Review of “The Classification of General Affine Connections in Newton–Cartan Geometry: Towards Metric-Affine Newton–Cartan Gravity”, by Philip K. Schwartz (*Classical and Quantum Gravity* 42 015010, 2024)

Eleanor March* & James Read†

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In recent years, there has been heightened interest in (at least) two threads regarding geometrical aspects of spacetime theories. On the one hand, physicists have explored a richer space of relativistic spacetime structures than that of general relativity, in which the conditions both of torsion-freeness and of metric compatibility are relaxed—this has led to the study of so-called ‘metric-affine theories’ of gravitation, on which see e.g. Hehl et al. (1995) for a masterly review. On the other hand, physicists have been increasingly interested in securing a rigorous and fully general understanding of the non-relativistic limit of general relativity—this has to novel version of Newtonian physics, potentially with spacetime torsion (‘Type II’ Newton–Cartan theory—see Hansen et al. (2022) for a systematic overview).

Only recently have physicists begun to bring these two threads into contact with one another. Read and Teh (2018) and Schwartz (2023) showed that the non-relativistic limit of ‘teleparallel gravity’ (a geometrical alternative to general relativity with spacetime torsion rather than spacetime curvature, and as such a special case of a metric-affine theory of gravitation) yields a novel non-relativistic spacetime theory, which (with suitable ‘gauge fixing’) yields standard, potential-based Newtonian gravitation theory; Wolf et al. (2024) then generalised this work by showing that the entire ‘geometric trinity’ of gravitational theories (on which see Beltrán Jiménez et al. (2019) for a review; the third node of this trinity is ‘symmetric teleparallel gravity’, which is a theory with non-metricity but no curvature or torsion)—all of which again are special cases of metric-affine theories of gravity—has a non-relativistic limit, yielding a novel, non-relativistic geometric trinity of gravitational theories. The common structure of this non-relativistic trinity was then identified by March et al. (2024).

*eleanor.march@philosophy.ox.ac.uk
†james.read@philosophy.ox.ac.uk

Up to the article under review here (Schwartz 2024), however, no *general* study of non-relativistic spacetime structures (with connections manifesting both torsion and non-metricity) has been developed; Schwartz rectifies the situation and presents the general result, which is that the connection coefficients of a fully general non-relativistic connection take the form

$$\Gamma^{\rho}_{\mu\nu} = v^{\rho}\partial_{\mu}\tau_{\nu} + \frac{1}{2}h^{\rho\sigma}(\partial_{\mu}h_{\nu\sigma} + \partial_{\nu}h_{\mu\sigma} - \partial_{\sigma}h_{\mu\nu}) \\ + \frac{1}{2}P^{\rho}_{\lambda}T^{\lambda}{}_{\mu\nu} - T_{(\mu\nu)}{}^{\rho} + \tau_{(\mu}\Omega_{\nu)}{}^{\rho} - \frac{1}{2}Q^{\rho}{}_{\mu\nu} + P^{\rho}_{\lambda}Q_{(\mu\nu)}{}^{\lambda} - v^{\rho}\hat{Q}_{\mu\nu},$$

where τ_{μ} and $h^{\mu\nu}$ are the temporal and spatial metrics (respectively) as usual for a Galilean structure, $T^{\mu}{}_{\nu\lambda}$ is the torsion tensor, $\hat{Q}_{\mu\nu} := \nabla_{\mu}\tau_{\nu}$ and $Q_{\rho}{}^{\mu\nu} := \nabla_{\rho}h^{\mu\nu}$ are the non-metricities, $\mathbf{\Omega}$ is the Newton–Coriolis 2-form with respect to a unit timelike vector v^{μ}, and $P^{\rho}{}_{\lambda} := \delta^{\rho}{}_{\lambda} - v^{\mu}\tau_{\nu}$ is the spatial projector along v^{μ}.

Before we proceed further, two *caveat emptors* which are not stressed explicitly by Schwartz. First: it is somewhat misleading to describe *any* connection as being specifically (i) 'relativistic', (ii) 'non-relativistic', or even (more on which below) (iii) 'ultra-relativistic'. This is because any connection can be decomposed into a combination of (a) a 'Levi-Civita-like' term, (b) torsion terms, (c) non-metricity terms, and (d) terms associated with the non-uniqueness of the compatible connection—for *any* of (i)–(iii)! This point is stressed by Vigneron et al. (2025), but bears emphasising in order to forestall any misconception that there is anything distinctively 'non-relativistic' about any connection. Second: as stressed by Schwartz (2024, p. 7), but also by Vigneron et al. (2025) and Wolf et al. (2024), for non-relativistic (and indeed ultra-relativistic—again, more on this below) connections with both torsion and non-metricity, these two geometric properties are *not* independent of one another.

In any case, after presenting this general result, Schwartz then moves on, in §4 of his article, to consider general non-relativistic connections from the principal fibre bundle point of view, demonstrating how the objects of a Galilean structure with connection can be defined from tetrads and connections on a principal bundle with the (orthochronous) homogeneous Galilean group as its structure group (Schwartz denotes this $G(M)$, which is the general linear frame bundle $F(M)$ with appropriate reduction of the structure group). Interestingly, the connection with which one works here is not a connection form, because it takes values in the wrong Lie algebra (one is dealing with a $\mathfrak{gl}(n+1)$-valued one-form on $G(M)$)—we return to this below.

