Quantum Logics in Cognition: A Proposal

José Alejandro Fernández Cuesta*
Complutense University of Madrid
Department of Logic and Theoretical Philosophy
Plaza Menéndez Pelayo, 28040, Madrid, Spain
josealef@ucm.es 0000-0002-9637-9291

Michele Piazzai
The University of Edinburgh
University of Edinburgh Business School
29 Buccleuch Place, Edinburgh, EH8 9JS, Scotland
michele.piazzai@ed.ac.uk 0000-0001-9676-7422

Umberto Rivieccio
National University of Distance Education
Department of Logic, History, and Philosophy of Science
Paseo de la Senda del Rey 7, 28040, Madrid, Spain
umberto@fsof.uned.es 0000-0003-1364-5003

* Corresponding author

Abstract

Quantum logics are non-classical logics defined from the mathematical formalism of quantum mechanics. While they are conventionally used to model inferential processes in physics, their scope of application is potentially much broader. We argue that quantum logics can serve as a framework to model human cognition, as their semantics seem able to capture not only how people make inferences about quantum mechanics, but also how they reason in general. We begin by defining quantum logics from an algebraic perspective in a classical first-order setting. Next, we present findings from cognitive science that suggest these logics are apt to characterize human reasoning. We then consider how such a connection between quantum logics and cognition contributes to longstanding philosophical debates about the epistemological status of logic and the problem of adoption. Finally, we discuss how cognitive applications of quantum logics could advance our understanding of human psychology and even quantum foundations.

Keywords Quantum logic, quantum cognition, applied logic, adoption problem, philosophy of logic

Acknowledgments The authors are grateful to members of the Buenos Aires Logic Group at the National Scientific and Technical Research Council of Argentina and the University of Buenos Aires, and to participants of the Metis seminar at the National University of Distance Education in Madrid.

Statements and Declarations

Funding José Alejandro Fernández Cuesta received funding from a UCM-Santander grant (CT15/23). Michele Piazzai received funding from the Spanish Ministry of Science and Innovation (RYC2021-032325-I). Umberto Rivieccio received funding from a UNED-Santander grant (2023-PUNED-0052) and from the Spanish Ministry of Science and Innovation (PID2022-142378NB-I00).

Competing Interests The authors have no relevant financial or non-financial interests to disclose.

Availability of Data and Materials Not applicable.

Authors' Contributions The authors equally contributed to the conception and writing of this paper.

1 Introduction

The expression "quantum logic" can take on a variety of meanings. It can refer in a general and informal fashion to the distinguishing features of quantum mechanics, but also to particular interpretations of this theory (Griffiths, 2003), or more technically to the study of quantum-logical gates (Dalla Chiara et al., 2018). In this paper, we follow a tradition dating back to Birkhoff and Von Neumann (1936), and take quantum logic (QL) to mean a formal logical system, ideally equipped with syntax, semantics, and associated calculi (see also Stachel, 1976). Because there can be many such systems, each with its own semantic variations, we speak whenever appropriate of QLs in the plural. The purpose of this paper is to argue that, while QLs have been traditionally used to formalize inferential processes in physics, they can serve as a modeling framework beyond this domain. In particular, we consider the use of QLs in models of human cognition and examine philosophical consequences of this interdisciplinary application.

In launching their quantum-logical program, Birkhoff and Von Neumann (1936) were moved by similar aims. They wanted to generalize the mathematical formalism of quantum mechanics, which is essentially a non-classical theory of probability, to make it independent of physical phenomena it was originally meant to describe. In doing so, they hoped to solve problems connected to the interpretation of physical results, exposing them as problems not inherent to the theory but rather stemming from the use of classical logical tools. With this in mind, they set out to "discover" new logical structures that could play a role in quantum mechanics and potentially other physical theories intractable for classical logic. They identified suitable structures in the algebras arising from self-adjoint operators in complex Hilbert spaces. Therefore, they resolved "to find a calculus of propositions which is formally indistinguishable from the calculus of linear subspaces with respect to *set products*, *linear sums*, and *orthogonal complements*—and resembles the usual calculus of propositions with respect to *and*, *or*, and *not*" (Birkhoff and Von Neumann, 1936, p. 823). However, their research program was prematurely abandoned because of technical and formal difficulties (Dapprich, 2016), and interest in QL waned until the 1950s.

Around this time, the launch of new axiomatic projects in physics (Mackey, 1963) and the diffusion of philosophical frameworks like metascientific structuralism (Balzer et al., 1987) made QL intriguing again from the perspective of philosophers and physicists. Philosophers were looking for a standardized approach to quantum theory that would help them pursue ontological programs consistent with quantum results. At the same time, physicists sought a more rigorous understanding of quantum mechanics to further develop its mathematical formalism and clarify issues connected to its interpretation. The converging needs of these communities became especially urgent after hidden-variable accounts of quantum mechanics were rejected on formal and experimental grounds (Giuntini, 1991). The second half of the century hence witnessed a revival of QL, among other systems arising from philosophically grounded non-classical semantics, e.g., fuzzy, intuitionistic, paraconsistent, and non-monotonic logics (Cattaneo et al., 1993; Da Costa and Krause, 1994; Engesser et al., 2007b).

Today, QLs are actively studied but remain primarily concerned with modeling statements about physical systems, often translating them directly from natural language. For example, these logics would be used to formalize the statement that a photon has a particular spin, momentum, position, range of positions, etc. This focus on physical applications has remained practically unchanged since QL's inception: indeed, the state of a physical system as determined by experimental measurement is what Birkhoff and Von Neumann (1936) originally intended quantum propositions to express. The authors aimed to generalize the formalism of quantum mechanics to make it applicable to other theories, but the theories they had in mind as candidates for application were invariably physical. Their goal was "to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic" (Birkhoff and Von Neumann, 1936, p. 823).

In this paper, we take a different stance. We frame QLs as a general formalism where propositions do not concern a physical system *ab initio*, but more broadly a system whose behavior is non-classical. We define a standard formalism for QLs using classical first-order logic (FOL) as a metalanguage, which makes it easier to expand the scope of possible applications. To illustrate just how far this can extend beyond physics, we review recent developments in cognitive science, where quantum probability is at the core of a fast-growing research program (Pothos and Busemeyer, 2022), and consider the use of QLs to formalize cognitive processes. We report key results from field and laboratory experiments that are difficult or even impossible to reconcile with classical logic, whereas QLs could provide a unified modeling framework.

Our proposal to apply QLs outside physics has interesting philosophical implications. It is especially relevant to debates connected to the epistemology of logic, which focus on the question of whether logic is empirical. Birkhoff and Von Neumann's (1936) effort to extract a logic from quantum mechanics suggests this is the case, as the authors aimed to infer logical principles from empirical evidence. The philosophical discussions that followed, accompanying and motivating QLs, hinged precisely on the fact that these logics were "discovered" from the results of physical experiments (Bacciagaluppi, 2009; Dummett, 1976; Putnam, 1969, 1974). In his contributions to these exchanges, Putnam (1974) argued it is possible, if not outright necessary, to "think quantum-logically" about certain physical results. Thus, some classical logical principles on which people rely to make inferences in everyday life, according to classical logicians, should be unexpectedly suspended to reason specifically about propositions in quantum mechanics.

The question of whether logic is empirical can now be considered settled, owing to Kripke's (2024) oft-cited argument, dating back to 1974, that people are bound to think classically and it is impossible to depart from classical logic by adopting a non-classical one, including QL. This argument gives rise to what is known as the adoption problem (Birman, 2024; Boghossian and Wright, 2024) and stymied Putnam's (1974) proposal. In recent years, however, cognitive science suggested that the notion of quantum-logical reasoning is not so outlandish (e.g., Aerts, 2009). If it were the case that people think quantum-logically, and in a sense even more radical than envisioned by Putnam (1974), then the adoption problem could be readily settled: QLs need not be adopted because we already reason according to them. They are already part of our logical endowment, though we might be unaware of it, so the question of whether they can be adopted is moot. This gives us an opportunity to revisit longstanding epistemological issues and decouple them from questions that purely concern the behavior of physical systems. The prospect of deploying QLs to formalize cognitive processes, reopening debates that are widely considered settled, represents the most important contributions of this paper to the logico-philosophical literature.

Section 2 presents what we refer to as the standard formalism of QLs. We define the algebraic structure and the main logical connectives, among which implication requires special attention. We outline this formalism in a deliberately neutral way, using axiomatic definitions within a classical first-order setting, so as to leave the door open to philosophical positions aligned with either logical monism (Kripke, 2024; Suszko, 1976) or pluralism (Beall and Restall, 2000). Section 3 moves to cognitive science and reviews the core motivations and findings of the quantum cognition program, suggesting possible routes to applications of standard QLs. Section 4 delves into philosophical debates arising from QLs' interpretation: in particular, we discuss the contrasting positions of Putnam (1969, 1974) and Kripke (2024), hoping to reignite conversations about the empirical character of logic, the adoption problem, and the "quantumness" of logical languages built from physical results. Finally, Section 5 makes a case for applying QLs to human cognition and offers concluding remarks.

2 Standard QLs

2.1 Preliminaries

The relation between QLs and quantum mechanics can be easily understood in analogy to that between classical logic and Newtonian mechanics. In the classical case, any physical system maps to a point σ in a phase space Σ, with each $\sigma \in \Sigma$ representing a unique state of the system. We usually assume $\Sigma = \mathbb{R}^6$, so that σ is a real sextuple with three values for position and three for momentum components. A system's state can always be ascertained by experimental measurement, with pure states corresponding to maximal information about position and momentum. A classical experimental proposition p captures a natural language statement such as "momentum along the vertical axis lies in the $[0, 1]$ interval," or "position on the horizontal plane is exactly $(1.5, -3.6)$." These propositions are what Birkhoff and Von Neumann (1936, p. 828) termed "physical qualities." In the language of probability theory (e.g., Kolmogorov, 1933) they are more generally referred to as events. Every p is associated with a subset $x \subseteq \Sigma$ consisting of all the pure states in which p holds true. The powerset $\mathcal{P}(\Sigma)$ is taken to represent all the possible experimental propositions.[1]

[1] Because some states are practically indistinguishable from one another, we usually restrict $\mathcal{P}(\Sigma)$ to a subset with empirical meaning, such as the set of all measurable subsets of Σ. This

This classical setup enables an easy translation between physical measurements and logical propositions. We reserve, as usual, the symbols $\wedge, \vee$ for classical conjunction and disjunction, and $\cap, \cup$ for the associated set-theoretic operations of intersection and union. For quantum-logical operations, we introduce $\sqcap, \sqcup$ to represent the meet or infimum, which corresponds to set-theoretic intersection, and the join or supremum, which does not correspond to set-theoretic union. Finally, we use $\neg, \perp$ to represent classical and quantum negations, the latter being defined, as we shall see, by way of orthocomplementation. The set of all classical physical propositions p yields a Boolean algebra

$$\langle \mathcal{P}(\Sigma), \cap, \cup, \neg, \subseteq, 1, 0 \rangle.$$

We can choose any standard axioms for these operations: if we choose, as is common, Zermelo-Fraenkel set theory with the axiom of choice (ZFC), then the operations are based on classical logical connectives: $a \cup b \coloneqq \{x : x \in a \vee x \in b\}$, $a \cap b \coloneqq \{x : x \in a \wedge x \in b\}$, and $a \subseteq b \coloneqq \{x : (x \in a \rightarrow x \in b)\}$.[2]

We can switch seamlessly between the propositional logic associated with this structure and its physical interpretation. Moreover, logical propositions and their interpretation can be directly transferred into a Kolmogorovian probability space. There is always a balance in these transitions because all information and properties of operations are preserved. The law of excluded middle holds and pure states decide any event, because every proposition is associated with either the subspace x of the probability space or its complement.

What is the counterpart of this formalism in quantum theory? As explained by, e.g., Dalla Chiara and Giuntini (2002), QL is based on linear algebraic operations that constitute the main mathematical machinery of quantum mechanics. For a quantum system, pure states no longer represent events in which we have maximal information about the state of a system but rather probability distributions over such events.[3] Wave functions determine which events occur upon measurement, assigning truth values to experimental propositions. This raises a number of critical questions: (i) how to represent experimental propositions in this quantum setting; (ii) how to handle the transition between the physical interpretation of set-theoretic operations and their logical counterparts; (iii) whether information is lost in this transition, and if so, what this loss implies. With respect to (i), the answer was already provided by Birkhoff and Von Neumann (1936). In QL, experimental propositions correspond to closed linear subspaces of a complex Hilbert space $\mathcal{H}$, which replaces the classical phase space Σ. With regard to (ii) and (iii), the answer requires us to first introduce the algebraic structure that serves as the basis for a new propositional logic (Svozil, 1998).

distinction, however, is not relevant to our present purposes.

