HOW THE TORTOISE CAN BEAT ACHILLES: A PARADOX ON CURVES OF INFINITE LENGTH

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Abstract

Achilles and the tortoise compete in a race where the beginning (the start) is at point O and end (the finish) is at point P. At all times the tortoise can run at a speed that is a fraction Λ of Achilles' speed at most (with Λ being a positive real number lower than 1, $0 < \Lambda < 1$), and both start the race at $t = 0$ at O. If the trajectory joining O with P is a straight line, Achilles will obviously win every time. It is easy to prove that there is a trajectory joining O and P along which the tortoise has a strategy to win every time, reaching the finish before Achilles.

1. Introduction.

The term "paradox" is sometimes used in a narrow sense, as a contradiction or inconsistency. However, there is also a broader and more interesting sense. This is very clearly characterised by Kleiner and Movshovitz-Hadar (1994) when they explain: "We will use the term "paradox" in a broad sense to mean an inconsistency, a counterexample to widely held notions, a misconception, a true statement that seems to be false, or a false statement that seems to be true". The paradox on curves of infinite length presented in this paper (when the fastest runner (Achilles) makes the least progress) should be understood in this broad sense. In particular, it will be seen that there are circumstances where this paradox is true even though it always appears to be clearly false. An elementary, mathematical (kinematic) description of space invader evolution leads to the surprising outcome that, in a race along locally rectifiable curves of infinite length, the slower runner (the tortoise) can be bound to beat the faster runner. In such cases, the faster runner advances faster (by definition) yet, paradoxically, makes less progress.

2. A preliminary step. Emerging from Infinity.

With sufficient generality for present purposes, hereafter $f(t)$ denotes any function of t whose domain of definition is some open interval along real straight line $\mathfrak{R}$ (in particular, this may be all of $\mathfrak{R}$) which is continuous and differentiable therein.

A function, $x = f(t)$, describes the world line of a point particle moving in one spatial dimension. If function $f(t)$ exists before $t = t^{\Delta}$ and has a vertical asymptote at $t = t^{\Delta}$, $f(t)$ therefore describes the world line of a particle that "escapes to infinity" at that instant. For example, $x = f(t) = t/(1 - t)$ (with interval $(-\infty, 1)$ as the domain of definition) describes the world line of a particle that "goes to infinity" at $t = t^{\Delta} = 1$: $\lim_{t \to 1-} (\frac{t}{1-t}) = +\infty$. To be more precise, it can be said that it escapes to infinity $+\infty$ at $t = 1$ (a particle can also escape to infinity $-\infty$ at $t = 1$). There are "realistic" examples of escaping to infinity, i.e. examples of this type of evolution in the framework of Newtonian gravitational theory of point mass particles. Diacu (2001) mentions several of these, which are all mathematically complex outcomes published in mathematics journals (e.g. Gerver 1991 or Xia 1992) rather than physics journals. Clearly (Earman 1986), the time reversal of an escape-to-infinity process describes a process in which a particle appears from spatial infinity. If function $f(t)$ exists after $t = t^{\Delta}$ and has a vertical asymptote at $t = t^{\Delta}$, it therefore describes the world line of a particle that "emerges from infinity" at that instant. Such unexpectedly emerging particles have sometimes been referred to as "space invaders". For example, a simple space invader world line is the following: $x = f(t) = (t - 1)/t$ (with interval $(0, +\infty)$ as the domain of definition). Since here $\lim_{t \to 0+} (\frac{t-1}{t}) = -\infty$, the world line of a particle emerging from infinity at $t = t^{\Delta} = 0$ is described. To be more precise, it can be said that it emerges from infinity $-\infty$ at $t = 0$ (a particle can also emerge from infinity $+\infty$ at $t = 0$). Note that a particle emerging from infinity at $t = 0$ does not exist (it is not in space) at $t = 0$, but rather in some non-empty interval $(0, +a)$.

