On coordinate-based and coordinate-free approaches to Maxwellian spacetime

Eleanor March*

Abstract

I discuss and clarify the relationship between the recent wave of 'intrinsic' coordinate-free approaches to Maxwell gravitation and the coordinate-based discussions of Saunders (2013) and Wallace (2020).

1 Introduction

In recent years, philosophers of physics have considered afresh the question of the appropriate spacetime setting for Newtonian gravitation theory. At the center of this debate have been two apparently conflicting proposals for what one should take this geometry to be: on the one hand, Saunders' (2013) proposal that Corollary VI to the Laws of Motion in Newton's *Principia* reveals that Maxwellian spacetime is the correct setting for Newtonian physics, and on the other hand, Knox's (2014) proposal that Corollary VI motivates a transition to a geometrized formulation of Newtonian gravitation, known as Newton-Cartan theory (NCT). Their claims have sparked a series of discussions of theories of Newtonian gravitation set on Maxwellian spacetime, and their relationship to NCT (Weatherall 2016; Teh 2018; Jacobs 2023; March, Wolf, and Read 2024; March 2024; Dewar 2018; Chen 2023).

One focus of these discussions has been on how Maxwellian spacetime—which is supposed to be equipped with a standard of rotation, but not a standard of absolute acceleration—should best be characterized. Earman (1989) originally defined the standard of rotation as an equivalence class of derivative operators, and Dewar (2018) also adopted this definition. But a number of authors have voiced concerns about this approach. For example, Weatherall (2018, 34) notes that it "makes reference to structure that one does not attribute to spacetime," Jacobs (2022) argues that it is not suitably "intrinsic,"[1] and Wallace (2019, 2020) even suggests that the awkwardness of differential-geometric presentations of Maxwellian spacetime obscures the similarities between NCT and theories of Newtonian gravitation set on Maxwellian spacetime, and (more generally) shows that coordinate-free differential geometry is not an intuitive way of characterizing certain spacetime structures. In response to these concerns, Weatherall (2018) developed an 'intrinsic' characterization of the standard of rotation, and Chen (2023) and March (2023) have recently shown that this object can be used to write down dynamics for Newtonian gravitation on Maxwellian spacetime, a.k.a. Maxwell gravitation (MG).

However, this new wave of (coordinate-free differential-geometric) presentations of MG are somewhat removed from Saunders' original coordinate-based

*Faculty of Philosophy, University of Oxford. eleanor.march@philosophy.ox.ac.uk

1. See also Dürr and Read (2019, 1094–96).

'vector relationism' (VR). It would be of interest to see how these fit together. It also remains unclear how Wallace's (2020) own (also coordinate-based) discussion of VR and NCT relates to the approaches outlined above.

In this paper, I aim to fill in these remaining pieces of the puzzle, by (a) making precise the relationship between VR and MG, and (b) translating Wallace's argument into the language of coordinate-free differential geometry. I thereby (i) clarify how Wallace's argument relates to other arguments concerning the (in)equivalence of MG and NCT in the literature, and (ii) address Wallace's concern that coordinate-free presentations of MG obscure its similarities to NCT. Indeed, I will argue, the same similarities Wallace discusses can be seen very naturally from a coordinate-free differential-geometric standpoint. Finally, this (iii) gives us the resources to connect up to Teh's (2018) discussion of Wallace and VR, in which he also claims put Wallace into the language of coordinate-free differential geometry.

In more detail, the structure of this paper is as follows. In §2, I recall some basic details of MG and NCT. I then turn to the task of connecting these coordinate-free approaches with the work of Saunders (2013) and Wallace (2020). In §3, I present Saunders' VR, and make precise its relationship to MG. §4 reconstructs Wallace's argument that VR and NCT are theoretically equivalent; §5 aims to dispel the remainder of Wallace's concerns about coordinate-free presentations of Maxwellian spacetime by showing that the same argument can be made in the language of coordinate-free differential geometry. To end, in §6, I compare my approach to that of Teh (2018). §7 concludes.

2 Background: Maxwell Gravitation and Newton-Cartan Theory

This section reviews some basic details of the coordinate-free approaches to MG and NCT as presented in e.g. Chen (2023), March (2024), and Malament (2012)—readers familiar with this material should feel free to skip to the next section. Let M be a smooth four-manifold (assumed connected, Hausdorff, and paracompact). A temporal metric t_a on M is a smooth, closed, non-vanishing one-form;[2] a spatial metric h^{ab} on M is a smooth, symmetric, rank-$(2,0)$ tensor field which admits, at each point in M, a set of four non-vanishing covectors $\overset{i}{\sigma}_a$, $i = 0, 1, 2, 3$, which form a basis for the cotangent space and satisfy $h^{ab}\overset{i}{\sigma}_a\overset{j}{\sigma}_b = 1$ for $i = j = 1, 2, 3$ and 0 otherwise. A spatial and temporal metric are orthogonal iff $h^{an}t_n = \mathbf{0}$. A vector field σ^a is spacelike iff $t_n\sigma^n = 0$, and timelike otherwise. Given the structure defined here, t_a induces a foliation of M into spacelike hypersurfaces, and relative to any such hypersurface, h^{ab} induces a unique spatial derivative operator D such that $D_a h^{bc} = \mathbf{0}$.[3] h^{ab} is flat just in case for any such spacelike hypersurface, D commutes on spacelike vector fields, i.e. $D_{[a}D_{b]}\sigma^c = \mathbf{0}$ for all spacelike vector fields σ^a.

Given a structure $\langle M, t_a, h^{ab}\rangle$, with t_a and h^{ab} orthogonal, we can consider two further pieces of structure on $\langle M, t_a, h^{ab}\rangle$: a compatible connection ∇,[4]

2. Here and throughout, abstract indices are written in Latin script; component indices are written in Greek script, with the exception of i, j, k, which are reserved for the spatial components of tensor fields in some coordinate basis; and the Einstein summation convention is used. Round brackets denote symmetrization, square brackets antisymmetrization.

3. See Weatherall (2018, 37–38) and Malament (2012, §4.1) for details.

4. Recall that a connection is compatible with the metrics just in case $\nabla_a t_b = \mathbf{0}$ and $\nabla_a h^{bc} = \mathbf{0}$.

and a compatible standard of rotation $\circlearrowright$.[5] In what follows, we will often want to consider connections and standards of rotation which 'agree' with one another in the following sense: for any vector field η^a on M, $\nabla^{[a}\eta^{b]} = \circlearrowright^a \eta^b$. In this case, following March (2024), I will say that the connection and standard of rotation are *compatible*.[6] Likewise, a connection ∇ is compatible with a spacetime $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ just in case it is compatible with the metrics and $\circlearrowright$. Finally, a spacetime $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ is *rotationally flat* just in case h^{ab} is flat and there exists a unit timelike vector field ξ^a on M such that $\circlearrowright^a \xi^b = \mathbf{0}$ and $\pounds_\xi h^{ab} = \mathbf{0}$,[7] or equivalently, just in case some flat derivative operator is compatible with $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ (Weatherall 2018, Proposition 1). I will call a structure $\langle M, t_a, h^{ab}, \circlearrowright\rangle$, with M diffeomorphic to $\mathbb{R}^4$, and $\langle M, t_a, h^{ab}\rangle$ complete (in the sense that every spacelike hypersurface is geodesically complete with respect to D),[8] a Maxwellian spacetime, and a structure $\langle M, t_a, h^{ab}, \nabla\rangle$ (under the same conditions) a Newton-Cartan spacetime.