[2]Note that $\subseteq$ denotes an algebraic operation and not, as usual, a relation. This notation is conventional in the literature on QLs, including Foulis and Randall (1981), Svozil (1998), and Dalla Chiara and Giuntini (2002), among others.

[3]What these distributions mean exactly and what they represent philosophically are some of the thorniest issues related to the interpretation of quantum mechanics and QLs. However, the mathematical framework we are about to introduce, which is used to formalize them, is one of the few points largely agreed upon by philosophers of physics.

In the first paragraphs of their foundational paper, Birkhoff and Von Neumann (1936) emphasized that their effort to build a new logic was motivated not so much by the novelty of particular logical notions but rather by the fact that these notions were "presupposed" by a formal physical theory. Philosophically, this remains the key distinction between QLs and other non-classical logics: QLs were not constructed *ad hoc* to model an ideal situation or to study their properties as mathematical objects. Instead, their non-classical semantics were "discovered" within the natural interpretation of a formalism developed in response to the demands of experimental data.

Consider two pure states ψ_1, ψ_2 and their linear combination or wave function $\Psi = c_1\psi_1 + c_2\psi_2$, where $c_1, c_2 \in \mathbb{C}$. A system in state Ψ verifies with probability $|c_1|^2$ the experimental propositions associated with ψ_1 and with probability $|c_2|^2$ those associated with ψ_2. Suppose that both ψ_1 and ψ_2 assign probability 1 to a proposition p, and that Ψ is also a pure state. In this case, Ψ also assigns probability 1 to p and experimental propositions are read, under arbitrary linear combinations, as closed subspaces of $\mathcal{H}$ (Dalla Chiara and Giuntini, 2002). But how does one move from these physical states to their algebraic representations? The natural way is to take the set of all closed subspaces $x \in \mathcal{H}$ as the representation of all possible events and take the class $\mathcal{C}(\mathcal{H})$ as the domain for the quantum algebra. Now states are no longer points, as in the classical case, but complex functions acting as vectors over $\mathbb{C}$. Among these vectors we define addition, scalar product, and an inner product $\langle \cdot | \cdot \rangle$ which, following Svozil (1998), is a complex function in $\mathcal{H} \times \mathcal{H}$ such that:

1. $\langle x|x \rangle = 0$ iff $x = 0$.
2. $\forall x \in \mathcal{H}\, (\langle x|x \rangle \geq 0)$.
3. $\forall x, y, z \in \mathcal{H}\, (\langle x + y|z \rangle = \langle x|z \rangle + \langle y|z \rangle)$.
4. $\forall x, y \in \mathcal{H}, c \in \mathbb{C}\, (\langle cx|y \rangle = c\langle x|y \rangle)$.
5. $\langle x|y \rangle = \langle y|x \rangle^*$.
6. If $x_n \in \mathcal{H} \wedge n \in \mathbb{N} \wedge \lim_{n,m\to\infty} \langle x_n - x_m | x_n - x_m \rangle = 0$, then
$$\exists x \in \mathcal{H} \left(\lim_{n\to\infty} \langle x_n - x | x_n - x \rangle = 0 \right).$$

In the resulting algebraic structure, logical operations require a reading that is not always set-theoretic. In particular, negation is defined via *orthocomplementation*: if x is a subspace of $\mathcal{H}$, then its orthocomplement $x^\perp$ is the set of all vectors orthogonal to every element of x. Formally, $\psi \in x^\perp \equiv \psi \perp x \equiv \forall \psi' \in x(\langle \psi, \psi' \rangle = 0)$. Let $H_{x^\perp}$ be the closed linear subspace associated with $x^\perp$. We can introduce the quantum negation of x as the class $\{y : \langle y|x \rangle = 0 \wedge x \in H_{x^\perp}\}$ of subspaces $y \in \mathcal{H}$. Conjunction is defined by set-theoretic intersection: the conjunction of two propositions p, q corresponding to subspaces x, y is represented by the infimum or meet, $x \sqcap y$, given by the class $H_{x \sqcap y} := \{z : z \in H_x \wedge z \in H_y\}$. Disjunction is defined not by set-theoretic union but by the geometric operation of *span*. This is because the union of two closed subspaces is not necessarily a closed subspace. Hence, the disjunction of two propositions p, q is represented by the span or join $x \sqcup y$ of associated subspaces, which is not $x \cup y$ but rather the smallest subspace that includes $x \cup y$. The closure of the span is given by the class $H_x \sqcup H_y \coloneqq \{z : z = c_1 a + c_2 b : c_1, c_2 \in \mathbb{C} \wedge a \in H_x \wedge b \in H_y\}$. A closed subspace representing a system in $\mathcal{C}(\mathcal{H})$ can be an element of $H_x \sqcup H_y$ without belonging to either one of H_x and H_y, because $H_x \sqcup H_y$ includes not only vectors in the union of these two subspaces but also their linear combinations. This causes the failure of distributivity between disjunction and conjunction in the set of linearly closed subspaces of $\mathcal{H}$. Thus, in QLs, meet and join in general do not distribute over one another.[4]

In the classical case, the logical operations of conjunction, disjunction, and negation provide an adequate basis for logical connectives from which all other operations can be defined. In particular, they can be used to define a classical material conditional $\rightarrow$ which is needed to represent conditional statements. In QLs, the classical definition of implication in terms of negation and disjunction is not satisfactory. Indeed, how to represent conditionals is an open question in this literature (Dalla Chiara and Giuntini, 2002; Hardegree, 1974). For now, it suffices to note that we must start from the partial order relation underlying the definition of $\mathcal{C}(\mathcal{H})$. In the next section, we look at the algebraic structure obtained from the operations above, and then, following Dalla Chiara and Giuntini (2002), we consider some minimal criteria for a notion of quantum conditional.

2.2 Standard QLs and conditionals

To introduce QLs algebraically, we begin with a bounded lattice, impose some constraints to obtain an *ortholattice*, and then restrict distributivity to end up with an *orthomodular* lattice, which serves as the algebraic counterpart of $\mathcal{C}(\mathcal{H})$.[5] We begin by defining B as a poset, i.e., a set endowed with a partial-order relation $\sqsubseteq$, having top and bottom elements $1, 0$, and satisfying $\forall a \in B(0 \sqsubseteq a \wedge a \sqsubseteq 1)$. Any pair of elements in B has an infimum and a supremum, so the partial order corresponds to a bounded lattice. We define a new orthocomplementation operation $\perp$ to obtain the algebra

$$\langle B, \sqsubseteq, \perp, 1, 0 \rangle.$$

This is an ortholattice. For some authors (e.g., Dalla Chiara and Giuntini, 2002), an ortholattice is already sufficient to define a simple QL, termed *orthologic*.

Although it can be easy to work with an ortholattice, orthologic is not characteristically quantum because it cannot be extracted directly from the closed subspaces we use to represent physical systems. Moreover, we must introduce

[4] In recent literature, there have been attempts to reconstruct the formalism of quantum mechanics while retaining the classical meet and join operations, and therefore distributivity (Griffiths, 2003, 2014). Other authors took a different route, proposing to relinquish even more classical principles than distributivity, motivated by a heuristic approach to quantum mechanics. These proposals have been developed at the metalogical set-theoretic level (e.g., Krause, 2010), but they depart from the standard approach to QLs we take here.

[5] We do not consider Hilbert spaces of infinite dimensionality, but see Engesser et al. (2007a) for a suitable definition as quasilattices. We also note that an alternative way to introduce QLs is using the set of projectors over $\mathcal{H}$ isomorphic to the orthomodular lattice we define here.

a weaker form of distributivity to capture the behavior of the span operator. Therefore, we require $\forall a, b \in B \left(a \sqcap \left(a^{\perp} \sqcup (a \sqcap b)\right) \sqsubseteq b\right)$, and its equivalents:

1. $a \sqsubseteq b \Rightarrow b = a \sqcup (a^{\perp} \sqcap b)$.
2. $a \sqsubseteq b \Leftrightarrow a \sqcap (a \sqcap b)^{\perp} = 0$.

This makes the lattice orthomodular and, in general, non-Boolean. What we call "standard QL" is the propositional logic determined by this lattice. It is already clear at this point why it can be convenient to speak of QLs in the plural: we have not yet finished presenting the basic structure and we already have two logics. Even more reasons to use the plural will materialize soon, when we come to the issue of conditionals. As we shall see, any interpretation of conditionals we can obtain from the poset relation in $\mathcal{H}$ turns out to be anomalous in some way (Dalla Chiara and Giuntini, 2002; Svozil, 1998).

With this in mind, we are now in a position to define the quantum algebra analogous to the classical one introduced in Section 2.1:

$$\langle \mathcal{C}(\mathcal{H}), \sqcap, \sqcup, \perp, \sqsubseteq, 1, 0 \rangle.$$

Given this structure, or equivalently, given the isomorphic structure generated over $P(\mathcal{H})$ based on projection operators and their lattice-theoretic counterparts, we obtain the nondistributive orthomodular lattice. This allows us to represent the measurement of quantum properties like spin and to algebraically model experiments conducted with interferometers, double slits, or other devices.

It is worth remarking, as Randall and Foulis (1981, Example 9.7) previously did, that there are significant difficulties in providing a direct lattice-theoretic interpretation of certain operations required to capture quantum phenomena, such as the tensor product, which is needed to model entanglement. Dvurečenskij (1995) offered a valuable perspective on the technical discussions involved. Still, there exists a vast literature on this subject (Svozil, 1998) and formal tools for characterizing the tensor product within our quantum algebra do exist. The simplest among them is the following: assuming that each subspace in $\mathcal{C}(\mathcal{H})$ corresponds to a physical system and defines a lattice, we can use direct product between lattices as algebras and characterize the tensor product operation through the categorical product, in the category of lattices, just as horizontal sum would represent the categorical coproduct. First, we define a pasting operation between blocks, i.e., quasi-classical Boolean algebras we identify with subalgebras of the lattice. Second, we distinguish between local and global measures within the global lattice, so that every block is locally measurable but non-commeasurable blocks are not globally definable. Non-commeasurable blocks can then be grouped together or pasted into a larger propositional structure that is generally non-Boolean. This corresponds to what in quantum mechanics is obtained via the tensor product of Hilbert spaces.

Other difficulties arise with the introduction of a suitable conditional. Let us pause for a moment here to consider a philosophical nuance. It is common in QLs to try and force an interpretation of the conditional defined via the poset relation that somehow resembles an if-then clause in natural language. However, we do not find this desirable. In fact, such an interpretation is questionable even in classical logic. The paradoxes of material implication have been extensively discussed (Ajdukiewicz, 1956; Anderson and Belnap, 1975; Grice, 1989; Jackson, 1991), spurring the development of non-classical logics that aim to better capture the features of natural-language conditionals (e.g., Cooper, 1968). Yet we do not necessarily want standard QLs to model natural language. Recall that Birkhoff and Von Neumann's (1936) philosophical motivation was precisely that one "discovers" these logics, rather than building them for a purpose. Therefore, we strive to detach the conditional connective based on the partial order relation generated in the lattice from its interpretation as an implication.

With this in mind, the simplest way to introduce a conditional would be via the class $H_{x\to y} \coloneqq \{H_x \sqsubseteq H_y\}$. This conditional, however, is not truth-functional, and the classical inter-definition $\phi \to \psi \equiv \neg\phi \vee \psi$ does not hold. If we were to force this and define the conditional by $\phi \to \psi := \neg\phi \vee \psi$, we would not end up with a reasonable implication-type connective. Orthomodular lattices need not be even relatively pseudocomplemented: given two elements $a, b \in B$, a maximum d such that $a \sqcap d \sqsubseteq b$ does not necessarily exist in B. In light of this, Kalmbach (1983) proposed five possible binary algebraic operations $\curvearrowright$ satisfying $a \sqsubseteq b \Leftrightarrow a \curvearrowright b = 1$, each leading to a different logic. We view these as versions of the same standard logic, so we call them standard QLs. Among these operations, it has become common to introduce the Sasaki hook (Smets, 2001) as the most suitable candidate:

$$a \to b \coloneqq a^{\perp} \sqcup (a \sqcap b) \,.$$

We call two elements a, b of the orthomodular lattice *compatible* if and only if $a = (a \sqcap b^{\perp}) \sqcup (a \sqcap b)$ and the subalgebra generated by $\{a, b\}$ is a Boolean algebra. We then can add the conditional criterion: for a and b compatible, $a \to b = a^{\perp} \sqcup b$. Finally, we introduce a weaker version of the import-export principle: for a and b compatible, $c \sqcap a \sqsubseteq b \Leftrightarrow c \sqsubseteq a \to b$.

