April 10, 2023

Forthcoming in *Philosophia Mathematica*.
DOI: 10.1093/philmat/nkac032. Please cite the published version.

APPLIED MATHEMATICS WITHOUT NUMBERS

JACK HIMELRIGHT

Abstract. In this paper, I develop a "safety result" for applied mathematics. I show that whenever a theory in natural science entails some non-mathematical conclusion via an application of mathematics, there is a counterpart theory that carries no commitment to mathematical objects, entails the same conclusion, and the claims of which are true if the claims of the original theory are "correct": roughly, true given the assumption that mathematical objects exist. The framework used for proving the safety result has some advantages over existing nominalistic accounts of applied mathematics. It also provides a nominalistic account of pure mathematics.

1. Introduction

Many philosophers have developed nominalistic accounts of applied mathematics. Some have given a nominalistic treatment of mathematics in the context of its application within a specific scientific theory. Others have given a nominalistic treatment of pure mathematics, in part or in whole, and then utilize what they developed as a tool to nominalize applications of mathematics in a variety of areas. While these accounts have been significant advances for the nominalist project, they each face their own limitations.

In this paper, I offer a new nominalistic account of applied mathematics that has some advantages over current ones. I prove a "safety result" that shows that, for entirely nominalistic reasons, many theories that apply mathematics to the physical world never lead to false conclusions. I conclude that there is a nominalistic explanation of the utility of applied mathematics. In addition, the apparatus used to prove the safety result staightforwardly produces a nominalistic treatment of pure mathematics, which I will argue is a further point in its favor.

I will begin by describing the present situation of nominalism about mathematics, using Hartry Field and Geoffrey Hellman as representatives of the different approaches mentioned above.[1] I describe what limitations their approaches face and explain the distinct advantages of my approach. Following this, I will provide an overview of my approach and introduce the necessary philosophical machinery. Next, I will describe the safety result, leaving some details to be filled out in later sections. Then I will discuss the philosophical significance of the safety result: it shows that mathematical entities are dispensable from natural science. Afterwards, I will prove the safety result. Finally, I will discuss pure mathematics, discuss applied mathematics for other possible physical realities, and give concluding remarks.

2. The State of Play for Nominalism about Mathematics

One of the early and most prominent attempts to nominalize applied mathematics comes from Hartry Field. In *Science without Numbers*, Field (1980/2016) develops a nominalistic version of Newtonian gravitational theory (NGT) (pp. 61-91). Field also shows that mathematics is "conservative" with respect to nominalistic theories (pp. 7-20). This entails that the theory that results from adding the mathematics of platonistic NGT (and classical mathematics in general) to his nominalistic version does not prove any nominalistic conclusions that his nominalistic version does not already prove. His result suggests that, contrary to what one might expect, mathematics is dispensable from science. Since the supposed indispensability of mathematics from science is often taken to be one of the main reasons for accepting platonism in mathematics, at first glance, Field's project seems to undermine much of the motivation for accepting platonism.[2]

Unfortunately, there are difficulties with Field's approach. His proof that mathematics is conservative is not itself nominalistic, and his strategy for nominalizing NGT does

[1]Another noteworthy example of the second approach is Charles S. Chihara (1990). As an aside, my primary objection to Chihara's system is that I do not think that there is a satisfactory way for a nominalist to understand his constructibility quantifiers. I favor using only standard modal operators.

[2]The literature on this topic is vast, but for an overview of the "Quine-Putnam indispensability argument", see Field (1989), pp. 14-20. For a defense of the position that the only way to successfully undermine the argument is to show that mathematical objects are dispensable from scientific theories, see Mark Colyvan (2010).

not straightforwardly extend to some other important scientific theories.[3] Later, Geoffrey Hellman (1996) showed that fourth-order Peano Arithmetic (PA^4) can be developed in a nominalistic way using plural quantification, fusions, and a continuum of mereological simples (pp. 112-113). This is important for two reasons. First, spacetime itself is a witness of this system: spacetime points constitute a continuum of simples. Second, as has been discovered within the project of reverse mathematics, the vast majority of applied mathematics can be constructed even within PA^2.[4] PA^4 is sufficient for all known applications of mathematics, and it is reasonable to expect that no applications of mathematics are likely to exceed its resources.

One might think that the contingency of the structure of our spacetime constitutes a significant limitation of developing PA^4 in this way. That the prospect of applying mathematics to the physical world should depend on actual spacetime having the structure that it has seems odd. Perhaps some possible spacetimes are discrete, or perhaps they have a relationalist nature as opposed to a substantivalist one (in which case there is no continuum of concrete points). In either case, PA^4 could not be developed out of them in the Hellman way. Happily, this contingency is not an insurmountable difficulty. No matter how the concrete world is, it is logically possible for there to be another spacetime with a continuum of points that co-exists with the original concrete world but does not overlap it.[5] Then PA^4

[3]For the former point, see Chihara (1990), pp. 162-163. For a rebuttal, see Field (1992). For the latter point, see David Malament (1982), pp. 532-534.

[4]See especially Stephen Simpson (1999), pp. 6-23, and Solomon Feferman (1998), pp. 249-283.

[5]The possible scenarios I will discuss are second-order logical possibilities: they invoke full plural quantification. It might be thought that metaphysical possibility is required for my purposes rather than logical possibility on the grounds that quantification into and out of modal operators for logical possibility is ill-behaved. For example, it seems logically possible that Hesperus is not Phosphorus (since not contradictory) but logically impossible that Hesperus is not Hesperus. Introducing quantifiers then leads to disaster: since Hesperus is Phosphorus, this means that there is something such that it is logically possible and logically impossible for it to be Hesperus. Here several points can be made. First, a constraint may be put on logical possibility such that something is logically possible only if it is consistent with all true identity claims. Since 'Hesperus' and 'Phosphorus' rigidly designate the same thing, with such a constraint in place, that Hesperus is not Phosphorus turns out to be logically impossible. Second, in my proposal, the translation of mathematical sentences will eliminate names for mathematicals in favor of logical variables and quantifiers governing them. Since the trouble is raised by names, these problem cases will not arise in my system. (Applied mathematical sentences do sometimes refer to particular physical objects, but for scientific purposes names of concreta could be replaced with definite descriptions.) And third, Sharon Berry (2022) has recently introduced novel operators for conditional logical possibility that avoid the problems posed by quantification

can be developed in the manner of Hellman no matter how the concrete world is: one need only consider what would be true were there a spacetime like ours in addition to whatever concrete things existed in the original scenario.

There are, however, some limitations on restricting ourselves to the resources of PA^4. One of these is technical. Although all foreseeable applications of mathematics do not depend on anything more, it is at least an epistemic possibility that some applications of mathematics might depend on something more powerful. Hellman (1989) himself points out that because higher-level mathematics is not conservative with respect to lower-level mathematics, it might turn out that some empirical facts—in Hellman's example, the structure of the periodic table of elements—are the case by mathematical necessity from the perspective of higher-level mathematics, but can only be taken as brute from the perspective of PA^4 (pp. 120-122). The significance of this point is even greater if we are worried about having an account of applied mathematics that is overly contingent: it seems even more likely that some possible spacetime has features only explainable from the perspective of a higher-level mathematical theory than that actual spacetime has this feature.[6] Consider that some philosophers have thought, on the grounds of highly permissive modal recombination principles, that for every cardinal k, there are possible spacetimes with k occupants.[7] PA^4 is not enough to describe the features of such mathematically rich spacetimes.

Another limitation is philosophical. While it is good to nominalize applied mathematics, one might wish to go further and nominalize pure and applied mathematics in one fell swoop. One might maintain, as David Lewis (1993) did, that it is presumptuous to hold that set theory is simply false on philosophical grounds (pp. 14-15). The achievements of mathematics are many, whereas the achievements of philosophy are few. Even worse, given

in modal contexts (pp. 46-48). While I prefer using familiar modal operators, these might be utilized. In any case, provided that there is a coherent notion of metaphysical possibility, the scenarios that I regard as logically possible seem metaphysically possible as well.

[6]I do not myself believe in mere possibilia, but it is more convenient to speak this way than to phrase everything with sentential modal operators. My remarks here should be taken as shorthand for the equivalents stated with operators.

[7]See Daniel Nolan (1996). Nolan argues that David Lewis in particular should believe this, but anyone who accepts a modal form of the recombination principles that Lewis endorses should also believe this if Nolan's argument is successful.

philosophy's historical tendency to lead people to believe various bizarre, contrary theses, philosophy seems to reliably produce false beliefs about certain matters. As Lewis remarked (p. 14):

> How would *you* like to go and tell the mathematicians that they must change their ways, and abjure countless errors, now that *philosophy* has discovered that there are no classes? Will you tell them, with a straight face, to follow philosophical argument wherever it leads? If they challenge your credentials, will you boast of philosophy's other great discoveries: that motion is impossible, that a being than which no greater can be conceived cannot be conceived not to exist, that it is unthinkable that anything exists outside the mind, that time is unreal, that no theory has ever been made at all probable by evidence (but on the other hand that an empirically ideal theory can't possibly be false), that it is a wide-open scientific question whether anyone has ever believed anything, *ad nauseam*? Not me!

