

Equivalent Gravities and Equivalence Principle: Foundations and experimental implications

Christian Mancini^{1,2}, Guglielmo Maria Tino^{3,4,5},
Salvatore Capozziello^{1,2,6}

¹Scuola Superiore Meridionale, Largo San Marcellino 10, I-80138,
Napoli, Italy.

²Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Complesso
Universitario di Monte S. Angelo, Via Cinthia Edificio 6, I-80126,
Napoli, Italy.

³Department of Physics and Astronomy, Università di Firenze, via
Sansone 1, I-50019, Sesto Fiorentino, Italy.

⁴European Laboratory for Non-Linear Spectroscopy (LENS), via Nello
Carrara 1, I-50019, Sesto Fiorentino, Italy.

⁵Istituto Nazionale di Fisica Nucleare, Sezione di Firenze, via Sansone 1,
I-50019, Sesto Fiorentino, Italy.

⁶Dipartimento di Fisica "E. Pancini", Università di Napoli "Federico
II", Via Cinthia Edificio 6, I-80126, Napoli, Italy.

Contributing authors: christian.mancini@na.infn.it;
guglielmo.tino@unifi.it; capozziello@na.infn.it;

Abstract

The so-called Geometric Trinity of Gravity includes General Relativity (GR), based on spacetime curvature; the Teleparallel Equivalent of GR (TEGR), which relies on spacetime torsion; and the Symmetric Teleparallel Equivalent of GR (STTEGR), grounded in nonmetricity. Recent studies demonstrate that GR, TEGR, and STTEGR are dynamically equivalent, raising questions about the fundamental structure of spacetime, the under-determination of these theories, and whether empirical distinctions among them are possible. The aim of this work is to show that they are equivalent in many features but not exactly in everything. In particular, their relationship with the Equivalence Principle (EP) is different. The EP is a deeply theory-laden assumption, which is assumed as fundamental in constructing GR, with significant implications for our understanding of

spacetime. However, it introduces unresolved conceptual issues, including its impact on the nature of the metric and connection, its meaning at the quantum level, tensions with other fundamental interactions and new physics, and its role in dark matter and dark energy problems. In contrast, TEGR and STEGR recover the EP, in particular in its strong formulation, but do not rely on it as a foundational principle. The fact that GR, TEGR, and STEGR are equivalent in non-trivial predictions, but the EP is not necessary for TEGR and STEGR, suggests that it may not be a fundamental feature but an emergent one, potentially marking differences in the empirical content of the three theories. Thus, the developments within the Geometric Trinity framework challenge traditional assumptions about spacetime and may help to better understand some of the unresolved foundational difficulties related to the EP.

Keywords: Theories of Gravity, Equivalence Principle, Spacetime Structure

Contents

1	Introduction	2
2	General Relativity: assumptions and shortcomings	4
3	Equivalent Gravities	9
3.1	Equivalence of Lagrangians	11
3.2	Equivalence of the field equations	12
3.3	The role of the Strong Equivalence Principle	13
4	Epistemological considerations on Equivalent Gravities	15
4.1	High degree of under-determination	15
4.2	The debate on the Equivalence Principle	17
5	Discussion	20
5.1	Implications and limits of the argument	20
5.2	Experimental perspectives	21
6	Conclusions	23
	Bibliography	25

1 Introduction

Despite General Relativity (GR) being considered the “standard theory” of gravity, there has never been a period in its history without serious alternatives being proposed and developed, with different approaches and motivations [1–4]. Particularly in the last four decades, in the so-called “precision cosmology era”, advancements in cosmological observations, gravitational wave physics and high precision tests have

highlighted significant shortcomings, both at theoretical and observational level, both at small (UV) scales, and at large (IR) scales. [1].

Several authors find it natural and elegant to consider improving the gravitational components of field equations [1–3], by exploring extended or alternative theories of gravity. There are many types of approaches to modified gravity theories [1, 2]: straightforward extensions like $f(R)$ gravity, Vector-Scalar-Tensor theories, where geometry can non-minimally couple to new fields; higher-order theories, where derivatives of metric components higher than second order can appear; theories with modified geometry; theories based on different principles, such as MODified Newtonian Dynamics (MOND); unifying theories attempting to quantize gravity, and so on.

A general approach can be inserted in the context of the so-called metric-affine theories of gravity, which fall into the category of approaches where geometry is enlarged and improved [5]. In particular, in 1919 Palatini showed that the metric tensor and the affine connection, which constitute GR, can be considered as two different geometric structures and can be varied independently [6]. These considerations led to the development of theories where the field equations can be formulated in terms of other geometric invariants: the torsion tensor and the non-metricity tensor. Together with the curvature tensor, considering these three geometric objects, we can build up different theories, where GR is a particular case in a lake of more general metric-affine theories, where torsion and non-metricity are set to zero. Among these theories, it is of particular interest the so-called Geometric Trinity of Gravity [5, 7–9], which comprises GR, built upon the metric tensor and grounded on the curvature of spacetime; the Teleparallel Equivalent of GR (TEGR), formulated in terms of torsion of spacetime and relying on tetrads and spin connection; and the Symmetric Teleparallel Equivalent of GR (STEGR), built on nonmetricity and constructed from metric tensor and affine connection. For this reason, in this case we can speak of *modified spacetime* rather than modified gravity, as some authors suggested [10].

Significantly, these three theories have been found to be dynamically equivalent to GR, as the names suggest, and the Geometric Trinity recently gained a lot of attention in both the theoretical [5, 7–9, 11] and the philosophical literature [12–16]. From these recent theoretical developments, many questions arise. Should we consider TEGR and STEGR as proper alternative theories to GR or merely different dynamical formulations? Is it possible to empirically discriminate among them? If they are dynamically equivalent, is gravitation given by curvature, torsion, or non-metricity of spacetime at some fundamental level?

In this work, it will be shown that a closer inspection on the equivalent features reveals crucial differences among the theories, that is, their relation with respect to the Equivalence Principle (EP). The EP is one of the most important assumptions of GR, which has been famously called as the “midwife” who allowed Einstein to develop the theory [17, 18]. As we will see in detail, there are many open conceptual difficulties, which are direct or indirect consequences of the EP imposition at the foundation of the theory. Some of them puzzled Einstein himself until the very end of his life, such as the coincidence between the geodesic and the causal structure, or the fact that the fundamental object of the theory is the Riemannian metric $g_{\mu\nu}$ instead of the connection $\Gamma_{\mu\nu}^\rho$. The fact that the EP is such a theory-laden principle has led some

authors to describe it as a “beast” [18], or a bunch of beasts, given all its different formulations.

Significantly, the Geometric Trinity suggests that there are viable theories of gravity which have not the necessity to impose the EP. The fact that these three representations of gravity are dynamically equivalent, and that the EP can be recovered but not at the foundation of TEGR and STEGR, suggests that the EP could be not a fundamental principle, but an emergent feature related to some symmetry or gauge [11, 19, 20]. Therefore, it will be argued that, at some level, the EP might constitute a direct difference in the empirical content between the three representations and could allow us to discriminate among them.

This result is relevant because, if it is the case, it could allow to relax this theory-laden assumption at fundamental level and address some of the open problems. This, in some sense, is in line with the intuition by Synge, who considered it only a midwife and not a fundamental feature of the world [17].

In the EP discussion, we follow the approach by Lehmkuhl [18], as well as the recent contribution by Read and Teh on the EP in teleparallel gravity [21]. We adopt their framework and extend the discussion to the full Geometric Trinity.

The paper is organized as follows. In Sec. 2, the open fundamental difficulties entailed by the EP are discussed. In Sec. 3, recent results on the equivalent features in the Geometric Trinity of Gravity, relevant for this work, are presented. Then, in Sec. 4, the relation between the EP and the Geometric Trinity will be analysed. In Sec. 5, the epistemological and experimental implications will be discussed. In Sec. 6, conclusions are drawn.

2 General Relativity: assumptions and shortcomings

There are many significant conceptual difficulties that emerge from the assumption and the application of the EP, which are known in the physical and foundational literature, and which are sufficient to give to the EP the reputation of being “a beast” [18].

Without entering in historical details, GR is built upon the EP, as well as upon other fundamental assumptions. In the following, we adopt the Lehmkuhl framework [18], and we follow, in particular, the definitions provided by Read and Teh in a recent discussion of the EP in TEGR [21].

In its weaker form (WEP), the EP states that the gravitational mass of a body is equal to its inertial mass (called WEP2 in Ref. [18, p. 5] and [21, p. 3484]). The empirical indistinguishability between gravitational and inertial effects was a crucial starting point for Einstein in formulating his version of the EP, which is now known as the Einstein Equivalence Principle (EEP) [18, p. 10]:

EEP: Gravity and inertia are the same in their very essence (“wesensgleich”).

Mathematically, it means that both gravitational and inertial effects are represented by the components of the same compatible connection [21, p. 3486].

Finally, the link between the EEP and the local validity of Special Relativity (SR), first underlined by Pauli in 1921 [18, pp. 21-22], constitutes the basis for the

strong formulation of EP (SEP). According to the definition by Pauli, for every infinitely small region of the world, there always exists a coordinate system such that gravitational effects can be neglected. Einstein did not call it as the SEP, but he considered the EEP and the local validity of SR as two strictly related concepts [18]. Read and Teh state the SEP as follows [21, p. 3486]:

SEP: At any point $p \in M$, a manifold, one can find an orthonormal frame in which gravito-inertial effects, as represented by connection coefficients, vanish.

Then, Read and Teh define a version of the SEP which do not assume the EEP, which will be useful in relation to TEGR:

SEP- \neg EEP: At any point $p \in M$, a manifold, one can find a frame in which inertial effects, as represented by connection coefficients, cancel gravitational effects, as represented by some tensor quantity.

Adding that the frames in which the principle holds are related by Lorentz transformations, we arrive to what Red and Teh call SEP_2 [21, p. 3487].

The standard frame where the physics of SR is locally recovered, the metric tensor $g_{\mu\nu}$ takes the Minkowski form $\eta_{\mu\nu}$, and the connection coefficients $\Gamma_{\mu\nu}^\rho$ vanish, is called the Local Inertial Frame (LIF). The requirement that such a frame exists at every point $p \in M$, demands the metric compatibility condition $\nabla_\lambda g_{\mu\nu} = 0$ [22, p. 313].

Note that there are also other different formulations of the SEP, as the extension of the EEP also to bodies with non-negligible self-gravitational interactions and gravitational experiments [4, p. 76]. However, the relation between these different definitions is not yet clear and this issue is not addressed in this work.

