What Price Fiber Bundle Substantivalism?
On How to Avoid Holes in Fibers

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Abstract

On a mathematically foundational level, our most successful physical theories (gauge field theories and general-relativistic theories) are formulated in a framework based on the differential geometry of connections on principal bundles. After reviewing the essentials of this framework, we articulate the generalized hole and point-coincidence arguments, examining how they weight on a substantivalist position towards bundle spaces. This question, then, is considered in light of the Dressing Field Method, which allows a manifestly invariant reformulation of gauge field theories and general-relativistic theories, making their conceptual structure more transparent: it formally implements the point-coincidence argument and thus allows to define (dressed) fields and (dressed) bundle spaces immune to hole-type arguments.

Keywords: Gauge theories, Hole argument, Gauge-invariance, Dressing Field Method, Dressed spaces.

Contents

1 Introduction

One of the central aims of philosophy of science, and philosophy of physics especially, is to determine how scientific theories relate to reality. A basic goal, in particular, is to identify the fundamental ontology suggested by our best physical theories. Sometimes, this task is relatively straightforward—or at least appears to be—as in the case of classical, or even special relativistic, mechanics. At other times, it is highly non-trivial and continues to puzzle even the most careful thinkers for decades. The prime example is quantum mechanics. For instance, whether the wave function is ontologically real, and what exactly it represents (if anything), remains one of the most debated questions in philosophy of physics. In between these extremes, we have the time-honored subject of the nature of spacetime in general-relativistic (GR) physics, and in the past 25 years a growing interest in the status and meaning of gauge symmetries, which are at the heart of the Standard Model (SM) of particle physics.

Both these topics are actually deeply related. The SM, as a gauge field theory (GFT), is built on the principle of *invariance* under *gauge transformations*, i.e. the idea that the Lagrangian must be invariant under local "internal" transformations of its field variables. This is the *Gauge Principle*. GR is built on the principle of *covariance* under *diffeomorphisms*, i.e. the idea that the Lagrangian must be covariant under local "space-time" transformations of its field variables. This is the *General Covariance Principle*. Together, GFT and GR physics constitute the framework of general-relativistic gauge field theory (gRGFT), based on principles of local symmetries.

Regarding the philosophical endeavor mentioned in the opening paragraph, these symmetries pose a significant challenge. They complicate the task of identifying which parts of the formalism refer *directly* to basic physical entities. Indeed, it is well-established in physics that physical magnitudes (and observables) ought to be invariants under local transformations—i.e. the latter are *symmetries* of the former. However, in GFT, the SM in particular, none of the fundamental field variables are gauge-invariant: if they refer to physical fields, physical degrees of freedom (d.o.f.) of distinct types, they do so in a non-trivially redundant way. This raises the question of what the meaning of this mathematical over-abundance is, and whether it is possible to describe the relevant physics without it (see, e.g., Berghofer et al. (2023); Earman (2004); Healey (2007)).

Idem for GR, with the added complication that its local transformations, diffeomorphisms, also act on the mathematical object taken to refer to spacetime: the manifold on which the field variables "live". This is the starting point of the literature on the *hole argument* in GR, and the way it gives a new spin on the traditional philosophical dispute between substantivalist and relationalist views of space(time). What should be clear from GR is that the ontological status of the fields and that of spacetime are tightly interconnected: this is the fundamental insight of the *point-coincidence argument* (see Stachel (1986) and Giovanelli (2021)).

What is less well appreciated is that the same holds true in GFT: the ontological status of the fields is better understood when put in relation with that of the space in which they "live" and on which gauge transformations act as well: fiber bundle spaces. Despite being as omnipresent (at least tacitly) in GFT as standard manifolds are in GR, the question of how much of a realist one can be toward fiber bundles has received little attention. Even scarcer is the literature that articulates or discusses generalizations of the hole and point-coincidence arguments and examines their implications for a realist view of bundles.[1]

This is the issue we aim to address in a systematic and unified way in this paper. We start by reviewing in section 2 the hole argument (defending it against objections) and its implications in GR physics. In section 3, we provide a dense presentation of the mathematical framework underpinning gRGFT: the differential geometry of connections on principal bundles. This sets the stage for us to articulate precisely the generalized hole and point-coincidence arguments in section 4, and show how they weight in on the substantivalist *vs* relationalist views of principal bundles. Finally, in section 5 we show that the Dressing Field Method (DFM) allows an invariant reformulation of gRGFTs that makes their conceptual structure more transparent, and their philosophical analysis much easier: In particular, it allows to define *dressed* principal bundles and *dressed* fields which are, by construction, immune to hole-type arguments. In section 6, we conclude with a few remarks on the interpretive resources of the view presented here.

[1] Exceptions that treat this topic are Lyre (1999), Healey (2001), Stachel (2014), and François and Ravera (2025).

2 The hole argument in general-relativistic theory

The traditional opposition between the substantivalist and relationalist views on space(-time), famously a topic of the Leibniz-Clarke debate, hinges upon the question as to wether space(-time) has an existence autonomous from the physical objects it "contains": Substantivalists (with Newton and Clarke) affirm the proposition, relationalists (with Leibniz) deny it.[2] It should be remarked that the debate is not necessarily one opposing realist *vs* anti-realist views on space(-time), the substantivalist being the realist and the relationalist being the anti-realist. The relationalist can be as much a realist as the substantivalist, affirming the existence of space(-time) simply not at a primary ontological level, but rather at a secondary, emergent or effective one: it *supervenes* on ontologically primary physical objects.[3]

This debate must be reconsidered in light of the advent of General Relativity (GR), and general-relativistic physics more broadly, which not only imposes the notion of spacetime—with a dynamics coupled to that of matter-energy—but also forces us to confront the fact that this notion is no longer simple and primitive, but a layered one: In the best practice of physics, a general-relativistic spacetime is defined as a Lorentzian manifold (M, g), where M is a smooth differential manifold endowed with g, a Lorentzian metric. The bare manifold M may a priori be understood to represent the totality of "spatiotemporal events" in their topological and differential relations only.[4] To define their spatiotemporal relations, in particular the causal structure, g is needed. This leads to both an update and a "lifting of degeneracy" of the traditional debate, which now comes in two related variants: the substantivalist *vs* relationalist views on *spacetime*, and the substantivalist *vs* relationalist views on the *manifold of events*.[5]

It should be no surprise that both variants are crucially informed by Einstein's dual "hole argument" ("*Lochbetrachtung*") and "point-coincidence argument", as they were key to his timely understanding of the meaning of diffeomorphism covariance leading to the final completion of GR, and to his definite views on spacetime. It is unfortunate that subsequently most physicists misunderstood these as philosophical meanderings distracting Einstein from the straight technical path, when they did not just ignore or outright forget them. Their deep significance was rediscovered and brought to a wider attention by Stachel in 1980 (published in Stachel (1989)), and the arguments were given a modernized form in Earman and Norton (1987). The literature on the subject is now considerable, and even involves physicists engaged in the open problem of the quantum nature of gravity/spacetime. Let us give a brief description of the issue, drawing from the account of François and Ravera (2025).

General-relativistic physics describes the coupled dynamics of matter-energy and gravity, both fundamentally understood as *fields*. Their mathematical description starts with defining a smooth manifold M, representing the totality of spatio-temporal events. It is then endowed with a Lorentzian metric field g representing the gravitational field: the couple (M, g), a Lorentzian manifold, is taken to represent a relativistic spacetime. Other fields φ are then defined on M, representing matter and interaction fields. The latter may be thought of as the "content" of spacetime, or all fields $\phi = \{g, \varphi\}$ may be seen as the "content" of the manifold of events M. These fields satisfy (coupled) field equations $E(\phi) = E(g, \varphi) = 0$ on M or on a finite region $U \subset M$. Usually they are derived from an action functional $S(\phi) = \int_U L(\phi)$, where $L(\phi) \in \Omega^m(M)$, with $m = \dim(M)$, is a Lagrangian differential top-form on M.

The mathematical object M has a natural group of automorphisms: the group of diffeomorphisms $\text{Diff}(M)$. It acts by pullback on fields on M, so that $\phi \mapsto \phi^\psi := \psi^*\phi$ for any $\psi \in \text{Diff}(M)$. Any field ϕ thus has a $\text{Diff}(M)$-orbit $\mathcal{O}_\phi$, and the space of all possible fields $\Phi = \{\phi\}$ is partitioned into distinct orbits. The space of $\text{Diff}(M)$-orbits is the quotient space $\mathcal{M} := \Phi/\text{Diff}(M)$, called a *moduli space*, and there is a natural projection $\pi : \Phi \to \mathcal{M}$, $\phi \mapsto \pi(\phi) =: [\phi]$, s.t. $[\phi] = [\phi^\psi]$, i.e. there is a 1:1 correspondence $\mathcal{O}_\phi \leftrightarrow [\phi]$.

A defining feature of GR physics is the requirement of covariance of (the Lagrangian and thus of) the field

[2] Of course, the word "contains" is already loaded and tacitly skews towards an a priori substantivalist view.

[3] The space(-time) relations among ontologically primary physical objects being analogous to that of, say, the temperature of a gas or the fluidity of a liquid; an idea resonant with current physics' speculations about an "emergent" spacetime.

[4] The "bare" manifold is itself not a simple notion. We have first the set-theoretic level, the collection of points as an unstructured set. Then we have the topological one: endowing the set with a topology (setting contiguity relations among points and subsets of points) resulting in a topological manifold. Finally, endowing the latter with a smooth structure (establishing the possibility of differential calculus) results in a differential manifold. The next level would be endowing it with a connection, to get a connected manifold (where geodesics, curvature and torsion may be defined), and/or endowing it with a metric, to get a metric manifold. Then one may ask compatibility between the connection and metric structures, or remark that a metric uniquely induces a connection, as is often the case in standard GR textbooks.