For these reasons, a nominalist might hold that set theory is false in its platonic interpretation but still feel the need, out of epistemic humility, to say that set theorists are engaged in a practice that reliably correlates with significant truths of some sort.[8] If one can do so for set theorists, by extension, one can do so for pure mathematicians as a whole. PA^4 is not up to this task.

Hellman offers another approach toward mathematics that has the hope of avoiding both of these limitations. This is the Hellman (1989) "global approach" toward applied mathematics (pp. 99-101, 118-124). Hellman begins by nominalizing the notion of a relation, using mereological simples, fusions and plural quantification.[9] He then gives a nominalistic

[8]Lewis would go further and say that the theorems of set theory express significant truths, not just reliably correlate with them. However, one could share Lewis' conservative attitude without taking it to its extreme. I will leave open the possibility that the claims of set theory really had the structure of my nominalist counterparts of them all along, but I do not want to commit to such a strong thesis myself. More will be said about this when I discuss pure mathematics.

[9]Originally, Hellman (1989) used a primitive pairing predicate to generate relations (pp. 49-52), but he notes this improvement in Hellman (1996), pp. 104-105.

treatment of Zermelo set theory with urelements + an axiom stating that there is only one limit ordinal (Z^+). This treatment can be extended to full ZFC with urelements as desired.[10]

Hellman's strategy is as follows. Let 'α' be a name for the fusion of actual spacetime and its actual material occupants. Let '$\Box^@$' be a modal operator that quantifies over all possible worlds in which α is just how it actually is. Then for arbitrary applied mathematical sentence φ, Hellman's replacement is: $\Box^@\forall X\forall f[(\wedge\mathsf{Z}^+\wedge\mathcal{U})^{X[f/\in]} \rightarrow \varphi^{f/\in}]$. Here,'$X$' is a plural variable; '$\mathcal{U}$' is a claim to the effect that the parts of α constitute the urelements; '$^{X[f/\in]}$' indicates that f is being substituted for $\in$ in the statements of Z^+ and $\mathcal{U}$ and X satisfy the axioms of Z^+ and $\mathcal{U}$ under that substitution; and $\varphi^{f/\in}$ is the result of substituting f for $\in$ in the set-theoretic articulation of φ.[11] If one wants to capture full ZFCU, one only needs to substitute it for Z^+ in Hellman's replacements. The result of doing so is a theory that is capable of expressing pure mathematics and accounting for arbitrary applications of lower-level and higher-level mathematics to α. By swapping out '$\Box^@$' for restricted necessity operators that quantify over possible worlds with different spatiotemporal facts, it can also express applications of mathematics to other possible spacetimes as well.

There are two problems, however. The first is that it is not clear that the operator '$\Box^@$' can be satisfactorily understood nominalistically. The second is that, for all that has been said so far, there might be no possible worlds in which some things satisfy the axioms of Z^+ or ZFCU and α is just how it actually is. I begin with the former, then turn toward the latter.

[10]Hellman himself denied this proposal was nominalist, preferring the label 'modal neutralism' (Hellman (1989), pp. 115-117). This is because it exceeds the resources of what can be coded in a spacetime with the structure of $\mathbb{R}^4$. Physical or material nominalism, which confines itself to the limits of physical realities like ours, is what Hellman had in mind when using the term 'nominalism'. However, the view is still nominalist in the following logical or conceptual sense: it is consistent with everything being a fusion of concrete, spatiotemporal objects, and it is sufficient to define all mathematical vocabulary (in a structuralist manner). The former condition holds both because it might be logically possible for there to be spacetimes with a richer structure than actual spacetime and because it is logically possible for there to be more than continuum-many disconnected spacetimes. These points will come up in subsequent discussion. In any case, it is logical or conceptual nominalism that I intend to support when arguing for nominalism.

[11]Here I am overlooking some use-mention issues in connection with 'f', f, '$\in$', and $\in$ for ease of exposition. Hellman himself simply uses '$\Box$' in his formulation, but I added the '$^@$' to make clear that it is a restricted necessity operator.

The explanation of '$\Box^@$' I gave above is obviously platonistic. One might try to define it this way:

$$\Box^@\varphi := \Box(\alpha \text{ is just how it actually is} \rightarrow \varphi)$$

But this only passes the buck. The question now becomes how to understand the phrase 'is just how it actually is'. The only way I can see to define the phrase is by quantifying over properties, as follows:

$$x \text{ is just how it actually is} :=$$
$$\forall y(y \text{ is an intrinsic property} \rightarrow [x \text{ instantiates } y \leftrightarrow @(x \text{ instantiates } y)])$$

In terms of understanding the phrase nominalistically, that definition is of limited use. Standard conceptions of properties are not nominalistically acceptable. That said, there is a tradition stemming from Aristotle that identifies properties with causally-involved, concrete property instances, and some nominalists may wish to avail themselves of it (see also Hilary Putnam (1969), pp. 247-254). In any case, the approach toward nominalizing mathematics I will develop here, framed in terms of causally isolated mereological simples, sidesteps an issue that such an account of properties raises: finding a criterion that individuates between physical properties in Aristotle's (and Putnam's) sense. Perhaps a nominalist might instead allow themselves to take the phrase as a primitive, but I am not sure it is an acceptable one. It is one thing to use it when speaking loosely, but it is another to treat it as a primitive in a theory properly speaking.[12]

Hellman does not leave matters here, and some of his comments give rise to a more satisfying approach to defining '$\Box^@$' (pp. 129-135). Say an applied mathematical sentence φ is "correct" if the nominalistic states of affairs obtain that are sufficient for φ to be true if there are mathematical objects. For example, if the Eiffel Tower is 1,063 feet tall, then '1,063 measures the Eiffel Tower's height in feet' is correct. Suppose one could introduce a

[12]That said, Cian Dorr (2008) uses this expression without defining it (p. 37). For my part, I think it's too ambiguous to be an acceptable primitive. There's some reading of 'just how it actually is' on which I am not just how I actually am if I stand in different relations to things. Context makes clear that in the preceding discussion, that is not the intended reading, but the intended reading can only be readily distinguished from the unintended reading by utilizing platonic resources.

language $\mathcal{L}$ that is nominalistic and rich enough that it contains some sentences that together determine, for each applied mathematical sentence, whether it is correct. Now let $\mathcal{C}$ be a conjunction of some such sentences of $\mathcal{L}$ that are true. Then one can define '$\Box^{@}$' in this way:

$$\Box^{@}\varphi := \Box(\mathcal{C} \rightarrow \varphi)^{13}$$

Hellman gives an extended discussion regarding the development of such a language that merits considerable attention. But the core difficulty that I want to highlight is this. We do not, in fact, have $\mathcal{L}$. Moreover, nominalists, being nominalists, cannot say that in some possible language, there is a suitable interpretation of '$\Box^{@}$'. Until $\mathcal{L}$ is fully developed and an appropriate plurality of sentences can be identified, this definition of '$\Box^{@}$' will always be either incomplete or platonistic.

The same sort of problem arises for another proposal of Hellman's (pp. 124-129). Suppose we could identify some correct applied mathematical sentences that together determine the correctness or incorrectness of each applied mathematical sentence. Let $\mathcal{D}$ be a conjunction of them. Then one might, instead of defining '$\Box^{@}$' as an operator, take the whole of Hellman's replacements to be of the form '$\Box\forall X \forall f[(\wedge \mathsf{Z}^{+} \wedge \mathcal{D} \wedge \mathcal{U})^{X[f/\in]} \rightarrow \varphi^{f/\in}]$', where '$\Box$' is just the normal necessity operator. However, we are not in a position to identify the relevant collection of applied mathematical sentences. There are too many unsolved problems in science to say. Nor can we simply say that there are such sentences, without identifying them, and declare $\mathcal{D}$ to be a conjunction of them: that involves unacceptably platonistic talk about sentence-types.[14]

In view of these problems, it is not clear that '$\Box^{@}$' can be nominalistically defined, at least for the foreseeable future. Suppose, however, that this difficulty can be overcome. That leads to the second problem. What assurance is there that an incorrect applied mathematical

[13] Hellman's formulation is more complex (p. 130), but this gets at the essence of the matter and should deliver the appropriate truth-conditions.

[14] Another option, using Chihara's constructibility quantifers (Chihara (1990)), is to say that some correct sentence-*tokens* are constructible that collectively determine the correctness or incorrectness of each constructible applied mathematical sentence-token. Then one can identify $\mathcal{D}$ with a constructible conjunction of them and say that Hellman's proposal above yields adequate constructible replacements for applied mathematical sentences. The question of whether constructibility quantifers are nominalistically acceptable will be set aside in the present work (see fn. 1).

sentence's replacement under Hellman's scheme is not true, one for which the appropriate nominalistic states of affairs do not actually obtain? To be sure of that, we need to be sure that it is logically possible for there to be some X and f such that $(\wedge \mathsf{Z}^+ \wedge \mathcal{U})^{X[f/\in]}$ while α is just how it actually is. But from a nominalist point of view, there is no obvious guarantee that this is logically possible. It isn't self-evident that even if there is no abstract realm, some things and relation (in the Hellman sense) could satisfy Z^+, let alone satisfy it in worlds where α has the same character that it actually has. It isn't just a matter of there being enough objects: they also need to stand in the right relations to each other. So there is a danger that every Hellman replacement is vacuously true.