In both MG and NCT, we will assume that matter fields are associated with a symmetric rank-$(2,0)$ tensor T^{ab},[9] called the mass-momentum tensor, which is assumed to satisfy the *Newtonian mass condition*: whenever $T^{ab} \neq \mathbf{0}$, $T^{nm}t_n t_m > 0$. This captures the idea that the matter fields we are interested in are massive, in the sense that there can only be non-zero mass-momentum in spacetime regions where the mass density $\rho := T^{nm}t_n t_m$ is strictly positive.[10] Since T^{ab} is symmetric, the Newtonian mass condition guarantees that whenever $T^{ab} \neq \mathbf{0}$, we can uniquely decompose T^{ab} as

$$T^{ab} = \rho\xi^a\xi^b + \sigma^{ab},$$

where $\xi^a := \rho^{-1}t_n T^{na}$ is a smooth unit timelike future-directed vector field (interpretable as the net four-velocity of the matter fields F), and σ^{ab} is a smooth symmetric rank-$(2,0)$ tensor field which is spacelike in both indices (interpretable as the stress tensor for F).

We are now in a position to introduce MG and NCT. I will begin with MG.[11] Let $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ be a Maxwellian spacetime, and let T^{ab} be the mass-momentum tensor for whichever matter fields are present. Then $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab}\rangle$ is a model of *Maxwell gravitation* just in case

(i) $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ is rotationally flat; and

5. This was introduced by Weatherall (2018): if t_a, h^{ab} are orthogonal temporal and spatial metrics on M, a standard of rotation $\circlearrowright$ compatible with t_a and h^{ab} is a map from smooth vector fields ξ^a on M to smooth, antisymmetric rank-$(2,0)$ tensor fields $\circlearrowright^b \xi^a$ on M, such that (i) $\circlearrowright$ commutes with addition of smooth vector fields; (ii) given any smooth vector field ξ^a and smooth scalar field α, $\circlearrowright^a (\alpha\xi^b) = \alpha \circlearrowright^a \xi^b + \xi^{[b}d^{a]}\alpha$; (iii) $\circlearrowright$ commutes with index substitution; (iv) given any smooth vector field ξ^a, if $d_a(\xi^n t_n) = \mathbf{0}$ then $\circlearrowright^a \xi^b$ is spacelike in both indices; and (v) given any smooth spacelike vector field σ^a, $\circlearrowright^a \sigma^b = D^{[a}\sigma^{b]}$.

6. See Weatherall (2018, proposition 1); any connection determines a unique compatible standard of rotation, but a standard of rotation does not determine a unique compatible connection.

7. Here and throughout, $\pounds$ denotes the Lie derivative.

8. These conditions could be dropped; I adopt them here to ease comparison with Saunders' vector relationism in §3.

9. One might take the symmetry of T^{ab} as a postulate, as in e.g. Malament (2012), or to follow from a variational definition—see e.g. Duval and Künzle (1978) and Weatherall (2019).

10. For example, Weatherall (2012, 211) suggests that "[one] might take [the Newtonian mass condition] to be a benign and unsurprising characterization of what we mean by 'massive particle' in Newtonian gravitation."

11. For details on the relationship between this way of presenting MG and the approach of Dewar (2018), see Chen (2023).

(ii) For all points $p \in M$ such that $\rho \neq 0$, the following equations hold at p:

$$\pounds_\xi \rho - \frac{1}{2}\rho \hat{h}_{mn} \pounds_\xi h^{mn} = 0 \tag{1a}$$

$$\frac{1}{3}\sum_{i=1}^{3} \overset{i}{\lambda}_r \xi^n \Delta_n(\xi^m \Delta_m \overset{i}{\lambda}{}^r) = -\frac{4}{3}\pi\rho - \frac{1}{3}D_m(\rho^{-1}D_n\sigma^{nm}) \tag{1b}$$

$$\pounds_\xi(\circlearrowright^c \xi^a) + 2(\circlearrowright^n \xi^{[c})\hat{h}_{nm}\pounds_\xi h^{a]m} + \circlearrowright^c(\rho^{-1}D_n\sigma^{na}) = \mathbf{0}, \tag{1c}$$

where $\hat{h}_{ab}$ is the spatial metric relative to ξ^a,[12] the $\overset{i}{\lambda}{}^a$ are three orthonormal connecting fields for ξ^a, and Δ is the "restricted derivative operator" defined in Weatherall (2018, 36–37). This acts on arbitrary spacelike vector fields σ^a at a point p according to

$$\eta^n \Delta_n \sigma^a := \pounds_\eta \sigma^a + \sigma_n \circlearrowright^n \eta^a - \frac{1}{2}\sigma_n \pounds_\eta h^{an}$$

where η^a is a unit timelike vector at p (the Lie derivative is taken with respect to any extension of η^a off of p). It also has the property that $\eta^n\Delta_n\sigma^a = \eta^n\nabla_n\sigma^a$ for any derivative operator ∇ compatible with $\circlearrowright$ (37).

For NCT, let $\langle M, t_a, h^{ab}, \nabla\rangle$ be a Newton-Cartan spacetime, and T^{ab} the mass-momentum tensor for whichever matter fields are present. Then $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of *Newton-Cartan theory* just in case

$$\nabla_n T^{na} = \mathbf{0} \tag{2a}$$

$$R_{ab} = 4\pi\rho t_a t_b \tag{2b}$$

$$R^a{}_b{}^c{}_d = R^c{}_d{}^a{}_b \tag{2c}$$

$$R^{ab}{}_{cd} = \mathbf{0}. \tag{2d}$$

The relationship between MG and NCT is summarized by the following pair of propositions (Chen 2023; March 2023):

Proposition 1. *Let $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ be a model of Newton-Cartan theory. Then there exists a unique standard of rotation $\circlearrowright$ such that ∇ is compatible with $\circlearrowright$ and $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab}\rangle$ is a model of Maxwell gravitation.*

Proposition 2. *Let $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab}\rangle$ be a model of Maxwell gravitation. Then there exists a derivative operator ∇ compatible with $\circlearrowright$ such that $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of Newton-Cartan theory. Moreover, the derivative operator ∇ is not unique. If ∇ is such a derivative operator, then so is $(\nabla, t_b t_c \sigma^a)$,[13] where σ^a is any spacelike, twist-free, and divergence-free vector field such that $\rho\sigma^a = \mathbf{0}$.*

Corollary 2.1 (Chen, 2023)**.** *Let $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab}\rangle$ be a model of Maxwell gravitation such that $\rho \neq 0$ throughout some open region O. Then there exists a unique derivative operator ∇ compatible with $\circlearrowright$ such that $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of Newton-Cartan theory.*

12. That is, the unique symmetric tensor field on M such that $\hat{h}_{an}\xi^n = \mathbf{0}$ and $h^{an}\hat{h}_{nb} = \delta^a{}_b - t_b\xi^a$.