[5] Famously, Earman and Norton (1987) endorsed the view that M represents spacetime. Yet, when physicists are careful and self-aware, they use the word "spacetime", correctly, to refer to (M, g). Only when they are loose with language – as is often the case in the specialized literature – do they tend to (mis)use the term to refer to M. We might be well advised to follow physics best practice, and use a terminology that reflects it. We shall then care to distinguish the "manifold of events" and "spacetime" variants of the substantive *vs* relationalist debate.

equations under $\text{Diff}(M)$: The field equations then satisfy $E(\phi)^{\psi} := E(\psi^*\phi) = \psi^* E(\phi) = 0$. The space of solutions $\mathcal{S} := \{\, \phi \in \Phi \text{ s.t. } E(\phi) = 0 \,\}$ is therefore also partitioned into distinct $\text{Diff}(M)$-orbits $\mathcal{O}_\phi$, s.t. $\pi : \mathcal{S} \to \mathcal{M}_\mathcal{S}$, $\phi \mapsto [\phi]$, with $\mathcal{M}_\mathcal{S}$ the moduli space of solutions. The immediate dual consequence is that the theory cannot distinguish solutions in the same $\text{Diff}(M)$-orbit $\mathcal{O}_\phi$, and is blind to the points and regions of M.

An a priori realism toward the formalism, i.e. a pre-critical ontological commitment to mathematical objects—which is what e.g. manifold M substantivalism is predicated upon—would thus imply that the theory underdeterminates its ontology. It amounts to accepting a metaphysical multiplicity of fields $\phi \in \mathcal{O}_\phi$ that are physically indistinguishable—something that is disreputable in the natural sciences, and in physics especially, which tends to abide by a form of Ockham's principle of parsimony.

The hole argument is meant to further highlight how this position appears to lead also to an ill-defined Cauchy problem, and can be thus formulated: Consider $\phi, \phi^{\psi} \in \mathcal{O}_\phi \subset \Phi$ s.t. $\psi \in \text{Diff}(M)$ has compact support $D \subset U \subset M$, so that $\phi = \phi^{\psi}$ on M/D, but $\phi \neq \phi^{\psi}$ on D. Then, given boundary conditions for the fields ϕ at ∂U, the field equations do not uniquely determine a solution in U since $\phi, \phi^{\psi} \in \mathcal{O}_\phi \subset \mathcal{S}$. This in particular implies a failure of determinism if instead one starts from Cauchy data and D is in the future of the Cauchy surface $\Sigma \subset U \subset M$. This would render the framework essentially useless as a physical theory.

The way out of these predicaments is Einstein's point-coincidence argument—Norton (1987) credits Stachel for the name—which consists in the observation that the physical content of the theory (in particular its possible verifications) is exhausted by the pointwise coincidental values of fields, e.g. $g(x)$ and $\varphi(x)$, which are $\text{Diff}(M)$-invariant and represent *physical* events/magnitudes. This has two natural dual implications.

The first is immediate: points and regions of M are completely unphysical. *Physical events and regions* are encoded as the $\text{Diff}(M)$-invariant network of pointwise relations among the fields ϕ. As Norton (1989) says, "the full force of the hole argument is directed against manifold [M] substantivalism": The manifold M is there only to bootstrap one's ability to erect a description of the relevant physics of fields, like a scaffold for a building. And like a scaffold, it is removed from the physical picture by the requirement of $\text{Diff}(M)$-covariance of the field equations. The *physical manifold of events* has no autonomous existence apart from the physical fields on which it supervenes.

The second, which Earman and Norton called "Leibniz equivalence", is that a whole $\text{Diff}(M)$-orbit $\mathcal{O}_\phi \subset \mathcal{S}$ represents one and the same physical solution, so that *physical field* degrees of freedom (d.o.f.) are not best represented by any single $\phi \in \mathcal{O}_\phi \subset \mathcal{S}$, but by a point (a $\text{Diff}(M)$-equivalence class) $[\phi] \in \mathcal{M}_\mathcal{S}$ in the moduli space of solutions. This is indeed the standard view in physics. But the argument seems to further underlie that what is $\text{Diff}(M)$-invariant and physical are the *relations* among the d.o.f. of the fields ϕ, so that the physical fields d.o.f. mutually co-define each other: The d.o.f of the physical metric, i.e. of the gravitational field (redundantly described by g), are only well-defined in relation to the d.o.f. of matter and/or interaction fields (redundantly described by φ).[6]

The articulation of the hole and point-coincidence arguments makes a compelling case for a relationalist view of the *physical* manifold of events, on which the physical field d.o.f. "live". In particular, the *physical metric /gravitational field* is as autonomous an entity as it is possible in the general-relativistic framework, physical field d.o.f. and the co-defining relations in which they participate being coextensive. Seeing that the gravitational field is the vessel of most of the essential properties usually attributed to spacetime, one may indeed consider that it embodies a substantival notion of spacetime, insofar as the "substantival" notion gets suitably "relationalized". This is not far from a family of views called "sophisticated substantivalism", see Pooley (2013); Stachel (2014).

Let us add two comments before concluding this section. First, it was argued by Norton (1989) that defining spacetime as "M + further structure" ("*mpfs*") is a form of sophisticated substantivalism that may still be vulnerable to a hole-type argument if the "further structure" has symmetries, i.e. a non-trivial Killing group. As we noted, the consensus in physics is to define spacetime as (M, g). Furthermore, all exact solutions g of Einstein's equations have a Killing group $\mathsf{K}(g)$, so substantivalism towards these symmetric spacetimes is undermined according to Norton (1989). Yet, realistic physical solutions certainly have only approximate symmetries, so substantivalism towards these spacetimes would be possible. In line with the above paragraph, the discussion of the physical significance of Killing symmetries by François and Ravera (2025) dissent from this view by denying that the (sophisticated) substantivalist has to see Killing related models as physically distinct *in the fully dynamically coupled regime*.[7]

[6] Considering a region of the Universe free of matter and interaction fields (without φ), one may submit that the d.o.f. of the physical metric/gravitational field are to be instantiated as invariant relations among those of the mathematical field g.

[7] Killing symmetries of g being symmetries of the (energy-momentum tensor of) φ by Einstein equation $E(g, \varphi) = G(g) - \kappa T(g, \varphi) = 0$.

Relatedly, Weatherall (2018) and Halvorson and Manchak (2022) have argued that, spacetime being described as a Lorentzian manifold (M, g), mathematical practice suggests that the only admissible transformations in GR are the morphisms of the Category Lor of Lorentzian manifold, taken to be isometries: $\psi \in \text{Diff}(M)$ s.t. $\psi^* g = g$ on M, i.e. the Killing group $\mathsf{K}(g)$. The hole argument is then rejected, for "hole diffeomorphisms" are not isometries. If we were to grant this stricture of the categorical language, there would be in general no notion of symmetry/morphisms in GR, since as we just remarked, most (realistic) metrics have no non-trivial Killing group, $\mathsf{K}(g) = \{\text{id}_M\}$—which bears on discussions of how GR extends the Special Relativity (SR) principle. But it is doubtful that we have to, as it is undeniable that $\text{Diff}(M)$ is the covariance group of the field equations $E(\phi) = 0$, a fact whose physical significance needs to be understood in the theory's own terms. Preemptively restricting the covariance group of the theory on (debatable) mathematical grounds to avoid one of its key physical consequence seems unwise.

If we hold that the hole argument stands in general, its resolution by the point-coincidence yields the physical insights summarized above. These insights seem to be those Einstein took to be the final word of the general-relativistic framework. In the note to the 15th edition of his book "Relativity: The Special and the General Theory", Einstein (1952) states that he added a new appendix, entitled "Relativity and the Problem of Space", to present his "*views on the problem of space in general and on the gradual modifications of our ideas on space resulting from the influence of the relativistic view-point*", which he summarizes in the following few words: "*I wished to show that space-time is not necessarily something to which one can ascribe a separate existence, independently of the actual objects of physical reality. Physical objects are not in space, but these objects are spatially extended. In this way the concept 'empty space' loses its meaning.*" The last paragraphs of this appendix are worth quoting extensively:

> "We are now in a position to see how far the transition to the general theory of relativity modifies the concept of space. In accordance with classical mechanics and according to the special theory of relativity, space (space-time) has an existence independent of matter or field. In order to be able to describe at all that which fills up space and is dependent on the co-ordinates, space-time or the inertial system with its metrical properties must be thought of at once as existing, for otherwise the description of 'that which fills up space' would have no meaning.[8] On the basis of the general theory of relativity, on the other hand, space as opposed to 'what fills space', which is dependent on the co-ordinates, has no separate existence. Thus a pure gravitational field might have been described in terms of the g_{ik} (as functions of the co-ordinates), by solution of the gravitational equations. If we imagine the gravitational field, i.e. the functions g_{ik}, to be removed, there does not remain a space of the type (1) [i.e. of SR], but absolutely *nothing*, and also no 'topological space'. For the functions g_{ik} describe not only the field, but at the same time also the topological and metrical structural properties of the manifold. A space of the type (1), judged from the stand-point of the general theory of relativity, is not a space without field, but a special case of the g_{ik} field, for which – for the co-ordinate system used, which in itself has no objective significance – the functions g_{ik} have values that do not depend on the co-ordinates. There is no such thing as an empty space, i.e. a space without field. Space-time does not claim existence on its own, but only as a structural quality of the field.
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> Thus Descartes was not so far from the truth when he believed he must exclude the existence of an empty space. The notion indeed appears absurd, as long as physical reality is seen exclusively in ponderable bodies. It requires the idea of the field as the representative of reality, in combination with the general principle of relativity, to show the true kernel of Descartes' idea; there exists no space 'empty of field'."

This is a masterful synthesis of the general-relativistic update of the substantivalist *vs* relationalist debate. Yet, it must be reconsidered once more, this time in light of a framework that is a direct heir of GR (cf. O'Raifeartaigh (1997), François (2021), François (2023)): Gauge Field Theory (GFT). This framework, of which Maxwell electromagnetism, the electroweak model and chromodynamics are models, offers *prima facie* a new view of spacetime. Taken together, general-relativistic field theory and GFT form the broad framework of *general-Relativistic Gauge Field Theory* (gRGFT). Its mathematical foundation is the differential geometry of connections on fiber bundles, of which we give a dense review in the next section. It provides the necessary background to appreciate the notion of "enriched spacetime" that naturally arises from gRGFT, and how *generalized* hole and point-coincidence arguments inform a realist view towards such an enriched spacetime.

[8] "If we consider that which fills space (e.g. the field) to be removed, there still remains the metric space in accordance with (1) [the Minkowski metric], which would also determine the inertial behaviour of a test body introduced into it."