It is interesting to note that if two particles emerge from infinity at $t = t^{\Delta}$ moving at different velocities in the same direction (at least initially, so both emerge from infinity $+\infty$ or both emerge from infinity $-\infty$), but in such a way that one is always faster than the other, it follows that for all instants after $t = t^{\Delta}$ but sufficiently close to t^{Δ}, the faster particle lags behind the slower particle: the latter is in the lead. For present purposes, rigorous proof will be sufficient in the case where the slow particle velocity v_{slow} is always a fraction ε ($0 < \varepsilon < 1$) of the fast particle velocity v_{fast}. Hence, $v_{fast} = x'_{fast} = f'(t)$ and $v_{slow} = x'_{slow} = \varepsilon f'(t)$. Therefore, $x_{fast} = f(t) + A$ and $x_{slow} = \varepsilon f(t) + B$, where A and B are arbitrary constants. The following chain of equivalences applies:

$$x_{fast} < x_{slow} \leftrightarrow f(t) + A < \varepsilon f(t) + B \leftrightarrow f(t) < (B - A)/(1 - \varepsilon).$$

Now assume that both particles emerge from infinity $-\infty$ at $t = t^{\Delta}$ (the infinity $+\infty$ case is completely analogous). Since A, B and ε are constants ($\varepsilon < 1$) and $\lim_{t \to t^{\Delta +}} f(t) = -\infty$ (this is what is meant by both particles emerging from infinity $-\infty$ at $t = t^{\Delta}$), for all instants after $t = t^{\Delta}$ but sufficiently close to t^{Δ}, $f(t) < (B - A)/(1 - \varepsilon)$ is satisfied and, consequently, $x_{fast} < x_{slow}$. The faster particle lags behind the slower particle: it trails the slower particle. Indeed, in some cases, the fast particle will always trail the slow particle. For example, if $x_{fast} = -2/t$ and $x_{slow} = -1/t$, therefore $A = B = t^{\Delta} = 0$, $f(t) = -2/t$ and $\varepsilon = 1/2$. Consequently, $f(t) < (B - A)/(1 - \varepsilon)$ is trivially satisfied for every $t > 0$, i.e. $x_{fast} < x_{slow}$ will be permanent.

3. The infinite in the finite.

The analysis above has shown that if two particles emerge from infinity at a certain instant, with one moving faster than the other in the same direction, there are points in space where the slow particle will arrive before the fast particle. In order to turn this into an interesting paradox, infinity needs to be "zoomed in" (placed in a finite context) by making use of curves of infinite length. Consider a hyperbolic spiral H_S (Figure 1), its origin O and any point P ($\neq$ O) on it. The arc length between O and P is infinite (the curve gives a number increasing to infinity of turns as it approaches O). Nevertheless, it is locally rectifiable everywhere except at O (any closed fragment of the curve that does not contain O has finite length). We parameterize the curve with the parameter s so that $s(P) = 0$ and $s(O) = +\infty$. There is therefore a natural metric associated with s, d_s. The arc length $d_s(P_i, P_j)$ between any two points P_i and P_j ($P_i, P_j \in H_S$, $P_i, P_j \neq O$) in Figure 1 is finite, yet $d_s(P, O) = \infty$. Moreover, point O on the hyperbolic spiral H_S plays a role that is analogous to one of the points of infinity on $\overline{\Re}$ (usually denoted as $-\infty$). This means that $d_s(P, O) = \infty$ for every $P \neq O$, analogous to how, intuitively speaking, the usual Euclidean distance d between the point at infinity $-\infty$ of the extended real straight line $\overline{\Re}$ and a point $X \in \Re$ is also ∞. However, this analogy is only partial because there is a clear way in which point O has been "zoomed in" on H_S from infinity by being placed in a finite context. In effect, H_S exists on a plane equipped with the usual Euclidean metric in two dimensions, d^2. Therefore, in the sense of natural metric d^2, O is at a finite distance from the other points on curve H_S despite being at an infinite distance from all of them in the sense of metric d_s induced by the arc length. By approaching infinity in this way, the use of a non-rectifiable curve (curve of infinite length) enables the paradox ("when the fastest runner makes the least progress") to be revealed in all its glory.