13. The notation here follows Malament (2012, proposition 1.7.3): $\nabla' = (\nabla, C^a{}_{bc})$ iff for all smooth tensor fields $\alpha^{a_1\ldots a_r}{}_{b_1\ldots b_s}$ on M,

$$(\nabla'_n - \nabla_n)\alpha^{a_1\ldots a_r}{}_{b_1\ldots b_s} = \alpha^{a_1\ldots a_r}{}_{mb_2\ldots b_s}C^m{}_{nb_1} + \ldots + \alpha^{a_1\ldots a_r}{}_{b_1\ldots b_{s-1}m}C^m{}_{nb_s} - \alpha^{ma_2\ldots a_r}{}_{b_1\ldots b_s}C^{a_1}{}_{nm} - \ldots - \alpha^{a_1\ldots a_{r-1}m}{}_{b_1\ldots b_s}C^{a_r}{}_{nm}.$$

3 Maxwell Gravitation and Vector Relationism

In §2, I have reviewed the recent coordinate-free approaches to MG, and their relationship to NCT. But as noted in §1, these presentations of MG are rather distant from Saunders' original (2013) discussion of Newtonian gravitation on Maxwellian spacetime. This distance has three sources, which will occupy us for the rest of this section:

- Saunders' preferred characterization of the appropriate setting for his vector relationist dynamics is as an affine space which he calls Newton-Huygens spacetime, rather than a differentiable manifold with differential-geometric objects defined thereon.
- Saunders' dynamics are presented in the coordinate-based framework.
- Saunders' theory concerns only the dynamics of point particles, rather than fields.

The first of these is easily dealt with—as Saunders notes, the idea of Newton-Huygens spacetime is just that (rotationally flat) Maxwellian spacetime can be redescribed as an affine space, albeit one in which affine structure is appropriately restricted to spacelike hypersurfaces.[14] For the second two bullet points, we need to recall some details of Saunders' theory. Saunders presents vector relationism as a theory of the displacement vectors between point particles, formulated with reference to some Maxwellian coordinate system. The dynamics are specified by the following pair of equations:

$$\mathbf{r}_{ij} = \mathbf{X}_i - \mathbf{X}_j \tag{3a}$$

$$\frac{d^2\mathbf{r}_{ij}}{dt^2} = \frac{1}{m_i}\sum_{k\neq i}\mathbf{F}_{ik} - \frac{1}{m_j}\sum_{k\neq j}\mathbf{F}_{jk}, \tag{3b}$$

where $\mathbf{X}_i(t)$ denotes the position of particle i at time t with respect to such a coordinate system, m_i its mass, and the $\mathbf{F}_{ij}$ denote interparticle forces. These are taken to be antisymmetric in i and j (this is the import of Newton's third law) and functions of $\mathbf{r}_{ij}$ only. The equations (3) are invariant under the Maxwell group (Wallace 2020)—transformations of the form

$$t \to t + \tau$$
$$x^i(t) \to R^i{}_j x^j(t) + a^i(t),$$

where $R^i{}_j$ is an arbitrary 3D rotation matrix, $a^i(t)$ an arbitrary vector-valued function of time, and τ an arbitrary scalar.

With this in hand, I will now address the second two bullet points by examining the relationship between the dynamics (1) and (3) of MG and VR, respectively. First, following Wallace (2020, 11), we can decompose the forces in (3b) into 'universal' and 'non-universal' components—characterized, respectively by whether the ratio q_i/m_i is constant for that force, where m_i is the inertial mass of a particle and q_i its charge. For the case of only potential

14. Saunders (2013) only discusses Earman's (1989) characterization of Maxwellian spacetime, but this is equivalent to the definition of a rotationally flat Maxwellian spacetime as presented here—see Chen (2023, proposition 1). See Wallace (2020) for discussion of why rotationally flat Maxwellian spacetime is the appropriate setting for vector relationism.

forces, (3) may then be written as

$$\frac{d^2\mathbf{X}_i}{dt^2} - \frac{d^2\mathbf{X}_j}{dt^2} = -\sum_{k\neq i}\nabla\phi(\mathbf{X}_i - \mathbf{X}_k) + \sum_{k\neq j}\nabla\phi(\mathbf{X}_j - \mathbf{X}_k)$$
$$- \frac{q_i}{m_i}\sum_{k\neq i}\nabla V(\mathbf{X}_i - \mathbf{X}_k) + \frac{q_j}{m_j}\sum_{k\neq j}\nabla V(\mathbf{X}_j - \mathbf{X}_k), \quad (4)$$

where ϕ is the potential associated with the universal force, and V the potential for the non-universal force (there could be multiple such; I omit them for simplicity). Now consider the continuum limit, where point-particle trajectories are parametrized by some continuous spatial parameter $\mathbf{x}$. In this limit, (4) becomes

$$\partial_i\left(\frac{d^2\mathbf{X}(\mathbf{x},t)}{dt^2}\right)\delta x^i = -\partial_i\int d^3\mathbf{x}'\nabla\phi(\mathbf{x}-\mathbf{x}',t)\delta x^i$$
$$- \partial_i\int d^3\mathbf{x}'\tilde{\rho}(\mathbf{x},t)\rho^{-1}(\mathbf{x},t)\nabla V(\mathbf{x}-\mathbf{x}',t)\delta x^i,$$

where $\rho(\mathbf{x},t)$ is the mass density, and $\tilde{\rho}(\mathbf{x},t)$ the charge density associated with the non-universal interaction, so that

$$\partial_i\left(\frac{d^2X^j(\mathbf{x},t)}{dt^2}\right) = -\partial_i\int d^3\mathbf{x}'(\partial^j\phi(\mathbf{x}-\mathbf{x}',t) + \tilde{\rho}(\mathbf{x},t)\rho^{-1}(\mathbf{x},t)\partial^j V(\mathbf{x}-\mathbf{x}',t)). \quad (5)$$

When ϕ is the familiar gravitational potential, we have

$$\phi(\mathbf{x}-\mathbf{x}',t) = \frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|},$$

so that

$$\partial_i\left(\frac{d^2X^j(\mathbf{x},t)}{dt^2}\right) = -\partial_i\int d^3\mathbf{x}'\rho(\mathbf{x}',t)\partial^j(|\mathbf{x}-\mathbf{x}'|)^{-1}$$
$$- \partial_i\int d^3\mathbf{x}'\tilde{\rho}(\mathbf{x},t)\rho^{-1}(\mathbf{x},t)\partial^j V(\mathbf{x}-\mathbf{x}',t). \quad (6)$$

We have seen that the appropriate spacetime setting for vector relationism is a rotationally flat Maxwellian spacetime, $\langle M, t_a, h^{ab}, \circlearrowleft\rangle$. Since we can always (since M was assumed diffeomorphic to $\mathbb{R}^4$) find a globally defined scalar field t such that $d_a t = t_a$, we can then set up an arbitrary Maxwellian coordinate system x^μ on M as follows: we take $x^\mu = (t, x^i)$, where t is as above and the x^i are three smooth scalar fields such that the vector fields $(\partial/\partial x^i)^a$ are spacelike, orthonormal, rigid, and twist-free (with respect to $\circlearrowleft$).[15]

Let x^μ be such a coordinate system, and let ∇ be the coordinate derivative operator on M canonically associated with x^μ.[16] ∇ is flat (since it is a coordinate derivative operator); it is compatible with t_a by construction, and is compatible with h^{ab} since the $(\partial/\partial x^i)^a$ are spacelike and orthonormal. Moreover, since the $(\partial/\partial x^\mu)^a$ are all twist-free with respect to $\circlearrowleft$ and $\circlearrowleft$ is rotationally flat, ∇ is also compatible with $\circlearrowleft$.[17] Now consider a smooth unit timelike vector field ξ^a on M. The integral curves ξ of any such field can always be parametrized by their

15. Existence of such fields follows from the fact that $\langle M, t_a, h^{ab}, \circlearrowleft\rangle$ was assumed complete and rotationally flat.