3 Fiber bundles and basics of gauge theory

The geometry of principal fiber bundles, and their connections, provides the kinematics of gRGFTs. Yet, despite bundles being so fundamental to gauge theories, only a few mathematical physicists endorsed some form of fiber bundle *realism*—see next section. Here, we shall give a fairly technical, but dense and synthetic, review of the basics of the bundle geometry underpinning gRGFT, doing so in a careful logical progression. Pragmatically, this will fix notations and set up a clear conceptual picture, both needed for our subsequent sections on the generalized hole and point-coincidence arguments, and of the Dressing Field Method (DFM). Still, our presentation is overcomplete with regard to this goal: We hope it serves a wider pedagogical purpose by introducing a wider audience to the mathematical foundations of classical gRGFT, enhancing its appreciation of the geometric origin of concepts they have likely encountered mostly in a field-theoretic context, which hopefully benefits philosophical discussions. The reader whose interest is piqued and who wishes to go further will profitably consult the book by Hamilton (2017), as well as the shorter text of François (2021) (sections 1-4).

A principal bundle is a smooth manifold P supporting the smooth right action of a Lie group H called its *structure group*, $P \times H \to P$, $(p, h) \mapsto ph$. The right action is also denoted $R_h : P \to P$, $p \mapsto R_h p := ph$. The orbits of the action by H are the *fibers* of P: by definition this action is free and transitive, so fibers are identical and isomorphic to H *as manifolds* but unlike H have no group structure. The *moduli* space of all fibers is itself a manifold M, called the *base* of P: we have $P/H := M$, and there is a canonical projection $\pi : P \to M$, $\pi(ph) = \pi(p) = x$, i.e. s.t. $\pi \circ R_h = \pi$. One often says that the point $x \in M$ has a fiber $P_{|x}$ *attached* to it, or one calls $P_{|x}$ the fiber *over* x. Sometimes a bundle P over M with structure group H is denoted $P \xrightarrow{\pi} M$.

The local structure of a bundle P is trivial in that, given a finite region $U \subset M$, $P_{|U} \simeq U \times H$. A bundle is said trivial if $P = M \times H$. Bundles are meant to generalize the trivial case. Local sections of P over U are smooth maps $\sigma : U \to P_{|U}$, $x \mapsto \sigma(x)$, s.t. $\pi \circ \sigma = \mathrm{id}_U$. They provide *bundle charts/coordinates*.

The tangent bundle TP of P has a canonical *vertical subbundle* $VP \subset TP$ whose sections, the "vertical" vector fields, are tangent to the fibers of P; they may be defined as belonging to $\ker \pi_*$, where $\pi_* : TP \to TM$ sends vectors on P to vectors on M. The infinitesimal action of H induces *fundamental vertical vector fields*: For $X \in \mathfrak{h}$ an element of the Lie algebra of H, let $c(\tau) := ph(\tau) = p \exp\{\tau X\}$ be a curve in the fiber through $p = c(0)$, then $X^v_{|p} := \frac{d}{d\tau} c(\tau)\big|_{\tau=0}$ is tangent to the fiber at p.

Cartan calculus applies on P, so one may define the de Rham complex of differential forms $(\Omega^\bullet(P), d)$ with d the de Rham, or exterior, derivative. Forms that vanish when evaluated on vertical vector fields are said *horizontal*. The pullback of a form α by the right action of H defines its "equivariance": $R^*_h : T_{ph}P \to T_pP$, $\alpha_{|ph} \mapsto R^*_h \alpha_{|ph}$ for $h \in H$. It describes how α changes when moved along the fibers via H. Given a representation of H, $\rho : H \to GL(V)$, $h \mapsto \rho(h)$, with V a vector space, one defines the space of representation-valued *equivariant forms* as those whose equivariance is simple and controlled by the representation, $\Omega^\bullet_{\text{eq}}(P, V) := \{\alpha \in \Omega^\bullet(P, V) \,|\, R^*_h \alpha_{|ph} = \rho(h^{-1})\alpha_{|p}\}$. Forms that are both horizontal and equivariant are said *tensorial*, for reasons soon disclosed. Forms with trivial equivariance, i.e. s.t. $R^*\alpha_{|ph} = \alpha_{|p}$, are said *invariant*. Forms that are both horizontal and invariant are said *basic*: they are important as they induce forms on M, hence their name, and may be defined as belonging to $\operatorname{Im}\pi^*$, the image of $\pi^* : \Omega^\bullet(M) \to \Omega^\bullet(P), \beta \mapsto \alpha := \pi^*\beta$.

The maximal group of transformations of the bundle P is its group of automorphisms,

$$\mathrm{Aut}(P) := \{\Xi \in \mathrm{Diff}(P) \,|\, \Xi(ph) = \Xi(p)h\}, \tag{1}$$

i.e. its elements are those diffeomorphisms of P that respect its fibration structure by sending fibers to fibers, and thus induces well-defined diffeomorphisms of the base M: we have the natural surjection $\mathrm{Aut}(P) \to \mathrm{Diff}(M)$. The subgroup of vertical automorphisms is $\mathrm{Aut}_v(P) := \{\Xi \in \mathrm{Aut}(P) \,|\, \pi \circ \Xi = \pi\}$, i.e. its elements are those automorphisms of P that induce the identity transformation id_M on M. It is a *normal* subgroup of $\mathrm{Aut}(P)$, which is noted $\mathrm{Aut}_v(P) \triangleleft \mathrm{Aut}(P)$, and implies that their quotient $\mathrm{Aut}(P)/\mathrm{Aut}_v(P)$ is itself a group, which is isomorphic to $\mathrm{Diff}(M)$. All this is synthesized in the *short exact sequence* of groups associated to a bundle P:

$$\mathrm{id}_P \;\to\; \mathrm{Aut}_v(P) \simeq \mathcal{H} \;\xrightarrow{\triangleleft}\; \mathrm{Aut}(P) \;\to\; \mathrm{Diff}(M) \;\to\; \mathrm{id}_M. \tag{2}$$

It also features the fact that the subgroup $\mathrm{Aut}_v(P)$ is isomorphic to the *gauge group* of P, which is defined as $\mathcal{H} := \{\gamma : P \to H \,|\, R^*_h\gamma = h^{-1}\gamma h\}$, i.e. its elements are s.t. $\gamma(ph) = h^{-1}\gamma(p)h$, and the isomorphism is $\Xi(p) = p\gamma(p)$.

The action by pullback of $\mathrm{Aut}_v(P)$ on a form α defines its *gauge transformation*, $\Xi^* : \Omega^\bullet(P)_{|\Xi(p)} \rightarrow \Omega^\bullet(P)_{|p}$, $\alpha \mapsto \Xi^*\alpha =: \alpha^\gamma$. The defined notation α^γ is justified by the above isomorphism, by which the gauge transform is expressible in terms of $\gamma \in \mathcal{H}$ generating $\Xi \in \mathrm{Aut}_v(P)$. It describes how α changes when moved along the fibers via $\mathrm{Aut}_v(P) \simeq \mathcal{H}$. For tensorial forms, gauge transformations are entirely controlled by their equivariance, thus by the representation, $\alpha^\gamma = \rho(\gamma)^{-1}\alpha$: they are then homogeneous,"gauge-tensorial", hence their name. Remark that, since $\eta \in \mathcal{H}$ is also $\eta \in \Omega^0_{\text{tens}}(P, H)$, the gauge transformation of a gauge group element is: $\eta^\gamma = \gamma^{-1}\eta\gamma$ for $\eta, \gamma \in \mathcal{H}$. For basic forms, gauge transformations are trivial $\alpha^\gamma = \alpha$, as expected for a form inducing a well-defined object on M where the action of $\mathrm{Aut}_v(P)$ is trivial. Both tensorial and basic forms are essential for gauge field theory: the latter capture gauge-invariant ("physical") d.o.f., while e.g. tensorial 0-form ϕ describe *matter fields*.

Up to now, all structures described on P are canonical – even the representation (ρ, V) ingredients, which arguably come along with the structure group. The first and most important non-canonical object to be described is a *connection 1-form*, $\omega \in \Omega^1_{\text{eq}}(\Omega, \mathfrak{h})$: by definition its equivariance is controlled by the adjoint representation $\mathrm{Ad} : H \rightarrow GL(\mathfrak{h})$, so that $R^*_h\omega = \mathrm{Ad}(h^{-1})\,\omega := h^{-1}\omega\, h$, and it is required to satisfy $\omega(X^v) = X \in \mathfrak{h}$ on fundamental vertical vector fields. These two properties imply: First, that the space of connections is an affine space modeled on the vector space $\Omega^1_{\text{tens}}(P, \mathfrak{h})$, meaning that while addition of connections is impossible, still $\omega' - \omega = \alpha \in \Omega^1_{\text{tens}}(P, \mathfrak{h})$. Secondly, that the gauge transformation of a connection is $\omega^\gamma = \gamma^{-1}\omega\gamma + \gamma^{-1}d\gamma$, i.e. it is "gauge pseudo-tensorial".

The above is known as an Ehresmann (or principal) connection. It is the one underlying "Yang-Mills type" GFT. Gauge theories of gravity are based on *Cartan connections*. They are 1-forms $\varpi \in \Omega^1_{\text{eq}}(P, \mathfrak{g})$, where $\mathfrak{g} \supset \mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h} = V$ supports a left-action of H, satisfying the same defining properties as ω, plus a third distinctive one: $\varpi : TP \rightarrow \mathfrak{g}$ is a linear isomorphism. This is the key feature of Cartan geometry, making it the foundation of gauge gravity, as it means that the geometry of P encodes that of M. In many cases, it implies that a Cartan connection splits as $\varpi = \omega \oplus \theta$, where ω is an Ehresmann connection and $\theta \in \Omega^1_{\text{tens}}(P, V)$ is a *soldering form*. The properties of Cartan connections imply, first that they form an affine space modeled on the vector space $\Omega^1_{\text{tens}}(P, \mathfrak{g})$, and secondly that their gauge transformation is $\varpi^\gamma = \gamma^{-1}\varpi\gamma + \gamma^{-1}d\gamma$, which splits into $\omega^\gamma = \gamma^{-1}\omega\gamma + \gamma^{-1}d\gamma$ and $\theta^\gamma = \gamma^{-1}\theta$.

