# Can Singularities Be Avoided in Lorentzian-Euclidean Black Holes?

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**Abstract**

We critically examine the revised general relativity (GR) framework proposed by Capozziello, De Bianchi, and Battista [1, 2]. The authors introduce a Lorentzian-Euclidean Schwarzschild metric with a signature change at the event horizon ($r = 2M$), claiming that radial geodesics halt at $r = 2M$ with infinite proper time, avoiding the singularity at $r = 0$. We argue that their framework lacks physical justification, producing unphysical dynamics in the Lorentzian region ($r > 2M$), where the metric is identical to Schwarzschild. Their revisions violate fundamental GR principles—including the equivalence principle, energy conservation, geodesic well-definedness, and consistency with the metric's geometry—without empirical or theoretical grounding. Notably, their modified energy definition and geodesic equation yield an infinite proper time, contradicting GR's finite result. We address the potential defense that these violations are expected in a revised GR, demonstrating that their framework's deviations are ad hoc and undermine its validity as a physically meaningful extension.

## 1 Introduction

Capozziello et al. [1, 2] propose a revised general relativity (GR) framework featuring a Lorentzian-Euclidean Schwarzschild metric that transitions from a Lorentzian signature for $r > 2M$ to a Euclidean one for $r < 2M$, with a degenerate metric at the event horizon ($r = 2M$). They claim this metric, regularized via the Hadamard *partie finie* technique, is a vacuum solution of an extended GR and introduces "atemporality"—a mechanism where time becomes imaginary inside the black hole, preventing observers from reaching the singularity at $r = 0$. A key result is that radial geodesics halt at $r = 2M$ with infinite proper time, contrasting with standard GR's finite proper time crossing.

We argue that their framework lacks physical and theoretical justification, producing unphysical dynamics in the Lorentzian region ($r > 2M$), where the metric is identical to Schwarzschild. Their revisions violate fundamental GR principles: the equivalence principle, energy conservation, geodesic well-definedness, and consistency with the metric's geometry. These violations lead to an infinite proper time to reach $r = 2M$, an unphysical result not supported by the metric's geometry or field equations. We address the potential objection that these violations are expected in a revised GR, demonstrating that their ad hoc modifications undermine the framework's validity.

We use geometric units ($G = c = 1$) and focus on radial geodesics for an observer starting at rest at $r = r_i > 2M$.

## 2 The Authors' Framework

The Lorentzian-Euclidean Schwarzschild metric is (Equation 1 in [1]):

$$ds^2 = -\varepsilon\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2, \tag{1}$$

where $\varepsilon = \text{sign}\left(1 - \frac{2M}{r}\right) = 1$ for $r > 2M$, $\varepsilon = 0$ at $r = 2M$, and $\varepsilon = -1$ for $r < 2M$. The revised GR framework includes:

- **Degenerate Metrics (Section 3.3)**: The metric is degenerate at $r = 2M$, where $\det g_{\mu\nu} = -\varepsilon(r^2 \sin\theta)^2 = 0$, claimed to be permissible in an extended GR [3].
- **Signature Change**: The metric transitions from Lorentzian to Euclidean at $r = 2M$, with the Riemann tensor regularized (Section 3.1).
- **Modified Normalization (Section 2.2)**: The four-velocity satisfies $g_{\mu\nu}u^\mu u^\nu = -\varepsilon$ (Equation 14), unlike GR's $g_{\mu\nu}u^\mu u^\nu = -1$.
- **Energy Definition (Section 4.1)**: The energy is $E = -\varepsilon g_{\mu\nu}\xi^\mu u^\nu = \varepsilon^2\left(1 - \frac{2M}{r}\right)\dot{t}$, set as $E = \alpha\varepsilon^2$, with $\alpha^2 = 1 - \frac{2M}{r_i}$ (text before Equation 31).
- **Geodesic Equation (Section 4.1)**: For radial motion (Equation 26):

$$\dot{r}^2 = \varepsilon\left(\frac{2M}{r} - \frac{2M}{r_i}\right). \tag{2}$$

- **Atemporality (Section 4)**: Time becomes imaginary for $r < 2M$, and radial observers halt at $r = 2M$ with infinite proper time (Figure 3).

The authors assert this framework solves Einstein's equations without a stress-energy tensor at $r = 2M$ and avoids the singularity via atemporality.

# 3 Violations of Fundamental GR Principles

We evaluate the authors' framework, focusing on violations that produce unphysical dynamics in the Lorentzian region ($r > 2M$).

## 3.1 Equivalence Principle

The equivalence principle requires a non-degenerate metric to define local Minkowski frames [4]. The authors' metric is degenerate at $r = 2M$, where:

$$\det g_{\mu\nu} = -\varepsilon(r^2 \sin\theta)^2 = 0 \quad \text{(Equation 8)}, \tag{3}$$

violating this principle (Section 3.3). While the metric is non-degenerate for $r > 2M$, their $\varepsilon$-dependent energy and geodesic equation impose unphysical effects (e.g., halting at $r = 2M$) stemming from this degeneracy, affecting the Lorentzian region where no such deviation is warranted.