16. That is, the unique derivative operator such that all the $\nabla_a(\partial/\partial x^\mu)^b = \mathbf{0}$.

17. Note that $(\partial/\partial t)^a$ is twist-free by construction, since t_a is closed.

temporal length, which differs from t by at most an arbitrary additive constant. Then on any such curve ξ, we have

$$\xi^a = \frac{dx^\mu(\xi(t))}{dt}\left(\frac{\partial}{\partial x^\mu}\right)^a$$

so that, since ∇ is flat

$$\xi^n \nabla_n \xi^a = \frac{d^2 x^\mu(\xi(t))}{dt^2}\left(\frac{\partial}{\partial x^\mu}\right)^a.$$

Clearly, the only non-vanishing d^2x^μ/dt^2 are the d^2x^i/dt^2. Moreover, if σ^{ab} is a (symmetric) tensor field which is spacelike in both indices, then we can write

$$D_n \sigma^{na} = \partial_\mu \sigma^{\mu\nu}\left(\frac{\partial}{\partial x^\nu}\right)^a$$

where the only non-vanishing $\partial_\mu \sigma^{\mu\nu}$ are the $\partial_\mu \sigma^{\mu i}$. If we now take ξ^a to represent the four velocity field of a fluid, and σ^{ab} the stress tensor for that fluid, then these suggest the following identifications:

$$\xi^n \nabla_n \xi^m (d_m x^i) = \frac{d^2 X^i(\mathbf{x}, t)}{dt^2} \tag{7a}$$

$$\rho^{-1} D_n \sigma^{nm} (d_m x^i) = \int d^3\mathbf{x}' \tilde{\rho}(\mathbf{x}, t)\rho^{-1}(\mathbf{x}, t)\partial^i V(\mathbf{x} - \mathbf{x}', t). \tag{7b}$$

Why? Take (7a). We are looking for something with which to identify the (non-zero) components of the acceleration vector field of a fluid $\xi^n \nabla_n \xi^m (d_m x^i)$ with respect to the coordinate derivative operator canonically associated with some Maxwellian coordinate system x^μ. Not only is this precisely what the $d^2 X^i(\mathbf{x}, t)/dt^2$ represent, we have also seen that when ∇ is such a derivative operator, the $\xi^n \nabla_n \xi^m (d_m x^i) = d^2 x^i(\xi(t))/dt^2$ take this same form. Now consider (7b). The left hand side of this equation are the (non-zero) components of a spacelike vector field which is supposed to describe the acceleration due to non-gravitational interactions—think of (the geometrized version of) Newton's second law

$$\rho \xi^n \nabla_n \xi^a = -\nabla_n \sigma^{na}.$$

And this is precisely the role of the term on the right hand side. We can then write (6) as

$$\nabla_r (\xi^n \nabla_n \xi^m)(d_m x^j)\left(\frac{\partial}{\partial x^i}\right)^r = -\partial_i \int d^3\mathbf{x}' \rho(\mathbf{x}', t)\partial^j (|\mathbf{x} - \mathbf{x}'|)^{-1} - D_r(\rho^{-1} D_n \sigma^{nm})(d_m x^j)\left(\frac{\partial}{\partial x^i}\right)^r. \tag{8}$$

Now consider the case where $i = j$. In this case, carrying out the differentiation in the right hand side of (8) gives

$$\nabla_m (\xi^n \nabla_n \xi^m) = -4\pi\rho - D_m(\rho^{-1} D_n \sigma^{nm})$$

where we have used the fact that $\xi^n \nabla_n \xi^a$ and $\rho^{-1} D_n \sigma^{na}$ are both spacelike. This immediately yields (1b). Meanwhile, if we take $i \neq j$ in (8), then differentiating and raising indices we have

$$\nabla^r (\xi^n \nabla_n \xi^m)(d_m x^j)(d_r x^i) = \int d^3\mathbf{x}' \rho(\mathbf{x}', t)\left(3\frac{(x^j - x'^j)(x^i - x'^i)}{|\mathbf{x} - \mathbf{x}'|^5}\right) - D^r(\rho^{-1} D_n \sigma^{nm})(d_m x^j)(d_r x^i),$$

so that, since $\circlearrowright^a (\xi^n \nabla_n \xi^b)$ is spacelike in both indices,

$$\nabla^a(\xi^n \nabla_n \xi^b) - \nabla^b(\xi^n \nabla_n \xi^a) = -D^a(\rho^{-1} D_n \sigma^{nb}) + D^b(\rho^{-1} D_n \sigma^{na}),$$

which, given the continuity equation (1a) and the fact that ∇ is flat by construction, entails (1c) (see the proof of Chen (2023, proposition 2)). For (1a) itself, note that in Newtonian point particle mechanics, mass is transported only by particles along their (continuous) worldlines, and is a fortiori locally conserved. So we have obtained (1) from (3), i.e. MG from VR.

Conversely, it is also possible to recover the dynamics (3) from (1), i.e. VR from MG. Given the identifications (7), we can use (1) to derive expressions for $\partial_i(d^2X^i/dt^2)$ and $\partial^{[i}(d^2X^{j]}/dt^2)$ in any Maxwellian coordinate system x^μ on M. These are sufficient to specify (5) uniquely, providing that $\partial_i(d^2X^i/dt^2)$ and $\partial^{[i}(d^2X^{j]}/dt^2)$ fall off at least as $1/r^2$ at spatial infinity. If we then specialize to the case of a point-particle distribution (which justifies making the above assumptions about d^2X^i/dt^2), this gives $\phi(\mathbf{x}-\mathbf{x}',t) \to \phi(\mathbf{x}-\mathbf{x}',t)\sum_i \delta^3(\mathbf{x}'-\mathbf{X}_i(t))$ and analogously for V. Hence,

$$\partial_i\left(\frac{d^2X^j(\mathbf{x},t)}{dt^2}\right) = -\partial_i \sum_k \partial^j \phi(\mathbf{x}-\mathbf{X}_k,t) - \partial_i \sum_k \tilde{\rho}\rho^{-1}\partial^j V(\mathbf{x}-\mathbf{X}_k,t). \quad (9)$$

Since $\tilde{\rho}\rho^{-1} = \sum_i q_i/m_i \delta^3(\mathbf{x}-\mathbf{X}_i(t))$, (4) then follows from integrating along any path between $\mathbf{X}_i(t)$ and $\mathbf{X}_j(t)$.

So despite their surface-level differences, there is a close relationship between MG and VR. Both are set on rotationally flat Maxwellian spacetime. Moreover, the dynamics of MG emerge naturally in the continuum limit of VR, whilst VR is precisely what results from restricting MG to the point particle sector. In turn, this supports Dewar's (2018) claim that "Maxwell gravitation ... represents the natural extension of Saunders' remarks to the field-theoretic context." Dewar argues for this on the basis that MG, like VR, collapses the distinction between materially identical models of NCT. However, the fact that MG can be recovered in the continuum limit of VR, and vice versa, provides a more direct route to this conclusion.