In the following, we will list eight foundational problems relevant for this work.

i) Coincidence of the causal and the geodesic structure

As a consequence of the imposition of the SEP, the Christoffel symbols $\Gamma_{\mu\nu}^\rho$ coincide with the Levi-Civita connection. Assuming the SEP, the unique possible affine symmetric and metric compatible connection is the Levi-Civita one [22, p. 314]:

$$g_{\mu\nu}(p) = \eta_{\mu\nu}, \quad \Gamma_{\mu\nu}^\rho(p) = 0, \quad \nabla_\lambda g_{\mu\nu} = 0, \quad \Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho$$

\Downarrow

$$\Gamma_{\mu\nu}^\rho = \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} g^{\rho\gamma} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \quad (1)$$

Consequently, by construction, the Levi-Civita connection has no dynamics, but it is a by-product of the metric $g_{\mu\nu}$, containing its derivatives. Physically, it represents the apparent forces acting on the body due to the curved geometric background. This means that the metric $g_{\mu\nu}$ determines, at the same time, the causal structure (light cones with rods and clocks) and the geodesic structure (the free fall of test particles) [22, 23]. However, it is important to underline that *a priori* there is no relation between the connection $\Gamma_{\mu\nu}^\rho$ and the metric tensor $g_{\mu\nu}$, but it is a consequence of the imposition

of SEP. In fact, this coincidence does not work anymore for extensions of GR as $f(R)$ [24]. This unjustified coincidence is a first conceptual problem, which has been widely discussed in the literature (see e.g. [25]).

ii) The metric as the fundamental object of the theory

The second problematic direct consequence of this picture is the following. As mentioned, in GR the real fundamental dynamical object is the metric tensor $g_{\mu\nu}$. However, it is not the gravitational field, but a set of potentials. The proper gravitational field is represented by the connection, as Einstein himself insisted many times [18, 26, 27]. It can be directly seen from the geodesic equation:

$$\frac{d^2 x^\rho}{d\tau^2} + \Gamma_{\mu\gamma}^\rho \frac{dx^\mu}{d\tau} \frac{dx^\gamma}{d\tau} = 0, \quad (2)$$

where $\Gamma_{\mu\gamma}^\rho \frac{dx^\mu}{d\tau} \frac{dx^\gamma}{d\tau}$ represents the generalization of the Newtonian forces.

This problem puzzled Einstein until the very end of his life, when he remarked again that the fundamental element should be the connection, and only indirectly the Riemannian metric $g_{\mu\nu}$ [28, pp. XVIII-XIX] (see also [29, p.9]). Moreover, in experiments, what we really measure are the forces (or the accelerations), which are represented by the connection $\Gamma_{\mu\nu}^\rho$. And as we have seen, the connections are the first *derivatives* of the metric and the second derivatives of the local inertial coordinates:

$$\Gamma_{\mu\gamma}^\rho = \frac{dx^\rho}{d\xi^\sigma} \frac{d^2 \xi^\sigma}{dx^\mu dx^\gamma}. \quad (3)$$

Therefore, the quantity we observe in typical experiments is not related to the metric but to the connection, which in GR gains its dynamics from the former.

There were also other reasons for Einstein to not consider the metric $g_{\mu\nu}$ as the fundamental object, as the fact that it seemed to him too similar to the concept of Newtonian absolute space, the overcoming of which was one of his first objectives [18].

Therefore, thanks to the SEP, spacetime is described by the double $\{M, g_{\mu\nu}\}$, i.e. the Riemannian manifold, where M is the manifold and $g_{\mu\nu}$ the metric tensor, that is the fundamental object.

iii) Tensions with new physics predictions

As a general consideration, it is possible to state that “new physics” naturally predicts the violation of the SEP at some level. The simplest case is in Scalar-Tensor Theories, where, in the Einstein Frame, the mediation of the “fifth force” causes a difference in the free fall between different objects [1], which violates the WEP, and therefore the SEP. Then, in theories featuring Quintessence, where the cosmological constant is replaced by a slowly evolving scalar field, one expects that the coupling of this field with matter induces gravitational forces that depend on the composition of the body. This clearly violates the WEP. Scalar fields violating the WEP are also predicted by theories involving extra dimensions, such as String Theory (ST). As argued in Ref. [30], current precision levels of WEP tests (today 10^{-15} , as we will see in Sec. 5) should not

discourage further research, as ST could imply WEP violations even further. For this reason, in Ref. [30], it is suggested that the WEP experiments are the most sensitive tools that we have to test new physics. Similarly, some authors argue that quantum properties of gravity could have observable experimental consequences at low energies, such as the dependence of geodesic motion on the mass of test particles, as explored in [31, 32]. Other authors [33] derive a direct violation of WEP at finite temperature from Quantum Field Theory. Then, famously, there is MOND (MOdified Newtonian Dynamics), which from observational constraints, should induce violations of the SEP [34]. Data from open clusters show a smaller mass discrepancy than would be required if MOND obey the SEP, even though it seems that these predictions consider the definition of SEP as the extension of EEP also to bodies with non-negligible self-gravitational interactions [4, p. 76], which is not the same concept under consideration in this work. Finally, it can be shown that MOND is just a particular case of some field theory (e.g. $f(R)$ gravity [35, 36]) so SEP can be questioned as soon as one relaxes the hypothesis that GR is the “only” viable theory of gravity.

On this line, in Ref.[30], it is suggested that any bias towards metric theories is entirely unjustified, both historically and from the perspective of contemporary fundamental physics.

iv) Validity of the SEP at quantum level

A further significant issue is that we do not know if the WEP, and therefore the SEP is valid at quantum level. At the moment, we are only assuming its validity. We are not even sure if this principle could be generalized by the quantum formalism. There are attempts in this direction, but there are conflicting opinions among physicists [23, 37, 38]. Since at quantum level particles behave like wave packets, it is difficult to make sense of the concepts of free fall universality or the identity between the gravitational and the inertial mass. Anyway, given quantum mechanics, we have no a priori reasons to postulate the WEP validity.

v) Dark Energy and Dark Matter as possible geometric issues

There is also the problem of Dark Energy (DE) and Cold Dark Matter (CDM), on which there is a big debate in the community. As already mentioned, several authors find more elegant to consider altering the gravitational component of the field equations [1] in order to address the dark phenomenology as a geometric problem, instead of a fluid one [39]. The main practical reason is that, up today, there is no final indication that dark side components could be addressed by new fundamental particle sector [40–43]. This debate between the dark fluid hypothesis and modified gravity approaches is impressively widespread in the community of philosophers of physics [3, 44–46], with particular focus on the metric postulate. Authors argued that since any viable metric theory of gravity finds dark matter in the sieve [47], it could be the case that a non-metric theory of gravity could help in better understanding this problem.

The interesting thing is that, similarly to GR, where we can extend it to $f(R)$ gravity, $f(T)$ and $f(Q)$ gravity are the extensions of TEGR and STEGR, respectively. Where R is the Ricci curvature scalar, T is the torsion scalar and Q the non-metricity

scalar, while $f(R)$, $f(T)$ and $f(Q)$ are more general functions of them. The dynamical equivalence in the Geometric Trinity holds only for theories linear in the scalar invariants and not for the extensions, for different reasons [5, 9, 48]. Firstly, the extensions give rise to dynamics with different degrees of freedom. In particular, in $f(R)$ gravity, we have field equations of fourth order, in metric representation, whereas $f(T)$ and $f(Q)$ still remains of second-order. In addition, in $f(T)$ and $f(Q)$, we cannot choose, in general, a gauge to simplify the calculations, as in the cases of TEGR and STEGR. The point is that similarly to the fact that $f(R)$ theories are being studied to resolve shortcomings of GR at different scales (see [1]), also the extensions of TEGR and STEGR show interesting features in this direction. Physicists are already exploring $f(T)$ and $f(Q)$ gravities to study not only DE but also large structures, bouncing cosmologies, quantum cosmology, relativistic MOND theories, cosmography, inflation and gravitational waves (see for instance [48, 49]).

Therefore, the exploration of these alternatives to GR may provide a promising path to a successor and more fundamental theory. We can say “more fundamental” because TEGR and STEGR have not to postulate the EP in any form, as we will see, so they could be regarded as generalizations of GR, as Einstein already guessed.

vi) Difference with other fundamental interactions

The SEP sets gravity apart from other fundamental interactions of Nature. In today theoretical physics, it is believed to be very important to formulate theories as *gauge* theories, since it works so well with other fundamental interactions [50]. The interesting fact is that gravitation can be reformulated as a gauge theory properly without assuming the EP in any form [5]. See Ref. [51] for a philosophical discussion on TEGR and STEGR as gauge theories.

vii) Epistemic justification of the coincidence between m_G and m_I

There is the foundational problem of how we can justify the coincidence between gravitational and inertial mass, which is the basis of the formulation of the WEP. For Newton, the mass of any body, understood as the property of the body itself to respond to a force, corresponded to its “weight”, which is its property to respond to gravity. In modern terms, we would say that inertial mass m_I is equal to passive gravitational mass m_G , terms coined by Bondi [52]. Einstein said that it was precisely the famous Eötvös experiments on the equivalence between m_I and m_G that directly inspired him in the formulation of the EEP [18] and which, in fact, constitutes one of its cornerstone.

Today, for many people, this equivalence might seem obvious, but back then, it was not, and it would not be even today if we “forgot” to acknowledge this principle. In other words, *prima facie*, there are no reasons to postulate the identity $m_G \equiv m_I$, and it is not related to some fundamental symmetry. Einstein himself embraced it from empirical reasons. So apart from observations, how one could even imagine this equivalence?

viii) *Curvature over torsion of spacetime*

It is important to underline that in GR, torsion of spacetime is set to zero a priori, since with the imposition of the SEP, Einstein chose the symmetric connection, the Levi-Civita one, and so curvature of spacetime. In 1922, Cartan explored a different direction, considering a natural extension of GR constituted not only by the Levi-Civita connection, but also by the torsion tensor, that is the antisymmetric part of a metric compatible affine connection. In this way, he developed a geometric formulation where he suggested that torsion can be physically related to the intrinsic (quantum) angular momentum of matter and it vanishes in vacuum [5]. Einstein himself, in the period 1923 – 1933, tried different geometries for the construction of a unified field theory [29, p.57]. In 1928, he published his first paper on “fernpaallelism”, or teleparallelism [53]. As we will see in more detail, since TEGR is found to be dynamically equivalent to GR, classical tests that were understood to confirm the curvature of spacetime can similarly be understood as confirming the torsion of spacetime. This is known as the problem of *geometric under-determination* [12, 14].

A priori, why prefer curvature over torsion of spacetime? Similar considerations can be applied also to non-metricity, but curvature and torsion are easier to think about metaphysically. It is natural to our mind to think that a massive object can cause curvature of spacetime. However, *prima facie*, it is natural, in a similar way, to think that a massive object could cause also torsion. Consider, for instance, a rotating black hole.

So GR assumes a priori the SEP, and therefore curvature. The Geometric Trinity challenges also this fundamental assumption on spacetime focusing only on equivalence of dynamics.

