The Identity of Indiscernibles). However, in the rough set approach indiscernibility is defined relative to a given set of functionals (attributes).

That is, the indiscernibility of RST is a consequence of not considering all possible properties for the characterization of its objects when using Leibniz's principle. The principle holds, but not all properties are taken into account.

Let's see how quantum indiscernibility and Leibniz's principle are linked within the framework of QST.

In [5, s.9], when analyzing Leibniz's Principle of Identity of Indiscernibles (LP), its authors ask *But how (under certain circumstances) can fermions behave at the same time as indistinguishable and Leibnizian particles?* Where it is understood that particles are *Leibnizian* if they satisfy LP; that is, if two particles are distinct, there exists a property that distinguishes them. Intuitively, one might observe that: if two electrons (according to LP) are distinguished by at least one property $P$, then they are distinguished and therefore cannot be indistinguishable. This observation would be correct in classical logic. However, in quantum logic it may happen that a sentence like $\exists P(P(a) \wedge \neg P(b))$ is true even if any possible choice of $P$ does not satisfy the formula $P(a) \wedge \neg P(b)$.

The authors consider the two electrons of a Helium atom in its ground state. Let $a, b$ be names for the two electrons of the Helium atom. $P^+$ represents the property *having spin up (in a certain direction)* and $P^-$, *having spin down (in the same direction)*. The following statement is physically true

$$(P^+(a) \wedge \neg P^+(b)) \vee (P^-(a) \wedge \neg P^-(b)).$$

Therefore, the following instance of LP is also true:

$$\neg(a = b) \longrightarrow \exists P \quad (P(a) \wedge \neg P(b)).$$

However, the truth status of each member of the disjunction $(P^+(a) \wedge \neg P^+(b)) \vee (P^-(a) \wedge \neg P^-(b))$ is indeterminate, and this holds for any other possible choice of $P$. Therefore, QST is compatible with the Leibniz principle.

We will analyze this principle from another point of view and prove that it is a theorem of QST.

**Definition 9.1** (Leibiniz's Law, LL)**.** [29]

$$\forall x \forall y(\neg(x = y) \rightarrow \exists_Q z(x \in z \wedge \neg(y \in z)).$$

First of all, we recall that $\neg(y \in z)$ is not equivalent to $y \notin z$ (see Axiom 2.2). Let us introduce the following additional axiom:

**Axiom 9.1** (Unitary quaset)**.** $\forall x \exists_Q y \forall z(z \in y \leftrightarrow z = x)$.

[29] We have modified the original LL of [5] since the particular case put by the authors, namely, $\neg(x = y) \rightarrow x \in \{x\} \wedge y \notin \{x\}$ is a theorem of QST as it results from what comes next.

We call such $y$ the *unitary quaset* of $x$ and denote it by $\{x\}$. It has the expected properties; for instance, one can easily prove that $\{x\} = \{y\} \leftrightarrow x = y$ and that $x \in \{y\}$ iff $x = y$. In particular, we have the following theorem:

**Theorem 9.1.** For every quaset $x$,

$$qext(\{x\}) = \{x\}. \tag{28}$$

Proof. Let us call $y$ the quasi-extension of $\{x\}$. According to the definition, it satisfies the following:

$$\forall z(z \in y \leftrightarrow z \in \{x\}) \wedge \forall z(z \notin y \leftrightarrow \neg(z \in \{x\})).$$

What we need to prove is that $x$ is the only element of $y$. Suppose there exists $w$ such that $w \neq x$ and $w \in y$. Hence $w \in \{x\}$ and (the second conjunct) $w \notin \{x\}$ iff $\neg(w \in \{x\})$. But this is a contradiction. ∎

Notwithstanding, we remark that the definition of weak membership (definition 7.1) does not forbid that something in addition to $x$ weakly belongs to $\{x\}$.