2.3 Interpretation problems

In standard QLs, elementary propositions can be informally read as natural language statements of the form "if we measure physical property A, we observe the result α" (Dalla Chiara and Giuntini, 2002; Stachel, 1976; Svozil, 1998). We call A an *observable* and define it as a self-adjoint operator in $\mathcal{H}$, i.e., a Hermitian matrix such that $\langle Ax|y\rangle = \langle x|Ay\rangle$. We call α an *eigenvalue* of A. We must remark that while this interpretation of propositions is widespread, it is not at all binding nor does it reflect the underlying motivations of standard QLs. An observable need not correspond to a physical measurement. This is often assumed to be the case, but only for historical reasons connected to QL's origin as the "logic of quantum mechanics" (Birkhoff and Von Neumann, 1936). We can carry out an analysis of the logical formalism without ever even settling on a conventional meaning for quantum propositions.

If we define propositions directly from the natural language we use to interpret quantum-mechanical results, we risk erroneously grounding them in experimental criteria, or worse, in purely psychological or phenomenological interpretations. For example, if we assert that "in a Mach-Zehnder interferometer, a photon does not follow any specific path" as an informal reading of particular properties of a quantum system, it becomes exceedingly difficult to construct an interpretation of the logical proposition. Such an endeavor would ultimately depend on contentious philosophical concepts, like intuition or phenomenology.

By contrast, in standard QLs we restrict ourselves to closed subspaces in $\mathcal{C}(\mathcal{H})$ as logical propositions or atoms, eschewing explicit natural-language readings. We share this position in the present paper. As noted before, we build standard QLs using no more than FOL, and neither QLs nor the underlying calculus of linear subspaces seem to provide an intuitive interpretation of the orthomodular lattice. The absence of a clear natural-language interpretation is one of the reasons why Birkhoff and Von Neumann (1936) spoke of logical notions presupposed by a mathematical formalism that waited to be "discovered." This led to the argument, to which we turn in greater detail below, that logic is an empirical discipline (Putnam, 1969). Indeed, clarifying what QL's presupposition involves is a suitable way to sketch the debate on whether logic is empirical.

For the purpose of interpretation, this agnostic approach seems not only convenient but also desirable. The original aim of Birkhoff and Von Neumann (1936) was to model inferential processes in physics that surprisingly presuppose non-classical notions, which is easier given a physical interpretation of QL's algebraic structure. However, our aim in this paper is different: we detach QLs from physics and recast them simply as non-classical logics linked to algebraic structures defined by axioms alternative to those of Boolean algebras. The development of these logics was historically motivated by physics, but the connection may end there. In this more general approach, propositions are formally represented as closed linear subspaces of $\mathcal{H}$ and remain strictly mathematical objects. This prevents interpretative issues that affect QLs' physical applications—for example, how to use closed linear subspaces within the definition of physical systems. Only if we ground QLs onto informal interpretations do we encounter an *ad hoc* characterization of non-classical semantics, as typically occurs in applications of non-classical logics. And only if we assume that the set $\mathcal{C}(\mathcal{H})$ reveals some privileged aspect of physical reality do we render the problem of the epistemological status of logic inherent to QLs. But taking $\mathcal{C}(\mathcal{H})$ rather than natural language as the propositional basis, and dissociating it from quantum mechanics, allows us to avoid both commitments. We find this approach especially appropriate given our intention to introduce QLs from FOL.

The important point above can be restated as follows: we face interpretative issues if we either use natural language as the foundation for quantum propositions or assume that, for some reason, the scope of QLs is restricted to physics. We sidestep these issues by characterizing QLs within FOL as non-classical logics based on a peculiar algebra, where distributivity is replaced by orthomodularity. These logics do not have any special relationship with physics. We limit the construction of QLs to the algebraic counterpart of the orthomodular lattice and avoid any natural-language interpretation. This is helpful because there are domains where QLs could be successfully applied and which have nothing to do with quantum mechanics. Only if we try and force a direct logical translation of natural-language statements—for example, by requiring that the satisfaction of a given property for a particular physical system correspond to some first-order sentence—do we incur interpretative problems. In our approach this does not happen, because the system and its properties are not simple elements of the new logic but complex formulas that already belong to FOL.[6]

When it comes to switching between the algebra and its interpretation, the greatest difficulties arise when trying to read basic operations through which logical connectives are defined. Some of these operations are set-theoretic and can be expressed directly in our chosen metalanguage, i.e., classical logic. For example, in FOL semantics, our quantum-logical conjunction $\phi \sqcap \psi$ would be the set-theoretic intersection of two subspaces, H_ϕ and H_ψ. Therefore, we can take it as an abbreviation of the first-order sentence: $\forall x\,((x \in H_\phi \wedge H_\psi) \leftrightarrow (x \in H_\phi \sqcap H_\psi))$. Other connectives, however, correspond to more complex geometric operations, such as orthocomplementation and span. Because these are not set-theoretic, they require us to extend the models in which we work. If we try to skip FOL and work directly with QLs, logical objects end up belonging to different model-theoretic levels and we have to make *ad hoc* decisions about how to adapt the language, as previously done to accommodate conditionals. By working with FOL at a metalevel, instead, we avoid this problem and the potential loss of information in the logical interpretation of the algebraic structure.

We remark that the fact $\mathcal{C}(\mathcal{H})$ can be interpreted as the algebraic characterization of a non-classical logic does not mean we must take it as some kind of logical atom or indefinite element. Its geometric interpretation over the field of $\mathbb{C}$ in Birkhoff and Von Neumann (1936) well illustrates this point. The meta-theory on which we build $\mathcal{C}(\mathcal{H})$ over $\mathbb{C}$ is ultimately the standard meta-theory consisting of FOL and ZFC. It is classical, not in an Aristotelian but in a Fregean way. We introduce algebraic operations that give rise to non-classical logical connectives, but in our view, these operations merely serve to abbreviate classical sentences. Perhaps this was not readily apparent one hundred years ago, when Birkhoff and Von Neumann (1936) introduced QL, because the mathematical tools upon which said operations depend were not very well understood. We are now aware that the application of an operator that rotates a quantum state is logically classical in all of its terms: the set of projectors over $\mathcal{H}$ is isomorphic up to our algebra based on $\mathcal{C}(\mathcal{H})$. The same holds for orthocomplementation and span, which remain within the operations of the probability space. In our view, it is crucial to recognize not only that classical logic is present at a metalevel (Finn, 2021), but also that quantum-logical operations are, in a sense, classical.

To be more specific, they are classical in that we define them as algebraic operations interpreted geometrically over linearly closed subspaces of $\mathcal{H}$ using nothing but classical logic and proceeding axiomatically. They may be considered non-classical because we allow the definition of nondistributive semantics. This is the root of the purported tension between QLs and classical logic that motivates

[6] In this sense, we would be situated within the models of any standard set theory, no matter which one we choose and whether we state it in second order to avoid axiomatic schemata.

the debate on whether logic is empirical (Putnam, 1969; Bacciagaluppi, 2009; Kripke, 2024). We believe this tension rests on this dual nature of quantum connectives: classical in the sense that they are reducible to first-order axioms, and non-classical or "quantum" in the sense that they are presupposed as elementary operations within the quantum formalism.

This claim is highly relevant to a philosophical debate involving Hjortland (2017), Williamson (2018), and Horvat and Toader (2024). For Williamson (2018), the rejection of classical mathematics, including Kolmogorovian probability theory, and of the classical logic that governs it may have internal or external motivations. In QLs, the motivation is external as classical mathematics fail to explain empirical results. An interesting exchange begins when the author attends to Hjortland's (2017) "isolationist strategy" (see also Field, 2008), according to which the principles of classical logic are embedded in mathematics, but domains that are not mathematized or mathematizable call for non-classical logics. The possibility of isolating these domains, at least theoretically, gives the strategy its name. But for Williamson (2018, p. 401) this strategy is "too optimistic about the prospects of isolating mathematics from logical deviance in non-mathematical discourse. [Isolationists] overlook the capacity of pure mathematics to be *applied*." He offers a few examples, such as the use of quantifiers in non-mathematical contexts, and then considers the possible isolationist response of limiting classical logic to pure mathematics. In the end, however, he finds that "[this] lazy strategy does not work: mathematics must be redeveloped from scratch within the non-classical framework" (Williamson, 2018, p. 415).

A response to this argument against non-classical logics, QLs included, can be found in Horvat and Toader (2024). These authors argue that there is no tension between the application of classical mathematics in a domain, e.g., quantum mechanics, and the use of non-classical logics in that domain. Such a tension could, for Williamson (2018), escalate to an inconsistency, forcing proponents of non-classical logics to either rebuild mathematics on non-classical foundations or give up classical mathematics entirely. But Horvat and Toader (2024) point out that the tension is only apparent and the escalation can be avoided, because the mathematical tools commonly applied in quantum settings, which follow classical logic, and the formalism of QLs are mutually compatible. This resonates with our claim in this paper that QLs can be defined using no more than FOL as a metalanguage, although Horvat and Toader (2024) stop short of admitting homogeneity between the two formalisms at a metalevel.

Our main point of divergence from Horvat and Toader's (2024) position is that precisely this compatibility generates interest in the relationship between QLs and classical logic. We add nuance to Williamson's (2018) argument by suggesting that the tension between the application of classical mathematics and the use of QLs is rooted in something deeper than just applied logic, and should be traced back to Birkhoff and Von Neumann's (1936) claim to a discovery. To Horvat and Toader's (2024) response, instead, we add that the compatibility of classical logic and QLs is the source of genuine philosophical surprise. If they were incompatible, we would be simply dealing with logical frameworks competing for quantitatively or qualitatively better applications.

In light of this debate, the present paper takes a further and somewhat more dramatic step. Having introduced QLs from a classical first-order setting, we entertain the possibility of using QLs to formalize inferential processes outside the domain of physics. More concretely, we focus on processes pertaining to human cognition. Such an interdisciplinary application is feasible in principle because we did not give $\mathcal{C}(\mathcal{H})$ any physical interpretation. This leads to a refutation of the working hypothesis, embraced by Williamson (2018) and shared by Horvat and Toader (2024), that we can "switch interchangeably between a logic's origin and its intended domain of application, as the two typically coincide" (Horvat and Toader, 2024, p. 2). Indeed, QLs were originally meant to formalize physical phenomena but we now consider applying them to something radically different. We also note that if QLs could be used to formalize cognitive processes, their application would not be limited to domains that are non-mathematized or non-mathematizable, as Hjortland's (2017) isolationist position suggests. We return to this point later: for the moment, it suffices to note that QLs, as abstract and domain-independent formalisms, are compatible with the mathematical tools commonly used in quantum mechanics, formalized via classical logic, regardless of differences in their motivations and origins.

3 Thinking quantum-logically

3.1 Quantum cognition

We mentioned that, in Williamson's (2018) view, the rejection of classical logical principles in QLs is motivated by an empirical fact, i.e., classical mathematical tools are unable to describe the behavior of some physical systems. We also mentioned that what distinguishes QLs from other non-classical logics is the fact that it was not deliberately built but "discovered." Birkhoff and Von Neumann's (1936) opening paragraphs emphasized not just the novelty of QL but also its necessity to quantum mechanics. In this sense, QLs were peculiar from their very inception, being formalisms that no one expected to find at the core of our best physical theory. This makes the question of whether logic is empirical (Putnam, 1969) philosophically legitimate. Indeed, QLs are motivated by the observation or "discovery" that the classical law of distributivity fails.

Though the debate on whether logic is empirical is considered settled, we contend it could be reopened. This would be especially germane if one happened to find a new domain of application that is far removed from quantum mechanics but where empirical results nonetheless point to the failure of distributivity. We could be surprised to learn that beneath these results lies a non-classical logic, just as Birkhoff and Von Neumann (1936) were surprised to find non-classicality at the core of a physical theory. The application of QLsin this new setting would support the argument that pure logic has an empirical character, and hopefully renew philosophers' interest in a substitutive project that replaces distributivity with a weaker law—orthomodularity—as envisioned by Putnam (1969, 1974). Interestingly, such a domain may have been recently identified in cognitive science (Aerts et al., 2013; Busemeyer and Bruza, 2012; Pothos and Busemeyer, 2022; Wang et al., 2013). Not only is this far removed from quantum mechanics, it is also surprisingly close to everyday experience.

The identification of this promising new domain was not happenstance. The "unreasonable effectiveness" (Wigner, 1960) with which quantum mechanics explain the behavior of physical systems caused many to wonder over the years whether similar effectiveness could be achieved by redeploying the underlying formalism to other fields of science. This is not just a philosophical whim: from a technical standpoint, non-physical applications of the quantum formalism have always seemed feasible, even to the founding fathers of quantum mechanics (Bohr, 1958; Schrödinger, 1944). Indeed, by the time it was apparent that quantum theory would be empirically successful, Von Neumann's (1932) axiomatization had already made this theory independent of physical phenomena. These axioms made it clear to physicists that what they had discovered was essentially a new theory of probability, different from the classical one (Kolmogorov, 1933) but just as broad in its scope. Since then, the interdisciplinary applications proposed for the quantum formalism have been many and extremely diverse, especially in the social sciences (Khrennikov, 2010; Wendt, 2015). Among these, perhaps none attained as high a degree of legitimacy and mainstream popularity as psychological research on quantum cognition (see Pothos and Busemeyer, 2022, for a review). We discuss this research program here as a testament to QLs' potential to apply far beyond the conventional domain of physics.