In this paper, I will provide an account of mathematics that satisfies all the above desiderata. It is inspired by Hellman's global approach, taking a modal-structuralist stance toward mathematics, but utilizes counterfactuals about a scenario describing an inaccessible infinity of simples. ZFCU can be developed from this scenario nominalistically, as all that is needed for the proof of ZFCU is the postulation of enough simples and some modest principles governing composition and size. The core insight will be similar to the point about adding a logically possible spacetime to whatever happens to exist in order to generate PA^4, but provides a stronger framework for doing mathematics. The account will yield replacements for applied mathematical claims and only makes use of the familiar '@' and '$\Box\!\!\rightarrow$' to get the right truth-conditions. Moreover, since the scenario is clearly possible (as will be evident), we can be assured that these replacements are not vacuously true.[15] The replacements will undergird an important safety result, showing that mathematical objects are dispensable from natural science. Additionally, this mathematical framework can do justice to the conservative impulse to affirm that set theorists are engaged in a practice that reliably tracks

[15]That the counterfactuals I use have possible antecedents and make no mention of abstract objects is an important feature of my approach. Richard Woodward (2010) and Lukas Skiba (2019) offer general safety results that could be used for applied mathematics, but Woodward's depends on the assumption that abstract objects only contingently do not exist, while Skiba's depends on either that assumption or the non-triviality of counterpossibles. Neither assumption is a desireable hill to die on for the sake of defending nominalism, and neither is necessary for my safety result.

significant truths. Finally, the framework can account for the safety of applied mathematics in other possible spacetimes.

3. A New Modal-Structuralism

To simplify the syntax, let ZFCU^+ be $\wedge\mathsf{ZFCU} \wedge \mathcal{U}$. $\mathcal{U}$ can be explicitly stated as $\forall x(x$ is an urelement $\leftrightarrow @(x$ is part of $\alpha))$, where 'α' retains its meaning from before. We work with ZFCU^+ because of the possible applications of higher-level mathematics mentioned earlier. To see the motivation for the approach I will take, it is best to begin by considering some alternatives. One initial idea for modifying Hellman's global strategy to avoid the need for '$\Box^@$' is to use counterfactuals. In particular, one might suggest using the following replacement for any mathematical sentence φ:

$$\exists X \exists f (\mathsf{ZFCU}^+)^{X[f/\in]} \mathrel{\Box\!\!\rightarrow} \forall Y \forall g [(\mathsf{ZFCU}^+)^{Y[g/\in]} \rightarrow \varphi^{g/\in}]$$

The symbols in the counterfactual are to be interpreted as expected in light of the preceding section. The thought is that instead of finding some explicit criteria that guarantee that α is just how it actually is and looking at at all possible worlds in which those criteria hold and some X and f satisfy the axioms of ZFCU^+, we might instead look at all the *nearest* possible worlds in which some X and f satisfy the axioms of ZFCU^+. The hope is that those worlds are ones in which α has not changed. If that hope is well-founded, then there is no need to find any such explicit criteria in order to hold facts about α fixed. Assuming the counterfactual is not vacuous, it will have the right truth-conditions even if nothing is explicitly mentioned about α.

Unfortunately, not all hopes are fulfilled. It seems plausible that at least some of the nearest worlds in which some X and f satisfy ZFCU^+ are ones in which a vast infinity of objects are added into α itself, provided that spacetimes do not necessarily have $\leq \mathfrak{c}$ points. Certainly in those worlds α is not how it actually is. Nor is there any reason to think that the actual material parts of α are unchanged in those worlds: actual material objects would be causally related to entirely new things.

The hope need not die here, however. We can add a statement to the antecedent of the counterfactual which is designed specifically to exclude those worlds. Let us introduce a sentence $\mathcal{A}$ which states the following: $\exists X(X$ compose $\alpha \wedge @[X$ compose $\alpha] \wedge \forall y \forall z[(y$ is a part of $\alpha \wedge y$ is spatiotemporally related to $z) \rightarrow z$ is a part of $\alpha])$, where 'X' is once again a plural variable. In English, $\mathcal{A}$ says that α is composed by whatever actually composes α and that no part of α is spatiotemporally related to anything disjoint from α. The last constraint is to rule out states of affairs in which α is embedded within a larger spacetime, supposing that is possible. With $\mathcal{A}$, the strategy can be refined to replace any mathematical sentence φ as follows:

$$[\mathcal{A} \wedge \exists X \exists f(\mathsf{ZFCU}^{+})^{X[f/\in]}] \mathrel{\Box\!\!\rightarrow} \forall Y \forall g[(\mathsf{ZFCU}^{+})^{Y[g/\in]} \rightarrow \varphi^{g/\in}]$$

The hope now is that all the nearest worlds in which α has all and only its actual parts and is spatiotemporally isolated and in which some X and f satisfy the axioms of ZFCU^{+} are worlds in which α is just how it actually is. Certainly not all the worlds in which those conditions obtain are worlds where α is just how it actually is. It seems possible for α to have a different history even when it is isolated and has just the parts it actually has. Since we are using counterfactuals, however, only the nearest worlds are relevant. And given these constraints, it is difficult to see how any alteration in α's history could be relevant to whether some X and f satisfy the axioms of ZFCU^{+}. But while it seems likely that these sentences have the right truth-conditions to serve as adequate replacements for applied mathematical claims, it is not guaranteed. I cannot see how any worlds in which $\mathcal{A}$ is true, some X and f satisfy the axioms of ZFCU^{+}, and α obeys different physical laws can be nearer to the actual world than ones in which $\mathcal{A}$ is true, some X and f do the same, and α is just how it actually is. Likewise for worlds in which $\mathcal{A}$ is true, there are some suitable X and f, and the protons of α are instead electrons; and worlds in which $\mathcal{A}$ is true, there are some suitable X and f, and Hobbes never attempted to square the circle. However, this difficulty in seeing might just reflect the limits of my faculties. The axioms of ZFCU^{+} are substantial, and the imagination only stretches so far.

The last strategy is on the right track, but only addresses one half of the problem: the problem of keeping α isolated and securing all and only its actual parts. The other half is starting with ZFCU^+. What is needed is to find some logically possible scenario such that we can be sure that the properties of α are irrelevant to whether it obtains (besides the number of its parts, which is held fixed by $\mathcal{A}$) and from which it can be proven that some X and f satisfy the axioms of ZFCU^+. That logically possible scenario is the existence of an inaccessible infinity of simples. The existence of an inaccessible infinity of simples is a purely numerical issue: so long as α is stipulated to have its actual number of parts and is spatiotemporally isolated, what α is like makes no difference to whether there exists an inaccessible infinity of simples. α has the simples that it has, and its spatiotemporal isolation ensures that it makes no causal difference toward anything outside of itself (such as bringing disjoint particles into existence). And it turns out that an inaccessible infinity of simples is sufficient to guarantee the existence of some X and f that satisfy the axioms of ZFCU^+. Letting $\mathcal{E}$ be a statement to the effect that there exists an inaccessible infinity of simples, the *modal-structuralist replacement* of any applied mathematical sentence φ is:

$$(\mathcal{A} \wedge \mathcal{E}) \mathrel{\Box\!\!\rightarrow} \forall X \forall f[(\mathsf{ZFCU}^+)^{X[f/\in]} \rightarrow \varphi^{f/\in}]$$

More needs to be said about these replacements.[16] We need some way to (a) formulate $\mathcal{E}$ nominalistically, as inaccessibility is usually defined in terms of set theory. We also need to (b) show how $\mathcal{E}$ entails that some X and f satisfy the axioms of ZFCU^+. For now, the important point is that if this can be done, these replacements for applied mathematical claims have the right truth-conditions to serve as replacements. Given $\mathcal{A}$, whatever happens in α is irrelevant to the existence of an inaccessible infinity of simples. Then the nearest possible worlds in which $\mathcal{A}$ is true and there exists an inaccessible infinity of simples are ones in which α is just how it actually is.[17] To be sure, as a general matter, a world's nearness

[16]Of course, to say that they are *the* modal-structuralist replacements is a bit misleading. There is more than one modal-structuralist way to approach mathematics. But these are the replacements I will use, and it is handy to have a definite article.

[17]One might suspect I am engaging in some slight of hand here. Earlier, I complained about the expression 'just how it actually is', and now I am using it to articulate the thought that my replacements have the appropriate truth-conditions. However, I mentioned I am fine with it being used in a loose way, just not as a

to the actual world is not exclusively a question of how similar α is in that world to how α actually is. Supposing so runs roughshod on our counterfactual intuitions. Nevertheless, that α is just how it actually is counts in favor of a world's nearness, and with α's causal isolation and the fixity of its parts, nothing is to be gained toward attaining the inaccessible infinity of simples by changing α.[18] Moreover, the fact that $\mathcal{E}$ entails that some X and f satisfy the axioms of ZFCU^+ ensures that the consequents of the replacements are not vacuously true at the $\mathcal{E}$ worlds. And, in connection with the second problem raised for Hellman's global approach, the fact that $\mathcal{A} \wedge \mathcal{E}$ is logically possible guarantees that the replacements are not vacuously true.