4 Wallace on Vector Relationism and Newton-Cartan Theory

With the relationship between VR and MG on a firmer footing, I will now turn to Wallace's (2020) discussion of VR and NCT. Here, Wallace claims to show that "mathematically speaking, there is no real distinction between Newton-Cartan theory ... and vector relationism" (24), and suggests that any differences between the two theories are partly an artefact of the awkwardness of standard differential-geometric presentations of Maxwellian spacetime (28). As a result, Wallace adheres to a coordinate-based presentation of both theories in setting out his argument.

Wallace's discussion of VR and NCT centers on the behavior of dynamically isolated subsystems of particles embedded in a larger universe—showing that within VR, such systems exhibit emergent inertial behavior which can be idealized in terms of test particles. This forms the basis of his argument that VR and NCT are equivalent. When non-gravitational interactions vanish, the equations governing the relative acceleration vectors of infinitesimally-separated test particles can be written to take the same form as the (coordinate-based) equation of geodesic deviation in NCT, and thus, Wallace claims, may equally well be interpreted as such (Wallace 2020, §8).

Wallace is not explicit about the standard of theoretical equivalence he is working with here. But it is fairly straightforward to reconstruct from his remarks what he may have in mind. Having recovered the Newton-Cartan equation of geodesic deviation within VR, Wallace claims of the two theories:

> [Both] are built using Maxwellian spacetime as a background; both have dynamics that can be expressed as a set of inertial trajectories defined by the matter distribution and in turn constraining the matter distribution via a matter dynamics according to which material particles follow those trajectories except when acted on by non-gravitational forces. (24)

Similarly, in his concluding remarks, Wallace argues that

> [There] is essentially no difference between Newton-Cartan theory ... and Saunders's relational version of Newtonian dynamics: at the formal level, the latter can be reformulated as the former; at the substantive level, the inertial structure of Saunders's theory is well defined and coincides with that defined by the Newton–Cartan connection. (28)

From these remarks, one can isolate three points which Wallace takes to bear on whether MG and NCT are equivalent:

1. They have the same background spacetime structure.
2. Their central dynamical equations can be (re)written so as to appear mathematically identical.
3. They have the same inertial structure.

For our purposes, we can elevate this to a criterion of theoretical equivalence, though it should be borne in mind both that Wallace does not explicitly endorse this, and that such a criterion may be more or less well-suited to theories other than VR and NCT. I will now make several comments on this criterion, all of which will indicate refinements of the points 1-3 above.

First, on point 1, what does the 'background spacetime structure' of a theory consist in? In the literature, there are various competing schools of thought about how this is to be identified. For example, one might take 'spacetime structure' to be objects of a certain object-type that appear between the angle brackets of a theory's models á la e.g. Earman (1989) or Friedman (1983), or one might invoke a criterion such as Knox's (2013) spacetime functionalism, according to which 'spacetime structure' is just whatever it is that encodes the local structure of inertial frames.[18] Again, Wallace's remarks give some hint as to what he may have in mind here:

> [In] Newton–Cartan theory, the connection does double duty, imposing both the rotation standard (a piece of absolute structure) and the inertial structure (something dynamical and contingent). ... the Newton–Cartan connection is naturally understood as an additional piece of structure added to Maxwellian spacetime; indeed, as the Maxwellian version of the affine connection. (Wallace 2020, 29)

18. Though note that Knox's spacetime functionalism cannot be the right criterion if we are looking to identify Maxwellian spacetime as the background spacetime structure of MG and NCT, since Maxwellian spacetime by itself lacks a full inertial frame structure.

This suggests, for the purposes of point 1, that we should take the 'background spacetime structure' of a theory to be its absolute objects—i.e., those which are the same in all its dynamically possible models, where 'sameness' is sameness up to isomorphism (see e.g. Earman (1989, 45)).[19] If this is the right precisification of 1, then MG and NCT do indeed have the same background spacetime structure as Wallace claims—see March (2024).

Second, on 2, one might worry about the restriction to the 'central' dynamical equations of a theory. Whilst I won't attempt to address the question of what it means for some equation or other to be 'central' to a theory here, note that this restriction is needed because Wallace does not explicitly consider all the equations of NCT in his analysis (and as we will see in section 5, not all the equations of MG and NCT (or VR and NCT, for that matter) can be written so as to appear mathematically identical).

Continuing with point 2, one might also ask what it means for the central equations of two theories to 'be rewritten so as to appear mathematically identical'. For our purposes, we can take this mean: we can re-express the dynamics of the theories so that they have some non-empty set of equations (the 'central' ones) in common, whilst preserving solutionhood.

Finally on point 3, how, according to Wallace, are we to identify the inertial structure of a theory? Here, Wallace closely follows Knox (2013): the inertial structure of a theory is whatever it is that encodes the local structure of inertial frames, i.e. those with respect to which gravitating but otherwise force-free bodies move with constant velocities, in which the equations governing non-gravitational interactions take their simplest form, and which are universal (in the sense that all bodies and interactions pick out this same class of frames). Crucially, in the case of non-relativistic theories, this means that if there exists a connection such that the geometrised version of Newton's second law (NII)—i.e. $\rho\xi^n\nabla_n\xi^a = -\nabla_n\sigma^{na}$—is satisfied, then this (amongst other things) qualifies it as encoding the inertial structure of that theory.

5 Understanding Wallace from a Coordinate-Free Perspective

Having presented Wallace's argument, I will now show that with MG in hand, the same argument can be made in the language of coordinate-free differential geometry. For point 1, we have already noted that MG and NCT have the same absolute object structure (March 2024). For point 2, following Wallace, let us compare the dynamical equations (1) and (2) of MG and NCT. Let $\langle M, t_a, h^{ab}, \circlearrowright \rangle$ be a Maxwellian spacetime. Then for any derivative operator ∇ compatible with $\langle M, t_a, h^{ab}, \circlearrowright \rangle$, the following implications hold (illustrated in Figure 1). Statements and proofs of these equivalences are contained in appendix A.

There are several features of Figure 1 worth noting. First, whilst (2d) is equivalent to the rotational flatness condition, there is no similarly sharp correspondence between (2c) and (1c). (2c) and NII jointly imply (1c), but (1c) and NII do not imply (2c). This points to the fact that rotational flatness plays double duty in relating the two theories. From (1c) and NII we can infer that

19. Note that I am not suggesting, here, that the notion of an absolute object in this sense has anything to do with 'substantive general covariance' (whatever that might mean), nor am I interested in whether 'having an absolute object' in this sense allows one to distinguish e.g. general relativity from some desired contrast class of theories (as Pitts (2006) points out, it does not). It is a precise definition one can consider, and seems to me a reasonable way of cashing out which degrees of freedom of which objects in the theory are "dynamical and contingent," in the sense that they may vary from model-to-model.