We underscore that our proposal to apply QLs to formalize cognitive processes is novel. Until now, interdisciplinary applications of the quantum-mechanical formalism, in cognition and elsewhere, have only ever been attempted with the raw mathematical formalism. Therefore, while the probability theory at the heart of quantum mechanics has been successfully exported to non-physical domains (see, e.g., Pothos and Busemeyer, 2013), the *logic* of quantum mechanics has not. To our knowledge, quantum-logical operations outside physics have never been explicitly outlined, nor have their philosophical consequences for the debates mentioned in the foregoing section ever been explored.

3.2 Motivations

Quantum cognition is a psychological research program that relies on quantum probability theory as a framework to better understand human rationality (Pothos et al., 2017). This contrasts with most research in psychology, which relies on classical notions of probability (Chater and Oaksford, 2000; Oaksford and Chater, 1998, 2007). We reiterate, however, that quantum cognition does not involve the use of QLs as an alternative to classical logic, and unlike quantum probability, QLs remain foreign to cognitive science. We find this surprising for several reasons. First, logic is a discipline explicitly devoted to the study of rational inference and its application to cognitive processes is widely acknowledged as valuable (Fodor and Pylyshyn, 1988). Second, there is a natural connection between quantum probability and QLs as they are based on the same algebraic structure. Indeed, the probabilistic interpretation of quantum mechanics, grounded in Born's (1926) rule and Gleason's (1957) theorem (see Pitowsky, 2006), is based on the set of projectors isomorphic to $\mathcal{C}(\mathcal{H})$. Various factors may have contributed to QLs remaining outside the toolbox of cognitive scientists, even as the currency of quantum formalisms increased, but we suspect that the historical styling of QLs as "logics of quantum mechanics" due to Birkhoff and Von Neumann (1936) contributed to making these logics less visible and interesting.

Even without QLs, cognitive psychologists have been increasingly witnessing the power of quantum theory to illuminate aspects of human rationality and inference that proved impervious to classical explanations. Before research on quantum cognition took off, there were two main routes to modeling cognitive processes. The first, termed rational analysis (Oaksford and Chater, 1998), builds on classical mathematical principles to characterize human thought processes. According to this view, reasoning boils down to a probabilistic calculus that is rational to the extent it conforms to Kolmogorov's (1933) axioms. This includes, for example, the law of total probability, which follows from distributivity.[7] This approach is attractive because it enables the formalization of cognitive phenomena on the basis of well-understood mathematical rules. However, there is a price to pay in that rationality becomes computationally demanding, which does not sit well with the longstanding psychological intuition that humans are cognitive misers and their capacity for rational inference is limited (Simon, 1990). This motivates the second route, known as the heuristic approach (Gigerenzer and Gaissmaier, 2011; Kahneman et al., 1982), which denies that rationality means conformity to a probabilistic calculus and considers inference "rational" if it is successful in everyday life (Gigerenzer, 2000).[8] In this view, people do not abide by the rigorous calculus of a formal theory. Rather than seeing demanding computations through, they resort to cognitive shortcuts and rules of thumb, making inferences that are probabilistically incoherent but fast, easy, and usually good enough (Gigerenzer and Goldstein, 1996; Kahneman, 2011).

This second approach has many merits. It explains why human thought processes sometimes violate classical rules of inference (e.g., Tversky and Kahneman, 1983), which from the perspective of rational analysis can only be seen as a failure of reasoning. It also echoes Simon's (1990) idea that rationality, in the classical sense, is rendered impractical by cognitive limits to which humans are subject (Gigerenzer and Goldstein, 1996; Kahneman, 2003). As an alternative to rational analysis, however, the heuristic approach can be criticized for being *ad hoc* and difficult to formalize.[9] It is not very parsimonious either, as dozens of cognitive shortcuts were proposed over time to account for various kinds of deviations from classical rationality. A model of cognition that makes so many provisions can feel uncomfortable. This is where quantum theory comes in, as a framework for cognitive modeling that is mathematically grounded, like rational analysis,

[7] Kolmogorovian probability is bound to Boolean algebras and thus to classical logic.

[8] From a logical point of view, this parallels the pragmatic turn and its criticism of classical logic's application to ordinary discourse, which led to the development of non-classical logics like Cooper's (1968).

[9] The application of logic to the heuristic proposal can only be effective on a case-by-case basis. Moreover, it will be constrained by issues pertaining to linguistic relativity.

but capable of accommodating deviations from classical principles (Busemeyer et al., 2011). These are explained by quantum phenomena like incompatibility, contextuality, entanglement, superposition, and interference.

The quantum approach to cognition aims to explain how people draw inferences by translating cognitive tasks into computational procedures, positioning itself at what cognitive scientists term the "algorithmic level of analysis" (Love, 2015; Marr, 1982). Crucially, no claim is made as to how these procedures are implemented in the brain. In particular, there is no assumption that quantum-mechanical interactions take place anywhere in neural substrates. The approach only presumes that experiments performed on a cognitive system, e.g., questions presented to a human subject in a laboratory, can be represented as observables in a Hilbert space just like measurements in quantum mechanics. To emphasize how different this is from actual quantum mechanics, where observables are given a physical interpretation, cognitive applications of the quantum formalism go by different names, such as generalized quantum models (Atmanspacher et al., 2002) or quantum-like models (Khrennikov, 2010). Over time, it has become common to group them under the header of quantum cognition (Pothos and Busemeyer, 2022). In what follows, we discuss in greater detail how this research program helped cognitive psychologists overcome persistent empirical problems. In doing so, we want to outline what it would mean to "think quantum-logically," not in the limited sense of how to make inferences in quantum mechanics (Putnam, 1974), but in the broader sense of how to make inferences in ordinary life.

3.3 The case for QLs

There are several reasons to consider quantum formalisms, including QLs, suitable for cognitive applications. In coming up with quantum mechanics, physicists had to grapple with new and unfamiliar ways to think about probability, abandoning the set-theoretic axiomatization of Kolmogorov (1933), which remains standard in Newtonian mechanics, for a new axiomatization based on self-adjoint operators (Von Neumann, 1932). An analogous development is underway in cognitive science, where quantum probability was hailed as a solution to problems that have long beset the discipline (Pothos and Busemeyer, 2013). This approach has been empirically successful, as we will see, but even before starting to yield promising results it received endorsements from the physics community. Bohr (1958) famously argued that quantum theory could be relevant to cognitive processes: the very concept of incompatibility, which he introduced to physics in 1928, may have originated in psychology some 40 years prior (Holton, 1970). Physicists' familiarity with mathematical formalisms enabled them to translate this concept into a formal theory, and in this more rigorous form, the concept is being reintroduced to cognitive research (Wang and Busemeyer, 2015).

The link suggested by Bohr (1958) between quantum theory and cognition did not only inspire psychologists, however. Even in physics it sowed new perspectives that eventually led to a deeper understanding of quantum foundations. Indeed, some of the key constructs of quantum mechanics, such as quantum states, notoriously lend themselves to different interpretations. One of these, which captures the imagination of lay audiences and physicists alike (Von Baeyer, 2013), holds that quantum states are a purely psychological construct. They do not objectively exist: instead, they should be taken to represent beliefs held by an experimenter about the outcomes of potential measurements. In a nod to psychology, proponents of this interpretation claim that "the physical law that prescribes quantum probabilities is indeed fundamental, but the reason is that it is a fundamental rule of inference—*a law of thought*" (Caves et al., 2002, p. 5). Human cognition, it seems, offers extraordinarily fertile ground for applications of QLs. These logical systems are uniquely suited for the analysis of inferential processes in a quantum setting. There are, in particular, five characteristics of human cognition that are difficult if not impossible to handle for classical logic or probability theory, but which fit very naturally in a quantum framework. We explain them below and provide examples from empirical literature.

3.3.1 Incompatibility

To begin with, human inference is subject to order effects (Wang et al., 2014). We can ask questions of a cognitive system, much like we can measure the position and momentum of a physical system, only to find out that the answers depend on the sequence in which questions were asked. Suppose we asked a college student whether she would like to go on a trip with her friends during the weekend, and immediately afterward asked whether she feels ready for an exam coming up next week. The answers she gives to this sequence of questions could differ from those she would give if we had asked first about the exam and then about the trip. This order dependence is the mark of incompatibility, i.e., non-commutativity of observables. As with measurements of position and momentum, events corresponding to possible answers to some pairs of questions are impossible to represent in the same probability space, so they cannot be observed simultaneously. They can only be observed sequentially, and there is no reason why different sequences should yield the same outcomes.

Order effects seem widespread in cognitive processes. In an oft-cited survey conducted by Moore (2002) about the perceived honesty of two politicians, Bill Clinton and Al Gore, people were more likely to answer yes to the question of whether Clinton is honest if they were previously asked the same question about Gore, compared to the case in which the Clinton question came first. This survey made ripples in psychology because of its societal implications: when interviewing subjects, pollsters seldom consider the order in which they ask questions, but the order they choose could end up influencing elections. Similar effects have been documented in the evaluation of medical evidence by doctors (Bergus et al., 1998) and legal evidence by members of a jury (Kerstholt and Jackson, 1999). It is unsettling to think that one can receive a different diagnosis or sentence depending on the order in which documents are arranged in a pile, but such seems to be the nature of human cognition.

Classical logic and probability theory cannot easily account for order effects because they can only represent sequential measurements in the same probability space through set-theoretic intersection, which is commutative. As a result, different sequences return the same outcomes. It could be possible to accommodate order effects in classical theory by adding events to the Boolean algebra that obtain only in the case of particular sequences, but this makes the algebra exponentially larger and unlikely to be useful for predictive purposes. Quantum formalisms, instead, elegantly account for order effects by representing sequential measurements as the product of self-adjoint operators, which do not generally commute. In QLs, non-commutativity is allowed by the orthomodular structure of the algebra, which may include various Boolean subalgebras where events behave classically, but events corresponding to the outcomes of different questions are not necessarily in the same subalgebra. Therefore, compatibility is possible but not required. This alone would be a compelling reason to argue that QLs can model human reasoning better than classical logic.

3.3.2 Contextuality

Human thought processes can be extremely sensitive to features of the environment, or context, in which they unfold (Bruza et al., 2023). In other words, asking someone the same question in different situations can lead to different answers. This sensitivity, termed contextuality, is a key principle of quantum mechanics, rooted in the fact that physical systems interact with an experimenter's measurement instruments. As a result of this interaction, the mere act of measuring changes the state of a system, making the state before measurement fundamentally unknowable. This behavior is consistent with psychological research on the constructive nature of human beliefs and attitudes (Lichtenstein and Slovic, 2006; Schwarz and Bohner, 2001). There is much empirical evidence that the way people respond to experimental questions depends on font (Oppenheimer and Frank, 2008), formatting (Alter and Oppenheimer, 2008), and other seemingly trivial factors that can be considered part of a psychologist's measurement apparatus. Changes in responses that are due to manipulation of these factors indicate that people form opinions on the fly, based on whatever information is available at the time, as opposed to having underlying opinions that are simply "read out" during measurement.

Evidence of contextuality can be found in Hatchett and Schuman's (1976) study of White Americans' attitudes toward African Americans. This revealed that subjects tend to express more positive opinions of African Americans when the interviewer is Black rather than White. In principle, this could mean the subjects are racist and feel greater pressure to disguise their beliefs before someone who is directly offended. It turns out, however, that including the names of popular African Americans such as Oprah Winfrey or Michael Jordan in a list of individuals shown to subjects in the course of an unrelated task is sufficient to elicit more positive responses (Bodenhausen et al., 1995). This occurs even if subjects stay completely anonymous when expressing their opinions. Changes in their reported attitudes can be attributed to exposure to the names of well-liked public figures, which changes the information they access when formulating a response. Thus, experimental questions do not bring to light stable properties of a cognitive system, or beliefs that already exist and simply wait to be revealed, but properties that emerge through the very act of measuring.

Models based on classical logic and probability theory cannot explain why context exerts a constructive influence, because from a classical standpoint asking a question does not create information but merely records it. Classical measurements detect properties of a system, so-called hidden variables, that are assumed to be already there, even though there might be uncertainty about them. In Section 2, we acknowledged this feature of classical models by saying that, in these models, pure states are points in a phase space that involve maximal information and decide all experimental propositions. In a quantum model, however, asking a question both creates and records information. Pure states are vectors in a Hilbert space and do not specify the values of variables but probability distributions over them. Specific values obtain if and when their measurements are taken, but never exist otherwise.