The reason to introduce an Ehresmann connection is to obtain a first-order differential operator on $\Omega^\bullet_{\text{tens}}(P, V)$: this space is not preserved by d, so one defines the *covariant derivative* $D\_ := d\_ + \rho_*(\omega)\_ : \Omega^\bullet_{\text{tens}}(P, V) \rightarrow \Omega^{\bullet+1}_{\text{tens}}(P, V)$, $\alpha \mapsto D\alpha$, where $\rho_* : \mathfrak{h} \rightarrow \mathfrak{gl}(V)$. Which means in particular that α and $D\alpha$ have the same gauge transformations, their "gauge-tensoriality" is preserved: $\alpha^\gamma = \rho(\gamma)^{-1}\alpha$ and $(D\alpha)^\gamma = \rho(\gamma)^{-1}D\alpha$. As applied to $\alpha = \phi \in \Omega^0_{\text{tens}}(P, V)$, this is the geometric underpinning of the *Gauge Principle*, or *gauge argument*.

The *curvature 2-form* of a connection is $\Omega = d\omega + \tfrac{1}{2}[\omega, \omega] \in \Omega^2_{\text{tens}}(P, \mathfrak{h})$. Its gauge transformation is thus $\Omega^\gamma = \gamma^{-1}\Omega\gamma$, it is a "gauge tensor". Its covariant derivative is trivial, which just expresses the Bianchi identity: $D\Omega = d\Omega + [\omega, \Omega] = 0$, where $[\omega, \_] = \mathrm{ad}(\omega)\_ = \mathrm{Ad}_*(\omega)\_$. One also easily proves that $D \circ D = \rho_*(\Omega)$ on $\Omega^\bullet_{\text{tens}}(P, V)$. The curvature of a Cartan connection is $\bar{\Omega} = d\varpi + \tfrac{1}{2}[\varpi, \varpi] \in \Omega^2_{\text{tens}}(P, \mathfrak{g})$, splitting as $\bar{\Omega} = \Omega \oplus \Theta$, where $\Theta = D\theta$ is the torsion 2-form of ϖ. Its gauge transformation is $\bar{\Omega}^\gamma = \gamma^{-1}\bar{\Omega}\gamma$, which splits into $\Omega^\gamma = \gamma^{-1}\Omega\gamma$ and $\Theta^\gamma = \gamma^{-1}\Theta$. It satisfies the Bianchi identity: $\bar{D}\bar{\Omega} := d\bar{\Omega} + [\varpi, \bar{\Omega}] = 0$. One may check e.g. that the Cartan geometry based on $(\mathfrak{g}, \mathfrak{h})$ the Poincaré and Lorentz Lie algebras, is just Lorentzian geometry, $\mathcal{H}$ being then the Lorentz gauge group.

General-relativistic Gauge Field Theory (gRGFT) is written on the base manifold M of a bundle P rather than on P itself, or even on a finite region $U \subset M$ over which the bundle is trivializable, $P_{|U} \simeq U \times H$. The fields of gRGFT are the *local representatives* on U of global objects "living" on P, the former being the pullback of the latter via any local section: given $\sigma : U \rightarrow P_{|U}$, we have σ^*: $\Omega^\bullet(P_{|U}) \rightarrow \Omega^\bullet(U), \beta \mapsto \sigma^*\beta =: b$. Said otherwise, the local representatives b are the "bundle coordinate" versions of the intrinsic global objects β.

Then, the local representatives of an Ehresmann connection ω and its curvature Ω are $A := \sigma^*\omega \in \Omega^1(U, \mathfrak{h})$ and $F := \sigma^*\Omega = dA + \tfrac{1}{2}[A, A] \in \Omega^2(U, \mathfrak{h})$, which represent respectively a *gauge potential* and its *field strength*. The local representatives of a tensorial form and its covariant derivative, $\alpha, D\alpha \in \Omega^\bullet_{\text{tens}}(P, V)$, are $a := \sigma^*\alpha \in \Omega^\bullet(U, V)$ and $Da := s^*(D\alpha) = da + \rho_*(A)a$. From which we have that F satisfies the (local) Bianchi identity $DF = dF + [A, F] = 0$. In particular, the local representative of $\phi \in \Omega^0_{\text{tens}}(P, V)$ is a *matter field* $\varphi := \sigma^*\phi$, and $D\varphi = d\varphi + \rho_*(A)\varphi$ is just its *minimal coupling* to the gauge potential – usually arrived at via the heuristic of the "gauge argument" in GFT.

The local representative of a Cartan connection $\bar{A} := \sigma^*\varpi \in \Omega^1(U, \mathfrak{g})$ represents the (generalised) gravitational gauge potential. It splits as $\bar{A} = A + e$, where $e := \sigma^*\theta \in \Omega^1(U, V)$ is the *vielbein* 1-form, a.k.a. the *co-tetrad* field. Given a (non-degenerate symmetric) bilinear form $\eta : V \times V \rightarrow \mathbb{R}$, $(v, w) \mapsto \eta(v, w)$, the vielbein e induces a metric on M by $g := \eta \circ e : TM \times TM \rightarrow \mathbb{R}$, $(\mathsf{X}, \mathsf{Y}) \mapsto g(\mathsf{X}, \mathsf{Y}) := \eta(e(\mathsf{X}), e(\mathsf{Y}))$ – in coordinates; $g_{\mu\nu} = \eta_{ab}e^a{}_\mu e^b{}_\nu$. The gravitational field strength is $\bar{F} := \sigma^*\bar{\Omega} = F + T$, with $T := \sigma^*\Theta = De \in \Omega^2(U, V)$ the torsion, and s.t. $\bar{D}\bar{F} = 0$.

General-relativistic physics requires covariance under Diff(M), while Gauge Field Theory requires invariance under the gauge group $\mathcal{H}$, or rather the local gauge group on M i.e.

$$\mathcal{H}_{\text{loc}} := \{\gamma = \sigma^*\gamma, \gamma \in \mathcal{H} \mid \eta^\gamma = \gamma^{-1}\eta\gamma\}. \tag{3}$$

The full group of local symmetries of gRGFT is thus Diff(M) $\ltimes \mathcal{H}_{\text{loc}}$ which is just the local description of Aut(P). The group Diff(M) acts on any fields or forms by pullback: Taking from now on as our elementary variables the Yang-Mills and/or gravitational gauge potentials and matter fields, $\phi = \{A, \varphi, e/g,\}$, for $\psi \in$ Diff(M) we have $\phi^\psi := \psi^*\phi$, i.e.

$$A^\psi := \psi^* A, \quad \varphi^\psi := \psi^*\varphi, \quad \text{and} \quad e^\psi := \psi^* e \;\rightarrow\; g^\psi := \psi^* g. \tag{4}$$

We write similarly for minimal couplings $D\varphi$ and for the Yang-Mills and/or gravitational field strengths F, $\bar{F}$. The action of the gauge group $\mathcal{H}_{\text{loc}}$ on b is obtained by pullback of the result of the action of $\mathcal{H}$ on β: $b^\gamma := \sigma^*(\beta^\gamma)$. So, for $\gamma \in \mathcal{H}_{\text{loc}}$ we have that ϕ^γ is:

$$A^\gamma = \gamma^{-1}A\gamma + \gamma^{-1}d\gamma, \quad \varphi^\gamma := \rho(\gamma)^{-1}\varphi, \quad \text{and} \quad e^\gamma := \gamma^{-1}e \;\rightarrow\; g^\gamma := g. \tag{5}$$

Similarly, or consequently, we have $(D\varphi)^\gamma = \rho(\gamma)^{-1}D\varphi$ and $F^\gamma = \gamma^{-1}F\gamma$, as well as $\bar{F}^\gamma = \gamma^{-1}\bar{F}\gamma$ with $T^\gamma = \gamma^{-1}T$. Consider that in (5), one should typically understand that $\mathcal{H}_{\textbf{loc}} = \mathcal{H}^{\text{YM}}_{\textbf{loc}} \times \mathcal{H}^{\text{grav}}_{\textbf{loc}}$: The Yang-Mills gauge group $\mathcal{H}^{\text{YM}}_{\textbf{loc}}$ acting only on the Yang-Mills gauge potential $A = A'$ and matter fields φ, but trivially on the gravitational potential $\bar{A} = A + e$ (thus on g), while the gravitational gauge group $\mathcal{H}^{\text{grav}}_{\textbf{loc}}$ acts trivially on a Yang-Mills potential $A = A'$, but non-trivially on matter fields φ – which are *spinorial* representations of $\mathcal{H}^{\text{grav}}_{\textbf{loc}}$ – and on gravitational potential $\bar{A} = A + e$ (yet still trivially on g if η is $\mathcal{H}_{\textbf{loc}}$-invariant).[9]

A general-relativistic gauge field theory is defined by a Lagrangian functional on the field space $\Phi = \{A, \varphi, e/g\}$,

$$\begin{aligned} L:\; \Phi &\rightarrow \Omega^m(U, \mathbb{R}), \\ \phi &\mapsto L(\phi) = L(A, \varphi, e/g), \end{aligned} \tag{6}$$

with $m = \dim M$, i.e. the Lagrangian is a top form on M. As said above, the Lagrangian is required to be covariant under Diff(M) and quasi-invariant – i.e. invariant up to a boundary term – under $\mathcal{H}_{\text{loc}}$:

$$L(\phi)^\psi := L(\phi^\psi) = \psi^* L(\phi) \qquad \text{and} \qquad L(\phi)^\gamma := L(\phi^\gamma) = L(\phi) + db(\gamma; \phi). \tag{7}$$

This ensures that the field equations $E(\phi) = E(A, \varphi, g) = 0$ are covariant under Diff(M) $\ltimes \mathcal{H}_{\text{loc}}$, that is:

$$E(\phi)^\psi = E(\phi^\psi) = \psi^* E(\phi) = 0 \qquad \text{and} \qquad E(\phi)^\gamma = E(\phi^\gamma) = \rho(\gamma)^{-1}E(\phi) = 0, \tag{8}$$

where $\rho = \{\rho, \text{Ad}, \dots\}$ denotes the various representations of H (thus $\mathcal{H}$), including the trivial one, to which the various fields under consideration belong to.

In gRGFT, the base manifold M is typically regarded as representing the manifold of (spatio-temporal) events, and (M, g) representing spacetime. On that view, the bundle structure P "over" M may be understood as meaning that to each spatio-temporal events $x \in M$ there are "attached" identical fibers $P_{|x}$ that represent an internal, i.e. non-spatio-temporal, space. The bundle P thus represents an *enriched* manifold of events whose "points" are not structureless, as in general-relativistic physics, but endowed with an *internal* structure. The various gauge fields of gRGFT then have access to ("probe") this enriched space, and have thus both spatiotemporal and internal d.o.f.