In conclusion, there are many conceptual difficulties as direct or indirect consequences of the EP. As it will be argued in the next sections, the framework of metric-affine theories and the relaxation of the assumption of the EP could help in addressing these issues, apparently maintaining the same consolidated successes.

3 Equivalent Gravities

We will now summarize the main achievements of Geometric Trinity which are relevant for the present discussion. We refer to some recent works [5, 7, 11, 54].

As we have previously seen, the GR spacetime is assigned by the double

$$\{M, g_{\mu\nu}\}, \quad (4)$$

due to the imposition of the SEP. On the contrary, following the Palatini approach [55], the metric $g_{\mu\nu}$ and the connection $\Gamma_{\mu\nu}^\rho$ can be varied independently. In this case, spacetime is assigned by the triple:

$$\{M, g_{\mu\nu}, \Gamma_{\mu\nu}^\rho\}, \quad (5)$$

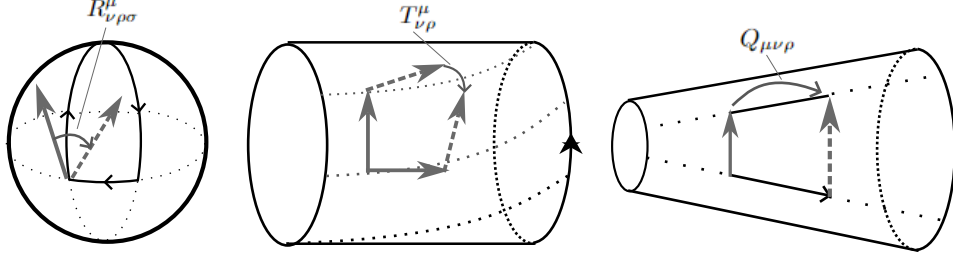


Figure 1 Typical way of defining the three geometrical invariants [5, 8]. There is curvature of spacetime if after having paralleling transported a vector along a closed loop, there is a non-zero angle between the initial and final vectors. There is torsion of spacetime if by paralleling transporting two vectors one along the other, it is not possible to close the parallelogram. Finally, non-metricity occurs if there is a change in the length of the vector when it is moved along a curve.

where $g_{\mu\nu}$ determines the causal structure while the connection $\Gamma^\rho_{\mu\nu}$ determines the free fall [7].

Einstein himself recognized as significant the Palatini method, since it represents a simplification of the relativistic formalism [28, p. XXIII], or a generalization, we would say. With the Palatini approach, the connection $\Gamma^\rho_{\mu\nu}$ can be written in a more general form considering the *affine* connection [7, 8]:

$$\Gamma^\rho_{\mu\nu} = \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\} + K^\rho_{\mu\nu} + L^\rho_{\mu\nu}. \quad (6)$$

Where $\left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\}$ is the Levi-Civita connection, $K^\rho_{\mu\nu}$ and $L^\rho_{\mu\nu}$ are the contortion and the disformation tensors, respectively [8]:

$$K^\rho_{\mu\nu} = \frac{1}{2}(T^\rho_{\mu\nu} + T^\rho_{\nu\mu} - T^\rho_{\mu\mu}) \quad (7)$$

$$L^\rho_{\mu\nu} = \frac{1}{2}(Q^\rho_{\mu\nu} - Q^\rho_{\nu\mu} - Q^\rho_{\nu\nu}). \quad (8)$$

$T^\mu_{\nu\rho}$ and $Q_{\mu\nu\rho}$ are the torsion and the non-metricity tensors, and $R^\mu_{\nu\rho\sigma}$ is the curvature tensor [8]:

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\tau\rho} \Gamma^\tau_{\nu\sigma} - \Gamma^\mu_{\tau\sigma} \Gamma^\tau_{\nu\rho}, \quad (9)$$

$$T^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu} - \Gamma^\mu_{\nu\rho} \neq 0, \quad (10)$$

$$Q_{\mu\nu\rho} = \nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\lambda\nu} \neq 0. \quad (11)$$

As one can see in Fig. 1, the curvature tensor encodes the variation of the angles in a parallel transport along a closed curve on a manifold; the torsion tensor encodes how the tangent space twists around a curve when we parallel transport two vectors along each other; non-metricity encodes the variation of vectors' length when they are moved along a curve [8].

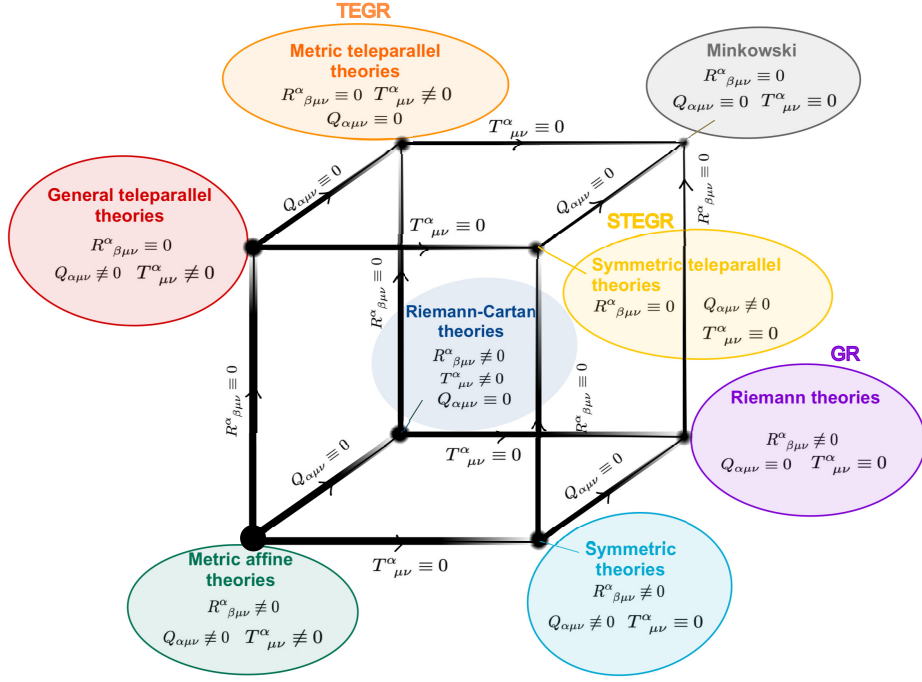


Figure 2 A map of the possible metric-affine theories. GR is a particular theory where torsion and non-metricity are set to zero (see also [5, 8]).

With these three geometrical objects, we can build all the possible metric-affine theories, as can be seen in Fig. 2.

As anticipated, several authors [5, 7, 11, 54] claim that TEGR and STEGR can be formulated to be dynamically equivalent to GR in multiple features. Thus, in the following we are going to analyse the features in which the equivalence arises.

3.1 Equivalence of Lagrangians

Firstly, there is an equivalence at the Lagrangian level. In fact, GR dynamics can be derived from the Hilbert-Einstein action [5]:

$$S_{GR} = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} (\mathcal{L}_{GR} + \mathcal{L}_m), \quad (12)$$

where $\mathcal{L}_{GR} = R$ and R is the Ricci curvature scalar; \mathcal{L}_m is the matter Lagrangian.

In TEGR and STEGR the same dynamics can be recovered, up to a boundary term. In TEGR, with $\mathcal{L}_{TEGR} = -T$, i.e. the torsion scalar, we have [5]:

$$S_{TEGR} = \frac{c^4}{16\pi G} \int d^4x e \mathcal{L}_{TEGR} + \int d^4x e \mathcal{L}_m \quad (13)$$

$$R = -T - \frac{2}{e} \partial_\mu (e T^\mu) \quad (14)$$

$$T = \frac{1}{2} S_A^{\mu\nu} T_{\mu\nu}^A = \frac{1}{2} (K_A^{\mu\nu} - e_A^\nu T^\mu + e_A^\mu T^\nu) T_{\mu\nu}^A, \quad (15)$$

where $T_\alpha^{\alpha\mu} = T^\mu$ is the torsion vector and K_{BA}^C the contortion tensor. $S_A^{\mu\nu}$ is the superpotential $S_A^{\mu\nu} = K_A^{\mu\nu} - e_A^\nu T^\mu + e_A^\mu T^\nu$. And e denotes the determinant of e_A^μ .

Similarly, in STEGR, with $\mathcal{L}_{STEGR} = Q$, the non-metricity scalar, we have [7, 56]:

$$S_{STEGR} = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} (\mathcal{L}_{STEGR} + \mathcal{L}_m) \quad (16)$$

$$Q = g^{\mu\nu} (L_{\beta\mu}^\alpha L_{\nu\alpha}^\beta - L_{\beta\alpha}^\alpha L_{\mu\nu}^\beta) = R + \nabla_\mu (Q^\mu - \bar{Q}^\mu), \quad (17)$$

where $Q_\alpha = Q_{\alpha\lambda}^\lambda$ and $\bar{Q}_\alpha = Q_{\alpha\lambda}^\lambda$.

First of all, the equivalence among the three theories is evident at Lagrangian level, up to a boundary term. As mentioned, this equivalence does not hold for extensions like $f(R)$, $f(T)$, and $f(Q)$ [9].

3.2 Equivalence of the field equations

Secondly, the same comparison can be developed at the level of field equations. We can start from the Bianchi identities, which have the important role to link the field equations with the conservation laws of the gravity tensor invariants and the energy-momentum tensor. Also in this case, the equivalence of the three formulations can be achieved.

The most general second Bianchi identity is the following [8]:

$$\nabla_\lambda R_{\beta\mu\nu}^\alpha + \nabla_\mu R_{\beta\nu\mu}^\alpha + \nabla_\nu R_{\beta\lambda\mu}^\alpha = T_{\mu\lambda}^\rho R_{\beta\nu\rho}^\alpha + T_{\nu\lambda}^\rho R_{\beta\mu\rho}^\alpha + T_{\nu\mu}^\rho R_{\beta\lambda\rho}^\alpha. \quad (18)$$

In GR, since we have no torsion and non-metricity, we derive the Einstein field equations (EFE) in vacuum:

$$\nabla_\mu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0, \quad (19)$$

where the notation $\overset{\circ}{R}$ stands for quantities built up on the Levi-Civita connection, i.e. in this case the Ricci tensor.

In TEGR, having vanishing curvature and non-metricity, via the Weitzenböck gauge, we obtain an equivalent expression of the EFE which is [5]:

$$R_{\beta\mu\nu}^\alpha = \overset{\circ}{R}{}^\alpha_{\beta\mu\nu} + K_{\beta\mu\nu}^\alpha \quad (20)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -K_{\mu\nu} + \frac{1}{2} g_{\mu\nu} K \quad (21)$$

$$K_{\mu\nu} - \frac{1}{2} g_{\mu\nu} K = 0 \quad (22)$$

Similarly, in STEGR, since there is no curvature and torsion, and via the coincident gauge, we obtain equivalent field equations [5]:

$$R_{\beta\mu\nu}^\alpha = \overset{\circ}{R}_{\beta\mu\nu}^\alpha + L_{\beta\mu\nu}^\alpha \quad (23)$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -L_{\mu\nu} + \frac{1}{2}g_{\mu\nu}L \quad (24)$$

$$L_{\mu\nu} - \frac{1}{2}g_{\mu\nu}L = 0 \quad (25)$$

Therefore, despite being built on different principles and geometric objects, equivalent formulations of the EFE can also be derived in TEGR and STEGR.