Contextuality is not only a key feature of quantum probability but also of QLs, where the possibility of hidden variables was ruled out very early by Von Neumann (1932). Indeed, "the denial that there are hidden variables is fundamental to quantum logic" (Stairs, 1983, p. 578). Hence, QLs are better suited than classical logic to capture the constructive nature of inferences. The choice to model cognitive states as vectors in a Hilbert space rather than points in a phase space also resonates with current approaches to knowledge representation and natural language processing, which characterize meaning by way of semantic spaces. These are effectively Hilbert-space models (Bruza and Cole, 2005).

3.3.3 Entanglement

Psychological research finds it extremely challenging to study cognitive systems by decomposing them—that is, by reducing them to constituent parts or subsystems to be examined in isolation. For example, it is difficult to decouple perception from memory (Barsalou, 2008, 2010), or to activate particular areas of memory without simultaneously activating others, even if the information they store is distant and unrelated (Bruza et al., 2009). Similar issues of non-decomposability arise in quantum mechanics: a photon passing through a beam splitter breaks down into two parts, of which one is reflected and the other transmitted, but it is impossible to analyze these parts independently. Measurements performed on the one immediately affect the other, which is surprising, especially if in the meantime these parts traveled far away from each other. Einstein et al. (1935) dubbed this phenomenon "spooky action at a distance," and expected the sheer absurdity of it to disprove quantum mechanics. But as was subsequently shown, first theoretically (Bell, 1964) and then empirically (e.g., Aspect et al., 1981, 1982), subsystems of a quantum system can be bound by nonlocal correlations that are forbidden in classical models and which can produce action at a distance. These correlations are referred to as entanglements.

If parts of a physical system are entangled, analyzing them as if they were independent makes the system's behavior appear inexplicable. Like photons, cognitive systems consist of entangled subsystems, as evinced by studies on word associations. It was found, for example, that mentioning the word "trout" causes people to think about trouts, but also about related concepts like fishing and fish in general (Nelson et al., 2004). Likewise, mentioning "right" brings to mind the concept of right, but also related concepts of correct, left, and wrong. Nelson et al. (2003) mapped thousands of such free associations to test two competing hypotheses: one based on the classical assumption that correlations are local, and another allowing nonlocality. The local hypothesis requires human thought to travel along word-to-word links and to do so more slowly when links are indirect, because there are related concepts in the middle. The nonlocal hypothesis, instead, allows thought to traverse associated words in synchrony and without going through links, i.e., at a distance, as a result of which indirect links matter just as much as direct ones. Experimental results supported the nonlocal hypothesis, suggesting that concepts are entangled.

Classical logic does not deal well with this evidence. If the algebraic structure of experimental propositions is Boolean, correlations can only be local (Pitowsky and Svozil, 2001). In contrast, no provision is made about locality in QLs, where the absence of distributivity makes the structure generally non-Boolean. This is because experimental propositions form an orthomodular lattice, not a distributive one. In Section 2.2, we introduced an operation that pastes together distributive sublattices or blocks into a single orthomodular structure, where distributivity no longer applies. This operation allows us to model the tensor product of Hilbert spaces, on which physicists and cognitive scientists rely to model entangled quantum systems (Aerts, 2009; Aerts and Sozzo, 2014; Sozzo, 2015). While locality remain possible in blocks, which correspond to Boolean subalgebras, it is not required everywhere. This makes a rather strong indictment of classical logic. It is not just that QLs can formalize cognitive processes in a simpler and more natural fashion; rather, some cognitive processes can be formalized only by QLs because they are classically intractable.

3.3.4 Superposition

Much like quantum processes in physics, everyday human inference can be fundamentally and inherently random (Busemeyer et al., 2020). Both classical and quantum models allow for randomness, but in classical ones, randomness is only epistemic. It reflects lack of knowledge about a system's true state, which is at any rate assumed to be definite. In principle, epistemic randomness could be completely eliminated by increasing the precision of measurement instruments, making a classical system deterministic. This is impossible in quantum systems because randomness is connatural to a quantum state and will be part of the system for as long as this is undisturbed by measurement, no matter how precise the instruments. If we fire a photon gun at a screen, for example, the photon hits the screen at a random location, but this is not merely because we are uncertain about its trajectory. Rather, the photon behaves as a wave that is everywhere with different amplitudes. It only behaves as a particle when it hits the screen, but before then, its position and therefore its state are indefinite.

It turns out that eliminating randomness is not possible in cognitive systems either. Picture someone considering a choice between two attractive options, say, two good restaurants or two lucrative investment opportunities. In one scenario, this person is explicitly asked what her choice is, and some time later she is asked how confident she is a given option is superior. In another scenario, she is only asked to rate her confidence in one of the options, without being previously asked to make a choice. This is the general setup of an experiment performed by Kvam et al. (2015), who found that people's confidence in a particular option differs across the two scenarios. In other words, asking people to make a choice affects the degrees of confidence they subsequently report.

This behavior cannot be explained from a classical perspective because the dynamics of classical systems require a process of evidence accumulation in favor of one or another option to follow a particle-like evolutionary path. At any point in time, a person may be asked if she favors one option or the other, but this should have no impact on the future trajectory of her beliefs. Her current level of confidence in a given option is simply read out from the underlying state of her cognitive system, i.e., its location along the trajectory, which may be unknown and thus epistemically random but is nonetheless definite. From a classical perspective, if a projectile is launched against a target, photographing it while it is in flight does not change where it eventually lands.

This is different in quantum mechanics, where a trajectory simply does not exist. A process of evidence accumulation will not yield a definite level of confidence about an option that can be read out, but a distribution over many levels of confidence. One of these will realize, just like rolling a die will allow only one of its faces to show up. In the terminology of quantum theory, this indefinite state of the system is referred to as a superposition. Asking the subject what her current confidence is by instructing her to make a choice, as in Kvam et al.'s (2015) first scenario, forces the superposition to collapse to a definite state, and if the process of evidence accumulation continues, the system evolves from there. The superposition in which it previously was is destroyed. If the subject is not asked to make a choice, however, the superposition persists. This is why the second scenario results in different levels of confidence.

The capacity to represent this special kind of uncertainty makes quantum formalisms uniquely suited to represent cognitive processes in which people feel ambiguous or conflicted about a prospective choice, but suddenly become confident once a choice is made. In QLs, superposition is represented via the join operation, which yields the closure of the span of subspaces corresponding to alternative outcomes. This includes not only the union of these subspaces, but also the space generated by linear combinations of vectors belonging to said subspaces, i.e., superposed states. No analogous representation is possible in classical logic, where the join corresponds to set-theoretic union.

3.3.5 Interference

Last, but not least in importance, cognitive processes cannot be considered rational in a classical sense because they appear to violate axioms of classical probability (Busemeyer et al., 2011). The rules of inference prescribed by classical axioms, e.g., Kolmogorov's (1933), are so often violated in psychological experiments that "irrational" behavior has been argued to be the norm rather than the exception (Gigerenzer and Goldstein, 1996; Kahneman et al., 1982). One of the most frequently cited examples involves alleged violations of Savage's (1954) sure-thing principle, one of the tenets of classical decision theory. This holds that, if an outcome is preferable in a certain state of the world and it is also preferable in the complementary state of the world, then it should be preferable if the state of the world is unknown. Consider again our earlier example about a college student and her exam. Suppose the student already took the exam but does not know yet if she passed or failed, and now she is being offered the opportunity to buy tickets for a vacation at a discounted price. The offer expires soon, so the student must decide quickly. Classical logic would lead us to expect that if she would buy the tickets knowing she passed and knowing she failed—though possibly for different reasons—then she should buy the tickets even if she does not know the exam's result.

Tversky and Shafir (1992) analyzed this hypothetical situation in experiments with actual college students and found that most would buy the tickets knowing they passed and knowing they failed, but not if they do not know. The same behavior was observed in a two-stage gambling game where players were instructed to make a first bet and then asked whether they would make another. Before making this decision, some players were told that they won the first bet, others were told that they lost it, and others yet that the result was still pending. Most players opted to make a second bet knowing the outcome of the first, whatever this would be, but not if the outcome was pending. In a separate study (Shafir and Tversky, 1992), the authors considered a Prisoner's Dilemma game in which players must decide to cooperate or defect given knowledge or uncertainty about the action of their opponent, and obtained similar results. While recent research questioned whether these findings violate the sure-thing principle (Gelastopoulos and Le Mens, 2024),[10] they have long been regarded as counterintuitive. The reason is that, in a classical model, the probability of a disjunctive outcome cannot be smaller than that of either outcome. By Kolmogorov's (1933) axioms, probabilities are monotonic in the size of events.

Other situations documented by cognitive psychologists suggest violations of monotonicity in the case of conjunctive rather than disjunctive outcomes. Here, the problem occurs when the probability of a conjunction exceeds that of one or both conjuncts. This is precisely what Tversky and Kahneman (1983) found in famous experiments where subjects were shown two scenarios involving fictitious individuals: in one of them, a man named Bill was described as intelligent and capable in mathematics, but unimaginative, compulsive, and generally lifeless. Subjects were then presented a series of statements about Bill and asked which would be most likely true. The series included, among others, the following statements: (a) Bill is an accountant, (b) Bill plays jazz, and (c) Bill is an accountant and plays jazz. Surprisingly, (c) was consistently regarded as more

[10]This criticism concerns the fact that informing players they won or lost the first bet, as opposed to telling them nothing, constitutes an experimental manipulation and it makes subsequent outcomes incomparable. The criticism is well-founded but does not apply to other experiments that also suggested violations of classical reasoning, as discussed below.

probable than either (a) or (b). In the second scenario, a woman named Linda was described as single, outspoken, very bright, and concerned with issues of discrimination and social justice. In this case, statements pinned Linda as (a) a feminist, (b) a bank teller, and (c) a feminist and a bank teller, among various other options. Again, subjects found (c) more likely than either (a) or (b).

These apparent failures of classical rationality were replicated by later studies, which attempted many possible disambiguations of experimental instructions, like clarifying the intended reading of the word "and" (Sides et al., 2002; Tentori and Crupi, 2012). Follow-up studies also tried to ensure that subjects in these experiments do not erroneously interpret the joint probability of two outcomes as a conditional one (Tentori et al., 2004). However, the puzzling results persisted. Moreover, similar results were reported by cognitive scientists preoccupied with different questions and experiments. For example, Hampton's (1988b; 1988a) early studies on conceptual combinations suggested that people break the monotonicity rule when categorizing objects as instances of disjunctive or conjunctive concepts. Similar results were obtained by Sozzo (2015). In these cases, there was no two-stage betting or Prisoner's Dilemma game, nor made-up scenarios involving fictitious individuals. Subjects faced single-stage choices about real-world objects and concepts, but their behavior still seemed to deviate from Kolmogorov's (1933) classical axioms.

While nonmonotonic reasoning represents a critical problem for classical notions of rationality, it should not be automatically taken as an indication that human thought processes are irrational. They can still be rational, or at least not fallacious, if modeled through the formalism of quantum mechanics, because this entails a different probability calculus (Aerts, 2009; Aerts et al., 2015; Franco, 2009). We could be simply dealing with a different model of rational inference: one for which correctness presumes consistency with quantum axioms (Pothos et al., 2017). In this framework, the probabilities assigned to conjunctive or disjunctive outcomes can deviate from what would be expected under classical axioms because of a characteristic phenomenon termed interference. This is a phenomenon "impossible, *absolutely* impossible, to explain in any classical way, and which has in it the heart of quantum mechanics" (Feynman, 1963, Section 37–1). Interference, which can be positive or negative, is a behavior typical of waves, but landmark experiments in physics demonstrated that it also applies to particles. Cognitive research now seems to suggest that it applies to something even more unlikely—human thought (e.g., Aerts et al., 2013).

The notion that interference plays a role in human decision-making is not easy to digest. Cognitive research in the tradition of rational analysis sought to provide a less exotic explanation for conjunction and disjunction errors by suitably extending classical models, as opposed to changing the underlying axioms. For example, Costello and Watts (2014) proposed a probability-plus-noise explanation in which random variation around classically rational behavior produces alleged fallacies. Zhu et al. (2020) offered a similar but more theoretically-grounded account by pinning this random variation on a sampling process, the result being a fully classical model that can explain why people are liable to erroneous judgments. This model was recently re-evaluated by Huang et al. (2024), who compared its performance to that of a more general quantum sampler. This similar to Zhu et al.'s (2020), but with an extra term that represents interference. The authors found that this more general model is better able to explain empirical results, although there are situations where people's behavior can be adequately approximated by classical assumptions.

On top of being consistent with the probabilistic calculus of quantum mechanics, interference is consistent with the propositional calculus of QLs, where it arises from the definition of quantum conditional (Friedman and Putnam, 1978). Therefore, while classical logic struggles to model experimental results that deviate from Kolmogorov's (1933) axioms, QLs accommodates them naturally. It thus seems that a formal theory of inference based on quantum-logical principles could capture features of human rationality better than one based on classical logic. But is it possible to abandon the laws of classical logic and "adopt" QLs on empirical grounds? This question is not philosophically trivial, as logic is not commonly regarded as an empirical science, and in any case, QLs were originally developed to formalize a theory of mechanics. Nonetheless, we believe the answer to this question is affirmative. We argue that QLs can be adopted and this does not even entail a complete substitution of classical logic because, in our framework, QLs are defined from FOL. In what follows, we delve into the problem of adoption and examine important philosophical consequences.