Since that QST does not involve the Axiom of Regularity, we can introduce the notion of a *fixed thing*, inspired in Quine's individuals, this way (we do not use the word 'individual' since the sense we attribute to it is different from Quine):

**Definition 9.2** (Fixed thing)**.** We call $x$ a fixed thing iff $x = \{x\}$.

Alternatively, we could say, like Quine, that $x$ is a fixed thing (to him, an individual) iff $\forall y(y \in x \leftrightarrow y = x)$ [50, p.32]. Of course a fixed thing is a quaset.

Now, let us turn to the validity of LL in QST for the case of quasets, that is, when both $x$ and $y$ are quasets.[30] A natural candidate for the quaset $z$ in the definition is in fact $\{x\}$, so that we arrive at the particular instance of LL as follows:

**Theorem 9.2** (Leibiniz's Law)**.** In QST, we have

$$\forall x \forall y(\neg(x = y) \rightarrow x \in \{x\} \wedge \neg(y \in \{x\})).$$

Proof: It is sure that $x \in \{x\}$, and $x$ is the only element that belongs to $\{x\}$ by the above results. Hence, if $\neg(y = x)$, $y$ cannot be also an element of $\{x\}$. ∎

A version of LL that *does not* hold in QST is this:

**Theorem 9.3** (Failure of a particular LL)**.** This sentence does not hold in QST:

$$\forall x \forall y \exists z(\neg(x = y) \rightarrow x \in^- z \wedge y \notin z)).$$

Proof: Assume that $\neg(y = x)$ and that $x \in^- z$. We need to show that $\neg(y \notin z)$. But this is immediate since $x \in^- z$ entails $\neg(x \in z) \wedge \neg(x \in z)$ by definition 7.1. Being $y \neq x$, $\neg(x \notin z)$ implies $\neg(y \notin z)$, what we need. ∎

$\mathfrak{Q}$ and $\mathfrak{Q}^-$are characterized by admitting genuinely indiscernible entities without being identical, that is, non-individuals (entities without identity). For non-individuals, Leibniz's principle is not even applicable. It is not that it is false, but that it is not even applicable to non-individuals. Entities without identity can be different *only in number*. That is, they can be numerically distinct (count as more than one), but share all their properties. A classic example of this is presented by Bose-Einstein condensates [27,51]. Thus, neither $\mathfrak{Q}$ nor $\mathfrak{Q}^-$satisfy STI.

## 10. Rigid and deformable structures

We will discuss here the mathematical concept of *rigid structure*, which is closely linked to the topics discussed above.

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30 We remark that Dalla Chiara and Toraldo di Francia do not postulate any axiom that can give the unitary quaset, but they speak of it anyway. Here we try to circumvent this fact by introducing it.

As we said, standard quantum mechanics is compatible with an ontology of genuinely indiscernible entities or non-individuals. If we want our formalisms to capture the essence of non-individuals, they must depart from standard mathematics, since the latter is adapted from the start to deal with entities with identity, entities for which the Leibniz principle applies.

In mathematical terms, if we wanted to capture this property of quantum entities, we should not be able to distinguish things by a property or a relation. If we use a standard mathematical framework such as a standard set theory (ZF, NBG, NF, etc.), then the distinction is always possible. The problem with using a standard framework is that we can consider indiscernible elements only within a deformable (non-rigid) structure, that is, a structure that admits non-trivial automorphisms [26]. But this is a fictitious solution, since it is proved that every structure inside ZF (or NBG, NF, etc.) can be extended to a rigid structure, where apparently indistinguishable elements are discernible. Moreover, the entire universe of sets in a theory like ZF is rigid [26, p.66], i.e. standard set theories are theories of individuals. Their variables are individual and their quantifiers always range over domains of individuals. In this sense, it can be said that, despite being formal theories, they have an ontological commitment to individuals.