Notice that even if there is a size limit on how many occupants a spacetime can have, the logical possibility of $\mathcal{E}$ is not undermined. It is logically possible for there to be an inaccessible infinity of electrons distributed throughout an inaccessible infinity of disconnected spacetimes. Since the occupants of different spacetimes do not causally interact with each other, different spacetimes do not limit each other in terms of number or quality. There might be a limit as to how many parts a spacetime can have, but there is no limit on the number of spacetimes themselves. Moreover, if the laws of nature turn out to be logically necessary—thereby imposing some significant constraints on the possible properties of spacetimes—it is still logically possible for a spacetime to exist alongside inaccessibly-many disconnected copies of itself. That is enough to guarantee that $\mathcal{E}$ is logically possible. By the same reasoning, $\mathcal{A} \wedge \mathcal{E}$ is logically possible. Since spacetimes isolated from α do not constrain

part of a theory properly speaking. That is how I am using it now. Ultimately, all that matters is that one gets the intuition that the truth-conditions of the replacements are correct when the time comes to evaluate the matter. Loose talk can sometimes help one see the precise picture, but the theory itself had better be precise. The same goes for my use of 'property' and 'possible world'.

[18] In the preceding I have assumed that the spatiotemporal isolation of α entails its causal isolation, but that can be questioned. Perhaps it is logically possible for there to be simple, atemporal disembodied minds that influence multiple spacetimes. To address this, we can directly add to $\mathcal{A}$ a claim that α is causally isolated in a strong sense: nothing disjoint from α causally influences α, and were something to try to influence α, it would fail. Of course, this will not help if there is in fact such a mind. If such is the case, perhaps in the nearest world in which α has all and only its actual parts, is spatiotemporally isolated, and is causally isolated from the mind that actually influences α, α does not exhibit the fine-tuning needed for organic life. For present purposes I will simply aim to provide a nominalistic account of mathematics that succeeds if there is no such mind. I leave it as an open question what would need to be said about such a mind (or minds) in $\mathcal{A}$ for the account to work if such a mind exists.

what properties α has, there are some possible worlds in which α has an inaccessible infinity of copies, all and only its actual parts, is spatiotemporally isolated, and—important for the counterfactuals—all and only its actual intrinsic properties.

One way to complete tasks (a) and (b) comes from David Lewis. In "Mathematics is Megethology", the Lewis (1993) follow-up to John P. Burgess, A. P. Hazen, and Lewis' Appendix to *Parts of Classes*, Lewis defines inaccessibility without platonistic resources and shows that the existence of inaccessibly-many simples entails that there are singleton relations (in the nominalized sense of relation also used by Hellman) (pp. 18-21). From the existence of singleton relations, Lewis is able to show, using the logic of plurals and mereology, that some X and f satisfy a theory of classes strong enough to derive ZFCU^+ (pp. 21-23). In fact, Lewis shows that the existence of inaccessibly-many simples guarantees that there is a model of $\mathsf{ZF}^2\mathsf{CU}^+$, or the combination of second-order ZFCU and the $\mathcal{U}$ axiom.[19] While Lewis' formulation of $\mathcal{E}$ and derivation of $\mathsf{ZF}^2\mathsf{CU}^+$ would work for present purposes, the need to start with singleton relations and a theory including proper classes is undesirable. Fortunately, acting as a reviewer, Geoffrey Hellman pointed out that it is possible to construct $\mathsf{ZF}^2\mathsf{CU}^+$ out of an inaccessible infinity of simples using a more direct method. This construction directly develops V_U, the iterative hierarchy of impure set theory, in a manner more in keeping with standard mathematical practice. All of the necessary principles are shared with Lewis, so there is no philosophical disadvantage *vis a vis* his megethology.[20]

To begin, we observe some definitions, switching our notation for plurals so that capital letters can be used for fusions later:

$x \prec xx :=$ x is among xx

$xx \prec yy :=$ xx are among yy

[19]The remark about the $\mathcal{U}$ axiom follows from Lewis showing that for any plurality of things that is not inaccessibly infinite in size, there exists a singleton relation that treats those and only those things as urelements.

[20]Much of the material before the construction of the hierarchy V^M, detailed later, is similar to the Hellman (2003) "theory of large domains" (pp. 148-154).

$\bigvee xx :=$ the yy such that $\forall y(y \prec yy \leftrightarrow \exists zz[zz \prec xx \wedge y$ is the fusion of $zz])$,
i.e. the fusions of the xx

$\mathbf{Part}(x, y) := x$ is a proper part of y

With these in hand, we can now define what it is for some xx to be infinite.

xx are infinite :=
$\exists yy(yy \prec xx \wedge \forall z[z \prec \bigvee yy \rightarrow \exists v(\mathbf{Part}[z, v] \wedge v \prec \bigvee yy)])$

Informally, that xx are infinite tells us that some of the fusions of xx are such that each one is a proper part of another. This is only possible if xx are at least countably infinite, though we do not yet have the tools to describe the notion of countability.

We can now describe the process of nominalizing talk of relations mentioned earlier in greater detail. With the aid of an infinity of objects, it is possible to code ordered pairs. Several methods for doing so are detailed in the Appendix to *Parts of Classes*. Burgess' Method of Double Images is the simplest (pp. 121-127). The essential idea is to use two-simple fusions to code two one-one correspondences between all the simples and two disjoint subpluralities of them. By stringing together quantifiers in a suitable order, we can describe one of the correspondences as coming first and the other as coming second. The ordered pair of simples a and b is the fusion of the first image of a and the second image of b. From this, we can give the general criterion that the ordered pair of any fusions of simples A and B is the fusion of the first images of the simple parts of A and the second images of the simple parts of B. Adding a further assumption that all the simples can be divided into three equal parts ensures that the ranges of the one-one correspondences do not overlap: two-simple fusions are now treated as coding one-one correspondences between any two of those parts and a part of the third. That the simples are infinite ensures that the simples can be partitioned into three equal parts.

Using ordered pairs, it is possible to recover claims about relations (and thus functions). Any claim about a relation can be taken as tacit plural quantification over ordered pairs.

This interpretation will be assumed hereafter, but left implicit. Now that we have explained how we can talk about relations, we introduce a principle of Global Choice.

Global Choice: For any xx, if there is a relation that relates each of xx to at least one thing and does not relate any two of xx to the same thing, then there are some yy and there is some f such that f is a bijection from xx to yy.

Since Lewis already accepts such a principle, we are no worse off than Lewis is with megethology.[21] Crucially, Global Choice allows us recover "ordinals" prior to developing set theory. Say oo are "ordinals" with respect to xx, abbreviated as '$oo\Omega xx$', iff xx are simples, oo are among xx, and there is a well-ordering of oo such that any well-ordering of simples among xx is order-isomorphic to a segment of that well-ordering of oo. By entailing that every plurality of simples has a well-order, Global Choice entails that all pluralities of simples contain subpluralities of "ordinals" with respect to them.[22]

Being inaccessible can be now defined as being uncountably infinite, supporting the Power Set operation, and supporting Replacement. More precisely, using '$xx \not\twoheadrightarrow yy$' to mean there is no surjection from xx to yy:

xx are uncountably infinite :=

$\exists yy(yy \prec xx \wedge yy$ are infinite $\wedge\ yy \not\twoheadrightarrow xx)$

xx support the Power Set operation :=

$\forall yy([yy \prec xx \wedge yy \not\twoheadrightarrow xx] \rightarrow \bigvee yy \not\twoheadrightarrow xx)$

xx support Replacement :=

$\forall yy \forall oo \forall f([yy \prec xx \wedge yy \not\twoheadrightarrow xx \wedge oo\Omega xx \wedge f : yy \longrightarrow oo] \rightarrow$

$[\exists o(o \prec oo \wedge \forall y[y \prec yy \rightarrow o > f(y)])])$,

where '$>$' indicates the converse of the well-ordering of oo.

These conditions can be put informally as follows. That xx are uncountably infinite tells us that there are some infinite yy among xx that are smaller than xx in size. That xx

[21] This is effectively equivalent to the Lewis (1991) formulation of Choice, since that formulation applies to proper classes as well as sets (pp. 104-105).

[22] Alternatively, one could justify introducing "ordinals" by directly assuming that there is a well-ordering of all the simples. Lewis also does this (Lewis (1993), p. 19; Burgess, Hazen and Lewis (1991), p. 130).

support the Power Set operation tells us that xx contains, for every yy among xx smaller than xx in size, more simples than the power set of yy. And that xx support Replacement tells us that xx contains, for every yy among xx smaller than xx in size and function on yy, some "ordinals" larger than the "ordinals" that be put into one-one correspondence with the function's image.