4 Adopting QLs

Putnam (1969) was among the first to consider the philosophical implications of the fact that quantum mechanics, as a physical theory, "presupposes" a logic. He was also influenced by Quine (1951) in viewing science as an interconnected network of knowledge subject to empirical revision. At the centre of this network is logic, which is by no means exceptional among scientific disciplines, although its central position means that empirical tests of logical laws can be particularly difficult. However, difficult is not impossible, and Birkhoff and Von Neumann's (1936) historical effort to extract a logic from quantum mechanics suggests that logic is in fact revisable. In two famous papers, Putnam (1969, 1974) likened the revision process that gave birth to QL to another process that had previously occurred in geometry:[11] some Euclidean notions, which used to be considered aprioristic knowledge capable of generating necessary truths, were eventually set aside after the successful application of non-Euclidean geometries in physical theories, as in the case of relativity theory. By the same token, core principles of classical logic, such as distributivity, had been set aside after the application of a distinctly non-classical logical formalism.

Kripke (2024) offered a response to Putnam (1969), based on arguments

[11]This comparison can traced back to earlier pragmatist philosophers interested in logic, such as Peirce (1932) and Dewey (1938). These also had an empirical conception of logic and argued that, like geometry, logic should be revisable. However, they did not make this argument for similar reasons as Putnam (1969, 1974) and they were not acquainted with Birkhoff and Von Neumann's (1936) proposal.

he outlined in unpublished work as early as 1974. The objection he raised is that "logical principles cannot be *adopted* because, if a subject already infers in accordance with them, no *adoption* is needed, and if subject does *not* infer in accordance with them, no *adoption* is possible" (Birman, 2024, p. 39). This has become known in the epistemology of logic as the adoption problem (see also Boghossian and Wright, 2024; Padró and Barrio, 2022). The counterargument Kripke (2024) developed on the basis of this problem is rather straightforward. Let us first distinguish two possible meanings of the word "logic" (cf. Peirce, 1932): on one hand, we have *logica utens*, or Logic with a capital L, which is the set of rules we regard as canonical and apply by default in our reasoning; on the other hand, we have *logica docens*, which includes the variety of formal systems commonly studied and taught by logicians.[12] Given this distinction, QLs supposedly belong to *logica docens*, while *logica utens* is classical.

Putnam (1969) proposed to adopt QLs as a new canon for reasoning in the specific setting of quantum mechanics, and in doing so, he proposed to modify *logica utens*. Kripke (2024) found this impossible because we cannot simply take a system pertaining to *logica docens* and use it as if it were *utens*. In this sense, no adoption is possible. The best we can do is momentarily entertain ourselves with non-classical ways of reasoning, but in the end, we will inevitably defer to the one true Logic and its classical principles. Empirical findings from the quantum cognition program, however, point to an unforeseen possibility: we tend to think of QLs as something that may or may not be adopted, but if the quantum formalism plays a role in human thought processes, as this program suggests, one could argue that QLs already belong to *logica utens*. No adoption is needed because we already use the rules prescribed by these logics.

In all fairness, it is not at all obvious whether Putnam (1969) proposed to adopt QLs as an outright substitute for classical logic, even in the narrow context of quantum mechanics. His goal was simply to argue that some rules of inference deemed valid *a priori* and necessarily true by classical logic should be considered false for empirical reasons. There is sufficient ground to revise some aspects of classical logic in light of quantum results, but these revisions leave us with an improved version of the same *logica utens*, not a wholly different one, and certainly not two distinct logics coexisting within the same *logica utens*. For Putnam (1969), it was never a matter of pitting QL against classical logic, as if these systems belonged to the same theoretic level. On the contrary, there was ample room for reconciliation given his anti-exceptionalist view of logic as just another empirical science (see also Hjortland, 2017). In this paper, we aim to remain neutral about the epistemological status of logic. However, we believe that applying QLs to human cognition amounts to claiming that QLs are already part of *logica utens*. It means we think quantum-logically in a sense even broader than Putnam (1974) foresaw. This avoids the adoption problem.

We also believe that the process whereby a logic is adopted, or accepted as canonical for human reasoning, may have been formulated by Kripke (2024) and

[12]To further emphasize their difference from Logic, some argue that these formal systems are not logics themselves but merely algebras (e.g., Suszko, 1976).

other proponents of the adoption problem in a way that is unduly restrictive. For example, Birman (2024) defines adoption as a series of steps: first, a subject does not make inferences consistent with particular logical principles; second, the subject accepts these principles; third, she starts making inferences consistent with them. Crucially, these inferences are made because of the subject's knowledge and acceptance of new principles.[13] Yet requiring a subject to know and accept the logical principles she follows in her reasoning sounds very demanding. People may reason according to principles that they neither acknowledge nor accept. Therefore, we reject the claim that a subject needs to know or accept QLs before she starts reasoning according to them.

One reason for rejecting this claim is that, if we assume a subject only uses classical logic in the first step, as opposed to several possible non-classical logics, we preclude the possibility of her ever reasoning non-classically. This is because requiring the subject to know QLs before she is able to use them, and assuming this knowledge is governed by classical logic, may render Kripke's (2024) conclusion that QLs cannot be adopted a *petitio principii*. In other words, if the subject's faculty of knowing rests on classical logic, and everything she knows must be consistent with classical principles, how can she ever come to know something that violates these principles? Ultimately, this makes it difficult to accommodate empirical evidence from the quantum cognition program. The crux of the matter then shifts to justifying the premise that people only rely on classical logic to know anything. Quantum cognition directly challenges this premise, not unlike others did before when confronted with paradoxes arising from the use of classical logic in informal contexts (e.g., Cooper, 1968).

Some arguments can be mobilized in support of Kripke's (2024) formulation of the adoption problem, including one we report here in the form of thought experiment. Imagine a classical logician named Alice who is in a room with two other logicians, Bob and Charlie. Bob is just a neutral observer of the situation, while Charlie adopts—per our relaxed version of adoption, which does not require awareness—a paraconsistent logic. Suppose that Charlie is more specifically a dialetheist, i.e., he believes that certain contradictions are true, or at least valid, and in any case, the principle of non-contradiction is not a metaphysically valid and universal law. At the same time, Charlie rejects the explosion principle so that his inferences need not become trivial. Now suppose that Alice is an authority figure for Charlie and she tells him: "Evidence shows that p is true, so you must reject $\neg p$ due to the principle of non-contradiction." Charlie thinks paraconsistently, so he replies something like: "You may be right, and even if my intuition cautions me against this principle, I trust you as an epistemic authority. Your statement q, by which p is true, is also true. So, logically, p can also be false." Confronted with this situation, Bob's only hope of determining who is correct requires him to make explicit the logic he uses to evaluate Alice and Charlie's respective statements. But what logic would this be? According to Kripke (2024), it can only be classical.

This hypothetical exchange is meant to demonstrate the impossibility of

[13] See Fiore (2022) for a critical analysis and some considerations about this stepwise process.

adoption as long as we distinguish Logic with a capital L, or *logica utens*, from the swarm of alternative formal systems logicians like to study as mathematical abstractions. On this front, we completely agree with Kripke's (2024) conclusion that QLs cannot be adopted. This would be further confirmed by the negative outcome of Putnam's (1969) own substitution project: local substitution, whereby selected aspects of classical logic are corrected to align with QLs, failed for technical reasons; and global substitution, whereby classical logic is entirely replaced, never even looked viable. Where we disagree with Kripke (2024) is in the assumption that *logica utens* is restricted to classical logic. This is not something that can be inferred from the adoption problem. It could just as well be that *logica utens* includes QLs, and with these already part of our logical background, no adoption is necessary. Empirical findings from the quantum cognition program point to this possibility, though this literature only makes a case for quantum probability in cognitive processes, not QLs.

Despite their differences, Putnam (1969) and Kripke (2024) subscribed to a philosophical position known as logical monism. In this paper, we aimed to provide a new perspective on QLs by using FOL as a metalanguage, which means treating classical logic and various non-classical systems, including QLs, as belonging to different logical orders. While we characterize QLs as non-classical at the propositional level, we approach them from a purely algebraic point of view through classical reasoning. For example, we obtain orthomodular lattices and their operations by axiomatically restricting Boolean operations. Based on this, we could say that our approach partially agrees with Kripke's (2024) classical monist position. We note, however, that our approach is also compatible with logical pluralism (Beall and Restall, 2000). The reason we model QLs using FOL is not to defend the primacy of classical logic but to render QLs independent from physics, in line with cognitive research suggesting that people are capable of quantum-logical reasoning. Clinging to Kripke's (2024) classical monism just because we rely on FOL does not do justice to the quantum cognition program. This approach would make it impossible to understand the program's empirical results from a logical standpoint: they do not square with classical logic, so we can only see them as deviant. We think they are not.

Extending QLs to cognitive science could enable us to argue in favor of a logical pluralism that directly applies to *logica utens*. This seems especially interesting if we consider that the adoption problem challenges but does not ultimately preclude the coexistence of multiple logics in *utens*. Moreover, Putnam's (1969) own substitution project turned out to be unfeasible. Empirical findings from quantum cognition could enrich the philosophical debate by introducing the question of whether a pluralist stance about *logica utens* is viable. Such a stance would help making sense of well-known challenges in the application of classical logic to informal reasoning, but at the same time, it would be consistent with the success of classical logic in defining standard theories. Remarkably, the conjecture of *logicae utentes* need not rest on an empiricist conception of logic, as was the case for the kind of pluralism embraced by philosophers in the pragmatist tradition (Dewey, 1938; Peirce, 1932). On the contrary, it can build on a view of logic as a discipline characterized by laws that are valid *a priori*. We can accept a softer version of the adoption problem whereby adoption is generally problematic, but as it turns out, in some circumstances we already reason under non-classical and specifically quantum-logical principles.

5 Conclusions

Although QLs were initially developed to model inferential processes in physics, recent research in cognitive science suggest that their scope of application is potentially much broader. A wealth of evidence from the quantum cognition program indicates that the semantics of QLs capture features of human rationality. As a result, these logics could be suited to model not just how people reason about quantum mechanics, as Putnam (1974) proposed, but how they reason in general. In this paper, we evaluated this claim by reviewing cognitive findings that point to QLs as a promising framework. We also presented a route to building standard QLs from FOL, which thanks to its generality facilitates interdisciplinary applications of logical formalisms. We believe that expanding the scope of QLs in this way is an interesting exercise for logicians, but perhaps more importantly, it can be useful to other areas of science. Cognitive applications would be particularly relevant to formal theories of human psychology, studies in the philosophy of logic, and research on quantum foundations.

From a psychological perspective, this endeavor is interesting because it can equip cognitive scientists with a new and improved understanding of human rationality. So far, this has been predicated largely on conformity to the rules of classical logic or probability theory (Chater and Oaksford, 2000). As these rules appear to be violated in cognitive experiments, however, many psychologists were compelled to concede that rationality cannot be based on mathematical axioms. The switch from classical to quantum probability in cognitive modeling started to change this perspective, as the probabilistic calculus of quantum theory offers a new criterion to define rationality which is both flexible and rigorous (Busemeyer et al., 2011; Pothos et al., 2017). Still, the quantum models currently deployed in cognitive applications are very diverse. For example, Busemeyer et al. (2011) used a static Hilbert-space model to explain common deviations from classical rationality and order effects; Kvam et al. (2015) used a quantum-walk model to account for evolving preferences, borrowing from research in quantum dynamics; and other research (Aerts, 2009; Aerts et al., 2015; Sozzo, 2015) used a Fock-space model from quantum field theory to represent the superposition of classical and quantum-logical reasoning. All of these models are "quantum" in the sense that they are consistent with Von Neumann's (1932) axioms, but they make different assumptions about the behavior of a cognitive system. Formalizing the assumptions and predictions of these models within a common logical language would help ensuring they are not only consistent with the axioms of quantum theory, but also with each other. From this standpoint, QLs hold potential to unify and consolidate various streams of research in quantum cognition.

Another way in which QLs can help cognitive research is by enabling qualitative predictions. The probabilistic formalisms previously deployed in quantum cognition generally require the specification of free parameters to accurately predict empirical results.[14] For example, Aerts's (2009) seminal models of conceptual conjunction require the modeler to specify the angles between subspaces corresponding to different concepts. For an arbitrary pair of concepts, there is no way to correctly set the angle without first looking at experimental data and estimating or deriving it from there. However, looking at experimental data in order to predict experimental data does not go very far in terms of providing useful predictions. Furthermore, there is no assurance that the angle between two arbitrary concepts is the same for any other pair. Therefore, the presence of free parameters limits the generality of a formal theory.