Yet, as reminded in section 2, the hole argument challenges "naive" substantivalism regarding both the manifold of events M and spacetime (M, g), and when paired with the point-coincidence argument, invites a more subtle view whereby the true manifold of physical spatio-temporal events arises in the network of relations among fundamental fields, with the physical metric/gravitational field playing a co-defining role. Physical spacetime is then represented in the general-relativistic framework in a way that is strictly concordant with neither the usual relational nor substantival views. It is then natural to expect, given that in gRGFT M arises as the quotient of the richer space P, that *generalized hole* and *point-coincidence arguments* similarly contribute to a better understanding of possible realist positions about the principal bundle.

[9] In established physics $\mathcal{H}^{\text{grav}}_{\textbf{loc}}$ is the Lorentz gauge group, so g is indeed invariant as η is the Minkowski form on $V = \mathbb{R}^n$. In speculative theories the action of $\mathcal{H}^{\text{grav}}_{\textbf{loc}}$ on g may be non-trivial: e.g. in conformal geometry/gravity, it induces a conformal transformation, $g \mapsto g^\gamma = z^2 g$ for $z \in C^\infty(U, \mathbb{R}/\{0\})$.

4 The generalized hole argument in general-relativistic gauge field theory

By bundle realism, we understand the view that the physical manifold of events has the structure of a fiber bundle, each spatio-temporal event having an internal structure represented by the fibers of the bundle. Given that our most fundamental physical theories are models of the gRGFT framework (and quantization thereof, in the case of the SM), one might expect that questions about the ontological status of fiber bundles would be among the most important topics discussed in the philosophy of physics. Yet, the literature on this is surprisingly sparse. Physicists and mathematical physicists may occasionally stress the technical importance of bundles, for example:

> "It is a widely held view among mathematicians that the fiber bundle is a natural geometrical concept. Since gauge fields, including in particular the electromagnetic field, are [connections on] fiber bundles, all gauge fields are thus based on geometry. To us it is remarkable that a geometrical concept formulated without reference to physics should turn out to be exactly the basis of one, and indeed maybe all, of the fundamental interactions of the physical world." Wu and Yang (1975)

> "For me, a gauge theory is any physical theory of a dynamic variable which, at the classical level, may be identified with a connection on a principal bundle." Trautman (1980)

> "What is gauge theory? It is not an overstatement to say that gauge theory is ultimately the theory of *principal bundles* [...]. The fundamental geometric object in a gauge theory is a principal bundle over spacetime [...]." Hamilton (2017)

But they almost invariably stop short of truly expressing any ontological commitment. In philosophy of physics, while the literature on the significance of gauge symmetries ($\mathcal{H}$) has considerably grown in the last 25 years, making it a well-established topic, the ontological status of the spaces of which they are automorphisms, bundle spaces, have comparatively received little attention.[10] Exceptions we know of, which at least raise or strongly hint at the question, being Stachel (1986), Lyre (1999) Guttmann and Lyre (2000); Lyre (2004a, 2004b), Healey (2001), Nounou (2003), Healey (2007), Maudlin (2007), Arntzenius (2012), Dewar (2019), Catren (2022), Jacobs (2023). One may consult Stachel (2014)'s extensive defense of the relevance of the bundle viewpoint both in GFT and GR.

We are now in a position to discuss substantivalist and relationalist approaches to the principal fiber bundle. Echoing the classical opposition, the former would contend that P represents a physical entity, the enriched manifold of events, which has an existence autonomous from the various gauge fields inhabiting it. The latter would deny this claim. As far as we can tell, Maudlin (2007) and Arntzenius (2012) lean toward a bundle substantivalist view. In the physicists' camp, Ne'eman is a rare exception on the record insisting that "physics selects the *realist* or *substantivist* view" Ne'eman (1996), with Lyre (2004a) reporting that in personal conversation Ne'eman explicitly characterized his position as a bundle substantivalism. On the contrary, Lyre (1999) and Healey (2001) would appear to be at least anti-substantivalists; a position they rest on the fact that bundle substantivalism is vulnerable to an "internal hole argument". Lyre (1999) saying that "*there exists a straightforward extension of the spacetime manifold hole argument to a generalized bundle space hole argument*", while Healey (2001) argues that "*fiber bundle substantivalism [...] is subject to an analogue of the "hole" argument against space-time substantivalism*".

We argue below for a form of sophisticated bundle substantivalism, which amounts to a realist but also relationalist view about bundles. At this point, we note that in the same way that the empirical success of GR warrants some commitment to a realist view of spacetime described as a curved Lorentzian manifold (M, g), modulo the necessary refinements brought by the hole and point-coincidence arguments, the empirical success of GFT warrants some commitment to a realist view of the enriched connected spacetime described as a bundle with connection (P, ω), modulo the necessary refinements brought by generalized hole and point-coincidence arguments, articulated next.

As reminded in section 3, GFT physics describes the coupled dynamics of matter and gauge interaction fields. The latter are the local, bundle coordinate, representatives of intrinsic objects on a principal bundle P. So, even if field theories are usually written on M, the fundamental fields are those on P: The field space is thus $\Phi = \{\omega, \alpha\}$, with α representing a set of tensorial forms (matter fields, soldering form) – so that if $\alpha = \theta$, then $(\omega, \theta) = \varpi$ is a

[10]Which is not to say that bundles are not often acknowledged in this literature as important mathematical structures. This is a curious situation; comparable to a fictitious one in which the community would have extensively discussed the physical meaning of the group of diffeomorphism Diff(M), but neglected the issue of the ontological status of the manifold M.

Cartan connection for a gauge theory of gravity. The gauge group, or the vertical automorphism group $\mathcal{H}\simeq\text{Aut}_v(P)$, partitions Φ into orbits $\mathcal{O}$. The moduli space of $(\mathcal{H}\simeq\text{Aut}_v(P))$-orbits is the quotient $\mathcal{M} := \Phi/\mathcal{H}$, and there is the projection $\pi : \Phi \to \mathcal{M}$, s.t. $[\omega, \alpha] = [\omega^\gamma = \Xi^*\omega, \alpha^\gamma = \Xi^*\alpha]$, i.e. there is a 1:1 correspondence $\mathcal{O} \leftrightarrow [\omega, \alpha]$.

The defining desideratum of GFTs is the $\mathcal{H}_{\text{loc}}$-covariance of field equations (8), whose solutions $\phi = (A, a)$ are the local representatives of global objects (ω, α). The space of global solutions $\mathcal{S} := \{(\omega, \alpha) \in \Phi \text{ s.t. } E(\phi) = 0\}$ is then also partitioned into distinct $\mathcal{H}$-orbits $\mathcal{O}$, so that $\pi : \mathcal{S} \to \mathcal{M}_{\mathcal{S}}$, $(\omega, \alpha) \mapsto [\omega, \alpha]$, where $\mathcal{M}_{\mathcal{S}}$ is the moduli space of global solutions. The immediate dual consequence is that GFT can neither distinguish solutions in the same $\mathcal{H}$-orbit $\mathcal{O}$, nor the points of the fibers, being indifferent to the distinction p *vs* $\Xi(p) = p\gamma(p)$, both in $P_{|x} \subset P$.

Here as in GR, an a priori realism toward the formalism amounts to accepting a metaphysical multiplicity of fields, in $\mathcal{O}$, that are physically indistinguishable. The "internal" hole argument further highlights that this position leads to an ill-defined Cauchy problem: Consider (ω, α) and $(\Xi^*\omega, \Xi^*\alpha) \in \mathcal{O} \subset \Phi$ s.t. $\Xi \in \text{Aut}_v(P) \sim \gamma \in \mathcal{H}$ has compact support $P_{|D} \subset P_{|U} \subset P$ (the "internal hole"), where $D \subset U \subset M$, i.e. $\Xi \sim \gamma \neq \text{id}$ on $P_{|D}$, so that $(\omega, \alpha) = (\omega^\gamma, \alpha^\gamma)$ on $P/P_{|D}$, but $(\omega, \alpha) \neq (\omega^\gamma, \alpha^\gamma)$ on $P_{|D}$. For the local representatives $\phi = (A, a)$ this means that $\gamma \in \mathcal{H}_{\text{loc}}$ has compact support on D, i.e. $\gamma \neq \text{id}$ on $D \subset M$, so that $\phi = \phi^\gamma$ on M/D, but $\phi \neq \phi^\gamma$ on D.[11] Then, given boundary conditions for the fields ϕ at ∂U, or for (ω, α) at $P_{|\partial U}$, the field equations do not uniquely determine a solution within U, therefore a global solution in $P_{|U}$ since both (ω, α) and $(\Xi^*\omega, \Xi^*\alpha) \in \mathcal{O} \subset \mathcal{S}$. This in particular implies a failure of determinism if the domain D is in the future of some Cauchy surface for the fields.

The natural way out, as in GR, is to appeal to an "internal point-coincidence argument", which emphasizes that the physical content of a GFT (in particular, its empirically accessible predictions) is exhausted by the pointwise coincidental values of fields' internal d.o.f., e.g. $\omega_{|p}$ and $\alpha_{|p} = \phi(p)$ (in their minimal coupling), which are $\text{Aut}_v(P) \simeq \mathcal{H}$-invariant and represent *physical* events/magnitudes. The immediate dual consequences are as follows.

First, points $p \in P_{|x}$ of the fibers, the internal space over/within $x = \pi(p) \in M$, are entirely unphysical. *Physical internal points* are encoded in the $\text{Aut}_v(P)$-invariant network of relations among the internal d.o.f. of the fields $\phi/(\omega, \alpha)$. Secondly, the argument implies the standard physics view and practice, which is to consider that *physical fields* d.o.f. are not represented by any single ϕ, or $(\omega, \alpha) \in \mathcal{O} \subset \mathcal{S}$, but by a point in a $\mathcal{H}_{(\text{loc})}$-equivalence class $[\phi]$, or $[\omega, \alpha] \in \mathcal{M}_{\mathcal{S}}$ in the moduli space of solutions. This extends Earman and Norton's "Leibniz equivalence" to GFT. The argument further suggests that what is physical are the $\text{Aut}_v(P) \simeq \mathcal{H}$-invariant relations among the internal d.o.f. of the fields $\phi/(\omega, \alpha)$, so that physical fields internal d.o.f. mutually co-define each other: e.g. the internal d.o.f. of the gauge fields (redundantly described by A/ω) are only well-defined in relation to those of the matter fields (redundantly described by φ/ϕ). Together, the internal hole and point-coincidence argument paint a relationalist view of the *physical internal space* of GFT.