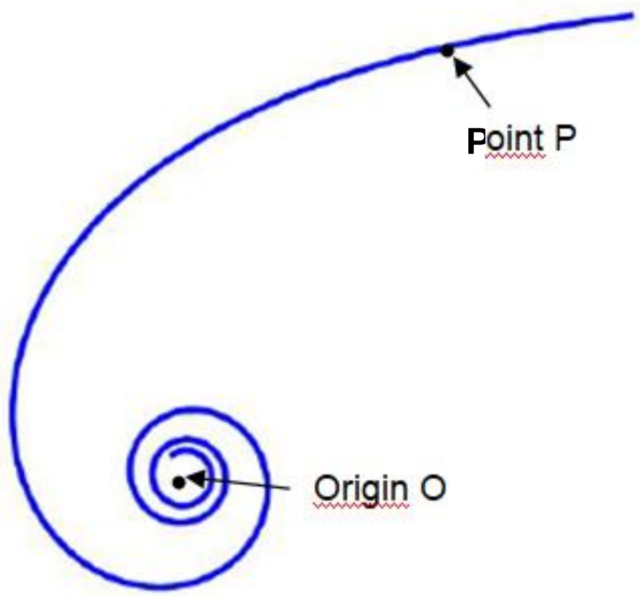


Figure 1

4. The argument

Rather than discuss particles emerging from infinity, the focus will now be on runners who, in a finite environment, follow non-rectifiable trajectories (travel around non-rectifiable curves). To singularise the context, the runners will be the swift Achilles and the slow Tortoise, the immortal characters in one of Zeno's classic paradoxes. It shall be seen below that, on a non-rectifiable curve (exemplified by H_S in Figure 1), the tortoise can always win a race against Achilles.

Turning now to the problem analysis, it can be assumed without loss of generality that Achilles completes the race in unit time. His law of motion when starting from O at t =

0 is $s_A(t) = F(t)$, at velocity $v_A(t) = F'(t)$. The argument hereafter bears many similarities to the argument in Section 2 above, but will, however, be formulated in more conceptual and geometric, and less algebraic, terms. $F(t)$ must be a strictly decreasing continuous function of t in interval $0 < t < 1$ (given the parameterisation performed) with $F(1) = 0$ (Achilles reaches P, $s = 0$, at $t = 1$) and $\lim_{t\to 0+}F(t) = +\infty$ (Achilles leaves O, $s = +\infty$, at $t = 0$). An example might be $F(t) = (1 - t)/t$, but our reasoning will be general. Figure 2 shows the Achilles world line in the one-dimensional space defined by parameter s.