The main problem with using standard mathematics (based on classical logic) to describe non-individuals is that any object $a$ can always be discerned from any other entity $b$. Consider this: take the singleton $\{a\}$ and define the identity of $a$ by $I_a(x) := x \in \{a\}$. This makes it so that only $a$ has this "property", so there will be a difference from any other object $b$, i.e. from any other entity that does not satisfy $I_a$. This is typical of standard mathematics, and is expected to be so, since the theory was designed to deal with individuals. Of course, we can mimic indiscernibles in such frameworks by confining them to deformable (non-rigid) structures. But this is a trick, since in set theories like the ZFC system, every structure can be extended to a rigid structure, one spanning only the trivial automorphism. In this extended structure, we realize that the supposedly indiscernible entity is actually an individual. We recommend [52] to the interested reader.

What can we say about rigidity in $\mathfrak{Q}$ and $\mathfrak{Q}^-$? Both of these qset theories allow us to build deformable structures that cannot be extended to rigid structures. In both of them, the automorphism $h(x) \equiv x$ is a non-trivial automorphism. The reason for this, as we said above, is that membership is a non-invariant relation under this automorphism. That is, if $x \in y$ and $x \equiv x'$, then nothing in the formalism ensures that $x' \in y$. Thus, they seem to be a good place to accommodate indiscernible entities.

With what has been said above about the extensionality and indiscernibility of RST, the reader will already suspect that this theory is rigid as ZF. Yes, since every rough set structure can be extended, including the relevant information, to a structure whose only automorphism is the identity. This, as we have already said, will be associated with the null area of each of the classes of the grain and with the equivalence relation (identity, associated with the automorphism $h(x) = x$).

The most delicate case with respect to this property is represented by QST and its extensions. The link between QST and rigid structures has hardly been touched upon in the specific literature on the subject. We will say a few words about it, without claiming that our reasoning is definitive. This topic will have to be further addressed in future articles.

In the original axiomatization of QST, the argument we applied to standard theories like ZF (using $I_a(x)$) is not applicable, since the theory does not have sufficient expressive power. On the one hand, it does not have a pair axiom for non-classical quasets, that is, for quasets outside the internal copy that QST has of ZF. Its authors possibly did not include such an axiom so that the same reasoning as in ZF cannot be followed. On the other hand, it should be noted that since QST is an intensional theory, its properties are not determined by their extensions. Therefore, if the formalism has to be able to discern distinct entities through properties, it is not enough to express them extensionally. Properties in QST are not simply expressed through the extension of a set. The issue of how to express properties in QST (and also in RST) could be related to what in the field of truth theories is known as *Underpill Solution* and *Overpill Solution* (see introduction of [53]).

Extensions of QST ($QST^+$, $\overline{QST}^+$) incorporate a pair axiom, but looking closely at 3.10, it can be noted that it does not admit the same treatment as in ZF. This is because the axiom does not guarantee that $x \in \{x\}$. It does guarantee that it belongs to its closure, that is, $x \in \overline{\{x\}}$. But the pair axiom does not prevent many other objects from belonging to this closure. Therefore, if a reasoning like ZF's is to be applied, it must undergo significant changes. It remains to carefully analyze what role the standardization axioms 3.4, (and antistandardization 3.5, 3.6) play, since these axioms generate many more sets when changing membership from standard to weak and vice versa. The expressive power of the theory increases notably with this axiom, which is why it has been left out of the first extension of QST.

## 11. Conclusions

We have shown strong links between QST, $\mathfrak{Q}^-$and RST. We think that these links can motivate new applications that, for some of them, take advantage of the developments already made in the others. The common structure inherited by their domains, due to the membership relations, can be used in fields such as non-deterministic semantics of Nmatrices, where QST has already been introduced. Equivalently, several topological developments in the RST framework can be transferred (with the necessary care) to $\mathfrak{Q}^-$or QST. We also saw some important differences that these theoretical frameworks have in relation to rigid structures and the Leibniz principle. Several questions are still unresolved, for example: what is the maximum degree of structural similarity that can be established between these theories? How can we achieve the maximum similarity between their axiomatic presentations? What possible variants of these theories, of some particular interest, can arise? Can the concept of non-individuality be used in AI, where RST has several applications? The ontology of identityless entities motivated by quantum mechanics, which has strong implications for information theory and quantum computing, could be beneficial when applied to AI.

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