Since the same field equations have been obtained, the same exact solutions, under the same symmetries and boundary conditions, have to be achieved. The Schwarzschild solution, i.e. the spherically symmetric solution, and the validity of the Birkhoff theorem can be derived in all the three formulations [5, 54]. This is important, since if the field equations and their solutions are the same, we have the same empirical predictions. For instance, the classic tests of the trajectories of massive bodies and photons, according to the Schwarzschild solution of the field equations, confirm in the same way GR, TEGR and STEGR. So we cannot say anymore that these classic tests are corroborations of GR and of spacetime curvature [14].

3.3 The role of the Strong Equivalence Principle

Finally, theorists have found another significant dynamical equivalence, that is the recovery of the SEP [5, 54].

Briefly, TEGR is built upon tetrads e_μ^A , which describe gravity, and spin connections $\omega_{B\mu}^A$, which account for inertial effects. Tetrad fields are geometric constructions which establish a relation between the manifold and its tangent spaces as a soldering agent. On the other hand, the spin connections account for inertial effects in rotated frames. The coordinates of the two frames are related with Lorentz transformations.

One can write also GR in the tetrad formalism, where the Levi-Civita connection is translated into a Lorentz connection [5, 7]:

$$\overset{\circ}{\omega}_{B\mu}^A = e_\lambda^A e_B^\nu \Gamma_{\mu\nu}^\lambda + e_\sigma^A \partial_\mu e_B^\sigma = e_\nu^A \nabla_\mu e_B^\nu, \quad (26)$$

where $\overset{\circ}{\omega}_{B\mu}^A$ accounts for both gravitational and inertial effects in GR. From the General Covariance Principle in its active formulation, we can obtain the coupling gravitational prescription. This principle states that the laws of physics - initially written in a special-relativistic context - must retain their form also in the presence of gravity, requiring invariance under spacetime diffeomorphisms. In the case of GR, for a general field Ψ [5, 54], it is:

$$\partial_\mu \Psi \longrightarrow D_\mu \Psi = \partial_\mu \Psi + \frac{1}{2} \overset{\circ}{\omega}_{B\mu}^C S_C^B \Psi \quad (27)$$

where S_C^B are the Lorentz generators.

The analogous coupling in TEGR can be similarly derived. Transitioning from the special-relativistic to the case with gravity, incorporating the appropriate contortion tensor in the Lorentz covariant derivative, we obtain:

$$\partial_\mu \Psi \longrightarrow D_\mu \Psi = \partial_\mu \Psi + \frac{1}{2} (\omega_{B\mu}^C - K_{B\mu}^C) S_C^B \Psi \quad (28)$$

The combination of (27) and (28) yields the relation between the gravitational and inertial effects in TEGR and GR [5, 54]:

$$\omega_{B\mu}^C - K_{B\mu}^C = \overset{\circ}{\omega}_{B\mu}^C, \quad (29)$$

where $\omega_{B\mu}^C$ and $K_{B\mu}^C$ account respectively for inertia and gravitation in TEGR, and $\overset{\circ}{\omega}_{B\mu}^C$ for both gravitation and inertia in GR. Therefore, choosing a LIF, where the GR spin connection vanishes $\overset{\circ}{\omega}_{B\mu}^A = 0$, we obtain the identity between the inertial effects and gravitation in TEGR [5, 54]:

$$\omega_{B\mu}^C = K_{B\mu}^C, \quad (30)$$

which recover the second formulation of SEP defined in Sec. 2, *SEP* \neg *EEP*, which does not assume the EEP, or in the words by Read and Teh, it does not conceptually unify gravity and inertia [21, p. 3486]. In fact, here it is evident how in TEGR we have the separation of gravitational and inertial effects, the former identified by the contortion tensor and the latter by the spin connection. For some authors, this possibility of separation is one of the most important properties of TEGR [54].

For a recovery of the first formulation of SEP in the TEGR context, see [21, p. 3491].

With respect to STEGR, writing its connection by tetrads e_β^α [5]:

$$\Gamma_{\mu\nu}^\alpha = (e^{-1})_\beta^\alpha \partial_\mu e_\nu^\beta, \quad (31)$$

without curvature and torsion, and by a particular transformation of coordinates in a point p , we can arrive at the coincident gauge [5, 56]:

$$\Gamma_{\mu\nu}^\alpha(p) = \frac{\partial x^\alpha}{\partial \xi^\lambda} \partial_\mu \partial_\nu \xi^\lambda = 0. \quad (32)$$

The vanishing of the connection physically means that the origin of the tangent flat space and the one of the manifold are coincident, which is the formulation of the SEP defined in Sec. 2 [5]. The SEP is then recovered also in STEGR, via the coincident gauge.

Therefore, as we can see, in TEGR and STEGR, the SEP is not postulated but emerges as a result of a gauge choice.

In conclusion of this section, we can summarize the significant results in Geometric Trinity as:

- The equivalence at Lagrangian level (up to a boundary term) holds.
- The equivalence of field equations holds starting from the general second Bianchi identities.
- The same solutions of the field equations are recovered in all the three theories.
- The SEP is recovered in TEGR and STEGR, even though such a principle is not at their foundation.

In the following section, we are going to discuss these equivalences with particular focus on their epistemic and experimental implications.

4 Epistemological considerations on Equivalent Gravities

4.1 High degree of under-determination

We have seen that there is a high degree of under-determination among the Geometric Trinity of Gravity, as already pointed out both in the physical [5, 7, 54] and the foundational [12–14, 57] literature. This causes both various epistemological and meta-physical problems. In this work we focus particularly on *if* and *how* we can distinguish experimentally among them. Although the three theories are dynamically equivalent, and the recovery of SEP is often regarded as one of the key features through which TEGR and STEGR demonstrate their empirical equivalence with GR, we argue that the differing foundational status of SEP across the three theories may nonetheless lead to differences in empirical content. We think that this difference is strictly related to the assumptions of GR on the spacetime structure and on the fundamental objects of the theory.

However, before coming into the details of the central argument of this work, it is important to introduce the epistemic tools that will be exploited. They are extensively discussed among philosophers of physics and epistemologists in the context of high degrees of under-determination.

The landscape of theoretical physics has evolved in the last 2-3 decades, because for most theories beyond the Standard Model of Particles and Quantum Gravity, empirical data are either scarce or completely absent. Nevertheless, theories like ST, Supersymmetry or Cosmic Inflation have all been defended for decades, although none of the classical methodologies seem to straightforwardly apply. While experimental testing remains the gold standard, the use of analogue experiments and the so-called non-empirical ways of theory assessment, or meta-empirical, have been proposed, especially in fundamental physics and cosmology. The issue is particularly relevant because today, fundamental physics clearly is not driven by perspectives of technological utilization in a few years, and the typical time scale for that intermediate state has grown beyond one generation of scientists. During most of the 20th century, fundamental physics was perceived as a scientific field where theories typically could be empirically tested within a reasonable time frame. But today the situation is different. Moreover,

even concerning proper experimental tests, contemporary experiments are far more intricate, and the evaluation and interpretation of the data are subtle and by no means trivial matters.

These are some of the reasons why some philosophers of physics have delved into this idea and formulated theories of “confirmation” that make the corresponding intuition more rigorous. This approach (see for instance [58–60]) exploits Bayesian Confirmation Theory and deeply relies on the practice of exploring and constraining the theory space. In fact, when scientists find that despite substantial efforts, no alternative viable hypothesis are capable of explaining some scientific problem, they tend to place more trust in the existing theory. This is indeed called the “No Alternatives Argument”. These approaches are called “meta-empirical”, since they are not empirical in the common sense, but still involve observations. For a general Bayesian formalization of this argument and a proof that it counts as evidence, see Ref. [61], as well as for its limits.

Note that we have not to confuse the use of the concept of non-empirical “confirmation” as the confirmation of a theory in the traditional sense.

Anyway, in this framework of constraining theory space, authors are developing some interesting tools to address cases of under-determination and assess untested hypotheses among competing theories.

In Ref. [62], the author develops an epistemological reflection on the theoretical exploration of alternative theories, which fits perfectly with the material of this work, allowing for a more precise formulation of the argument. In the following, the general argument of [62] will be briefly introduced, and then it will be applied to our specific case. See Ref. [62] also for the limits of the approach.

Let us assume we are interested in whether we can trust the predictions of some theory Th . We have made a large set of observations which are in agreement with the prediction P_1 of Th and therefore confirm it. Suppose that Th also makes the predictions P_2 and P_3 . We usually will have some confidence in these predictions of Th , as it has so far been an empirically successful theory. So the previous empirical success warrants an increase in our trust regarding the novel predictions P_2 and P_3 of Th .

Now let’s assume that for some reason we will not be able to conduct experiments on P_2 and P_3 . In these circumstances, we cannot further assess these predictions based on empirical data. Now assume that someone comes up with an alternative theory, say Th' , which happens to also predict the set of observations P_1 and it is therefore similarly confirmed by it. In addition, Th' predicts P_2 but disagrees about P_3 . Let us denote the predictions by:

$$\begin{aligned}\text{Predictions}(Th) &= \{P_1, P_2, P_3 \dots\} \\ \text{Predictions}(Th') &= \{P_1, P_2, \neg P_3 \dots\}\end{aligned}$$

How will the existence of this additional theory impact ones believe regarding the predictions P_2 and P_3 ? The same available empirical data, i.e. P_1 , confirms two competing theories, which agree with respect to one prediction, P_2 , and disagree with respect to another prediction, P_3 . If we have no reason to trust one theory more than the other,

then the proposal of the competing theory Th' should lead to an increase in our trust regarding the prediction P_2 , while it leads to a decrease with respect to the prediction P_3 . Now imagine further, scientists come up with another theory Th'' , which agrees with respect to the prediction P_2 and disagree with respect to P_3 :

$$\begin{aligned}\text{Predictions}(Th) &= \{P_1, P_2, P_3, \dots\} \\ \text{Predictions}(Th') &= \{P_1, P_2, \neg P_3, \dots\} \\ \text{Predictions}(Th'') &= \{P_1, P_2, \neg P_3, \dots\}\end{aligned}$$

It is reasonable to assume that we would slowly become more and more certain about P_2 being a feature of the world we live in but not about P_3 . This is more evident when we have agreement on multiple non-trivial risky predictions:

$$\begin{aligned}\text{Predictions}(Th) &= \{P_1, P_2, P_3, P_4, P_5, \dots\} \\ \text{Predictions}(Th') &= \{P_1, P_2, P_3, P_4, \neg P_5, \dots\} \\ \text{Predictions}(Th'') &= \{P_1, P_2, P_3, P_4, \neg P_5, \dots\}\end{aligned}$$

Therefore, if this argument is right, this counts as an evidence, a meta-empirical evidence, against the hypothesis P_5 . This counts not as an *empirical* observation, but as a *meta*-empirical observation. Counting as an observation, it affects the posterior probability of the validity of P_5 , as other empirical evidence [61]. Again, if also all the other competing theories would have had P_5 as a prediction, this would have been counted as a meta-empirical evidence in its favor. To be clear, in no way this is a posterior evidence with the same strength of an empirical one, but it still counts as evidence.