We can now use Global Choice and the inaccessibility of xx to construct a cumulative hierarchy V^M that can be shown to entail the existence of a model of $\mathsf{ZF^2CU^+}$. First, we use supporting Replacement and the existence of "ordinals" to justify transfinite induction. (Hereafter, the scare-quotes will dropped.) Then we can use transfinite recursion to define a bijection F from ordinals to functions as follows:[23]

$F(0)$ is an arbitrary bijection f_0 from parts of α to some simples disjoint from α and from a different simple disjoint from α to an additional simple disjoint from α. For any successor ordinal $\beta + 1$, $F(\beta + 1)$ is an arbitrary bijection $f_{\beta+1}$ from the fusions of the simples in the image of f_β to simples. For any limit ordinal λ, $F(\lambda)$ is the identity function f_λ for the simples in the images of the functions F maps to from ordinals $< \lambda$. Now say that for any ordinal o, the simples in the image of f_o are the o atoms, and the fusion of the o simples is the level o. Then we add a further condition: for any successor ordinal $\beta + 1$ and fusion X in the domain of $f_{\beta+1}$, if there exists some f indexed by an ordinal $\leq \beta$ such that $f(X)$ is defined, then $f_{\beta+1}(X) = f(X)$; otherwise, $f_{\beta+1}(X)$ is a simple that is not a part of any level $\leq \beta$, nor in the domain of f_0. Note that for any ordinals o and o' such that $o < o'$, it might be that some of the o simples $\prec$ the o' simples (and if o' is a limit, it's all of the o simples), but it is always the case that $\neg(o \text{ level} = o' \text{ level})$. That there are enough simples to iterate this process through the successor ordinals is guaranteed by supporting the Power

[23]Some care is needed with F, since "functions" are understood as pluralities of ordered pairs. If the F pairs map one ordinal to each ordered pair that partly constitutes a function, they do not themselves constitute a function. However, just as pluralities of ordered pairs can be used to represent functions, so they can be used to represent multifunctions. Then we may say that the F pairs constitute a multifunction that maps each ordinal to all and only the ordered pairs that constitute the appropriate function. This will all be left implicit for ease of presentation. (See Burgess, Hazen and Lewis (1991), pp. 125-126 for a discussion of how ordered triples may be treated as ordered pairs of their first elements and the ordered pairs of their last two elements. I am suggesting using the same objects as the constituents of a multifunction.)

Set operation, while being uncountably infinite and supporting Replacement guarantee that the simples in a limit level are smaller than xx.[24] Given the required simples exist, both the functions indexed by ordinals and F are guaranteed to exist by the Method of Double Image's construction of ordered pairs (A, B) and (B, A) for any individuals $A, B \prec \bigvee xx$.[25]

From F, we can recover the notions of sethood and membership, and then justify the axioms of set theory. Let the empty set be the simple in the domain of f_0 that is disjoint from α. Let **Atomic**$(x, y) := x$ is a simple and part of y. (Part, not proper part.) Then define sethood and membership as follows, where 'o' is a variable for ordinals:

X is a set :=

$$X = \emptyset \vee \exists o \exists f \exists y (F[o+1] = f \wedge f[X] = y)$$

$x \in X$:=

$$X \text{ is a set } \wedge \exists f \exists y ([F(0) = f \vee \exists o (F[o+1] = f)] \wedge f[x] = y \wedge \textbf{Atomic}[y, X])$$

What we have done can be put informally as follows. We began by using a bijection to map the parts of α to simples and picked an arbitrary simple to count as the empty set. Next, we introduced a rule: any fusion of simples in the image of that function has as its members whatever was mapped to its simple parts. This entails that the simples in the image are singletons, and the fusions of those simples are their unions. Then we produced the hierarchy V^M using two techniques. The first is generating successor levels by using bijections to map fusions (unions of singletons) at their immediate predecessor levels to simples (singletons), with the first level having been generated by the initial bijection. The second is generating limit levels by collecting all the simples (singletons) in previous levels. The membership rule is repeated for each successor level, while at limits, the rule is that any inhabitant of a limit

[24]Supposing that the parts of α are $2^{\mathfrak{c}}$ in cardinality, being uncountably infinite and supporting the Power Set operation guarantee that the 0 simples are smaller than xx. (The spacetime points pp in α are $\mathfrak{c}$ in cardinality, making $\bigvee pp$ $2^{\mathfrak{c}}$ in cardinality, and there are no more occupants than fusions of points.) For every limit λ, we can consider a bijection that maps simples to levels $\leq \lambda$ and observe that the simples in the domain are smaller than xx, as there are many more simples introduced when extending the hierarchy to the λ level than there are ordinals used to index levels up to that point. By supporting Replacement, it follows that there are ordinals $> \lambda$. Since the simples in each level are smaller than xx, the levels themselves are smaller than xx, and xx are infinite, it is obvious that the simples in the λ level are smaller than xx. This is an application of the plural-mereological analogue of cardinal arithmetic (see Lewis (1993), p. 20).

[25]See Burgess, Hazen and Lewis (1991), p. 125.

has the members of its parts (as given by the lower membership rules). As will be shown, the defined notions of sethood and membership satisfy the axioms of $\mathsf{ZF^2CU^+}$. Thus, V^M yields a model of second-order set theory with urelements. The basic picture is like Lewis', but doesn't proceed by means of singleton functions—we started with membership and recovered singletons from it—and does not introduce proper classes.

Proving that the defined relations of sethood and membership satisfy most of the axioms of $\mathsf{ZF^2CU^+}$ is straightforward. It is obvious that Empty Set holds, that second-order Replacement follows from supporting Replacement, and that Choice is a consequence of Global Choice. Since Pairing and second-order Separation follow from other axioms, we need not demonstrate them independently. Extensionality follows from the uniqueness of fusions. Unions follows from plural comprehension and unrestricted composition: for every plurality of accessibly-many fusions of accessibly-many simples, there is a plurality of all the simples that are part of any of those fusions, and that plurality has a fusion.[26] Foundation is a consequence of the well-ordering of the levels of V^M. Urelements follows from the fact that the 0 simples are accessibly-many, and all the urelements are parts of α. Power Set falls out of supporting the Power Set operation and the addition of new simples at each successor level corresponding to fusions at the previous level.

Infinity is less trivial. To demonstrate it, say that an ordinal o is Dedekind-finite just in case there is no plurality of its predecessors that is in one-one correspondence with the ordinals up to o, and Dedekind-infinite otherwise. By plural comprehension, there is a plurality $oo^{\mathbb{N}}$ of the Dedekind-finite ordinals. Then it need only be shown that $oo^{\mathbb{N}}$ are in one-one correspondence with a subplurality of any Dedekind-infinite plurality among the inaccessibly-many atoms. The proof of this would approximately follow the proof that any infinite set has a countable subset, but using plurals instead of sets.

With Hellman's plural-mereological hierarchy in hand, we can now state that $\mathcal{E}$ is the claim that some simples are inaccessible in the sense defined above. Note that Global Choice is

[26]Here some simples or fusions thereof are accessibly-many just in case there is no surjection from them onto some simples that are inaccessibly-many.

necessitated by the existence of any infinity of simples, including an inaccessible infinity. It depends solely on plural and mereological conditions being sufficient to generate choice functions out of an infinity of simples, not on any properties of α. Indeed, given the power of the Method of Double Images to produce the ordered pairs (A, B) and (B, A) for any fusions A and B, it should be entailed by the Method of Double Images itself. The Method of Double Images plainly does not depend on anything other than plural and mereological facts. Likewise, the plural-mereological analogue of cardinal arithmetic mentioned in fn. 24 clearly does not depend on anything other than plural and mereological truths. Obviously, if some X and f must satisfy $\mathsf{ZF^2CU^+}$ given an inaccessible infinity of simples (returning to the use of 'X' as a plural variable), some X and f must satisfy the weaker $\mathsf{ZFCU^+}$. For applied mathematics the additional power of second-order set theory is not relevant, though it will have a role when we consider pure mathematics.

Having defined the modal-structuralist replacements of applied mathematical sentences, it is now possible to prove the safety result. In the course of doing so, the idea of getting the right truth-conditions will be made more precise. However, it is useful to define one further notion. Say that for any applied mathematical sentence φ, the consequent of φ's modal-structuralist replacement is the *mathematical nominalization* of φ, or nominalization$_M$ of φ for short; and say that the nominalization$_M$ of any base sentence ψ is ψ itself. (Base sentences are first-order, mathematically nominalistic sentences; a more explicit definition will appear in the next section.) The notion of a nominalization$_M$ will be useful in the proof.

4. Preservation and Nominalistic Safety Described

The first step toward establishing the safety result mentioned above is to outline it. To do that, I must first describe another result I will call "preservation". Preservation will be used in the proof of the safety result. To state it, several notions must be introduced. Let a *theory* be any collection of sentences, which can be thought of as the assertions of the theory. Let an *applied mathematical language* be any first-order language the mathematical vocabulary of which is restricted to the syntax of ZFCU.[27] For example, if the mathematical fragment of the language of NGT were to be reconstructed set-theoretically, the resulting language would be an applied mathematical language. In general, I assume that the mathematical fragment of scientific theories can be reconstructed set-theoretically: given our knowledge of foundations, this should not be controversial. Let a *base sentence* be a first-order sentence that does not contain any distinctively mathematical vocabulary and the quantifiers of which, if any, are restricted to non-mathematical objects.[28] An example is 'Tom is human'. A base sentence is a base sentence of a theory just in case it is a member of that theory. Finally, let a theory be *modest* if it is consistent with ZFCU^+ and does not prove any sentence of pure mathematics that is not already a theorem of ZFC.[29] The hypothetical set-theoretic reconstruction of NGT mentioned earlier is an example of a modest theory. While the articulation of Newton's law of universal gravitation in the language of set theory is not a theorem of ZFCU^+, NGT does not contradict ZFCU^+ and does not prove any new pure mathematical theorems.