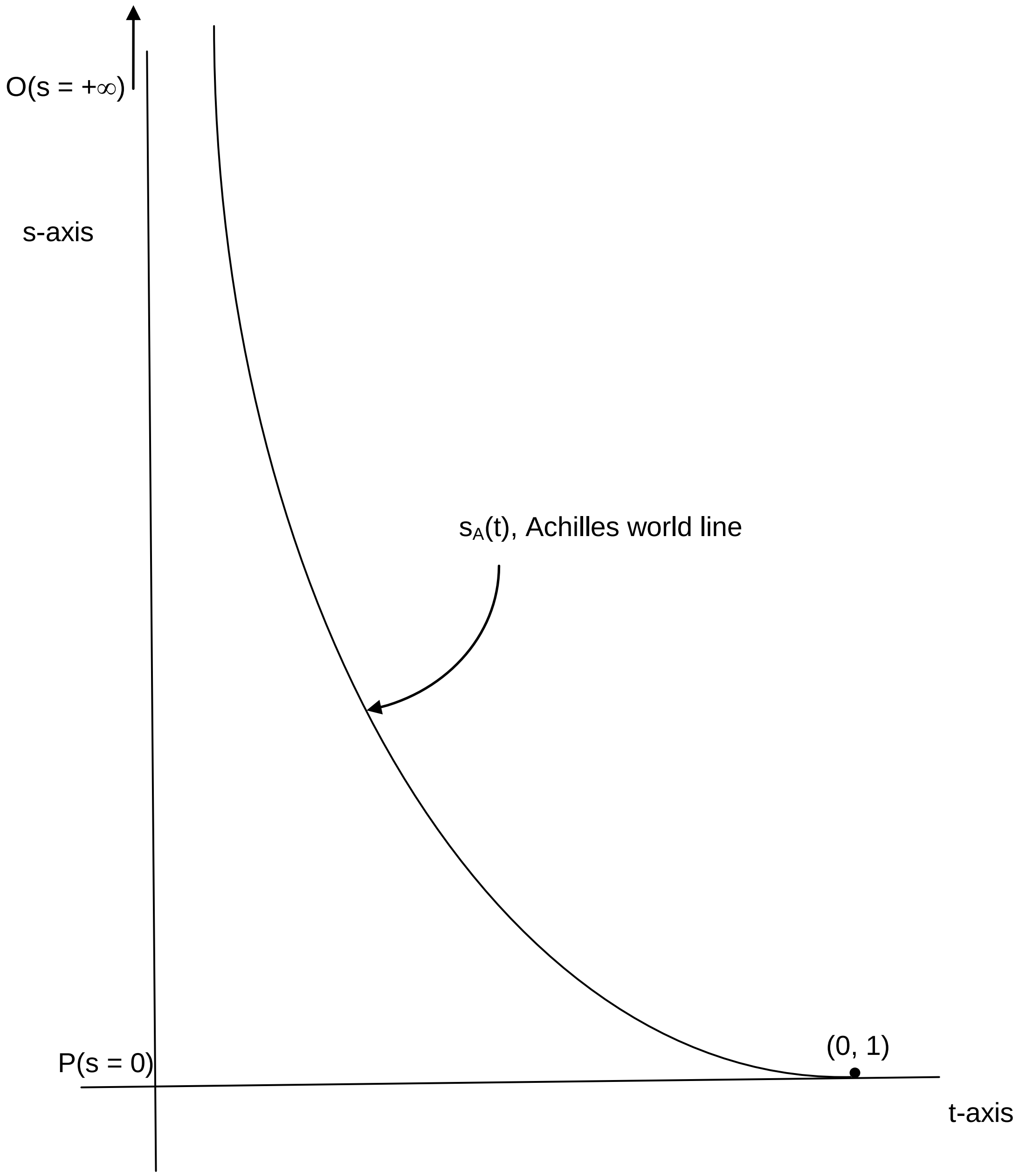


Figure 2

In order for the tortoise to execute its winning strategy every time, it takes note of Achilles' locations $s_A(t)$ at all instants during the race and decides to follow the law of motion $s_T(t) = G(t) = \Lambda F(t) - K = \Lambda s_A(t) - K$, where K is an arbitrary positive number and $0 < \Lambda < 1$. To appreciate the consequences of this, first consider a fictitious runner's movement, W, who moves according to the law of motion $s_W(t) = W(t) = \Lambda F(t) = \Lambda s_A(t)$, and whose world line is shown in Figure 3 ("intermediate world line"). As can be seen, they reach the finish line at the same time as Achilles even though their speed, $v_W(t) = W'(t) = \Lambda F'(t) = \Lambda v_A(t)$, is slower at all times.