In this way, the exploration of competing alternatives allows us to better assess the untested predictions of the theory. Therefore, the practice of exploring theory space is highly powerful especially in contexts with high degrees of under-determination [62]. As anticipated, the context of Geometric Trinity is properly one of them. In fact, one of the main problems of this debate is that, even conceptually, it turns out that it is difficult to sharply distinguish between predictions of different theories of gravity. This means that they often do not lead to clear observational differences. The result is that every proposal that is viable mimics every other proposal that is viable, both empirically and conceptually. This is primarily because modified gravity scenarios, both extensions and alternatives, are victims of the GR success, so they have to reproduce its phenomenology in many features.

4.2 The debate on the Equivalence Principle

As we have seen, despite the fact that TEGR and STEGR are built on different foundation principles with respect to GR, physicists claim that they both recover the SEP, which is of course a necessary condition for a consistent theory of gravity, at least at classical level. However, it is often overlooked that there is a crucial difference in their relation with the SEP. In GR, SEP is a fundamental assumption of the theory, while, in TEGR and STEGR, it is not postulated *a priori* but it is recovered *a posteriori*. As it will be argued here, this structural difference could represent a possible important

distinction in the empirical content of theories in Geometric Trinity, since if it is not fundamental, there is the possibility that, at some level, it could be not valid.

Moreover, this result seems sufficient to regard TEGR and STEGR as different proper theories, instead of mere mathematical reformulations of GR, as some authors suggested [57]. According to that perspective, it was possible that the under-determination among the Geometric Trinity would later turn out to be an ill-posed problem. On the contrary, the recent theoretical advancements seem to have clarified this point.

Given the results highlighted in Sec. 3, we can now apply the epistemological consideration just introduced above, that is:

$$\begin{array}{ll} \text{Pred.}(Th) = \{P_1, P_2, P_3, P_4, P_5\} & \text{Pred.}(GR) = \{L, FE, S, C, FEP\} \\ \text{Pred.}(Th') = \{P_1, P_2, P_3, P_4, \neg P_5\} & \text{Pred.}(TEGR) = \{L, FE, S, C, \neg FEP\} \\ \text{Pred.}(Th'') = \{P_1, P_2, P_3, P_4, \neg P_5\} & \text{Pred.}(STEGR) = \{L, FE, S, C, \neg FEP\} \end{array}$$

On the left, the general epistemic argument is shown given by [62] and explained before; on the right there is the application to Trinity Gravity. It seems that this case fits perfectly with the above general epistemological considerations.

Following the five predictions discussed in 3, L stands for the equivalence at the Lagrangian level, FE for the field equations derived from the second Bianchi identity, and S for the solutions of the FE . Then C stands for cosmological applications. In fact, cosmological observations can be considered very important evidence for GR, but since we can now build cosmological models also with TEGR and STEGR, their predictive power in cosmology should be taken into account as well. In fact, people are already studying cosmological applications of TEGR and STEGR (see for instance [48, 49]). So we cannot anymore say that cosmological observations are evidence for GR only.

Finally, FEP stands for *Fundamental Equivalence Principle*. As demonstrated before, in TEGR and STEGR, the SEP is not fundamental, and so we can write $\neg FEP$. In other words, we can say that TEGR and STEGR predict an *EMergent Equivalence Principle*, or *EMEP*, instead of a FEP . This does not mean a prediction for a violation of the SEP, but only that TEGR and STEGR do not have the SEP at their foundation, i.e. they do not share the prediction FEP . In fact, the prediction *EMEP* is, in some sense, shared by all the three theories, since an *EMEP* is contained in FEP . If the SEP is fundamental, it has to be always valid at any level, and so it is also valid at emergent levels, but the contrary is not necessarily the case. In other words, the FEP implies the *EMEP*, but the *EMEP* does not imply the FEP :

$$FEP \rightarrow EMEP \tag{33}$$

$$EMEP \not\rightarrow FEP \tag{34}$$

The meaning of this difference can be seen more clearly in the geometric definition of SEP. As observed, in GR, where the FEP holds, it must always be possible to find a LIF in which gravitational effects can be nullified. In contrast, in TEGR and STEGR, this is not necessary, although it remains possible.

So, there is now a new beast in the bunch of beasts [18], the *EMEP*.

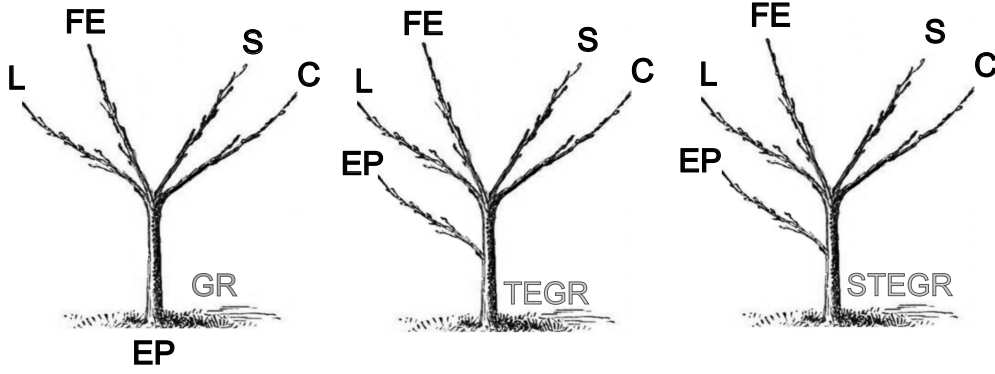


Figure 3 The figure shows how the three theories differ with respect to the EP (the SEP, in particular). In GR, EP is at the foundation of the theory and without it the other predictions are not possible; without the root, the branches would not exist. In TEGR and STEGR, on the contrary, the EP is not at the foundation, but it is a lateral branch, recoverable through the general covariance principle and via the coincident gauge, respectively. *L*: Lagrangians, *FE*: field equations, *S*: solutions of the *FE*, *C*: cosmological applications.

As argued, this situation decreases our confidence in that hypothesis on which the equivalent theories do not agree. That is to say, if the argument is correct, our confidence in the fundamentality of the SEP is decreased.

This argument is also independent of the history of the theory itself. In fact, imagine that both TEGR and STEGR would be built only with the SEP as a foundation; imagine that *L*, *FE*, *S* and *C* were derivable only if *FEP* was valid, i.e. only assuming the SEP. Then our confidence in the SEP as a necessary principle of any consistent theory of gravity would be increased. In the terms previously defined, this observation would have counted as a meta-empirical evidence for the hypothesis *FEP*, and consequently for a metric theory of gravity. It would have been considered evidence in favor of *FEP* because, after searching for an alternative hypothesis to *FEP* to build a viable and coherent theory of gravity, physicists would not have found it. Therefore, as shown above, the posterior probability of the *FEP* hypothesis would be increased. But the results are pointing to exactly the opposite. In fact, both TEGR and STEGR can recover the SEP, but it is not at their foundation, i.e. they do not predict *FEP*. See Fig. 3.

More precisely, any viable theory of gravity has to recover the SEP, at least at the scales and at the levels of accuracy of the present experiments. So the fact that TEGR and STEGR can recover the SEP is an important feature. This is one of the reason why it is difficult to distinguish empirically among the three theories, since TEGR and STEGR necessarily have to recover the experimental tests of GR. But properly the fact that the SEP is not at the foundation of them constitutes a possible important difference from the empirical content.

In other words, this argument shows that SEP is not a principle necessary present in all viable theories of gravity, but it is an assumption on which GR is built.

5 Discussion

5.1 Implications and limits of the argument

In the general example presented at the beginning of Sec. 4, we mentioned non-tested hypotheses. In this perspective, one could think that the SEP is instead a well tested hypothesis. However, the argument does not address the SEP itself, but rather the *fundamentality* of SEP, which has not been decisively tested — as other authors have suggested, questioning the *universality* of EP rather than its *fundamentality*, using similar arguments based on teleparallel gravity [63, ch. 9]. In fact, particularly at quantum level, the WEP, which is at the foundation of SEP, is tested but not decisively tested [64]. The situation would instead radically change if a fundamental theory, requiring some form of EP at any level, were formulated.

On the other hand, one could include, in the argument, all theories predicting the violation of the SEP, in order to show that also these theories disagree with the hypothesis *FEP*. However, these theories are not really dynamically equivalent to GR as in the case of TEGR and STEGR. Even if some of them would recover the GR phenomenology, at today status of art, they cannot be considered the final theory of gravity. They may have good empirical and observational evidences in different regimes and scales, as in the case of MOND, but they give not the same predictions P_1, \dots, P_n . Therefore, the same epistemic considerations would not be valid.

Then, we can briefly see how the eight conceptual difficulties presented in Sec. 2 are, at least, mitigated by these results.

The first two problems (i) and (ii) can be addressed thanks to the Palatini formalism, where the metric $g_{\mu\nu}$ and the connection $\Gamma_{\mu\nu}^\rho$ are independent, with $g_{\mu\nu}$ determining the causal structure and $\Gamma_{\mu\nu}^\rho$ the free fall, i.e. the geodesic structure. In this way, the connection becomes the true fundamental dynamical variable and the observable of the theory. Following the words by Einstein, the metric is “dethroned” and becomes an “ancillary variable” (see Ref. [6] and references therein).

The tension with new physics predictions (iii), with quantum mechanics (iv) and the unjustified *a priori* coincidence between m_I and m_G (vii) would be resolved, since the SEP, along with the WEP and the EEP, could be considered an emergent feature, recoverable but not postulated. So the only requirements would be the compatibility of theories with experimental constraints. Then, TEGR and STEGR allow gravity to be reformulated as a gauge theory (vi), and their extensions $f(T)$ and $f(Q)$ could be studied in order to search for a geometrical solution to DE and CDM problems (v).

Finally, without imposing the SEP, we are not privileging the spacetime structure over torsion or non-metricity *a priori* (viii). The geometric under-determination would remain, but the correct spacetime structure would no longer be determined by a postulate.