Arguably, one of the most significant results of the quantum cognition program was Wang and Busemeyer's (2013) derivation of an accurate and parameter-free prediction known as the quantum-question equality. This is a relation between quantities obtained from psychological experiments involving order effects, which is formally a consequence of quantum axioms (cf. Niestegge, 2008) and does not necessarily hold by classical ones. The reason why it constitutes an impressive result is that parameter-free predictions are rare, not just in psychology but in all social sciences, and can play a pivotal role in testing the validity of a theory. Formal logic, being concerned with qualitative as opposed to quantitative accounts of rational processes, is especially suited to derive relations that do not depend on free parameters. This could be exactly what research on quantum cognition needs to demonstrate that its exotic approach is worthwhile. For the moment, the quantum-question equality remains an exceptional result, but the deployment of QLs could lead to more predictions of the same kind and allow this research program to make significant leaps forward.

From a philosophical perspective, our proposed connection between QLs and cognition is interesting because it can inject new arguments into longstanding debates that were historically motivated by QLs. This includes discussions about the empirical character of logic (Bacciagaluppi, 2009; Putnam, 1969, 1974) and the adoption problem (Birman, 2024; Boghossian and Wright, 2024; Kripke, 2024). Our conclusion is that logic can be considered empirical to the extent we can discover new logical principles underlying human thought processes and consequently revise our understanding of *logica utens*. However, this kind of empiricism is far lighter than Putnam's (1969), as it allows for pluralistic and even aprioristic perspectives on *logica utens*. We find it surprising, but also very fitting, that a new application of QLs motivated by empirical evidence concerns precisely human thought. If at the time of Birkhoff and Von Neumann (1936) the main philosophical questions stemmed from the discovery of a non-classical logic at the core of a physical theory, now that we are considering applying the same logic to human cognition we must ask ourselves what would be the implications of discovering that people reason according to the quantum formalism. This is a new avenue for exploration within the philosophy of logic.

It seems reasonable to think that, if this formalism is applicable to contexts beyond physics, its empirical nature and "quantumness" are no more than histor-

[14]We are grateful to an anonymous reviewer for making this suggestion.

ical contingencies related to its initial discovery. Presenting QLs in FOL helps conveying this idea. Genuine surprise arises from the possibility of recovering the adoption problem in our framework, albeit in a relaxed version. Maybe our stance is ultimately compatible with logical monism, given the role played by classical FOL in our construction of standard QLs. However, it is also compatible with logical pluralism as *logica utens* comes to incorporate more than just classical logic in some inferential processes. This is at odds with both Putnam (1969) and Kripke (2024), who in spite of their many differences similarly upheld logical monism by giving a single Logic the privilege of being *utens*. We could avoid the adoption problem by admitting the primacy of classical logic, but we are drawn to pluralism. While QLs are non-classical, they can arise from a classical setting; at the same time, they demand we forget about some classical principles in particular contexts. The kind of pluralism we propose views different logics as coexisting within *logica utens*, but at different levels. In particular, we have FOL at a metalevel and QLs at a propositional one. This allows modeling inferences that depend on very different logical principles, e.g., orthomodularity as opposed to distributivity. From this angle, there is no need to adopt non-classical logics in the sense forcefully opposed by Kripke (2024). We already have them, and we need not be aware of them in order to use them.

Additional work is required to fully revisit philosophical debates on logical pluralism and exceptionalism in light of these considerations. Nonetheless, our foray into cognitive literature lends credibility to the claim that people think quantum-logically. This should already prompt us to revise what we call Logic with a capital L, if such a thing exists, and this revision may not occur exactly as Putnam (1969, 1974) intended, i.e., based on results from quantum mechanics. Instead, we flip the script by applying QLs outside quantum mechanics and then use results from these applications as groundwork for a discussion on what Logic should be. In our view, this should not be a logic that replaces or modifies the classical, but a formal system consisting of multiple logics with classical FOL as the metalogical foundation. We can consider logic empirical in a historical sense and embrace pluralism as we work toward a notion of *logicae utentes*. We can also move to an *a priori* pluralism, as opposed to an empirical one, which is capable of accepting several non-classical logics as valid models of rationality. This is possible because we soften the empiricism of QLs to the extent that it becomes a mere historical connotation rather than a property that qualitatively distinguishes QLs from other non-classical systems. While the cognitive arguments that drove us to this anti-exceptionalist and pluralist position are novel to the literature on QLs, they are consistent with Bacciagaluppi's (2009) conclusion that, if logic were empirical, it would be shaped by cognitive science or linguistics more than quantum mechanics. It is noteworthy that these disciplines converge.

Finally, the connection we make between QLs and cognition can potentially help physicists reach a deeper understanding of quantum foundations. Recent research in this field suggested that the rules of quantum theory are so fundamental and widely applicable because they concern what all physical experiments inevitably have in common: a human observer. They are laws of thought (Caves et al., 2002). They do not dictate how physical systems behave but rather how physicists should reason about a system's behavior. We take this perspective further by framing QLs as a general framework for modeling inferential processes, a tool to analyze how people reason about anything. This could shed light on questions pertaining to the philosophy of physics, such as the interpretation of particular elements of the quantum formalism (Fuchs and Schack, 2013; Reichenbach, 1944). As mentioned above, the adjective "quantum" should be considered only a byproduct of the circumstances in which the formalism was born. This adjective makes QLs retain the flavor of physical applications, but there is nothing strictly physical about these logics, nor about the formalism, as they are very general in nature. Many outside physics acknowledge the generality and usefulness of the quantum formalism, yet the same cannot be said for QLs. We hope our proposal contributes to changing this state of affairs.

References

Aerts, D. (2009). Quantum structure in cognition. *Journal of Mathematical Psychology*, 53(5):314–348.

Aerts, D., Gabora, L., and Sozzo, S. (2013). Concepts and their dynamics: A quantum-theoretic modeling of human thought. *Topics in Cognitive Science*, 5(4):737–772.

Aerts, D. and Sozzo, S. (2014). Quantum entanglement in concept combinations. *International Journal of Theoretical Physics*, 53:3587–3603.

Aerts, D., Sozzo, S., and Veloz, T. (2015). Quantum structure of negation and conjunction in human thought. *Frontiers in Psychology*, 6:1447.

Ajdukiewicz, K. (1956). Conditional statement and material implication. *Studia Logica*, 4:135–153.

Alter, A. L. and Oppenheimer, D. M. (2008). Effects of fluency on psychological distance and mental construal (or why New York is a large city, but *New York* is a civilized jungle). *Psychological Science*, 19(2):161–167.

Anderson, A. R. and Belnap, N. D. (1975). *Entailment: The logic of relevance and necessity*, volume 1. Princeton University Press, Princeton, NJ.

Aspect, A., Grangier, P., and Roger, G. (1981). Experimental tests of realistic local theories via Bell's theorem. *Physical Review Letters*, 47(7):460–463.

Aspect, A., Grangier, P., and Roger, G. (1982). Experimental realization of Einstein-Podolsky-Rosen-Bohm *gedankenexperiment*: A new violation of Bell's inequalities. *Physical Review Letters*, 49(2):91–94.

Atmanspacher, H., Römer, H., and Walach, H. (2002). Weak quantum theory: Complementarity and entanglement in physics and beyond. *Foundation of Physics*, 32:379–406.

Bacciagaluppi, G. (2009). Is logic empirical? In Engesser, K., Gabbay, D. M., and Lehmann, D., editors, *Handbook of quantum logic and quantum structures*, pages 49–78. Elsevier, Amsterdam, The Netherlands.

Balzer, W., Moulines, C. U., and Sneed, J. D. (1987). *An architectonic for science: The structuralist program.* D. Reidel, Dordrecht, The Netherlands.

Barsalou, L. W. (2008). Grounded cognition. *Annual Review of Psychology*, 59(1):617–645.

Barsalou, L. W. (2010). Grounded cognition: Past, present, and future. *Topics in Cognitive Science*, 2(4):716–724.

Beall, J. C. and Restall, G. (2000). Logical pluralism. *Australasian Journal of Philosophy*, 70:475–493.

Bell, J. S. (1964). On the Einstein Podolsky Rosen paradox. *Physics*, 1(3):195–200.

Bergus, G. R., Chapman, G. B., Levy, B. T., Ely, J. W., and Oppliger, R. A. (1998). Clinical diagnosis and the order of information. *Medical Decision Making*, 18(4):412–417.

Birkhoff, G. and Von Neumann, J. (1936). The logic of quantum mechanics. *Annals of Mathematics*, 37(4):823–843.

Birman, R. (2024). The adoption problem and the epistemology of logic. *Mind*, 133:37–60.

Bodenhausen, G. V., Schwarz, N., Bless, H., and Wänke, M. (1995). Effects of atypical exemplars on racial beliefs: Enlightened Racism or generalized appraisals. *Journal of Experimental Social Psychology*, 31(1):48–63.

Boghossian, P. and Wright, C. (2024). Kripke, Quine, the "adoption problem" and the empirical conception of logic. *Mind*, 133(529):86–116.

Bohr, N. (1958). *Atomic physics and human knowledge.* John Wiley & Sons, New York.

Born, M. (1926). Zur Quantenmechanik der Stoßvorgänge [Quantum mechanics of collision processes]. *Zeitschrift für Physik*, 37:863–867.

Bruza, P. D. and Cole, R. J. (2005). Quantum logic of semantic space: An exploratory investigation of context effects in practical reasoning. In Artemov, S., Barringer, H., d'Avila Gárcez, A., Lamb, L. C., and Woods, J., editors, *We will show them! Essays in honour of Dov Gabbay*, volume 1, pages 339–362. College Publications, Rickmansworth, UK.

Bruza, P. D., Fell, L., Hayte, P., Dehdashti, S., Obeid, A., Gibson, A., and Moreira, C. (2023). Contextuality and context-sensitivity in probabilistic models of cognition. *Cognitive Psychology*, 140:101529.

Bruza, P. D., Kitto, K., Nelson, D., and McEvoy, C. (2009). Is there something quantum-like about the human mental lexicon? *Journal of Mathematical Psychology*, 53(5):363–377.

Busemeyer, J. R. and Bruza, P. D. (2012). *Quantum models of cognition and decision*. Cambridge University Press, New York.

Busemeyer, J. R., Kvam, P. D., and Pleskac, T. J. (2020). Comparison of Markov versus quantum dynamical models of human decision making. *Wiley Interdisciplinary Reviews: Cognitive Science*, 11(4):e1526.

Busemeyer, J. R., Pothos, E. M., Franco, R., and Trueblood, J. S. (2011). A quantum theoretical explanation for probability judgment errors. *Psychological Review*, 118(2):193–218.

Cattaneo, G., Dalla Chiara, M. L., and Giuntini, R. (1993). Fuzzy intuitionistic quantum logics. *Studia Logica*, 52:419–442.

Caves, C. M., Fuchs, C. A., and Schack, R. (2002). Quantum probabilities as Bayesian probabilities. *Physical Review A*, 65(2):022305.

Chater, N. and Oaksford, M. (2000). The rational analysis of mind and behavior. *Synthese*, 122:93–131.

Cooper, W. S. (1968). The propositional logic of ordinary discourse. *Inquiry*, 11(1–4):295–230.

Costello, F. and Watts, P. (2014). Surprisingly rational: Probability theory plus noise explains biases in judgment. *Psychological Review*, 121(3):463–480.

Da Costa, N. C. A. and Krause, D. (1994). Schrödinger logics. *Studia Logica*, 53:533–550.

Dalla Chiara, M. L. and Giuntini, R. (2002). Quantum logics. In Gabbay, D. M. and Guenthner, F., editors, *Handbook of philosophical logic*, pages 129–228. Springer, Dordrecht, The Netherlands.

Dalla Chiara, M. L., Giuntini, R., Leporini, R., and Sergioli, G. (2018). *Quantum computation and logic: How quantum computers have inspired logical investigations*. Springer, Cham, Switzerland.

Dapprich, J. P. (2016). *Philosophy and logic of quantum physics: An investigation of the metaphysical and logical implications of quantum physics*. Peter Lang, New York.

Dewey, J. (1938). *Logic: The theory of inquiry*. Henry Holt & Co., New York.

Dummett, M. (1976). Is logic empirical? In Lewis, H. D., editor, *Contemporary British philosophy*, pages 45–68. George Allen & Unwin, London, U.K.

Dvurečenskij, A. (1995). Tensor product of difference posets and effect algebras. *International Journal of Theoretical Physics*, 34(8):1337–1348.

Einstein, A., Podolsky, B., and Rosen, N. (1935). Can quantum-mechanical description of physical reality be considered complete? *Physical Review*, 47(10):777–780.

Engesser, K., Gabbay, D. M., and Lehmann, D., editors (2007a). *Handbook of quantum logic and quantum structures: Quantum structures*. Elsevier, Amsterdam, The Netherlands.