In view of equation (2), the standard and internal hole arguments combine into a *generalized hole argument* which, articulated with a *generalized point-coincidence argument*, makes a strong case for the relationalist view of the principal bundle: The fibered manifold P only bootstraps our ability to erect a description of the relevant physics of fields with spatio-temporal and internal d.o.f.,[12] but is removed from the physical picture by the requirement (8) of covariance of the field equations under $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$, i.e. $\text{Aut}(P)$. To paraphrase Einstein (1952), the *physical fibered manifold of spatio-temporal and internal events* does not claim existence on its own, but only as a *structural quality of the fields* (ω, α).

One may define a notion of "enriched spacetime" as being represented by (P, ω), understood that $\omega = \omega' \oplus \varpi$, with ω' an Ehresmann connection describing Yang-Mills gauge interactions fields, and ϖ a Cartan connection describing the gravitational field. A physical gauge field, represented by ω, is as autonomous an entity as is possible in GFT, with the physical field d.o.f. and the co-defining relations in which they participate being coextensive. Then one may consider physical gauge fields to embody a substantival notion of enriched spacetime, insofar as the "substantival" notion gets suitably "relationalized". This would be a form of "sophisticated substantivalism" towards enriched spacetime, analogous to sophisticated substantivalism towards spacetime.[13]

[11] Notice that one works here with a given trivializing section $\sigma : U \to P_{|U}$. So we are comparing the local representatives on U of *distinct* global objects on $P_{|U}$, fields and their $\mathcal{H}$-transforms, through a unique local section. It is different from comparing local representatives on U of the *same* global objects, through distinct local sections σ and σ': this is the operation of "gluings" via transition functions of P, i.e. of bundle coordinate change, and represent *passive gauge transformations* akin to coordinate change in GR. These do not lead to ontological underdetermination, nor to indeterminism, as there is no hole argument holding.

[12] In the same way that in GR, Einstein (1952) stressed that fields are not "in spacetime", but have spacetime extension i.e. spatio-temporal d.o.f., in GFT one may say that the fields do not in part "live" or "probe" an independently existing internal space at each spatio-temporal event x – the fibers $P_{|x} \subset P$ – but have an "internal extension", i.e. internal d.o.f.

[13] Healey (2001) pointed out that, analogously to metric essentialism, one could define "connection essentialism" (though he rejects it).

It can be understood within a generalization of Norton (1989)'s "mpfs", which we may call "enriched manifolds plus further structures", "empfs". The further structure above being the connection ω. Emulating Norton (1989)'s argument, one could argue that empfs substantivalism is vulnerable to a hole-type argument if the "further structure" has a non-trivial Killing symmetry group. But we could first object that realistic fields, even in best cases, have only approximate symmetries, and secondly that a Norton (1989) type argument cannot hold in a fully *dynamically coupled regime*.[14]

Relatedly, one may mount an argument à la Weatherall (2018) and Halvorson and Manchak (2022): Given that the enriched spacetime is described by (P, ω), mathematical practice suggests that the only admissible transformations in GFT are the morphisms in the category Bund_ω of principal bundles with connections, taken to be connection preserving bundle morphisms, $\Xi : (Q, \omega_1) \to (P, \omega_2)$ s.t. $\Xi^*\omega_2 = \omega_1$, so that in the special case $\Xi : (P, \omega) \to (P, \omega)$, such transformations can only be Killing symmetries: $\mathsf{K}(\omega) := \{\Xi \in \text{Aut}(P) \text{ s.t. } \Xi^*\omega = \omega\}$. Internal and/or generalized hole arguments would be blocked since no "hole automorphism" can be Killing.

We may again object that adhering to the strictures of the categorical language would preemptively trivialize entirely the gauge symmetries of any realistic GFT, where the Killing group would be reduced to $\mathsf{K}(\omega) = \{\text{id}_P\}$ – and reduces them to finite-dimensional Killing gauge groups for idealized gauge field configurations. We may also notice that this would go against the heuristic logic of the Gauge Principle, which *starts* with matter fields without Killing gauge symmetries to motivate the introduction of a gauge potential to which they are (minimally) coupled. The dynamics of such a potential is also arrived at by the constraint of generic (non-Killing) gauge symmetries, which points towards the field strength F as the simplest covariant (gauge-tensorial) quantity built from derivatives of A – so that e.g. the kinematics of a YM potential A is given by the YM Lagrangian $L_{\text{YM}}(A) = \text{Tr}(F \wedge *F)$. More broadly, restricting admissible gauge transformations to Killing symmetries goes against decades of practice in theoretical particle physics whereby the constraints of gauge-invariance/covariance under various gauge groups $\mathcal{H}_{\text{loc}}$, was the cornerstone of model building – and was in particular successful for the SM. We therefore conclude that, like its analogue in GR, this argument remains unconvincing.

The significance of the covariance (8) of the field equation of gRGFT under $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$ should be analyzed in the framework's own terms. We take it to be decisively clarified by the generalized hole and point-coincidence arguments, yielding a relationalist view of P and a correspondingly relationalized sophisticated substantivalism towards the field of gRGFT. In the next section, we describe how the *Dressing Field Method* (DFM) allows a technical framing of these insights, through the definition of invariant dressed fields and dressed spaces.

5 Dressing Field Method: Dressed fields and spaces

As reflected in the above discussion, it is the established view in physics that physical magnitudes and observables must be gauge-invariant. This requirement is typically addressed by appropriately reducing the gauge symmetries of the theory. The *Dressing Field Method* (DFM) is an approach allowing to do so systematically by producing gauge-invariant variables out of the field space Φ of a theory. See François (2014) for early developments, François and Ravera (2024) for the most up-to-date technical accounts, and François and Ravera (2025a, 2025b) for recent applications to supersymmetric field theory, supergravity, and quantum mechanics.

What makes the DFM particularly attractive from a philosophical perspective is that by allowing to technically implement a manifestly invariant reformulation of gauge theories, it "wears its ontology on its sleeve", so to speak. Hence it gained some traction in the foundations of gauge theories, see e.g. Berghofer and François (2024); Berghofer et al. (2023); François and Ravera (2025).

In what follows, we remind the basics of the DFM, introducing dressing fields, dressed fields, and dressed spaces. We shall see how the DFM technically implements the point-coincidence argument, meaning that dressed spaces are by definition immune to the hole argument, thus providing a transparent formal implementation of the conceptual insights discussed in sections 2 and 4.

[14] Where e.g. no Killing gauge symmetries of a Yang-Mills gauge potential can fail to be symmetries of the matter fields due to the Yang-Mills equation $E(A, \varphi) = D * F - J(\varphi, A) = 0$, where $J(\varphi, A)$ is the current $(m-1)$-form of the matter field, and $*F \in \Omega^{m-2}(U, \mathfrak{h})$ is the Hodge dual of the curvature/field strength of A.

5.1 The case of internal gauge symmetries

Consider a GFT with field content $\phi = \{A, \varphi\}$ and (internal) gauge group $\mathcal{H}_{\text{loc}}$. A *dressing field* is a smooth map

$$u : U \subset M \to H, \quad \text{defined by} \quad u^\gamma = \gamma^{-1}u, \quad \gamma \in \mathcal{H}_{\text{loc}}. \tag{9}$$

Key to the DFM is the fact that a dressing field should be extracted from the d.o.f. in the field space Φ of the theory. This means that it should be a *field-dependent dressing field*, $u = u[\phi]$, so that its defining gauge transformation is

$$u[\phi]^\gamma := u[\phi^\gamma] = \gamma^{-1}u[\phi], \quad \gamma \in \mathcal{H}_{\text{loc}}. \tag{10}$$

Given a dressing field as above, one defines the $\mathcal{H}_{\text{loc}}$-invariant *dressed fields*, as "composite objects", as follows:

$$\phi^u = \{A^u, \varphi^u\} := \{u^{-1}Au + u^{-1}du, \ \rho(u)^{-1}\varphi\}. \tag{11}$$

To anticipate a contrast with the forthcoming discussion of dressings for Diff(M), notice that ϕ^u live on M still. This is the simplest illustration of the DFM "rule of thumb": To dress any object (fields, functional of fields), first compute its gauge transformation, then formally substitute in the resulting expression the gauge parameter γ by the dressing field u. The new expression is the "dressing" of the object and is by construction $\mathcal{H}_{\text{loc}}$-invariant.

The dressed fields $\phi^{u[\phi]} = \{A^{u[A,\varphi]}, \varphi^{u[A,\varphi]}\}$ can be understood as resulting from a reshuffling of the d.o.f. of the "bare" fields ϕ, which mixes physical d.o.f. and non-physical pure gauge modes, whereby pure gauge modes are eliminated – completely, or partially as we shall observe below.

It must be stressed that since the dressing field *is not* an element of the gauge group, as it does not have the defining $\mathcal{H}_{\text{loc}}$-transformation of such an element—see eq. (3)—the dressed fields (11) are therefore not gauge-transformed fields, and, in particular, a dressing via the DFM *is not* a gauge-fixing. See Berghofer and François (2024); François and Ravera (2025b) for a detailed discussion of this point.

Relatedly, notice that, with ϕ-dependent dressing fields, the DFM has a very natural *relational* interpretation: The dressed fields $\phi^{u[\phi]} = \{A^{u[A,\varphi]}, \varphi^{u[A,\varphi]}\}$ represent the *gauge-invariant* ("physical") *relations* among the d.o.f. embedded in the bare fields $\phi = \{A, \varphi\}$: they are "relational observables". See François and Ravera (2024, 2025). In other words, the dressed fields (11) formally implement the *internal point-coincidence argument* discussed in section 4, and are immune to the *internal hole argument*. Observe that this is achieved at the kinematical level.