As already mentioned above, this article is a significant contribution to the literature, delivering a substantially more general perspective on possible non-relativistic geometries than was hitherto available. The results of Schwartz (2024) provide the foundations for explorations of possible non-relativistic spacetime structures (in the context of the non-relativistic 'trinity') which have already been undertaken by Wolf et al. (2024). But the work also invites quite naturally a number of further explorations, on which we now comment:

1. Construct a generalised metric-affine theory of Newton–Cartan gravity.

2. Appraise the extent to which metric-affine Newton–Cartan gravity (or sectors thereof—see below) can be understood in the framework of Cartan geometry (on which see e.g. March et al. (2025)).[1]

3. Explore the gauge freedom inherent in different sectors of metric-affine Newton–Cartan gravity.

4. Explore the extent to which these constructions carry over to the case of ultra-relativistic gravity.

Let's take these in turn.

Ad (1): Hehl et al. (1995, §5) consider general Lagrangians for field theories which are functions of relativistic (i.e., Lorentzian) structures and associated general connections and their torsions and non-metricities. This raises the question of whether it would be possible to construct and study analogous Lagrangians in the non-relativistic case—analogous in the sense that they be functions of Galilean structures and general connections, again with associated torsions and non-metricities. In general, we see no roadblocks to being able to do this—and, indeed, the analysis of e.g. Noether currents undertaken by Hehl et al. (1995, §5) should (one hopes) carry over to that context. That said, there will be subtleties: for standard textbook Newton–Cartan theory (what Hansen et al. (2019a) call 'Type I' Newton–Cartan theory), it is known that no action principle exists; this carries over to the entire non-relativistic geometric trinity of gravity (see Wolf et al. (2024)). Hence, an analysis of variational principles will *not* apply to all theories constructed using the general non-relativistic connections presented by Schwartz. On the other hand, it is also known that there are sectors of torsionful, non-metric Newton–Cartan gravitation—dubbed 'Type II' Newton–Cartan theory by Hansen et al. (2019b)—for which action principles *do* exist; therefore, one would expect to be able to recover (at least) these theories from a variational approach to metric-affine Newton–Cartan gravity.

In light of this, one might wonder about alternative strategies for pursuing (1)—in particular, for understanding the relationship between the non-relativistic geometric trinity of gravity and sectors of metric-affine Newton–Cartan gravity. For example, it is well-known that the general theory corresponding to the case of torsionful metric connections (with curvature)—often called torsional Newton–Cartan gravity—can be obtained through a process of 'gauging' the Bargmann algebra (on which, see, e.g., Andringa et al. (2011)). One then recovers the flat torsionful (respectively non-flat, torsion-free) sectors of that theory, and corresponding two nodes of the geometric trinity, by restricting to the cases where the curvatures (respectively torsions) associated with translations (respectively rotations) vanish. This provides a way of isolating the subgroups of the Bargmann group corresponding to the torsionful and

[1] Somewhat relatedly, it might be nice to consider the extent to which the fibre bundle perspective of non-relativistic spacetime structures offered by Schwartz (2024, §4) is amenable to geometrical reformulation *à la* Gomes (2024). But here is not the place to explore that possibility.

curvature-based nodes of the non-relativistic geometric trinity. In principle, this same strategy is available for general theories of metric-affine Newton–Cartan gravity. That said, the Bargmann algebra is not strictly a subalgebra of the Lie algebra of the general linear group—unlike in the relativistic case—rather, it is a central extension of such a Lie algebra. This is crucial for recovering the gravitational potential in torsional Newton–Cartan gravity as a torsion associated by the generator of mass translations—i.e., mass-torsion. So, while we don't see any conceptual roadblock to (1) *per se*, evidently there will be subtleties involved in the strategy chosen for realising (1), and there is further work to be done understanding e.g. the scope for a centrally-extended version of the Lie algebra for metric-affine Newton–Cartan gravity analogous to the Bargmann algebra for the metric sector of the theory.

Ad (2): As already discussed above, when Schwartz offers a fibre bundle perspective on general non-relativistic connections, he does so by way of a connection on $G(M)$ valued in a larger Lie algebra. This reminds one of the geometry of Cartan connections (see Sharpe (2000) for mathematical background, and March et al. (2025) for discussion in the context of geometric alternatives to general relativity)—and recall that the connections between metric-affine theories of gravity and Cartan geometry have already been noted in the relativistic context by François and Ravera (2025). As such, it would be worthwhile exploring the extent Cartan geometry can be brought into contact with these non-relativistic metric-affine theories. Note though that again there will be roadblocks here: in Schwartz' case, one Lie group is not a *normal* subgroup of the other, which seems to preclude reading in terms of standard Cartan geometry.

Ad (3): It is well-known (see, e.g., March et al. (2024) and Schwartz (2023)) that both the torsionful and non-metric nodes of the non-relativistic geometric trinity exhibit gauge freedom in relation to Newton–Cartan theory, in that there are multiple distinct (non-isomorphic) models of these theories which correspond to the same model of Newton–Cartan theory. (This is, of course, also true of the relativistic geometric trinity in relation to GR—but the situation is less straightforward in the non-relativistic case, because agreement on the metrics is not sufficient to recover the same model of the torsion-free metric sector of the theory as in the relativistic case.) It would be of interest to systematically characterise the extent to which various different sectors of metric affine Newton–Cartan theory exhibit this kind of gauge freedom (whether in relation to Newton–Cartan theory or one another).

Ad (4): This article also raises the question as to whether it would be possible to construct a result analogous to Schwartz' Theorem 4—i.e., a general theory of *ultra*-relativistic structures. (Recall that ultra-relativistic, or Carrollian, structures, are what arises when one takes the $c \to 0$ limit of relativistic spacetime structures—see March and Read (2025) for a primer.) This question was, indeed, already asked and answered by Vigneron et al. (2025). Naturally, many of the questions raised above would then carry over to the the context of ultra-relativistic gravity.

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