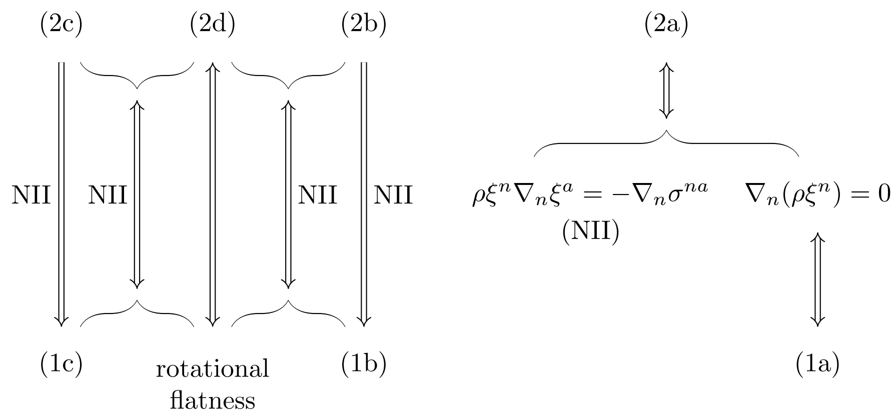

Figure 1: Relationships between the equations of Maxwell gravitation and Newton-Cartan theory. Labelled arrows are to be understood as in the scope of a conditional—so e.g. the first arrow from the left says that if NII holds, then (2c) implies (1c)

$\xi^n\xi^m({R^c}_n{}^a{}_m - {R^a}_m{}^c{}_n) = \mathbf{0}$; the rotational flatness condition allows us to further infer that $\xi^n h^{bm}({R^c}_n{}^a{}_m - {R^a}_m{}^c{}_n) = \mathbf{0}$, which yields (2c).[20]

Secondly, although (2b) and (1b) are not in general equivalent, they are equivalent on assumption of NII and rotational flatness. Likewise, given NII, rotational flatness and (1c) are equivalent to (2d) and (2c). As such, once NII has been fixed, we can then move freely between the remaining pairs of equations.

Now recall point 2 of Wallace's argument: for an idealized congruence of test particle trajectories, the dynamics (3) of VR can be rewritten so as to take the same form as the equation of geodesic deviation in NCT. But we have just seen that this has an obvious analogy for MG and NCT: by replacing (2b) with the expression for the average radial acceleration (1b), we can reformulate MG and NCT so that their central dynamical equations appear mathematically identical. Within NCT, (1b) encodes the relative acceleration of neighboring fluid elements due to both spacetime curvature and non-gravitational interactions, so represents the natural generalization of Wallace's geodesic deviation equation to non-test matter. And just as in Wallace's example, the only difference, as far as this pair of equations is concerned, is the interpretation of (1b)—in NCT, the $-4\pi\rho/3$ term is naturally understood as a manifestation of geodesic deviation in curved spacetime, whereas in MG it is not.

Moreover, once we move from VR to MG, the case for regarding this disagreement as merely verbal appears even stronger. After all, in VR, the gravitational field is explicitly represented elsewhere in the formalism. But in MG, we do not even have that. Of course, we are always free to ascribe the $-4\pi\rho/3$ term in (1b) to 'the gravitational field'—but without some further indication of what this is supposed to be, the gravitational field is simply that whereby neighboring test particles have non-zero relative acceleration. And since this is precisely the role of the Newton-Cartan spacetime curvature, the difference between the two begins to look insubstantive. As such, we seem to have in the relationship between (1b) and (2b) a coordinate-free realization of point 2 of Wallace's argument.

However, we can also say a little more about this reasoning. Given the relationships illustrated in Figure 1, not only are we free to replace (2b) with

20. Recall that $h^{dn}h^{bm}({R^c}_n{}^a{}_m - {R^a}_m{}^c{}_n) = \mathbf{0}$ in any classical spacetime.

(1b) in NCT, we can also replace (2c) with (1c), (2d) with the rotational flatness condition, and rewrite (2a) as the conjunction of NII and (1a). From this perspective, the only difference between these sets of equations is the presence of NII in NCT, whose role is essentially to provide a (partial) gauge fixing of the connection. This provides a further sense in which point 2 of Wallace's argument is strengthened when we move from VR to MG—*all* the dynamical equations of NCT, with the exception of NII, can be written so as to appear mathematically identical to the equations of MG.

Note that this also highlights why it is that NCT cannot be the continuum limit of VR. If one assumes that the dynamics for test particles in NCT are given by the geodesic equation, then it is possible to show that in both NCT and the continuum limit of VR, test particles satisfy the equation of geodesic deviation. But precisely what one cannot recover in the continuum limit of VR is the geodesic equation itself—or rather its generalization to non-test matter, NII.

Finally, this brings us to point 3, viz. the inertial structure of MG, such as it is. For this, it is helpful to recall proposition 2. Proposition 2 tells us that, providing there is sufficient matter in one's spacetime, there exists a unique Newton-Cartan connection which satisfies NII, i.e. such that massive test bodies follow geodesics. Moreover, providing that the test bodies of interest are sufficiently far from other massive matter (which we can idealize as meaning at spatial infinity), then this connection will, at least locally, be well-approximated by a flat connection. This allows us to recover (and expand upon) Wallace's claims about the emergence of inertial structure in MG, in three ways.

First, suppose that we say, with Wallace, that what it is to encode the inertial structure of a theory just is to be the unique connection such that massive test bodies follow geodesics. Then it follows that, whenever the conditions of corollary 2.1 are satisfied, a model of MG does indeed come equipped with an inertial structure, which coincides with the Newton-Cartan connection. So whilst Maxwellian spacetime lacks full inertial structure by itself, there is an emergent such structure to be had for those models in which there is sufficient matter available.

Second, continuing with the above theme, if we have antecedent reasons for adopting NII as an implicit definition of the inertial structure of a theory, then we might as well go ahead and add this as an extra condition to those models of MG in which there are open sets throughout which the mass density field is non-vanishing. In that case, one can also recover the dynamics of NCT from those of MG. So whilst one cannot rewrite the dynamics of MG to appear mathematically identical to those of NCT *by themselves*, there is a natural sense in which the dynamics of MG plus definitions *are* sufficient to recover the dynamics of NCT, again providing there is sufficient matter in one's spacetime. This provides a way of making sense of Wallace's claim that "Saunders's vector relational version of Newtonian dynamics ... can be reformulated as [Newton-Cartan theory]" (Wallace 2020, 28).

Third, corollary 2.1 clarifies just what is needed for the emergence of this inertial structure. In particular, sufficient for this is that there exist open sets throughout which the mass density field is non-vanishing. So providing that we are doing non-vacuum continuum mechanics (or even for certain point particle distributions—see March (2024)) then the above arguments can be made; one does not need to consider a full congruence of particle trajectories.