The dynamics of the theory being specified by a Lagrangian $L(\phi) = L(A, \varphi)$ that is $\mathcal{H}_{\text{loc}}$-quasi-invariant, eq. (7), given a dressing field u, by the DFM rule of thumb one defines the *dressed Lagrangian*

$$L^u := L(\phi^u) = L(\phi) + db(u; \phi), \tag{12}$$

which is just the Lagrangian expressed in terms of the dressed fields. If L is strictly $\mathcal{H}_{\text{loc}}$-invariant, i.e. $b = 0$, then $L(\phi^u) = L(\phi)$. In either cases, the field equations $\boldsymbol{E}(\phi^u) = 0$ for the dressed fields, obtained from (12), have the *same functional expression* as the field equations $\boldsymbol{E}(\phi) = 0$ for the bare fields, obtained from L. Manifestly, here they only differ by a boundary term: i.e. $\boldsymbol{E}(\phi^u) = \boldsymbol{E}(\phi) + dc(u; \phi)$, see eq. (391) in François and Ravera (2024). The dressed field equations are *deterministic*, in the sense that they uniquely determine the evolution of the relational d.o.f. ϕ^u of the theory, as expected from a scheme that solves the problem posed by the internal hole argument by technically implementing the internal point-coincidence argument in a manifest way. We may further observe that the theory confronted to experimental tests is always the dressed one, never the "bare" one.

5.1.1 Residual transformations

If the dressing field is s.t. $u^\eta = \eta^{-1}u$ for $\eta \in \mathcal{K}_{\text{loc}} \subset \mathcal{H}_{\text{loc}}$, i.e. it is a dressing for a subgroup of the full gauge group, the dressed fields (11) are expected to display *residual transformations* under what remains of the gauge group after reduction of $\mathcal{K}_{\text{loc}}$. These are called *residual transformations of the 1st kind*. Whenever possible, one must aim to avoid these by finding one or several dressing fields so as to completely reduce $\mathcal{H}_{\text{loc}}$. See François (2014) for the general scheme, and Attard and François (2017, 2018); François (2019) for applications to conformal Cartan geometry and twistor theory.

Furthermore, dressed objects may also exhibit residual transformations arising from a possible "ambiguity" in the choice/construction of the dressing field. Indeed, two dressing fields u, u' may a priori be related by $u' = u\xi$, where $\xi : U \to H$ is s.t. $\xi^\gamma = \xi$: the group (under pointwise product) of such maps we denote $\mathcal{G}$, and call it the group of *residual transformations of the 2nd kind*. Indeed, we may write its action on a dressing as $u^\xi := u\xi$, and while it does not act on bare variables, $\phi^\xi := \phi$, it does act naturally on dressed ones $(\phi^u)^\xi := \phi^{u\xi}$. When u is a ϕ-dependent dressing field, the constructive procedure may be expected to reduce $\mathcal{G}$ to a finite *discrete* group. In concrete situations where this happens, residual transformations of the 2nd kind have been shown to encode *physical reference frame covariance*, both at the classical and at the quantum level François and Ravera (2025a). Quite clearly, when $\mathcal{G}$ is discrete there is no possibility to mount a hole-type argument.

5.2 The case of diffeomorphisms

We now consider a general-relativistic theory, with field content $\phi = \{A, \varphi, e/g\}$ subject to the action of Diff(M). A Diff(M)-dressing field, introduced in François (2024), is a smooth map

$$\upsilon : N \to M, \quad \text{defined by} \quad \upsilon^\psi := \psi^{-1} \circ \upsilon, \quad \psi \in \text{Diff}(M). \tag{13}$$

Again, key to the DFM is the notion that the dressing field should be extracted from the d.o.f. of the theory, i.e. it should be a ϕ-dependent dressing field $\upsilon = \upsilon[\phi]$, so that its Diff$(M)$-transformation is

$$\upsilon[\phi]^\psi := \upsilon[\psi^*\phi] = \psi^{-1} \circ \upsilon[\phi]. \tag{14}$$

Given a dressing field υ as above, we define the Diff(M)-invariant dressed fields

$$\phi^\upsilon := \upsilon^*\phi, \quad \text{i.e.} \quad \{A^\upsilon, \varphi^\upsilon, e^\upsilon/g^\upsilon\} = \{\upsilon^*A, \upsilon^*\varphi, \upsilon^*e/\upsilon^*g\}. \tag{15}$$

Again, this is a simple case of the DFM rule of thumb: To dress any object—field or functional thereof—one writes its Diff(M)-transformation and then formally replaces ψ in the resulting expression with the dressing field υ. The new expression is the dressing of the object, and is Diff(M)-invariant by construction.

The dressed fields $\phi^{\upsilon[\phi]}$ again have a natural interpretation as relational variables: They result from a reshuffling of the d.o.f. of the bare fields where pure gauge modes are eliminated, so that what remains represents the Diff(M)-invariant relations among the d.o.f. embedded in ϕ. That is, the dressed fields (15) formally implement the *point-coincidence argument* discussed in section 2, and are immune to the *hole argument*. We stress again that this is achieved at the kinematical level.

The dynamics of the theory being specified by a Lagrangian $L(\phi)$ transforming under Diff(M) as in eq. (7), given a dressing field υ one defines the dressed Lagrangian as

$$L^\upsilon := L(\phi^\upsilon) = \upsilon^*L(\phi), \tag{16}$$

which is strictly Diff(M)-invariant by construction. This is yet another application of the DFM rule of thumb. The field equations $\boldsymbol{E}(\phi^\upsilon) = 0$ for the dressed fields have the same functional expression as those $\boldsymbol{E}(\phi) = 0$ for the bare fields: they are related by $\boldsymbol{E}(\phi^\upsilon) = \upsilon^*(\boldsymbol{E}(\phi) + dc(\upsilon; \phi))$, see eq. (391) in François and Ravera (2024). But contrary to the bare ones, the dressed field equations are deterministic. Again, remark that the dressed theory is the one confronted to empirical tests, not the bare one.

Contrary to the case discussed in section 5.1, observe that the dressed fields ϕ^υ *do not* live on "bare regions" $U \subset M$, but on *field-dependent dressed regions* defined as

$$U^\upsilon := \upsilon^{-1}(U) = \upsilon[\phi]^{-1}(U), \tag{17}$$

where υ^{-1} is the inverse map of υ, s.t. $\upsilon \circ \upsilon^{-1} = \text{id}_M$. These dressed regions are Diff(M)-invariant:

$$(U^\upsilon)^\psi = (\upsilon[\phi]^\psi)^{-1} \circ (U^\psi) = \upsilon[\phi]^{-1} \circ \psi \circ \psi^{-1} \circ (U) = U^\upsilon. \tag{18}$$

The ϕ-dependent Diff(M)-invariant regions $U^{\upsilon[\phi]}$ represent *physical regions* of the physical manifold of events, and are thus obviously immune to the hole argument. The definition (17) comes from integration theory, telling that integration is a Diff(M)-invariant operation; e.g. $S := \int_U L(\phi) = \int_{\psi^{-1}(U)} \psi^*L(\phi)$ yields by the DFM rule of thumb,

$$S = \int_U L(\phi) = \int_{\upsilon^{-1}(U)} \upsilon^*L(\phi) =: S^\upsilon. \tag{19}$$

In particular, a physical spatio-temporal event is a ϕ-dependent Diff(M)-invariant point $x^{\upsilon[\phi]} := \upsilon[\phi]^{-1}(x) \in U^\upsilon$, which is the technical implementation of the basic intuition underlying the point-coincidence argument. Remark that the latter was always partly tacitly baked into integration theory: Indeed, integration of a 0-form φ over the 0-dimensional manifold $x \subset M$ is the Diff(M)-invariant evaluation operation $\mathsf{ev}_x(\phi) := \int_x \varphi = \int_{\psi^{-1}(x)} \psi^*\varphi$ giving the invariant "value of φ at the point x", which can be understood as saying that the "value of φ" and "the point x" are co-defining, or relata coextensive with the relation they define. The DFM rule of thumb thus yields

$$\mathsf{ev}_x(\phi) = \int_x \varphi = \int_{\upsilon^{-1}(x)} \upsilon^*\varphi =: \mathsf{ev}_{x^\upsilon}(\phi^\upsilon). \tag{20}$$

Equations (19)-(20) implement in a mathematically concrete way the conceptual insight, described in section 2, according to which relational field d.o.f. "live" on the relationally defined physical regions. See François and Ravera (2024) for technical details, which involve the definition of integration as an operation on the *bundle of regions* (of M) associated to the field space bundle Φ.

One may then define $M^\upsilon := \text{Im}(\upsilon^{-1})$, which is identified as the *physical manifold of spatio-temporal events*. Notice that M^υ, its regions U^υ and points x^υ, simply *do no exist* without the fields ϕ, on which they are *supervenient*, and are thus the precise formal encapsulation of Einstein (1952)'s remark, quoted already at the end of section 2, saying "*If we imagine the [gravitational and other] field[s] [...] to be removed, there does not remain a space of the type (1) [of SR], but* ***absolutely nothing****, and also no 'topological space'. [...] Space-time does not claim existence on its own, but only as a structural quality of the field*" (our emphasis). While this is *tacitly* encoded by the Diff(M)-covariance of general relativistic-theories, it is made *manifest* via the DFM.

The *physical spacetime* may then be defined as the pair (M^υ, g^υ), which formally realizes the "relationalized" sophisticated substantivalism emphasized in section 2. Alternatively, defining spacetime as $(M^\upsilon, \mathsf{S}^\upsilon)$, for S any other structure besides, or instead of, a metric—most likely an Ehresmann (spin) connection $\mathsf{S} = A$, or $\mathsf{S} = (A, e)$ a Cartan connection—we obtain a "relationalized" sophisticated substantivalism of the "mpfs" kind. Clearly, Norton (1989)'s worry that "mpfs substantivalism" might be vulnerable to hole-type arguments cannot even get off the ground here.

Similarly to the internal case, potential ambiguities in the construction of the Diff(M)-dressing field υ are parametrized by a group $\mathcal{G} \subseteq \text{Diff}(N)$ of residual transformations of the 2nd kind, acting as $\upsilon^\xi := \upsilon \circ \xi$. In relevant realistic situations, it may be expected to reduce to a finite discrete group, just reflecting the spectrum of choices among available d.o.f. in Φ. Again, no hole-type argument can get off the ground in such cases.