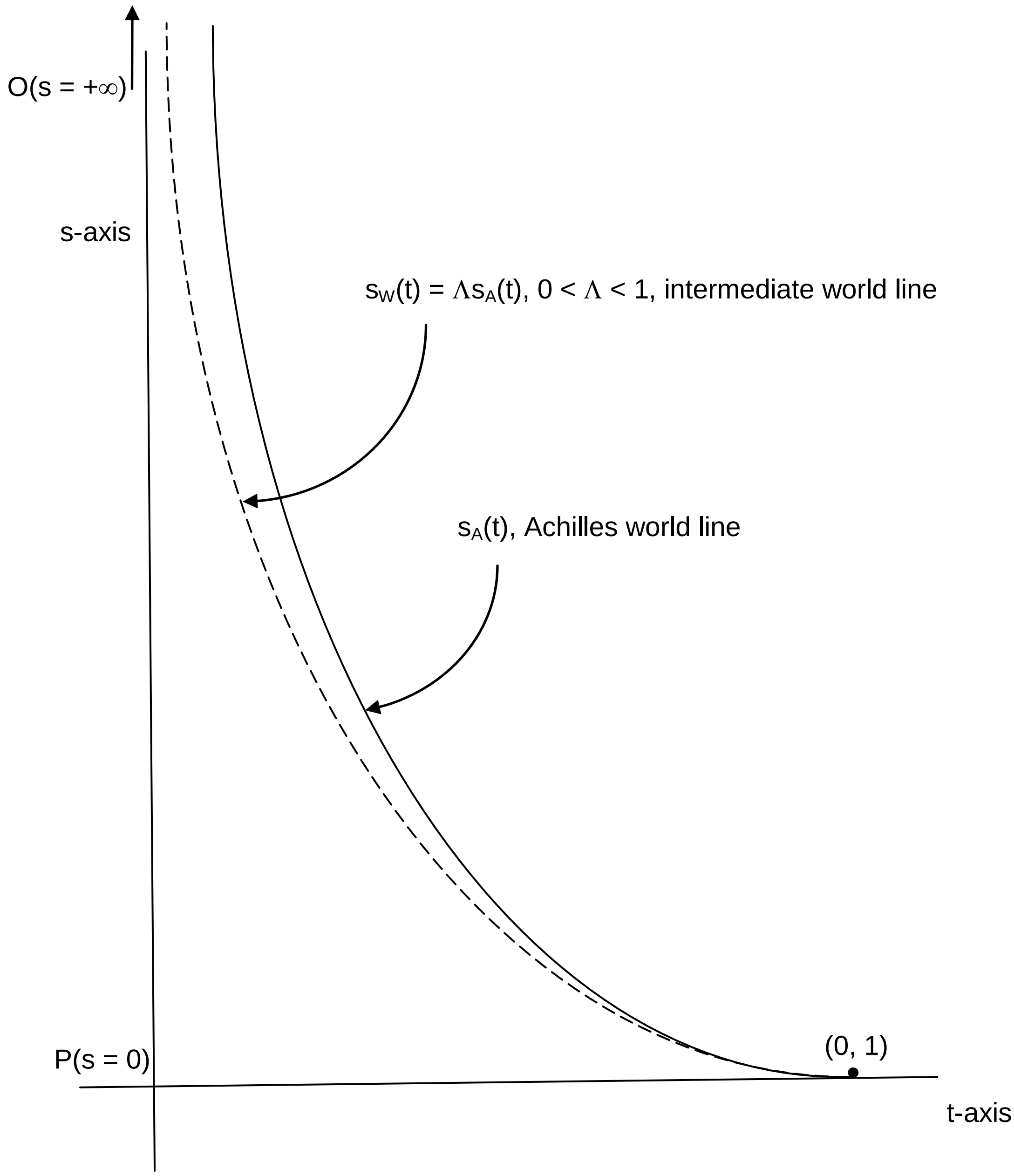


Figure 3

What is of interest, the tortoise world line, is now obtained from $s_W(t)$ by moving the latter K units vertically downwards, as shown in Figure 4. Thus, in effect, $s_T(t) = G(t) = \Lambda F(t) - K = \Lambda s_A(t) - K$.