With respect to the limits of the argument, there are a couple of useful considerations. First, the argument would not be valid if TEGR and STEGR were revealed as inconsistent theories, since we would no longer trust their predictions. A similar predictive power of the competing theories is a necessary condition for the soundness of

the argument. So if our trust in TEGR and STEGR would decrease for some reason, the argument would be not valid anymore.

Second, it could be objected that the number of true predictions which would be considered sufficient in order to trust the further predictions of the theories is somewhat arbitrary. Are *four* good predictions sufficient? Actually, in this case, there are five good predictions, since the *EMEP* is a true non-trivial and very significant prediction of the theories. One could point out that predictions *S* and *C* are actually consequences of prediction *FE*, since both the solutions and the cosmological applications depend on the field equations. This is true; maybe they are not independent predictions, but the interesting thing is again that with the extensions $f(T)$ and $f(Q)$, the equivalence is not anymore valid [9], so it is important to explore any possible different feature in solutions and applications [48, 49].

5.2 Experimental perspectives

As mentioned, physicists are not yet satisfied with the current precision of the tests of the different formulations of the EP, as we can see from the numerous experiments which are developed by many different research groups [23]. These tests are complex and require many years of work and experimental efforts, not to mention the proposed space missions. All this efforts are developed in order to increase the accuracy of the EP tests, demonstrating the unsatisfactory situation, especially at quantum level.

Our argument does not mean that SEP is false or that it has to be necessarily violated at some level, but surely it encourages experimentalists in the search for a possible violation to discriminate among concurring theories of gravity. If the SEP is not fundamental, it could be an emergent property. Therefore, it could be the case that it is violated at some level, for example at quantum level. The incoming experiments of free falling with quantum tests could be the straightforward approach to probe the above statement (see for instance [23, 65, 66]). Clearly these quantum tests are WEP tests, and \neg SEP does not imply \neg WEP, but \neg WEP does imply \neg SEP, i.e. if we detect a violation of the WEP, this would falsify also the first formulation of SEP given in Sec. 2. This is relevant because the finest experiments on EP we have at the moment are conceived for the weak formulation. Clearly, the free fall of a wave packet is something different with respect to the free fall of a classical test particle. This conceptual aspect needs further and deep investigations.

Currently, the highest accuracy on the Eötvös parameter, which quantify the violation of the WEP, has been reached by the MICROSCOPE space mission with a free-fall experiment performed with macroscopic classical masses. In 2017, they reached 10^{-14} [67] and, in 2022, 10^{-15} [68]. Other future space missions have been proposed, such as the Galileo Galilei (GG) [69] and the Satellite Test of the Equivalence Principle (STEP) [70], with the goal of achieving 10^{-17} and 10^{-18} , respectively. The atomic experiments have compared the free fall of different isotopes or atomic species such as ^{85}Rb and ^{87}Rb , ^{39}K and ^{87}Rb , the bosonic ^{88}Sr and the fermionic ^{87}Sr and also atoms in different spin orientations. In an experiment in Stanford [71], a precision of 10^{-12} was reached (see [23, 66] for a review). Similarly to the classical counterparts, the ultimate performance of atomic sensors for WEP tests can be reached in space, where tests with a precision of $10^{-15} \div 10^{-17}$ were proposed by the STE-QUEST

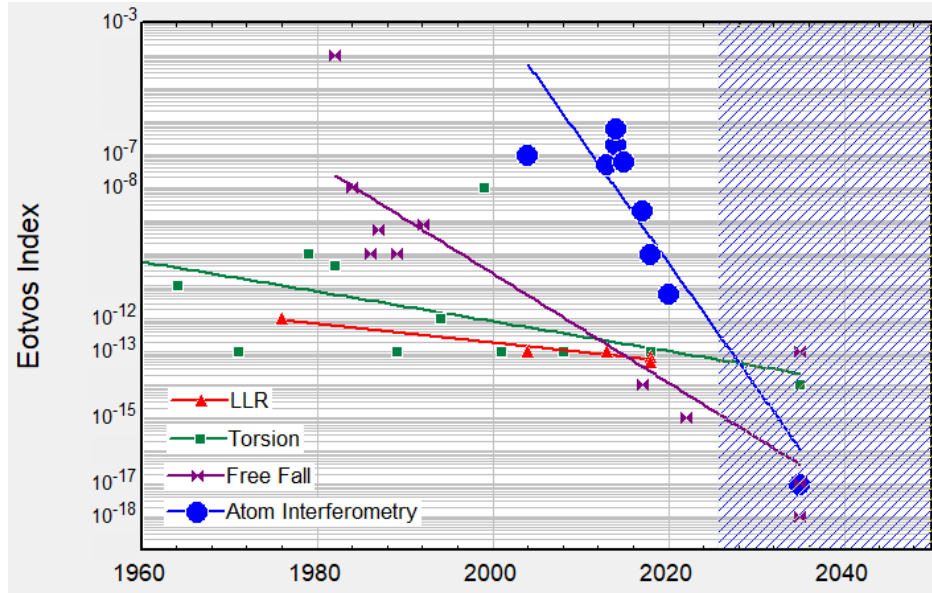


Figure 4 The most relevant WEP tests performed from 1960. Quantum WEP tests with atom interferometry are not yet at the same accuracy as their classical counterparts, but it is a novel technology (20-30 years) and its development is impressively faster than the classical tests. In the shaded area on the right there are the prospects of future missions and projects (for the data, see [23, 68, 69, 71–73]).

(Space–Time Explorer and Quantum Equivalence Space Test) mission [72, 73]. Experiments exploiting entangled atomic states aim to push the sensitivity beyond the so called Standard Quantum Limit [74]. Fig. 4 shows the limits set by the WEP tests performed with different methods, from 1960 until today and the future prospects.

For the SEP tests in the sense of Will [4, p. 76], the achieved accuracy limit is $\eta_{SEP} \approx 10^{-5}$ [23].

If a violation of some form of the EP were to be detected, it would constitute a falsification of GR at that level, while TEGR and STEGR would remain viable as theories [63, ch. 9]. Specifically, such a finding would suggest that the SEP is not fundamental (*FEP*) but rather an emergent property (*EMEP*) arising from a possible gauge choice, consistent with TEGR and STEGR. In this case, it would mean that the SEP is recoverable only at limited levels, with constraints highlighted by experiments. This is analogous to Einstein’s justification of the EEP on empirical grounds. In neither case do we have fundamental reasons to assume the given EP. However, GR explicitly assumes it as fundamental, with all the consequences we have discussed, whereas TEGR and STEGR do not.

Another interesting possibility, suggested by the Read and Teh taxonomy [21], concerns the fact that a falsification of WEP would not imply a falsification of $SEP \neg EEP$, although its recovery should still be constrained by experimental data. In this scenario as well, TEGR would be a more fundamental theory than GR. Both

these two scenarios would be in line with a recent suggestion by Weatherall that both TEGR and STEGR have more surplus structure than GR [51].

However, it is important to emphasize that this argument encourages testing the SEP but does not strictly predict its violation. In fact, even if the SEP is not fundamental, it is still possible that no violation will be found. In such a case, increasing the accuracy of traditional EP tests may be insufficient to discriminate among the three theories.

This consideration suggests the need to explore and design other types of experiments that could highlight the difference in *essence* between gravity and inertia, investigate the possible distinct dynamics of the metric and the connection, or determine in other way whether the SEP is fundamental or emergent. Consequently, these epistemic considerations point to the importance of developing experimental approaches beyond traditional EP tests, which might potentially differentiate among the three equivalent theories. This represents a subject for future research.

Then note that the continue corroboration of the SEP at any level poses also other conceptual problems. If after significant progress, no violation will be found, at what level of accuracy would we consider it satisfied? 10^{-18} ? 10^{-20} ? 10^{-50} ? What level of precision could be deemed sufficient to corroborate a metric theory? This is a perfect example of inductive risk, as one could always find an experiment which violates the EP, even if we will reach a precision of 10^{-50} . Is there an accuracy level where reasonable doubt would be mitigated? Maybe finding a more fundamental theory that explains why the SEP should be an *FEP* could help. However, justifying it experimentally seems difficult, as it is possible that its validity might turn out to be merely a contingent fact without any underlying fundamental reason. Philosophers of physics surely would enjoy the debate.

Finally, it is worth mentioning another issue which could be addressed with our epistemic model, that is, the problem of the number of degrees of freedom in theories of gravity [75–77]. The present approach could be exploited to extract the number of degrees of freedom and the dynamics of competing theories, in order to evaluate the equivalence among them and possible differences in their empirical content (see, e.g. [78, 79]).

6 Conclusions

We discussed that SEP is a non-trivial, theory-laden assumption in the framework of Equivalent Gravities. In fact, assuming SEP at the foundation of the theory, we are intrinsically stating that gravitation is given by the curvature of spacetime, rather than by torsion or non-metricity. This assumption also leads to a number of other problematic foundational consequences: *i*) the coincidence between the causal and the geodesic structure; *ii*) the fact that metric tensor is the fundamental variable of the theory instead of the connection; *iii*) the contrast with several new physics predictions; *iv*) the conceptual difficulties at quantum level; *v*) the relation with CDM and DE problems; *vi*) the conceptual difference of gravity with respect to other gauge theories; and, finally, *vii*) the unjustified coincidence between gravitational and inertial

mass. All these difficulties (or shortcomings) are direct or indirect consequences of the imposition of SEP, which is based on the assumption that gravitation and inertia are of the same essence (as expressed by EEP).

However, GR is just a particular case in the more general lake of metric-affine theories, which can be built not only with curvature of spacetime, but also with torsion and non-metricity. With TEGR, built upon torsion, and STEGR, built upon non-metricity, GR constitutes the so called Geometric Trinity of Gravity, because these three theories are found to be dynamically equivalent.

Then, on a closer inspection, we argued that there is a crucial hidden difference in relation with SEP. The significant fact is that, in both TEGR and STEGR, SEP can be recovered but without the necessity to postulate it at the foundation of the theory. If physicists would have found the SEP as a necessary fundamental principle also for TEGR and STEGR, this would have been considered as a meta-empirical evidence in favor of what we called the FEP, that is the Equivalence Principle as Fundamental. But, as we have seen, this is not the case. Therefore, given the equivalence among GR, TEGR and STEGR in non-trivial multiple predictions, and given the fact that SEP is not necessary for TEGR and STEGR, our confidence in the fundamentality of the SEP decreases.

If the argument is correct, this structural divergence could lead to differences in the empirical content between the three theories, because if the SEP is not a fundamental feature of reality (FEP), it is emergent (EMEP). And if it is emergent, it is possible that, at some level, it is not valid.

As argued, the relaxation of this theory-laden principle allows us also to address many of the aforementioned foundational problems. As Synge suggested, given its heuristic role, the EP could be considered as a midwife, but not a fundamental feature of the world [17].

These epistemic considerations encourage physicists to further enhance the accuracy of EP tests, especially at the quantum level, and to develop new experimental schemes to investigate potential differences in the empirical content of theories arising from their distinct relationships with the EP, such as discriminating between dynamics of metric and connection.