Engesser, K., Gabbay, D. M., and Lehmann, D. (2007b). *A new approach to quantum logic*. College Publications, London, U.K.

Feynman, R. P. (1963). *The Feynman lectures on physics*. Basic Books, New York.

Field, H. (2008). *Saving truth from paradox*. Oxford University Press, Oxford, U.K.

Finn, S. (2021). Limiting logical pluralism. *Synthese*, 198(Suppl. 20):4905–4923.

Fiore, C. G. (2022). What the adoption problem does not show. *Análisis Filosófico*, 42(1):79–103.

Fodor, J. A. and Pylyshyn, Z. W. (1988). Connectionism and cognitive architecture: A critical analysis. *Cognition*, 28(1–2):3–71.

Foulis, D. J. and Randall, C. H. (1981). What are quantum logics and what ought they to be? In Beltrametti, E. G., editor, *Current Issues in Quantum Logic*, pages 35–52. Springer, Boston.

Franco, R. (2009). The conjunction fallacy and interference effects. *Journal of Mathematical Psychology*, 53(5):415–422.

Friedman, M. and Putnam, H. (1978). Quantum logic, conditional probability, and interference. *Dialectica*, 32(3–4):305–315.

Fuchs, C. A. and Schack, R. (2013). Quantum-Bayesian coherence. *Review of Modern Physics*, 85(4):1693–1715.

Gelastopoulos, A. and Le Mens, G. (2024). The disjunction effect does not violate the law of total probability. https://doi.org/10.31234/osf.io/2xzf4. Accessed November 21, 2024.

Gigerenzer, G. (2000). *Adaptive thinking: Rationality in the real world*. Oxford University Press, New York.

Gigerenzer, G. and Gaissmaier, W. (2011). Heuristic decision making. *Annual Review of Psychology*, 62:451–482.

Gigerenzer, G. and Goldstein, D. G. (1996). Reasoning the fast and frugal way: Models of bounded rationality. *Psychological Review*, 103(4):650–669.

Giuntini, R. (1991). *Quantum logic and hidden variables.* Bibliographisches Institut, Leipzig, Germany.

Gleason, A. M. (1957). Measures on the closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics*, 6:885–894.

Grice, P. (1989). *Studies in the way of words.* Harvard University Press, Cambridge, MA.

Griffiths, R. B. (2003). *Consistent quantum theory.* Cambridge University Press, Cambridge, U.K.

Griffiths, R. B. (2014). The new quantum logic. *Foundation of Physics*, 44:610–640.

Hampton, J. A. (1988a). Disjunction of natural concepts. *Memory & Cognition*, 16(6):579–591.

Hampton, J. A. (1988b). Overextension of conjunctive concepts: Evidence for a unitary model of concept typicality and class inclusion. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 14(1):12–32.

Hardegree, G. M. (1974). The conditional in quantum logic. *Synthese*, 29(1–4):63–80.

Hatchett, S. and Schuman, H. (1976). White respondents and race-of-interviewer effects. *Public Opinion Quarterly*, 39(4):523–528.

Hjortland, O. T. (2017). Anti-exceptionalism about logic. *Philosophical Studies*, 174:631–658.

Holton, G. (1970). The roots of complementarity. *Daedalus*, 99(4):1015–1055.

Horvat, S. and Toader, I. D. (2024). An alleged tension between non-classical logics and applied classical mathematics. *The Philosophical Quarterly*, 125:pqad125.

Huang, J., Busemeyer, J. R., Ebelt, Z., and Pothos, E. M. (2024). Bridging the gap between subjective probability and probability judgments: The quantum sequential sampler. *Psychological Review*, forthcoming.

Jackson, F. (1991). *Conditionals.* Oxford University Press, Oxford, U.K.

Kahneman, D. (2003). Maps of bounded rationality: Psychology for behavioral economics. *American Economic Review*, 93(5):1449–1475.

Kahneman, D. (2011). *Thinking, fast and slow.* Farrar, Straus and Giraux, New York.

Kahneman, D., Slovic, P., and Tversky, A., editors (1982). *Judgment under uncertainty: Heuristics and biases.* Cambridge University Press, New York.

Kalmbach, G. (1983). *Orthomodular lattices.* Academic Press, London, U.K.

Kerstholt, J. H. and Jackson, J. L. (1999). Judicial decision making: Order of evidence presentation and availability of background information. *Applied Cognitive Psychology*, 12(5):445–454.

Khrennikov, A. (2010). *Ubiquitous quantum structure: From psychology to finance*. Springer, Berlin, Germany.

Kolmogorov, A. N. (1950 [1933]). *Foundations of the theory of probability.* Chelsea Publishing, New York.

Krause, D. (2010). Logical aspects of quantum (non-)individuality. *Foundations of Science*, 15:79–94.

Kripke, S. A. (2024). The question of logic. *Mind*, 133(529):1–36.

Kvam, P. D., Pleskac, T. J., Yu, S., and Busemeyer, J. R. (2015). Interference effects of choice on confidence: Quantum characteristics of evidence accumulation. *Proceedings of the National Academy of Sciences*, 112(34):10645–10650.

Lichtenstein, S. and Slovic, P., editors (2006). *The construction of preference.* Cambridge University Press, New York.

Love, B. C. (2015). The algorithmic level is the bridge between computation and the brain. *Topics in Cognitive Science*, 7(2):230–242.

Mackey, G. W. (1963). *Mathematical foundations of quantum mechanics.* W. A. Benjamin, New York.

Marr, D. (1982). *Vision: A computational investigation into the human representation and processing of visual information.* W. H. Freeman, San Francisco.

Moore, D. W. (2002). Measuring new types of question-order effects: Additive and subtractive. *Public Opinion Quarterly*, 66(1):80–91.

Nelson, D. L., McEvoy, C. L., and Pointer, L. (2003). Spreading activation or spooky action at a distance? *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 29(1):42–52.

Nelson, D. L., McEvoy, C. L., and Schreiber, T. A. (2004). The University of South Florida free association, rhyme, and word fragment norms. *Behavior Research Methods, Instruments, & Computers*, 36(3):402–407.

Niestegge, G. (2008). An approach to quantum mechanics via conditional probabilities. *Foundations of Physics*, 38:241–256.

Oaksford, M. and Chater, N., editors (1998). *Rational models of cognition.* Oxford University Press, Oxford, U.K.

Oaksford, M. and Chater, N. (2007). *Bayesian rationality: The probabilistic approach to human reasoning.* Oxford University Press, Oxford, U.K.

Oppenheimer, D. M. and Frank, M. C. (2008). A rose in any other font would not smell as sweet: Effects of perceptual fluency on categorization. *Cognition*, 106(3):1178–1194.

Padró, R. and Barrio, E. (2022). El problema de la adopción de reglas lógicas [The problem of adopting logical rules]. *Análisis Filosófico*, 41(1):33–41.

Peirce, C. S. (1932). *Principles of philosophy and elements of logic.* Harvard University Press, Cambridge, MA.

Pitowsky, I. (2006). Quantum mechanics as a theory of probability. In Demopoulos, W. and Pitowsky, I., editors, *Physical theory and its interpretation: Essays in honor of Jeffrey Bub*, pages 213–240. Springer, Dordrecht, The Netherlands.

Pitowsky, I. and Svozil, K. (2001). Optimal tests of quantum nonlocality. *Physical Review A*, 64(1):014102.

Pothos, E. M. and Busemeyer, J. R. (2013). Can quantum probability provide a new direction for cognitive modeling? *Behavioral and Brain Sciences*, 36(3):255–274.

Pothos, E. M. and Busemeyer, J. R. (2022). Quantum cognition. *Annual Review of Psychology*, 73:749–778.

Pothos, E. M., Busemeyer, J. R., Shiffrin, R. M., and Yearsley, J. M. (2017). The rational status of quantum cognition. *Journal of Experimental Psychology: General*, 146(7):968–987.

Putnam, H. (1969). Is logic empirical? In Cohen, R. S. and Wartofsky, M. W., editors, *Proceedings of the Boston colloquium for the philosophy of science 1966/1968*, volume 5 of *Boston Studies in the Philosophy and History of Science*, pages 216–241. Springer, Dordrecht, The Netherlands.

Putnam, H. (1974). How to think quantum-logically. *Synthese*, 29(1–4):55–61.

Quine, W. V. (1951). Two dogmas of empiricism. *The Philosophical Review*, 60:20–43.

Randall, C. H. and Foulis, D. J. (1981). Operational statistics and tensor product. In *Bibliographisches Inst.*

Reichenbach, H. (1944). *Philosophic foundations of quantum mechanics.* University of California Press, Berkeley, CA.

Savage, L. J. (1954). *The foundations of statistics.* John Wiley & Sons, New York.

Schrödinger, E. (1944). *What is life? The physical aspect of the living cell*. Cambridge University Press, Cambridge, U.K.

Schwarz, N. and Bohner, G. (2001). The construction of attitudes. In Tesser, A. and Schwarz, N., editors, *Blackwell handbook of social psychology: Intraindividual processes*, pages 436–457. Blackwell Publishers, Oxford, U.K.

Shafir, E. and Tversky, A. (1992). Thinking through uncertainty: Nonconsequential reasoning and choice. *Cognitive Psychology*, 24(4):449–474.

Sides, A., Osherson, D., Bonini, N., and Viale, R. (2002). On the reality of the conjunction fallacy. *Memory & Cognition*, 30:191–198.

Simon, H. A. (1990). Bounded rationality. In Eatwell, J., Milgate, M., and Newman, P., editors, *Utility and probability*, pages 15–18. Palgrave Macmillan, London, U.K.

Smets, S. (2001). On causation and a conterfactual in quantum logic: The Sasaki hook. *Logique et Analyse*, 44(173–175):307–325.

Sozzo, S. (2015). Conjunction and negation of natural concepts: A quantum-theoretic modeling. *Journal of Mathematical Psychology*, 66:83–102.

Stachel, J. (1976). The "logic" of "quantum logic". In Cohen, R. S., Hooker, C. A., Michalos, A. C., and Van Evra, J. W., editors, *PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association*, volume 1974, pages 515–526. D. Reidel, Dordrecht, The Netherlands.

Stairs, A. (1983). Quantum logic, realism, and value definiteness. *Philosophy of Science*, 50(4):578–602.

Suszko, R. (1976). The Fregean axiom and Polish mathematical logic in the 1920s. *Studia Logica*, 36(4):377–380.

Svozil, K. (1998). *Quantum logic*. Springer, Singapore.

Tentori, K., Bonini, N., and Osherson, D. (2004). The conjunction fallacy: A misunderstanding about conjunction? *Cognitive Science*, 28(3):467–477.

Tentori, K. and Crupi, V. (2012). On the conjunction fallacy and the meaning of *and*, yet again: A reply to Hertwig, Benz, and Krauss (2008). *Cognition*, 122(2):123–134.

Tversky, A. and Kahneman, D. (1983). Extensional versus intuitive reasoning: The conjunction fallacy in probability judgment. *Psychological Review*, 90(4):293–315.

Tversky, A. and Shafir, E. (1992). The disjunction effect in choice under uncertainty. *Psychological Science*, 3(5):305–310.

Von Baeyer, H. C. (2013). Quantum wierdness? It's all in your mind. *Scientific American*, 308(6):46–51.

Von Neumann, J. (1955 [1932]). *Mathematical foundations of quantum mechanics.* Princeton University Press, Princeton, NJ.

Wang, Z. and Busemeyer, J. R. (2013). A quantum question order model supported by empirical tests of an *a priori* and precise prediction. *Topics in Cognitive Science*, 5(4):689–710.

Wang, Z. and Busemeyer, J. R. (2015). Reintroducing the concept of complementarity into psychology. *Frontiers in Psychology*, 6:1822.

Wang, Z., Busemeyer, J. R., Atmanspacher, H., and Pothos, E. M. (2013). The potential of using quantum theory to build models of cognition. *Topics in Cognitive Science*, 5(4):672–688.

Wang, Z., Solloway, T., Shiffrin, R. M., and Busemeyer, J. R. (2014). Context effects produced by question orders reveal quantum nature of human judgments. *Proceedings of the National Academy of Sciences*, 11(26):9431–9436.

Wendt, A. (2015). *Quantum mind and social science: Unifying physical and social ontology.* Cambridge University Press, Cambridge, U.K.

Wigner, E. P. (1960). The unreasonable effectiveness of mathematics in the natural sciences. *Communications on Pure and Applied Mathematics*, 13(1):1–14.

Williamson, T. (2018). Alternative logics and applied mathematics. *Philosophical Issues*, 28(1):399–424.

Zhu, J.-Q., Sanborn, A. N., and Chater, N. (2020). The Bayesian sampler: Generic Bayesian inference causes incoherence in human probability judgments. *Psychological Review*, 127(5):719–748.