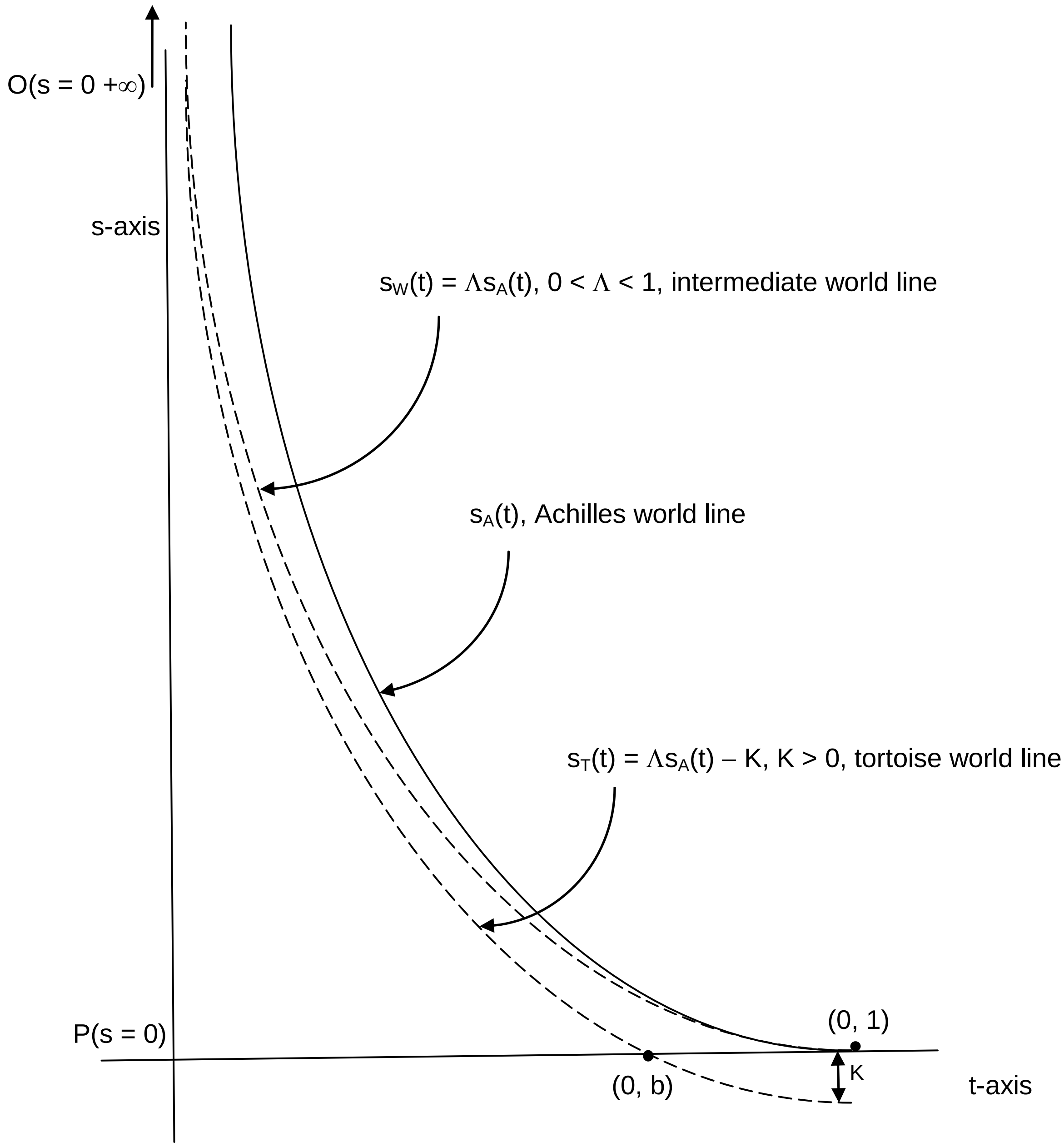


Figure 4

The situation will now be described more formally. Obviously $v_T(t) = G'(t) = \Lambda F'(t) = \Lambda v_A(t)$. As with F(t), G(t) is defined on (0, 1], is continuous and strictly decreasing in this interval. It therefore has one root at most. Since K > 0, it has exactly one root. Indeed, number b where G(b) = 0 is the solution of the equation $0 = \Lambda F(b) - K$, that is, $F(b) = K/\Lambda > 0$. We also know this b exists and is well-defined because F(t) (which is defined on (0, 1]) is continuous and strictly