Finally, the approach of Sec. 4 could be used also for other investigations. First of all, it should be used to evaluate other predictions in order to search for other possible relevant observables. For example, principles as the Local Lorentz Invariance, the Local Position Invariance, the Schiff conjecture, should be investigated also in TEGR and STEGR. Then, for instance, there could be also other equivalent formulations of gravity outside of the Geometric Trinity. Finding other dynamically equivalent theories would help in further constraining the configuration space of the theory. Having already found two of such theories, nothing precludes the fact that other equivalent representations of gravity could exist. Beside EP, theories of gravity could be compared also considering the number of degrees of freedom related to observables. This will be the argument of further studies.

Acknowledgements. C.M. and S.C. acknowledge the support of INFN sez. di Napoli, *iniziative specifiche* QGSKY and MOONLIGHT2. S.C. thanks the *Gruppo Nazionale di Fisica Matematica* of *Istituto Nazionale di Alta Matematica* for the

support. This paper is based upon work from COST Action CA21136 - *Addressing observational tensions in cosmology with systematics and fundamental physics* (CosmoVerse), supported by COST (European Cooperation in Science and Technology). The Authors thank Elena Castellani, Carmen Ferrara, and Flaminia Giacomini for the useful discussions and feedbacks.

References

- [1] Capozziello, S., De Laurentis, M.: Extended theories of gravity. *Physics Reports* **509**(4), 167–321 (2011) <https://doi.org/10.1016/j.physrep.2011.09.003>
- [2] Nojiri, S., Odintsov, S.D., Oikonomou, V.K.: Modified gravity theories on a nutshell: Inflation, bounce and late-time evolution. *Physics Reports* **692**, 1–104 (2017) <https://doi.org/10.1016/j.physrep.2017.06.001>
- [3] Smeenk, C., Weatherall, J.O.: Dark energy or modified gravity? *Philosophy of Science*, 1–10 (2023) <https://doi.org/10.1017/psa.2023.143>
- [4] Will, C.M.: *Theory and Experiment in Gravitational Physics*. Cambridge University Press, Cambridge (2018). <https://doi.org/10.1017/9781316338612>
- [5] Capozziello, S., De Falco, V., Ferrara, C.: Comparing equivalent gravities: common features and differences. *The European Physical Journal C* **82**(10), 865 (2022) <https://doi.org/10.1140/epjc/s10052-022-10823-x>
- [6] Ferraris, M., Francaviglia, M., Reina, C.: Variational formulation of general relativity from 1915 to 1925 “Palatini’s method” discovered by Einstein in 1925. *General relativity and gravitation* **14**, 243–254 (1982) <https://doi.org/10.1007/BF00756060>
- [7] Beltran Jimenez, J., Heisenberg, L., Koivisto, T.S.: The geometrical trinity of gravity. *Universe* **5**(7), 173 (2019) <https://doi.org/10.3390/universe5070173>
- [8] Bahamonde, S., Dialektopoulos, K.F., Escamilla-Rivera, C., Farrugia, G., Gakis, V., Hendry, M., Hohmann, M., Said, J.L., Mifsud, J., Di Valentino, E.: Teleparallel gravity: from theory to cosmology. *Reports on Progress in Physics* **86**(2), 026901 (2023) <https://doi.org/10.1088/1361-6633/ac9cef>
- [9] Capozziello, S., De Falco, V., Ferrara, C.: The role of the boundary term in $f(Q, B)$ symmetric teleparallel gravity. *The European Physical Journal C* **83**(10), 915 (2023) <https://doi.org/10.1140/epjc/s10052-023-12072-y>
- [10] Martens, N.C.M., Lehmkuhl, D.: Dark matter= modified gravity? Scrutinising the spacetime–matter distinction through the modified gravity/dark matter lens. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* **72**, 237–250 (2020) <https://doi.org/10.1016/j.shpsb.2020.08.003>

- [11] Capozziello, S., Ferrara, C.: The equivalence principle as a Noether symmetry. *Int. J. Geom. Meth. Mod. Phys.* **21**(10), 2440014 (2024) <https://doi.org/10.1142/S0219887824400140> [arXiv:2401.09737](https://arxiv.org/abs/2401.09737) [gr-qc]
- [12] Mulder, R., Read, J.: Is spacetime curved? Assessing the underdetermination of general relativity and teleparallel gravity. *Synthese* **204**(4), 126 (2024) <https://doi.org/10.1007/s11229-024-04773-y>
- [13] March, E., Wolf, W.J., Read, J.: On the geometric trinity of gravity, non-relativistic limits, and Maxwell gravitation. *arXiv preprint arXiv:2309.06889* (2023) <https://doi.org/10.48550/arXiv.2309.06889>
- [14] Wolf, W.J., Sanchioni, M., Read, J.: Underdetermination in classic and modern tests of general relativity. *European Journal for Philosophy of Science* **14**(4), 57 (2024) <https://doi.org/10.1007/s13194-024-00617-1>
- [15] Chen, L., March, E., Read, J.: Equivalence, reduction, and sophistication in teleparallel gravity (2024). <https://philsci-archive.pitt.edu/23834/>
- [16] Dürr, P., Read, J.: An invitation to conventionalism: a philosophy for modern (space-)times. *Synthese* **204**(1), 1 (2024) <https://doi.org/10.1007/s11229-024-04605-z>
- [17] Synge, J.L. (ed.): *Relativity: The General Theory*. Interscience Publishers, New York, (1960)
- [18] Lehmkuhl, D.: The equivalence principle (s). In: Knox, E., Wilson, A.r. (eds.) *The Routledge Companion to Philosophy of Physics*, pp. 125–144. Routledge, New York and London (2021). <https://doi.org/10.4324/9781315623818>
- [19] Capozziello, S.: Noether Symmetries in Quantum Cosmology: A Selection Rule for Observable Universes. *Fundam. Theor. Phys.* **216**, 269–292 (2024) https://doi.org/10.1007/978-3-031-61860-4_13
- [20] Bajardi, F., Capozziello, S.: *Noether Symmetries in Theories of Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, ??? (2022). <https://doi.org/10.1017/9781009208727>
- [21] Read, J., Teh, N.J.: Newtonian Equivalence Principles. *Erkenntnis* **88**(8), 3479–3503 (2023) <https://doi.org/10.1007/s10670-021-00513-7>
- [22] Misner, C.W., Thorne, K.S., Wheeler, J.A.: *Gravitation*. W. H. Freeman, San Francisco (1973)
- [23] Tino, G.M., Cacciapuoti, L., Capozziello, S., Lambiase, G., Sorrentino, F.: Precision gravity tests and the Einstein Equivalence Principle. *Progress in Particle and Nuclear Physics* **112**, 103772 (2020) <https://doi.org/10.1016/j.ppnp.2020.103772>

- [24] Allemandi, G., Capone, M., Capozziello, S., Francaviglia, M.: Conformal aspects of the Palatini approach in Extended Theories of Gravity. *General Relativity and Gravitation* **38**(1), 33–60 (2006) <https://doi.org/10.1007/s10714-005-0208-7>
- [25] Capozziello, S., De Laurentis, M., Francaviglia, M., Mercadante, S.: From Dark energy & dark matter to dark metric. *Foundations of Physics* **39**, 1161–1176 (2009) <https://doi.org/10.1007/s10701-009-9332-7>
- [26] Einstein, A.: Über friedrich kottlers abhandlung “über einsteins äquivalenzhypothese und die gravitation”. *Annalen der Physik* **356**(22), 639–642 (1916) <https://doi.org/10.1002/andp.19163562206>
- [27] Norton, J.: What was Einstein’s principle of equivalence? *Studies in History and Philosophy of Science Part A* **16**(3), 203–246 (1985) [https://doi.org/10.1016/0039-3681\(85\)90002-0](https://doi.org/10.1016/0039-3681(85)90002-0)
- [28] Pantaleo, M.: Cinquant’anni di Relatività, 1905-1955: Le Memorie Fondamentali di Albert Einstein e Il Valore delle Teorie Relativistiche Sotto L’aspetto Fisico, Matematico, Astronomico e Filosofico, Negli Scritti Originali di Eminenti Scienziati Italiani. Editrice universitaria, Firenze (1955)
- [29] Goenner, H.F.: On the history of unified field theories. Part II.(ca. 1930–ca. 1965). *Living reviews in relativity* **17**, 1–241 (2014) <https://doi.org/10.12942/lrr-2014-5>
- [30] Damour, T.: Theoretical aspects of the equivalence principle. *Classical and Quantum Gravity* **29**(18), 184001 (2012) <https://doi.org/10.1088/0264-9381/29/18/184001>
- [31] Fortier, T.M., *et al.*: Precision Atomic Spectroscopy for Improved Limits on Variation of the Fine Structure Constant and Local Position Invariance. *Phys. Rev. Lett.* **98**, 070801 (2007) <https://doi.org/10.1103/PhysRevLett.98.070801>
- [32] Lämmerzahl, C.: What Determines the Nature of Gravity? a Phenomenological Approach. In: Everitt, C.W.F., Huber, M.C.E., Kallenbach, R., Schäfer, G., Schutz, B.F., Treumann, R.A. (eds.) *Probing The Nature of Gravity: Confronting Theory and Experiment in Space*, pp. 551–572. Springer, New York (2009). https://doi.org/10.1007/978-1-4419-1362-3_35
- [33] Blasone, M., Capozziello, S., Lambiase, G., Petrucciello, L.: Equivalence principle violation at finite temperature in scalar-tensor gravity. *The European Physical Journal Plus* **134**(4), 169 (2019) <https://doi.org/10.1140/epjp/i2019-12682-2>
- [34] Milgrom, M.: Noncovariance at low accelerations as a route to MOND. *Phys. Rev. D* **100**, 084039 (2019) <https://doi.org/10.1103/PhysRevD.100.084039>
- [35] Capozziello, S., Jovanović, P., Jovanović, V.B., Borka, D.: Addressing the missing matter problem in galaxies through a new fundamental gravitational