All this serves to blunt the force of Wallace's (2019; 2020) recent claims that Maxwellian spacetime is not naturally characterized in coordinate-free differential-geometric terms, and that this is partly what obscures the similarities between MG and NCT. As I have shown, the same formal similarities which Wallace discusses can also be seen very naturally from a coordinate-free differential-geometric perspective. As such, one might suspect that the problem (such as there is) lies not with coordinate-free differential geometry per se, but with formulating a theory in terms of geometric objects which cannot be defined from the structure it ascribes to the world.[21]

6 The Link with Teh

Finally, I will consider the relationship between my discussion of Wallace and that of Teh (2018), who adopts a rather different strategy for diffusing Wallace's concerns about the coordinate-free framework. Teh's approach begins by noting that compatible connections on a classical spacetime can be represented by means of a special connection (for some unit timelike vector field ξ^a) and a two-form Ω_{ab} (see Malament (2012, propositions 4.3.4, 4.1.3)). Providing the connection of interest satisfies (2c), this two-form is closed, and so can (at least locally) be specified by a one-form A_a, defined up to exact one-form shifts. Since ξ^a is geodesic with respect to its special connection, one can therefore view ξ^a as encoding a 'background inertial structure', and Ω_{ab} as encoding the forces experienced by bodies relative to this inertial structure. Alternatively, one can view Ω_{ab} as encoding the force differences between different idealized congruences of particle trajectories, and so as realizing Saunders' vector relationist dynamics (Teh 2018, 207).

How does this allow one to make Wallace's argument, and in what ways does this address Wallace's concerns about the coordinate-free framework? On this, one can identify three points:

- As Teh himself (2018) notes, suppose we are given an equivalence class $[\nabla]$ of rotationally equivalent (not necessarily flat) connections which satisfy (2c).[22] Any such connection will be the special connection for some unit timelike vector field ξ^a. Now suppose that we are given another special connection ∇. Then all the ξ^a have the same rotation tensor with respect to ∇. This, Teh claims, furnishes the notion of rotational equivalence with a physical interpretation in terms of representations which share the same vorticity.

- Now suppose that the connections in this equivalence class are, in addition, flat. Then the choice of such a connection is equivalent to a choice of inertial frame (since the ξ^a in question must now be rigid). So the equivocation involved in defining the rotation standard of Maxwellian spacetime as an equivalence class of rotationally equivalent flat connections á la Earman (one might think) is no worse than that involved in equivocating between Maxwellian coordinate systems when writing down e.g. Saunders' vector relationist dynamics.

- Teh's framework emphasizes the way in which the two-form Ω_{ab} used to pick out the Newton-Cartan connection of interest can always be reinterpreted as encoding either forces experienced by test bodies relative to the inertial structure defined by ξ^a, or as encoding a connection relative to

21. c.f. Pitts (2012, 2022, 2006). For an extended discussion of other possible issues relating to this in the context of the interpretation vs. motivation and reduction vs. sophistication debates, see Jacobs (2022).

22. Recally that two connections ∇, ∇' are rotationally equivalent just in case, for all unit timelike vector fields ξ^a, $\nabla^{[a}\xi^{b]} = \mathbf{0}$ iff $\nabla'^{[a}\xi^{b]} = \mathbf{0}$.

> which those same test bodies exhibit geodesic motion. Or in other words, there is no mathematical difference between the universal forces of VR and the geodesic motion in curved spacetime of NCT, as Wallace argues.[23]

From this, it is clear that Teh's concern is not primarily to alleviate worries about taking equivalence classes simpliciter. Nevertheless, I suggest that his approach provides a complementary avenue to the one I have considered here—particularly on point three, which provides an alternative route to my earlier conclusion in §5 that there is little, if any, difference between the interpretation of (1b) in MG and NCT. On the other hand, insofar as Teh's framework highlights the fact that equivocating between rotationally equivalent flat connections is the same as equivocating between inertial frames, this might just seem like grist to Wallace's mill: wasn't one of the advantages of the coordinate-free approach supposed to be that it avoids all this need for equivocation, since we can just talk about the objects of interest directly?

Finally, note that the discussion of §5 highlights which of Teh's constructions carry over to Maxwellian spacetime characterized 'intrinsically' and which do not. In particular, Teh's 'proto-Maxwell spacetime'—which he defines using an equivalence class of rotationally equivalent connections all of which satisfy (2c)—cannot be defined using just Weatherall's standard of rotation.

7 Close

In this paper, my aim has been to connect up the recent wave of coordinate-free approaches to Maxwellian spacetime with the coordinate-based discussions of Saunders (2013) and Wallace (2020). By doing so, I have clarified the relationship between vector relationism and Maxwell gravitation (the latter is just the continuum limit of the former, and the former the point particle sector of the latter, as one would have hoped); I have also explained why Newton-Cartan theory is not the continuum limit of vector relationism, contra the appearance of Wallace's discussion. Finally, I have shown how the similarities between vector relationism and Newton-Cartan theory which Wallace discusses can also straightforwardly be seen using the coordinate-free approach, and used this both to assess Wallace's argument, and to connect up with Teh's discussion of Wallace.

In many ways, the upshot of all this is irenic. Maxwellian spacetime can be characterized in a variety of ways—whether in terms of privileged coordinates, equivalence classes of connections, or a primitive standard of rotation. And as far as the substantive things one can say about Maxwellian spacetime and Maxwell gravitation are concerned, I hope to have shown that there is little to choose between these three perspectives.

That is not to say that, depending on the context at hand, one may not have good reasons for preferring one approach over another—whether for calculational convenience, ease of presentation, physical (or mathematical, or metaphysical) perspicuity, etc. But I think it would be a mistake to conclude from this that there is some objective, once-and-for-all answer as to which approach fares better on any of these criteria. Put simply, what is most calculationally convenient, or easy to understand, or physically (or mathematically, or metaphysically) perspicuous for me need not be so for anyone else—and that is as it should be. The coordinate-based vs. coordinate-free debate may yet be fought and won on other grounds. But in any case, I hope to have laid to rest the

23. For a different take on this, see Weatherall and Manchak (2014).

idea that the example of Maxwellian spacetime provides a reason to prefer one approach over the other.

Acknowledgements

I would like to thank Adam Caulton, James Read, and two anonymous referees for helpful feedback on earlier versions of this manuscript. I am grateful to Balliol College, Oxford, and the Faculty of Philosophy, University of Oxford, for financial support.

A Statements and proofs of equivalences

Let $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ be a Maxwellian spacetime, ∇ any compatible connection, and T^{ab} the mass-momentum tensor, which we assume to satisfy the Newtonian mass condition. That (2d) holds iff $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ is rotationally flat is shown by Malament (2012, Proposition 4.2.4); that (2a) holds iff (3) and (1a) hold is shown by Malament (2012, 266), noting that $\xi^n\nabla_n\rho = \pounds_\xi\rho$ and $\nabla_n\xi^n = -1/2\hat{h}_{nm}\pounds_\xi h^{nm}$.