Preservation is as follows. We will define, for arbitary applied mathematical language $\mathcal{L}_A$ and modest theory Γ_A of $\mathcal{L}_A$, the *modal-structuralist counterpart* of Γ_A and $\mathcal{L}_A$, or $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$ for short. The details will be given later. For now, all that matters is that modal-structuralist counterparts are theories. They make use of modal-structuralist replacements and additional claims connecting counterfactuals and base sentences. It will turn out that for arbitrary base sentence Φ_A of $\mathcal{L}_A$:

[27] As I mentioned earlier, the framework I use is able to develop second-order ZFCU. The restriction on applied mathematical languages to be first-order is just because theories in natural science, my main target, don't require second-order quantifiers and consequence in first-order logic can be captured proof-theoretically.

[28] See Field (1980/2016) P-20 and P-21 for a parallel definition of a nominalistic sentence.

[29] Note that ZFCU^+ has the same mathematical vocabulary as ZFCU. The practical advantage of ZFCU^+ over ZFCU is that the former guarantees that we get the results we want impure set theory to deliver: sets of actual material things, and so on. Note as well that because ZFC is a fragment of ZFCU^+, every classical mathematical theorem is a theorem of ZFCU^+.

Preservation: If Γ_A proves Φ_A, then $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$ proves Φ_A in a counterfactual, plural-mereological logic that includes deduction within counterfactual conditionals, reflexivity, and sound rules for its fragment without the counterfactual operator.[30]

This means that for any applied mathematical language $\mathcal{L}$ and modest theory Γ of $\mathcal{L}$, Γ proves a base sentence of $\mathcal{L}$ only if $\mathcal{MS}(\Gamma, \mathcal{L})$ proves it in a weak counterfactual logic for plurals and mereology.

With preservation described, the last step before describing the safety result is introducing three additional expressions.

Say a base sentence φ is *focused* iff $\ulcorner \varphi \leftrightarrow ([\mathcal{A} \wedge \mathcal{E}] \;\Box\!\!\rightarrow \varphi)\urcorner$ is true. For example, for any number n, 'there are n electrons in α' is focused, while 'there are n electrons' is not: we cannot be sure that if there were an inaccessible infinity of simples spatiotemporally isolated from α, none of them would be electrons inhabiting spacetimes separate from our own. Having introduced the notion of a focused base sentence, we can say that a language is focused if it is an applied mathematical language and its base sentences are focused. An example of a focused language would be a two-sorted, first-order language with one sort for mathematical entities, another sort for parts of α, and every predicate of which is either shared with ZFCU^+ or fundamental.

Next, we revise the notion of "correctness" mentioned in §2 to extend beyond wholly nominalistic states of affairs while simultaneously nominalizing the concept itself.[31] Say that platonism$_M$ is the thesis that there are mathematical objects and a sentence is nominalistic$_M$ if it does not contain distinctively mathematical vocabulary and the quantifiers in it are restricted to non-mathematical objects. (Note that unlike base sentences, nominalistic$_M$ sentences need not be first-order.) Then any sentence φ is *correct* just in case (i) it is a true base sentence, (ii) it is a theorem of ZFCU^+, or (iii) there is some nominalistic$_M$ sentence

[30] Deduction within counterfactual conditionals is the principle that $\vdash \chi_1 \wedge ... \wedge \chi_n \rightarrow \psi$ entails $\vdash ((\varphi \;\Box\!\!\rightarrow \chi_1) \wedge ... \wedge (\varphi \;\Box\!\!\rightarrow \chi_n)) \rightarrow (\varphi \;\Box\!\!\rightarrow \psi)$, while reflexivity is the principle that $\vdash \varphi \;\Box\!\!\rightarrow \varphi$. Both are principles of even the weakest counterfactual logics.

[31] The importance of the former point is because the safety result has implications even for theories that have commitments to non-mathematical abstract objects. Base sentences can refer to non-mathematical abstracta, like tropes, *in re* universals, etc., though such entities are in no way used to develop nominalistic set theory out of $\mathcal{E}$. See §5 for discussion.

ψ such that ψ is true and it is *a priori* that if platonism$_M$ is true, then $\ulcorner\varphi \leftrightarrow \psi\urcorner$ is true. As an example of (iii), consider '195 numbers the countries'.[32] The combination of several factors make that sentence correct. First, 'there are one hundred and ninety-five countries' is true. Second, the latter sentence can be defined purely logically, and thus is nominalistic$_M$ in content. Third, one need not know how many countries there are to know that if platonism$_M$ is true, then 195 numbers the countries iff there are one-hundred and ninety-five countries. This conditional is *a priori*, and mathematical platonists and nominalists alike know it to be true.

With this revised notion of correctness, it is now possible to specify the exact sense in which the modal-structuralist replacements of applied mathematical sentences have fitting truth-conditions. First, as is evident from the previous section, the modal-structuralist replacements of the theorems of ZFCU^+ are true. Second, for any applied mathematical sentence φ, φ is correct iff φ's modal-structuralist replacement is true. In general, every applied mathematical sentence is such that it is *a priori* that if platonism$_M$ is true, then that sentence is true iff its modal-structuralist replacement is true. That in turn means that no applied mathematical sentence is incorrect simply for lacking an *a priori* conditional of the above sort for every nominalistic$_M$ sentence. These claims will be justified in a later section.[33]

I can now describe the safety result. I will show that the following is true for arbitrary focused language $\mathcal{L}_F$ and modest theory Γ_F of $\mathcal{L}_F$:

(I) Every member of $\mathcal{MS}(\Gamma_F, \mathcal{L}_F)$ is true if every member of Γ_F is correct.

(II) Every member of $\mathcal{MS}(\Gamma_F, \mathcal{L}_F)$ is nominalistic$_M$.

It trivially follows from preservation, (I) and (II) that, for arbitrary base sentence φ_F of $\mathcal{L}_F$:

[32] Since we are working with set theory, take sentences about 195 to have a set-theoretic interpretation.

[33] Something like this, as well as the basic insight behind the safety result I will provide, is anticipated by Mary Leng (2017), pp. 140-141. Specifically, Leng has the idea that if there is a way that α can be kept just how it actually is, modal-structuralist claims can be used to represent what applied mathematical sentences say about the concrete world. She also mentions that any nominalistic conclusion that can be drawn from an applied mathematical claim can be drawn from its modal-structuralist representation, provided the "non-interference" constraint is met, though nothing like the full safety result I will provide is offered.

Nominalistic Safety: If Γ_F proves φ_F and the members of Γ_F are correct, then $\mathcal{MS}(\Gamma_F, \mathcal{L}_F)$ proves φ_F in a counterfactual, plural-mereological logic that includes deduction within counterfactual conditionals, reflexivity, and sound rules for its fragment without the counterfactual operator; and every member of $\mathcal{MS}(\Gamma_F, \mathcal{L}_F)$ is true and nominalistic$_M$.

To put it colloquially, nominalistic safety tells us the following. Take any mathematically modest first-order theory of a language the truth or falsity of any non-mathematical sentence of which is not affected by the presence or absence of an inaccessible infinity of simples that are spatiotemporally isolated from α. If its claims are correct, then for every non-mathematical claim of that language that it proves, there is a proof of that claim from a true theory that lacks any commitment to mathematical objects. Applied mathematics always leads us to truths for entirely nominalistic reasons, at least in the case of theories like this.[34]

5. The Upshot of Nominalistic Safety

Nominalistic safety is important for its implications. Notice that all scientific theories can be expressed in focused languages: they are about our spacetime, not things outside of it. If it turned out that some other spacetime contained creatures with metabolism despite lacking any carbon, or planets larger than the size thought to be possible without collapsing into black holes, or intelligent human-like beings that could change their long-held psychological traits at a whim, that would not cause us to judge that our biology, physics, or psychology are false. Our universal generalizations about concreta in science were really restricted to our spacetime all along.[35] Since it's possible to reconstruct mathematics set-theoretically and good scientific theories are mathematically modest—they don't prove new mathematical

[34]Of course, one limitation is scientific theories are generally not axiomatized at all, let alone in a first-order language. But it seems that in principle, a theory in natural science ought to be axiomatizable in a first-order language if it does not make use of plurals, at least through schemas if not through a finite list of axioms. It is hard to see how a scientific theory could be true or false unless it is formulatable as a set of assertions. A scientific theory that utilizes plural quantification could likely be represented using singular quantification and impure sets.

[35]Scientists do sometimes speak of a multiverse, but a multiverse is still a single spacetime in the Lewisian sense of 'spacetime' that I have in mind.

theorems—nominalistic safety entails that every theory in natural science with true extra-mathematical commitments has a nominalistic$_M$ counterpart that makes only true claims. This counterpart entails the same non-mathematical conclusions as the original theory. The limitation to natural science comes from the fact that any theory that makes use of, e.g., propositional attitude verbs is likely not to be axiomatizable in a first-order language: it is likely to require additional connectives and term-forming operators. (By a first-order language, I have in mind one which has only the syntax of standard first-order logic with identity, plus some sorting where convenient. Of course, the predicates and constants have a variety of acceptable natural language interpretations. The key is that an interpretation can't smuggle in logical resources not available in standard first-order logic.)