## 3.2 Energy Conservation

The Schwarzschild metric's time-translation symmetry yields a conserved energy:

$$E = -\xi_\mu u^\mu = \left(1 - \frac{2M}{r}\right)\dot{t}, \tag{4}$$

$$E = \left(1 - \frac{2M}{r_i}\right)^{1/2}, \tag{5}$$

constant along the geodesic [4]. The authors define (Equation 16, text before Equation 31):

$$E = \varepsilon^2 \left(1 - \frac{2M}{r}\right)\dot{t}, \tag{6}$$

$$E = \alpha\varepsilon^2, \tag{7}$$

where $E \to 0$ as $\varepsilon \to 0$ at $r = 2M$. This violates Noether's theorem for $r > 2M$, where the metric is Schwarzschild, and leads to:

$$\dot{r}^2 = \varepsilon\left(\frac{2M}{r} - \frac{2M}{r_i}\right), \tag{8}$$

$$\dot{r} \to 0, \tag{9}$$

unlike GR's:

$$\dot{r}(r = 2M) = -\sqrt{\frac{r_i - 2M}{r_i}}. \tag{10}$$

### 3.3 Geodesic Well-Definedness

Geodesics should remain well-defined in non-singular regions [5]. In GR, radial geodesics cross $r = 2M$ in finite proper time. The authors' equation yields:

$$\tau = \int_{r_i}^{2M} \frac{\mathrm{d}r}{\sqrt{\varepsilon\left(\frac{2M(r_i - r)}{r r_i}\right)}}, \tag{11}$$

diverging as $\varepsilon \to 0$. The Kretschmann invariant is finite at $r = 2M$ (Equation 24):

$$\mathcal{K} = \frac{48M^2}{r^6} = \frac{3}{4M^4}, \tag{12}$$

indicating no singularity, so this halt is unphysical and contradicts GR's coordinate-invariant crossing.

### 3.4 Consistency with the Metric's Geometry

GR dynamics follow from the variational principle. The authors' geodesic equation and normalization $g_{\mu\nu}u^\mu u^\nu = -\varepsilon$ (Equation 14) are ad hoc, with $\varepsilon^3$, $\varepsilon^4$ terms (Equation 15) not derived from the metric's geometry for $r > 2M$, violating the principle that dynamics reflect the spacetime structure.

## 4 Discussion

The authors' infinite proper time result relies on a revised GR framework that violates fundamental principles, producing unphysical dynamics in the Lorentzian region. The degenerate metric, non-conserved energy, halted geodesics, and ad hoc dynamics are not justified by the Schwarzschild-like metric for $r > 2M$, where GR predicts finite proper time.

The authors may argue that violations of GR principles are expected in their extended framework, designed to accommodate degenerate metrics and achieve atemporality (Section 3.3). However, any revised GR must be physically justified through empirical evidence, consistency with known physics, or rigorous derivation from first principles [6]. The authors' modifications lack such grounding:

- **Lack of Physical Motivation**: The modified normalization and energy are not derived from a modified action or field equations, unlike other modified gravity theories (e.g., $f(R)$ gravity [7]). Their regularization (Section 3.1) ensures a vacuum solution, but the geodesic equation (Equation 26) is imposed ad hoc, producing unphysical results in a region where the metric is standard.

- **Unphysical Lorentzian Dynamics**: The infinite proper time contradicts GR's finite prediction in a non-singular region ($r > 2M$). This deviation is not justified by the metric's geometry or empirical observations, undermining the framework's validity.

- **Theoretical Inconsistency**: The energy's variation violates Noether's theorem without a modified symmetry, and the geodesic halt at a finite Kretschmann invariant (Equation 24) lacks physical basis. Unlike GR's coordinate-invariant predictions, supported by observations (e.g., gravitational waves [8]), their framework has no experimental support.

These flaws indicate that the authors' revisions are not a coherent extension of GR but an arbitrary modification to achieve atemporality, sacrificing physical consistency in the Lorentzian region.

# 5 Conclusion

The revised GR framework in [1] produces unphysical dynamics due to unjustified violations of GR principles. The infinite proper time to reach $r = 2M$ is an artifact of ad hoc modifications, not a robust feature of a physically meaningful theory. We urge the authors to provide empirical or theoretical justification for their revised dynamics and reconcile their results with GR's predictions in the Schwarzschild region.

# References


[1] S. Capozziello, S. De Bianchi, and E. Battista, "Avoiding Singularities in Lorentzian-Euclidean Black Holes: The Role of Atemporality," *Phys. Rev. D*, **109**, 104060 (2024).

[2] S. De Bianchi, S. Capozziello, and E. Battista, "Atemporality from Conservation Laws of Physics in Lorentzian-Euclidean Black Holes," *Found Phys*, **55**, 36 (2025).

[3] G. T. Horowitz, "Topology change in classical and quantum gravity," *Class. Quantum Grav.* **8**, 587 (1991).

[4] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[5] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973).

[6] C. M. Will, "The confrontation between general relativity and experiment," *Living Rev. Relativ.* **17**, 4 (2014).

[7] T. P. Sotiriou and V. Faraoni, "$f(R)$ theories of gravity," *Rev. Mod. Phys.* **82**, 451 (2010).

[8] B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), "Observation of gravitational waves from a binary black hole merger," *Phys. Rev. Lett.* **116**, 061102 (2016).