For the remaining four implications, assume that (3) holds. A straightforward computation shows that we can use ∇ to rewrite (1c) as

$$\xi^n\nabla_n(\omega^{ca}) = 2\omega^{n[c}\theta_n{}^{a]} - \nabla^{[c}(\rho^{-1}\nabla_n\sigma^{|n|a]})$$

where ω^{ab}, θ^{ab} are the rotation and expansion tensors for ξ^a, respectively. It follows that, given (3)

$$\xi^n\nabla_n(\omega^{ca}) = 2\omega^{n[c}\theta_n{}^{a]} + \nabla^{[c}(\xi^{|n|}\nabla_n\xi^{a]}). \tag{10}$$

Likewise (1b) can be rewritten as

$$\frac{1}{3}\sum_{i=1}^{3}\overset{i}{\lambda}_r\xi^n\nabla_n(\xi^m\nabla_m\overset{i}{\lambda}{}^r) = -\frac{4}{3}\pi\rho + \frac{1}{3}\nabla_m(\xi^n\nabla_n\xi^m). \tag{11}$$

Now we just need to do some calculations, which follow the proofs of propositions 4.3.6, 1.8.5, and 4.3.2 of Malament (2012) closely. First

$$\begin{aligned}\xi^n\nabla_n(\omega^{ca}) &= \nabla^{[c}(\xi^{|n|}\nabla_n\xi^{a]}) - (\nabla^{[c}\xi^{|n|})(\nabla_n\xi^{a]}) + (R^a{}_n{}^c{}_m - R^c{}_m{}^a{}_n)\xi^n\xi^m \\ &= 2\omega^{n[c}\theta_n{}^{a]} + \nabla^{[c}(\xi^{|n|}\nabla_n\xi^{a]}) + (R^a{}_n{}^c{}_m - R^c{}_m{}^a{}_n)\xi^n\xi^m\end{aligned}$$

where we have made use of the fact that ω^{ab} is spacelike in both indices. So if (2c) holds, (1c) immediately follows. Conversely, if (1c) holds then comparison with (10) yields that $(R^a{}_n{}^c{}_m - R^c{}_m{}^a{}_n)\xi^n\xi^m = 0$. Then to establish (2c), we just need to show that $(R^a{}_n{}^c{}_m - R^c{}_m{}^a{}_n)h^{nb}\xi^m = 0$ (since $(R^a{}_n{}^c{}_m - R^c{}_m{}^a{}_n)h^{nb}h^{md} = 0$ in any classical spacetime). This, in turn, follows from rotational flatness (using the symmetries of the Riemann tensor). Note that rotational flatness is needed here because ξ^a need not be twist-free. Next,

$$\begin{aligned}\frac{1}{3}\sum_{i=1}^{3}\overset{i}{\lambda}_r\xi^n\nabla_n(\xi^m\nabla_m\overset{i}{\lambda}{}^r) &= \frac{1}{3}\sum_{i=1}^{3}\overset{i}{\lambda}_r(\overset{i}{\lambda}{}^n\nabla_n(\xi^m\nabla_m\xi^r) + R^r{}_{nms}\xi^n\overset{i}{\lambda}{}^m\xi^s) \\ &= \frac{1}{3}\nabla_n(\xi^m\nabla_m\xi^n) - \frac{1}{3}R_{nm}\xi^n\xi^m\end{aligned}$$

so that if (2b) holds, so does (1b). Conversely, if (1b) holds, then $R_{nm}\xi^n\xi^m = 4\pi\rho$. If we then assume rotational flatness we also have that $R^a{}_n\xi^n = R^{ab} = 0$, which gives us (2b).

References

Chen, Elliott. 2023. "Newtonian Gravitation in Maxwell Spacetime." *Studies in the History and Philosophy of Science* 102:22–30.

Dewar, Neil. 2018. "Maxwell gravitation." *Philosophy of Science* 85 (2): 249–270.

Dürr, Patrick, and James Read. 2019. "Gravitational energy in Newtonian gravity: a response to Dewar and Weatherall." *Foundations of Physics* 49 (10): 1086–1110.

Duval, Ch., and H.P. Künzle. 1978. "Dynamics of continua and particles from general covariance of newtonian gravitation theory." *Reports on Mathematical Physics* 13 (3): 351–368. https://doi.org/10.1016/0034-4877(78)90063-0.

Earman, John. 1989. *World Enough and Space-Time.* MIT Press.

Friedman, Michael. 1983. *Foundations of space-time theories : relativistic physics and philosophy of science* [in eng]. Princeton: Princeton University Press. ISBN: 0691072396.

Jacobs, Caspar. 2022. "Invariance, intrinsicality, and perspicuity." *Synthese* 200 (2).

Jacobs, Caspar. 2023. "Are dynamical shifts dynamical symmetries?" *Philosophy of Science,* https://doi.org/10.1017/psa.2023.13.

Knox, Eleanor. 2013. "Effective spacetime geometry." *Studies in the History and Philosophy of Modern Physics* 44 (3): 346–356.

Knox, Eleanor. 2014. "Newtonian spacetime structure in light of the equivalence principle." *British Journal for the Philosophy of Science* 65 (4): 863–880.

Malament, David. 2012. *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory.* University of Chicago Press.

March, Eleanor. 2023. "Maxwell Gravitation without Reference to Equivalence Classes of Derivative Operators." eprint: http://philsci-archive.pitt.edu/id/eprint/22404.

March, Eleanor. 2024. "Are Maxwell Gravitation and Newton-Cartan Theory Theoretically Equivalent?" Forthcoming, *British Journal for the Philosophy of Science,* https://doi.org/10.1086/730863.

March, Eleanor, William J. Wolf, and James Read. 2024. "On the geometric trinity of gravity, non-relativistic limits, and Maxwell gravitation." *Philosophy of Physics* 2 (1): 15.

Pitts, J. Brian. 2006. "Absolute objects and counterexamples: Jones–Geroch dust, Torretti constant curvature, tetrad-spinor, and scalar density." *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* 37 (2): 347–371. https://doi.org/https://doi.org/10.1016/j.shpsb.2005.11.004.

Pitts, J. Brian. 2012. "The nontriviality of trivial general covariance: How electrons restrict 'time' coordinates, spinors (almost) fit into tensor calculus, and 716 of a tetrad is surplus structure." *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* 43 (1): 1–24. https://doi.org/https://doi.org/10.1016/j.shpsb.2011.11.001.

Pitts, J. Brian. 2022. "What Represents Space-Time? And What Follows for Substantivalism Vs. Relationalism and Gravitational Energy?" In *The Foundations of Spacetime Physics: Philosophical Perspectives,* edited by Antonio Vassallo. Routledge.

Saunders, Simon. 2013. "Rethinking Newton's *Principia.*" *Philosophy of Science* 80 (1): 22–48.

Teh, Nicholas. 2018. "Recovering recovery: on the relationship between gauge symmetry and Trautman recovery." *Philosophy of Science* 85 (2): 201–224.

Wallace, David. 2019. "Who's afraid of coordinate systems? An essay on the representation of spacetime structure." *Studies in the History and Philosophy of Modern Physics* 67:125–136.

Wallace, David. 2020. "Fundamental and emergent geometry in Newtonian physics." *British Journal for the Philosophy of Science* 71 (1): 1–32.

Weatherall, James Owen. 2012. "A brief remark on energy conditions and the Geroch-Jang theorem." *Foundations of Physics* 42 (2): 209–214.

Weatherall, James Owen. 2016. "Maxwell-Huygens, Newton-Cartan, and Saunders-Knox spacetimes." *Philosophy of Science* 83 (1): 82–92.

Weatherall, James Owen. 2018. "A brief comment on Maxwell(/Newton)[-Huygens] spacetime." *Studies in the History and Philosophy of Modern Physics* 63:34–38.

Weatherall, James Owen. 2019. "Conservation, inertia, and spacetime geometry." *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* 67:144–159. https://doi.org/10.1016/j.shpsb.2017.09.007.

Weatherall, James Owen, and John Byron Manchak. 2014. "The Geometry of Conventionality." *Philosophy of Science* 81 (2): 233–247. https://doi.org/10.1086/675680.