- radius. JCAP **06**, 044 (2017) <https://doi.org/10.1088/1475-7516/2017/06/044> [arXiv:1702.03430](https://arxiv.org/abs/1702.03430) [gr-qc]
- [36] Bernal, T., Capozziello, S., Hidalgo, J.C., Mendoza, S.: Recovering MOND from extended metric theories of gravity. Eur. Phys. J. C **71**, 1794 (2011) <https://doi.org/10.1140/epjc/s10052-011-1794-z> [arXiv:1108.5588](https://arxiv.org/abs/1108.5588) [astro-ph.CO]
 - [37] Giacomini, F., Brukner, Č.: Einstein’s Equivalence principle for superpositions of gravitational fields and quantum reference frames. arXiv preprint [arXiv:2012.13754](https://arxiv.org/abs/2012.13754) (2020) <https://doi.org/10.48550/arXiv.2012.13754>
 - [38] Torrieri, G.: The equivalence principle and inertial-gravitational quantum backreaction. Eur. Phys. J. ST **232**(20-22), 3505–3517 (2023) <https://doi.org/10.1140/epjs/s11734-023-01043-9> [arXiv:2210.08586](https://arxiv.org/abs/2210.08586) [gr-qc]
 - [39] Capozziello, S., Cardone, V.F., Troisi, A.: Dark energy and dark matter as curvature effects. JCAP **08**, 001 (2006) <https://doi.org/10.1088/1475-7516/2006/08/001> [arXiv:astro-ph/0602349](https://arxiv.org/abs/astro-ph/0602349)
 - [40] Bidin, C.M., Carraro, G., Mendez, R.A., Altena, W.F.: No evidence for a dark matter disk within 4 kpc from the Galactic plane. Astrophys. J. Lett. **724**, 122–126 (2010) <https://doi.org/10.1088/2041-8205/724/1/L122> [arXiv:1011.1289](https://arxiv.org/abs/1011.1289) [astro-ph.GA]
 - [41] Bidin, C.M., Carraro, G., Mendez, R.A., Smith, R.: No evidence of dark matter in the solar neighborhood. Bol. A. A. Astron. **54**, 289 (2011) [arXiv:1204.3919](https://arxiv.org/abs/1204.3919) [astro-ph.GA]
 - [42] Serpico, P.D.: Status of indirect dark matter detection. J. Phys. Conf. Ser. **375**, 012029 (2012) <https://doi.org/10.1088/1742-6596/375/1/012029>
 - [43] Serpico, P.D.: (Indirect) dark matter searches: Status and challenges. Int. J. Mod. Phys. E **30**(07), 2130002 (2021) <https://doi.org/10.1142/S0218301321300022>
 - [44] Martens, N.C., King, M.: Doing More with Less: Dark Matter & Modified Gravity. In: Philosophy of Astrophysics: Stars, Simulations, and the Struggle to Determine What Is Out There, pp. 91–107 (2023). <https://doi.org/10.1007/978-3-031-26618-8>
 - [45] Duerr, P.M., Wolf, W.J.: Methodological reflections on the mond/dark matter debate. Studies in History and Philosophy of Science **101**, 1–23 (2023) <https://doi.org/10.1016/j.shpsa.2023.07.001>
 - [46] Kashyap, A.: General Relativity, MOND, and the problem of unconceived alternatives. European Journal for Philosophy of Science **13**(3), 30 (2023) <https://doi.org/10.1007/s13194-023-00532-x>

- [47] Kosso, P.: EEP. Evidence of Dark Matter, and the Interpretive Role of General Relativity. *Studies in History and Philosophy of Modern Physics* **44**, 143–147 (2013) <https://doi.org/10.1016/j.shpsb.2013.01.001>
- [48] Cai, Y.-F., Capozziello, S., De Laurentis, M., Saridakis, E.N.: $f(T)$ teleparallel gravity and cosmology. *Reports on Progress in Physics* **79**(10), 106901 (2016) <https://doi.org/10.1088/0034-4885/79/10/106901>
- [49] Heisenberg, L.: Review on $f(Q)$ gravity. *Physics Reports* **1066**, 1–78 (2024) <https://doi.org/10.1016/j.physrep.2024.02.001>
- [50] Blagojevic, M.: *Gravitation and Gauge Symmetries*. CRC Press, Boca Raton (2001). <https://doi.org/10.1201/9781420034264>
- [51] Weatherall, J.O.: On (Some) Gauge Theories of Gravity. arXiv preprint arXiv:2504.05701 (2025)
- [52] Bondi, H.: Negative Mass in General Relativity. *Rev. Mod. Phys.* **29**, 423–428 (1957) <https://doi.org/10.1103/RevModPhys.29.423>
- [53] Einstein, A.: Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus. *Albert Einstein: Akademie-Vorträge: Sitzungsberichte der Preußischen Akademie der Wissenschaften 1914–1932*, 317–221 (1928) <https://doi.org/10.1002/3527608958.ch36>
- [54] Krššák, M., Van Den Hoogen, R., Pereira, J., Böhmer, C., Coley, A.: Teleparallel theories of gravity: illuminating a fully invariant approach. *Classical and Quantum Gravity* **36**(18), 183001 (2019) <https://doi.org/10.1088/1361-6382/ab2e1f>
- [55] Palatini, A.: Deduzione invariante delle equazioni gravitazionali dal principio di Hamilton. *Rendiconti del Circolo Matematico di Palermo* (1884-1940) **43**(1), 203–212 (1919) <https://doi.org/10.1007/BF03014670>
- [56] D’Ambrosio, F., Fell, S.D.B., Heisenberg, L., Kuhn, S.: Black holes in $f(Q)$ gravity. *Phys. Rev. D* **105**, 024042 (2022) <https://doi.org/10.1103/PhysRevD.105.024042>
- [57] Knox, E.: Newton–Cartan theory and teleparallel gravity: The force of a formulation. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* **42**(4), 264–275 (2011) <https://doi.org/10.1016/j.shpsb.2011.09.003>
- [58] Dawid, R.: *String Theory and the Scientific Method*. Cambridge University Press, Cambridge (2013). <https://doi.org/10.1017/CBO9781139342513>
- [59] Dawid, R.: Meta-empirical confirmation: Addressing three points of criticism. *Studies in History and Philosophy of Science* **93**, 66–71 (2022) <https://doi.org/>

- [60] Dawid, R.: The significance of non-empirical confirmation in fundamental physics. In: Dardashti, R., Dawid, R., Thebault, K. (eds.) *Why Trust a Theory?: Epistemology of Fundamental Physics*, pp. 99–119. Cambridge University Press, Cambridge (2019). <https://doi.org/10.48550/arXiv.1702.01133>
- [61] Dawid, R., Hartmann, S., Sprenger, J.: The no alternatives argument. *The British Journal for the Philosophy of Science* (2015) <https://doi.org/10.1093/bjps/axt045>
- [62] Dardashti, R.: Physics without Experiments? In: Dardashti, R., Dawid, R., Thebault, K. (eds.) *Why Trust a Theory?: Epistemology of Fundamental Physics*, pp. 154–172. Cambridge University Press, Cambridge (2019). <https://doi.org/10.1017/9781108671224>
- [63] Aldrovandi, R., Pereira, J.G.: *Teleparallel Gravity: An Introduction*. Springer, Dordrecht (2013). <https://doi.org/10.1007/978-94-007-5143-9>
- [64] Torrieri, G.: What is "quantum" about quantum gravity? (2024) [arXiv:2405.08192](https://arxiv.org/abs/2405.08192) [quant-ph]
- [65] Rosi, G., D'Amico, G., Cacciapuoti, L., Sorrentino, F., Prevedelli, M., Zych, M., Brukner, Č., Tino, G.M.: Quantum test of the equivalence principle for atoms in coherent superposition of internal energy states. *Nature Communications* **8**(1) (2017) <https://doi.org/10.1038/ncomms15529>
- [66] Tino, G.M.: Testing gravity with cold atom interferometry: results and prospects. *Quantum Science and Technology* **6**(2), 024014 (2021) <https://doi.org/10.1088/2058-9565/abd83e>
- [67] Touboul, P., *et al.*: MICROSCOPE Mission: first results of a space test of the equivalence principle. *Phys. Rev. Lett.* **119**, 231101 (2017) <https://doi.org/10.1103/PhysRevLett.119.231101>
- [68] Touboul, P., *et al.*: MICROSCOPE Mission: Final Results of the Test of the Equivalence Principle. *Phys. Rev. Lett.* **129**, 121102 (2022) <https://doi.org/10.1103/PhysRevLett.129.121102>
- [69] Nobili, A.M., Anselmi, A.: Testing the equivalence principle in space after the MICROSCOPE mission. *Phys. Rev. D* **98**, 042002 (2018) <https://doi.org/10.1103/PhysRevD.98.042002>
- [70] Overduin, J., Everitt, F., Worden, P., Mester, J.: STEP and fundamental physics. *Classical and Quantum Gravity* **29**(18), 184012 (2012) <https://doi.org/10.1088/0264-9381/29/18/184012>

- [71] Asenbaum, P., Overstreet, C., Kim, M., Curti, J., Kasevich, M.A.: Atom-Interferometric Test of the Equivalence Principle at the 10^{-12} level. *Phys. Rev. Lett.* **125**, 191101 (2020) <https://doi.org/10.1103/PhysRevLett.125.191101>
- [72] Altschul, B., *et al.*: Quantum tests of the Einstein Equivalence Principle with the STE-QUEST space mission. *Advances in Space Research* **55**(1), 501–524 (2015) <https://doi.org/10.1016/j.asr.2014.07.014>
- [73] Struckmann, C., Corgier, R., Loriani, S., Kleinsteinberg, G., Gox, N., Giese, E., Métris, G., Gaaloul, N., Wolf, P.: Platform and environment requirements of a satellite quantum test of the weak equivalence principle at the 10^{-17} level. *Phys. Rev. D* **109**, 064010 (2024) <https://doi.org/10.1103/PhysRevD.109.064010>
- [74] Salvi, L., Poli, N., Vuletic, V., Tino, G.M.: Squeezing on Momentum States for Atom Interferometry. *Physical Review Letters* **120**, 033601 (2018) <https://doi.org/10.1103/PhysRevLett.120.033601>
- [75] Magnano, G., Sokołowski, L.M.: Physical equivalence between nonlinear gravity theories and a general-relativistic self-gravitating scalar field. *Phys. Rev. D* **50**, 5039–5059 (1994) <https://doi.org/10.1103/PhysRevD.50.5039>
- [76] Belenchia, A., Letizia, M., Liberati, S., Casola, E.D.: Higher-order theories of gravity: diagnosis, extraction and reformulation via non-metric extra degrees of freedom—a review. *Reports on Progress in Physics* **81**(3), 036001 (2018) <https://doi.org/10.1088/1361-6633/aaa4ab>
- [77] Stabile, A., Capozziello, S.: Galaxy rotation curves in $f(r, \phi)$ gravity. *Phys. Rev. D* **87**, 064002 (2013) <https://doi.org/10.1103/PhysRevD.87.064002>
- [78] Bajardi, F., Blixt, D., Capozziello, S.: The Hamilton equations in $f(T)$ teleparallel gravity and in New General Relativity (2024) [arXiv:2412.20592](https://arxiv.org/abs/2412.20592) [gr-qc]
- [79] Capozziello, S., Finch, A., Said, J.L., Magro, A.: The 3+1 formalism in teleparallel and symmetric teleparallel gravity. *Eur. Phys. J. C* **81**(12), 1141 (2021) <https://doi.org/10.1140/epjc/s10052-021-09944-6> [arXiv:2108.03075](https://arxiv.org/abs/2108.03075) [gr-qc]