What this means is that mathematical objects are in principle dispensable from natural science, with scientific theories' nominalistic$_M$ counterparts being their in principle replacements.[36] This is of great importance to nominalism. As highlighted earlier, one of the most influential arguments against nominalism depends on the premise that mathematical objects are indispensable from science. The thought is that if mathematical objects are indispensable, then the only way to explain the success of science is by granting the existence of mathematical objects. Because spacetime itself is a witness to PA^4, we already can be reasonably confident that mathematical objects are dispensable from science. Still, the safety result offered here is broader in its implications. It chops the argument off at the root for all *possible* theories in natural science, some of which might employ a higher-level mathematical phenomenon to explain an empirical one.[37] It therefore puts the nominalist on even surer footing.

[36]Although I haven't drawn much attention to this fact, as will be seen, the counterparts are also nearly as formally elegant as the original theories. This distinguishes them from the unattractive nominalistic$_M$ counterparts that Craig's theorem (Craig (1956)) entails exist.

[37]This claim might seem too strong. In principle, there might be some phenomenon that can only be explained from the perspective of ZFCU + some large cardinal axiom. However, the modal-structuralist framework can be modified to account for these stronger theories (see fn. 50). The proof of safety and preservation would remain the same, but the articulation of $\mathcal{E}$ and the derivation of the set theory would change. If the framework is so modified, it will yield an explanation of the phenomenon.

To be sure, mathematics has applications outside of science. Some metaphysical theories might be stated in a language with modal operators and make mathematical claims, in which case the safety result I offer here will not be of use. Modal-structuralist counterparts are not provided for such theories since they are not first-order (in the relevant sense).[38] Moreover, as already mentioned, there are no modal-structuralist counterparts for many social scientific theories, plenty of which will make use of mathematics. Typically, however, people have thought that indispensability from natural science is particularly suggestive of reality, whereas indispensability from some metaphysical or social scientific theory is less so. For that reason, undermining the premise that mathematical objects are indispensable from natural science is the top priority for nominalists. With that in mind, from this point forward, when I mention science I will be referring to natural science.

However, there is something overlooked in my reasoning above. In my characterization of the non-mathematical realities that scientific theories concern, I discussed only concrete objects. But many scientific theories have ontological commitments to abstracta besides mathematical objects. Physics, for example, involves claims about units of measurement, shapes, and so on. This raises two points of significance.

First, I must mention that the truth-values of base sentences about non-mathematical abstract entities in reasonable scientific theories also are insensitive to the existence of an inaccessible infinity of simples spatiotemporally isolated from α, so they do not undermine the ability to formulate scientific theories in focused languages. They either express purportedly necessary truths, as with 'something is red iff it has the property of being red'; or else they concern how abstracta relate to entities in α directly, as with 'something (in α) can be alive without having an anatomical property'. Either kind is focused because the existence of abstract objects is not a contingent matter.

[38] This might seem to be inadequate for science, as science often involves counterfactuals, claims about what must be true, etc. However, these claims ought not to be thought of as parts of scientific theories themselves, but rather as justified on the basis of scientific theories. For example, a claim of the form 'were P, would be Q' can be justified on the grounds that it's a theorem of a scientific theory that if P, then Q, and a claim of the form 'it must be the case that P' can be justified on the grounds that it's a theorem of a scientific theory that P.

Second, nominalistic safety does not establish that all abstract objects are dispensable from science, only mathematical ones. If other abstract entities are indispensable from science, that poses a difficulty for nominalism that nominalistic safety does not address. In some cases, however, it is straightforward to imagine how to construct a counterpart of a scientific theory that is nominalistic aside from its commitments to mathematical objects. This can either be done directly, using nominalistic resources alone, or indirectly, with the assistance of impure sets. Take physics as an example. Following the direct approach, units of measurement could plausibly be replaced with comparisons to specific physical standards (e.g. "2 measures its mass in kg" becomes "2 measures how much more massive it is than the International Prototype kilogram"), and shapes might be replaced with claims about samely-shaped regions of spacetime. Following the indirect approach, at least some units of measurement, shapes, etc., might be capable of being represented as mixed sets of physical objects and mathematical entities.[39]

In any case, establishing nominalistic safety is a useful step on the path to a broader safety result for theories with additional abstract commitments.

6. Proof of Preservation

In order to prove preservation, we need a lemma.

Lemma: There is some sound (partial) proof system for the logic of plurals and mereology such that (a) if a first-order theory Γ proves a first-order sentence φ, then nominalizations$_M$ of the sentences of Γ combined with $\mathcal{E}$ prove the nominalization$_M$ of φ in that proof system, and (b) the proof system is at least as strong as natural deduction for propositional logic and transitive (i.e. if $\Gamma_1 \vdash \varphi$ and Γ_2 proves the members of Γ_1, then $\Gamma_2 \vdash \varphi$).[40]

[39] This highlights another advantage of working with ZFCU^+ over PA^4: it gives us access to impure sets, which might in turn be useful for reducing the commitments to abstracta besides the obviously mathematical ones. However, since it is unclear how often this is useful or necessary, I do not cite it as a major point in its favor.

[40] There is no complete proof system for plural logic, so the proof system must be partial.

Some clarification on (b) is in order. It is important that the proof system is at least as strong as natural deduction for propositional logic because I will appeal to propositional natural deduction rules when proving preservation. It doesn't matter whether the proof system is itself a natural deduction system, however: so long as the proof system agrees with propositional natural deduction's (positive) claims on what is provable from what (positive claims being opposed to silence), anything I show about provability from the natural deduction rules will apply to provability within the proof system. Likewise, I will appeal to transitivity of proof when proving preservation.

In defense of LEMMA, first note that the Lewisian megethological translation and the nominalization$_M$ of any first-order applied mathematical sentence φ are the same. While megethology proceeds by way of singleton relations, it ultimately derives a membership relation for a theory of classes. That theory of classes in turn entails there is a model of ZFCU^+. In the end, both the megethological translation and nominalization$_M$ state that for all X and f that satisfy ZFCU^+, $\varphi^{f/\in}$ is true (where $\varphi^{f/\in}$ is the result of substituting f for $\in$ in φ, as described in §2): even the megethologist has no need to refer to their full class theory to translate first-order mathematical sentences. Stipulating that the megethological translation of any base sentence is itself, the megethological translations of the sentences of Γ are its nominalizations$_M$. Second, observe that LEMMA is an expression of the proof-theoretic state of megethology as much as it is an expression of the proof-theoretic state of the framework we used to derive set theory from an inaccessible infinity of simples. It is straightforward to prove that principles used in Lewis (1993) to express the existence of an inaccessible infinity of simples—Hypotheses U, P, and I (p. 19; see also p. 22)—are equivalent to $\mathcal{E}$ when Global Choice and the plural-mereological analogue of cardinal arithmetic are assumed. So, given that megethological translations are nominalizations$_M$, LEMMA is effectively about megethology as much as it is about our plural-mereological framework.

The relevance of these points is this. A foundation for mathematics needs a proof system S such that every first-order mathematical derivation—whether it has premises or not—has a corresponding derivation in S *of* the foundational translation of the conclusion of the original derivation *from* the axioms of the foundation and the foundational translations of the premises occurring in the original derivation (if any; italics for clarity). If there is no such S, then the foundation is proof-theoretically deficient with respect to $\mathsf{ZFC(U)}$ when it comes to first-order mathematics. That is an unacceptable cost. If (a) fails for all proof systems, then megethology is proof-theoretically deficient in the manner just described. So far as I know, no one has complained that megethology is an inadequate foundation for mathematics. Yet if it is inadequate, one would expect to hear that complaint given that Lewis' goal with megethology was to provide a foundation.[41] Meanwhile, (b) reflects very weak properties any proof system for a foundation will almost certainly require for that foundation to be proof-theoretically adequate in the sense described.[42] So if the implicit consensus about megethology is right, LEMMA must be true.

Now we are ready to prove preservation. For every language $\mathcal{L}$, we can identify the *modal-structuralist maintenance set for* $\mathcal{L}$ ($MM_{\mathcal{L}}$) as $\{x \mid \exists y(y$ is a base sentence of $\mathcal{L}$ and $x = \ulcorner y \leftrightarrow ([\mathcal{A} \wedge \mathcal{E}] \;\Box\!\!\rightarrow\; y)\urcorner\,)\}$. Say that a *mixed sentence* is any applied mathematical sentence that is neither a theorem of ZFCU^{+} nor a base sentence. Then, lastly, for every theory Γ, we can identify the *mixed sentence replacement set for* Γ (R_{Γ}) as $\{x \mid \exists y(y$ is a mixed sentence, $y \in \Gamma$, and x is y's modal-structuralist replacement)$\}$. (Recall that the modal-structuralist replacement of an applied mathematical sentence φ is $(\mathcal{A} \wedge \mathcal{E}) \;\Box\!\!\rightarrow\; \psi$, where ψ is the nominalization$_M$ of φ.)

We can now define modal-structuralist counterparts using the arbitrary objects mentioned when describing preservation. Let B_A be the set of the base sentences of Γ_A. Then modal-structuralist counterparts can be defined as follows: $\mathcal{MS}(\Gamma_A, \mathcal{L}_A) = B_A \cup MM_{\mathcal{L}_A} \cup R_{\Gamma_A}$.

Finally, we can now introduce the logic I will use in my proof. It is defined as follows. It applies to languages consisting of the standard syntax for singular and plural logic and a

[41]With the exception of mathematics using large cardinal axioms; but see fns. 37 and 50.