The above is the general framework encompassing all manners of "scalar coordinatization" in GR: e.g. the Kretschmann-Komar (1958) approach where $\upsilon = \upsilon[g]$ and $g^{\upsilon[g]}$ is a case of self-dressing, or approaches à la Brown and Kuchar (1995) where $\upsilon = \upsilon[\varphi]$ and φ is some effective dust scalar field so that $g^{\upsilon[\varphi]}$ is the metric as seen in the "coordinate system" provided by the dust field, which is also a point of view developed by Rovelli (1991, 2002), and one that has been applied e.g. to cosmology by Giesel, Hofmann, Thiemann, and Winkler (2010).

5.3 The case of general-relativistic gauge field theories

Bringing together the results of sections 5.1 and 5.2, the DFM allows to reformulate gRGFTs in a manifestly invariant way: Defining a complete dressing field as the pair (υ, u) satisfying (9) and (13), the dressed fields

$$\phi^{(\upsilon,u)} := \upsilon^*(\phi^u) \tag{21}$$

are $(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}})$-invariant by construction. These are fully relational variables, representing invariant relations among the d.o.f. of the bare fields ϕ, formally implementing the *generalized point-coincidence* argument and thus immune to the *generalized hole argument*, as discussed in section 4. They represent *physical* spatio-temporal and internal d.o.f. of fields whose dynamics is given by the $(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}})$-invariant dressed Lagrangian

$$L^{(\upsilon,u)} := L(\upsilon^*(\phi^u)) = \upsilon^* L((\phi^u)), \tag{22}$$

yielding the fully deterministic (dressed) field equations $E(\phi^{(\upsilon,u)}) = 0$. Again, these are functionally the same as those of the bare fields – to which they relate by eq. (390) in François and Ravera (2024) – which explains why gRGFTs could be empirically tested *before* the issue of their observables could be cleanly settled: it is the dressed theories that were put to test. A fact e.g. stressed, in his own language, by Rovelli (2002) in the case of GR.

As previously, the fields $\phi^{(\upsilon,u)}$ live on the physical manifold of spatio-temporal events $M^{(\upsilon,u)} = M^{\upsilon}$ (or regions U^{υ} thereof), immune to the hole argument. One may rightly suspect that the above is a *local* description of a global intrinsic structure: that of global dressed fields on a dressed bundle space.

Indeed, in the same way that $(\psi, \gamma) \in \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$ is the local (coordinate bundle) version of a global automorphism $\Xi \in \text{Aut}(P)$, the pair (υ, u) is the local version of a global dressing field, which is a smooth map

$$\mathbf{u} : Q \to P, \quad \text{defined by} \quad \mathbf{u}^{\Xi} := \Xi^{-1} \circ \mathbf{u}, \quad \Xi \in \text{Aut}(P), \tag{23}$$

where Q is a "model" fibered manifold. Remark that $\mathbf{u}$ is not a *principal* bundle morphism. Such a dressing field is built from the d.o.f. of the global fields $\bar{\phi} = (\omega, \alpha)$ on P – with local representatives on $U \subset M$ the bare fields ϕ. It is a $\bar{\phi}$-dependent dressing field $\mathbf{u} = \mathbf{u}[\bar{\phi}]$, so that its defining Aut(P)-transformation is

$$\mathbf{u}[\bar{\phi}]^{\Xi} := \upsilon[\Xi^{*}\bar{\phi}] = \Xi^{-1} \circ \mathbf{u}[\bar{\phi}]. \tag{24}$$

It allows to define Aut(P)-invariant dressed fields

$$\bar{\phi}^{\mathbf{u}} := \mathbf{u}^{*}\bar{\phi}, \tag{25}$$

whose local representatives are (21), and like them are immune to the generalized hole argument. They live on the *physical manifold of spatio-temporal and internal events*, or *physical bundle space*, defined as $P^{\mathbf{u}} := \text{Im}(\mathbf{u}^{-1})$. Its regions are

$$V^{\mathbf{u}} := \mathbf{u}^{-1}(V) \tag{26}$$

for $V \subset P$, s.t. $\pi(V) = U \subset M$, and are Aut(P)-invariant: $(V^{\mathbf{u}})^{\Xi} = \left(\mathbf{u}[\bar{\phi}]^{\Xi}\right)^{-1} \circ (\Xi \circ (V)) = \mathbf{u}[\bar{\phi}]^{-1} \circ \Xi^{-1} \circ \Xi \circ (V) = V^{\mathbf{u}}$. Both dressed fields (25) and regions (26) are thus immune to the generalized hole argument. The reason for the definition (26) being, as previously, integration theory (on P). It gives e.g. formal substance to the generalized point-coincidence argument via the notion of the Aut(P)-invariant $\bar{\phi}$-dependent *physical* point $p^{u[\bar{\phi}]} := \mathbf{u}[\bar{\phi}]^{-1}(p)$, which is both a spatio-temporal and an internal physical event. Indeed, as a special case of (26) we have that the *physical internal space*, the dressed fibers, are $P^{\mathbf{u}}_{|x^{\mathbf{u}}} := \mathbf{u}^{-1}(P_{|x}) \subset P^{\mathbf{u}}$, realizing the internal point-coincidence argument. This means that $P^{\mathbf{u}}$ is a fibered manifold: the moduli space of physical fibers forming the physical manifold of spatio-temporal events $M^{\mathbf{u}} = M^{\upsilon}$, there is a smooth projection $\bar{\pi} : P^{\mathbf{u}} \to M^{\upsilon}$, $p^{\mathbf{u}} \mapsto \bar{\pi}(p^{\mathbf{u}}) =: x^{\mathbf{u}} = x^{\upsilon}$. Observe that the physical bundle $P^{\mathbf{u}}$ – its regions, fibers and points – does not exist without the fields $\bar{\phi}$ on which it supervenes. It formally realizes our paraphrasing of Einstein in section 4: the physical fibered manifold of spatio-temporal and internal events does not claim existence on its own, but only as a structural quality of the fields.

The physical enriched spacetime may then be defined as the pair $(P^{\mathbf{u}}, \omega^{\mathbf{u}})$, where $\omega^{\mathbf{u}} = \omega'^{\mathbf{u}} \oplus \varpi^{\mathbf{u}}$ for ω and ϖ Ehresmann and Cartan connections with local representatives A' and (A, e) respectively. It "projects" on the physical spacetime $(M^{\upsilon}, e^{(\upsilon,u)}/g^{\upsilon}, A^{(\upsilon,u)}, A'^{(\upsilon,u)})$, where field theory is usually written. This enriched spacetime thus realizes the " relationalized sophisticated substantival" view.

6 Conclusion

In this paper, we have outlined a framework for interpreting general-relativistic Gauge Field Theory (gRGFT) in which its covariance group of local transformations, $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$, is seen—via the articulation of the generalized hole and point-coincidence arguments—to tacitly encode its fundamental relational structure. Yet, the ontological picture presented here does not follow the canvas set by the traditional substantivalist versus relationalist dichotomy.

Rather, it strongly suggests a relationalist realist view of a principal bundle as representing the manifold of physical spatio-temporal and internal events, which supervene on—or supervene as a "structural quality" of—the fundamental physical fields of the theory. The d.o.f. of these fields are coextensive with the mutually co-defining network of their (invariant) relations (section 2 and 4). This nicely aligns with a "moderate" form of ontic structural realism regarding the fundamental fields of gRGFT; a form of OSR which, in contrast to French (2014), is not eliminativist regarding relata ("objects") and does not favor a primary ontology of relations (see Berghofer (2018); Ladyman (2023)) but regards relata and relations as inseparable and on equal-footing ontologically. This form of structural realism traces back to Eddington (1929), as noted by French (2003) and Rickles (2008), and has been further developed specifically in the context of fiber bundles under the label of "dynamic structural realism" by Stachel (2014).

Where we go beyond Stachel and others is by using the dressing field method (DFM), showing how dressed fields, bundles, and spaces are immune to any form of (generalized) hole argument. We indeed contend that the manifestly invariant and manifestly relational reformulation of gRGFTs via the DFM (section 2) is as close a technical realization of (moderate) ontic structural realist views as currently possible within the standard framework of differential bundle geometry (section 3). The dressed fields (11), (15), (25) and the dressed regions (17), (26) of the manifold of events M^{υ} and of the enriched manifold of events $P^{\mathbf{u}} \to M^{\upsilon}$, formally implement the generalized point-coincidence argument, and are by construction immune to the generalized hole argument. This, then, (re)opens the door for fiber bundle substantivalist views.

The "bundle-realism" elaborated here has several interesting interpretive implications, for example regarding the electroweak model (and the notion of spontaneous symmetry breaking), and the Aharonov-Bohm (AB) effect. See e.g. François and Ravera (2025) for a discussion of both. Indeed, the view elaborated above rests crucially on the insight of the generalized point-coincidence argument, and is then a completely local (in the field-theoretic sense) understanding of the physics of gRGFT. The AB effect is thus interpretable in a gauge-invariant local way, as a phenomenon of differential parallel transport *in the physical bundle space* $P^{\mathbf{u}}$ resulting from the local interaction of the physical matter field $\phi^{\mathbf{u}}$ and the physical electromagnetic potential $\omega^{\mathbf{u}}$—even though the effect is seen to be "displayed" effectively on the (quotient) space M^{υ} between the local representative fields $\varphi^{(\upsilon,u)}$ and $A^{(\upsilon,u)}$, which are "shadows" of the true play. The AB effect is thus not essentially different, and no more mysterious, than comparable parallel transport effects in GR inducing gravitational time dilation—such as the Shapiro effect, the gravitationally induced phase shift in photon, or even the Langevin twins effect—and resulting from the local interaction between physical matter and interaction fields ($\varphi^{(\upsilon,u)}/A^{(\upsilon,u)}$) and the physical gravitational potential (g^{υ}). Both types of effects can be understood in the same terms, and neither requires a form of "holism".[15]

Relatedly, while Healey (2007) took holonomies (i.e. integrals of potentials A along curves/loops in M) as a favorite candidate for the fundamental ontology of (gR)GFT, it is clear that on the view proposed here, holonomies are derived quantities, and—like the physical bundle space, its regions, points and curves—supervene on the fundamental relational ontology of physical (dressed) fields.

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[15]The term is used by, e.g., Healey (2007); Lyre (2004a); Nounou (2003) to denote a form of spatio-temporal non-locality or non-separability, and should not be confused with the better-known quantum variant often used in the context of entanglement.

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