decreasing, $\lim_{t\to 0+} F(t) = +\infty$, and F(1) = 0, from which it follows that it is invertible and its range of values is the interval $[0, +\infty)$. Moreover, it is $0 < b < 1$ because $F(b) = K/\Lambda \neq 0 = F(1)$.

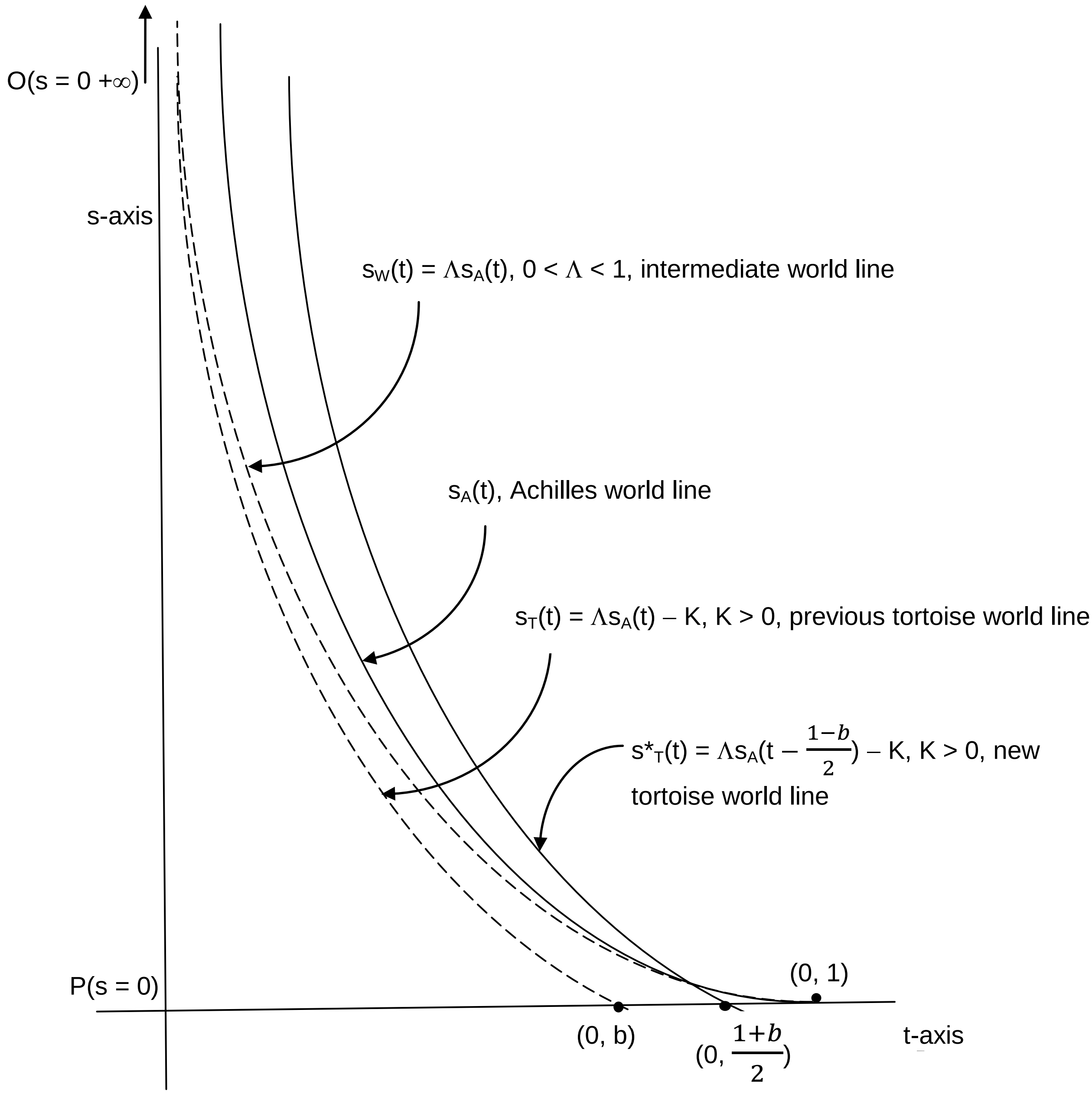
Therefore, the law of motion $s_T(t) = G(t) = \Lambda F(t) - K = \Lambda s_A(t) - K$ that the tortoise follows (with $K > 0$) provides the tortoise with a winning strategy. It reaches finish P at instant $t = b < 1$. Achilles, on the other hand, will finish at $t = 1$, as we know. Conclusion: the tortoise will always win.

The argument in context

Zeno's paradoxes have had extensive influence in philosophy, particularly in the philosophy of mathematics and philosophy of space and time physics. Huggett (2024) and Dowden provide a brief overview of Pythagoreanism, atomism, continuum, constructivism, infinitesimals, non-standard analysis and supertasks to this effect. Here I have sought to do something different. Rather than intervene in the controversies surrounding the meaning of Zeno's arguments, I intend to frame them in a different context from the usual. To this end, a paradox (veridical, in Quine's sense 1976) is presented which may help to enrich and broaden the current debate on space, time and motion. Two very simple examples in this respect are worth mentioning here:

1) In Zeno's original version, Achilles grants the tortoise a finite (spatial) advantage before starting the race. In my version of hyperbolic spiral H_S, no such thing is possible for purely logical reasons. There is no point on H_S that is at a finite distance from origin O when the distance is measured along the actual H_S.

2) It is clear that if Achilles grants the tortoise a time advantage (allowing it to start the race w time units before $t = 0$, for example at $t = -w$ with $w > 0$), its law of motion will obviously enable it to beat him, arriving at the finish line at instant $t = b - w < b < 1$. Simply move world line $s_T(t)$ in Figure 4 w time units to the left in order to see this. A more interesting case is when it is the tortoise that grants Achilles a time advantage by starting w time units after instant $t = 0$ (which is the instant at which Achilles starts). Barring this delay, causing the tortoise to start later, it is assumed that it moves according to the same continuous series of velocities as before. In order to see what will happen now, simply move world line $s_T(t)$ in Figure 4 w time units to the right. When $w = (1 - b)/2$, the new tortoise world line $s^*_T(t)$ in Figure 5 is obtained.



Indeed, since $s^*_T(t) = s_T(t - (1-b)/2) = \Lambda s_A(t - (1-b)/2) - K = \Lambda F(t - (1-b)/2) - K$, the tortoise will now reach the finish line ($s^*_T(t) = 0$) when $\Lambda F(t - (1-b)/2) - K = 0$, i.e. when $F(t - (1-b)/2) = \Lambda/K$. As we already know that $F(b) = K/\Lambda$, it must follow that $t - (1-b)/2 = b$, i.e. $t = (1 + b)/2$. This is the instant at which the tortoise will reach the finish line, as shown in Figure 5. It will do so before Achilles:

I) despite having started the race later

and

II) even though the magnitudes of velocities $v^*_T(t) = \Lambda F'(t- (1-b)/2)$ enjoyed by the tortoise from instant $t = (1-b)/2 > 0$, when it started the race, will always be a fraction Λ smaller ($0 < \Lambda < 1$) than the magnitudes of velocities $v_A(t) = F'(t)$ enjoyed by Achilles from instant $t = 0$, when he started the race.

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