[42]In fact, I strongly suspect that natural deduction for plural logic (an extension of natural deduction for first-order logic with additional introduction and elimination rules for plural quantifiers that mirror the ones for singular quantifiers, plural identity rules, and plural comprehension) supplemented with axioms for classical mereology, Global Choice, and the plural-mereological analogue of cardinal arithmetic meets the conditions described in LEMMA (see fn. 43 on adding axioms). However, nothing in the proof hangs on this point: as long as there is some proof system meeting those conditions, that is enough to prove preservation.

counterfactual operator. (Although I have not explicitly stated it, it is implied by my remarks that $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$ is a set of sentences of such a language.) For its fragment without the counterfactual operator, it has the standard natural deduction rules for first-order logic, the axioms of classical mereology, and whatever remaining rules are present in an arbitrary proof system satisfying (a) and (b) of LEMMA.[43] It also has the standard propositional natural deduction rules for the language as a whole, reflexivity, and deduction within counterfactual conditionals. The motivation for restricting the application of certain rules to the fragment without the counterfactual operator is to avoid any controversy that might arise from adding, e.g., quantifier rules with counterfactuals. Since the proof of preservation does not need such rules to apply without restriction, I do not include them without restriction in the logic I will use to prove it.

Having identified modal-structuralist counterparts and introduced the logic, we can now prove preservation. Suppose Γ_A proves Φ_A. Γ_A is the union of two sets: B_A and C_A, the set of theorems of ZFCU^+ and mixed sentences in Γ_A. Consider B_A. Every base sentence of B_A is in $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$. Moreover, every base sentence ψ of B_A is such that $\ulcorner \psi \leftrightarrow ([\mathcal{A} \wedge \mathcal{E}] \mathrel{\Box\!\!\rightarrow} \psi) \urcorner$ is in $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$. This means that by $\leftrightarrow$ elimination and modus ponens, for any base sentence ψ of B_A, $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$ proves $\ulcorner(\mathcal{A} \wedge \mathcal{E}) \mathrel{\Box\!\!\rightarrow} \psi\urcorner$. Now consider C_A. Conditional proof, $\wedge$ elimination, and deduction within counterfactual conditionals combined entail CLOSURE: for all $\chi_1, ...\chi_n, \psi, \phi$, if $\chi_1, ..., \chi_n \vdash \psi$, then $\vdash ((\phi \mathrel{\Box\!\!\rightarrow} \chi_1) \wedge ... \wedge (\phi \mathrel{\Box\!\!\rightarrow} \chi_n)) \rightarrow (\phi \mathrel{\Box\!\!\rightarrow} \psi)$.[44] By the fact that every theorem of ZFCU^+ is provable in ZFCU^+, the fact that the nominalizations$_M$ of the axioms of ZFCU^+ are provable from $\mathcal{E}$, LEMMA, CLOSURE, reflexivity, $\wedge$ elimination, repetition, and modus ponens: every φ that is a theorem of ZFCU^+ is such that $\ulcorner(\mathcal{A} \wedge \mathcal{E}) \mathrel{\Box\!\!\rightarrow} \psi\urcorner$ is a theorem, where ψ is the nominalization$_M$ of φ.[45] It trivially follows that any ϕ in C_A

[43] There is, of course, no problem with adding axioms to natural deduction systems if the arbitrary proof system happens to be axiomatic or with restricting certain inference rules of a logic to a fragment of the larger language to which the logic applies.

[44] *Proof.* If $\chi_1, ..., \chi_n \vdash \psi$, then $\chi_1 \wedge ... \wedge \chi_n \vdash \psi$: the earlier proof can be repeated, but additional lines can be added at the beginning using $\wedge$ elimination to arrive at $\chi_1, ..., \chi_n$ from $\chi_1 \wedge ... \wedge \chi_n$. With conditional proof, this means that $\vdash \chi_1 \wedge ... \wedge \chi_n \rightarrow \psi$. And with deduction within counterfactual conditionals, this entails $\vdash ((\phi \mathrel{\Box\!\!\rightarrow} \chi_1) \wedge ... \wedge (\phi \mathrel{\Box\!\!\rightarrow} \chi_n)) \rightarrow (\phi \mathrel{\Box\!\!\rightarrow} \psi)$.

[45] *Proof.* By the first fact and LEMMA, the nominalization$_M$ of any theorem of ZFCU^+ is provable from the nominalizations$_M$ of the axioms of ZFCU^+ and $\mathcal{E}$. By repetition, $\mathcal{E}$ proves itself. By this last observation

that is a theorem of ZFCU^+ is such that $\ulcorner(\mathcal{A} \wedge \mathcal{E}) \boxright \chi\urcorner$ is proven by $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$, where χ is ϕ's nominalization$_M$: a logical theorem is proven by any set of sentences. Additionally, any mixed sentence φ in C_A is such that $\ulcorner(\mathcal{A} \wedge \mathcal{E}) \boxright \psi\urcorner$ is a member of $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$, where ψ is φ's nominalization$_M$. All of this means, by the fact that the nominalization$_M$ of a base sentence is itself, that $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$ has the following feature: for any sentence φ in Γ_A, $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$ proves $\ulcorner(\mathcal{A} \wedge \mathcal{E}) \boxright \psi\urcorner$, where ψ is φ's nominalization$_M$. By $\wedge$ elimination, reflexivity, CLOSURE and modus ponens, $\ulcorner(\mathcal{A} \wedge \mathcal{E}) \boxright \mathcal{E}\urcorner$ is a theorem.[46] By the previous two observations, the facts that Γ_A proves Φ_A and Φ_A is its own nominalization$_M$ (since Φ_A is a base sentence), LEMMA, CLOSURE, $\wedge$ introduction and modus ponens, $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$ proves $\ulcorner(\mathcal{A} \wedge \mathcal{E}) \boxright \Phi_A\urcorner$.[47] Since $\ulcorner\Phi_A \leftrightarrow ([\mathcal{A} \wedge \mathcal{E}] \boxright \Phi_A)\urcorner \in MM_{\mathcal{L}_A}$, it is a member of $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$. Thus, by $\leftrightarrow$ elimination and modus ponens, $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$ proves Φ_A.

7. Proof of Nominalistic Safety

With preservation proved, all that remains to show nominalistic safety is to show (I) and (II). We can do so using the arbitrary objects introduced when describing nominalistic safety. To show (I), recall that $\mathcal{MS}(\Gamma_F, \mathcal{L}_F) = B_F \cup MM_{\mathcal{L}_F} \cup R_{\Gamma_F}$.

First, consider $MM_{\mathcal{L}_F}$. Since $\mathcal{L}_F$ is focused, by definition, all the members of $MM_{\mathcal{L}_F}$ are true. The correctness of the members of Γ_F is irrelevant.

---

and the second fact, $\mathcal{E}$ proves the nominalizations$_M$ of the axioms of ZFCU^+ and $\mathcal{E}$. Therefore, by the first conclusion, immediately preceding conclusion, and the transitivity of proof, $\mathcal{E}$ proves the nominalization$_M$ of any theorem of ZFCU^+. By $\wedge$ elimination, $(\mathcal{A} \wedge \mathcal{E})$ proves $\mathcal{E}$. So with the last two conclusions and transitivity of proof again, $(\mathcal{A} \wedge \mathcal{E})$ proves the nominalization$_M$ of any theorem of ZFCU^+. This fact and CLOSURE entail $\vdash ([\mathcal{A} \wedge \mathcal{E}] \boxright [\mathcal{A} \wedge \mathcal{E}]) \rightarrow ([\mathcal{A} \wedge \mathcal{E}] \boxright \psi)$. Reflexivity entails $\vdash (\mathcal{A} \wedge \mathcal{E}) \boxright (\mathcal{A} \wedge \mathcal{E})$. So, by modus ponens, $\vdash (\mathcal{A} \wedge \mathcal{E}) \boxright \psi$.

[46] *Proof.* By $\wedge$ elimination, $(\mathcal{A} \wedge \mathcal{E}) \vdash \mathcal{E}$. Then by CLOSURE, $\vdash ([\mathcal{A} \wedge \mathcal{E}] \boxright [\mathcal{A} \wedge \mathcal{E}]) \rightarrow ([\mathcal{A} \wedge \mathcal{E}] \boxright \mathcal{E})$. By reflexivity, $\vdash (\mathcal{A} \wedge \mathcal{E}) \boxright (\mathcal{A} \wedge \mathcal{E})$. So by the last two conclusions and modus ponens, $\vdash (\mathcal{A} \wedge \mathcal{E}) \boxright \mathcal{E}$.

[47] *Proof.* By LEMMA and the facts that $\Gamma_A \vdash \Phi_A$ and Φ_A is its own nominalization$_M$, $\chi_1, ..., \chi_n, \mathcal{E} \vdash \Phi_A$, where $\chi_1, ..., \chi_n$ are the nominalizations$_M$ of the members of Γ_A. The previous conclusion and CLOSURE entail $\vdash (([\mathcal{A} \wedge \mathcal{E}] \boxright \chi_1) \wedge ... \wedge ([\mathcal{A} \wedge \mathcal{E}] \boxright \chi_n) \wedge ([\mathcal{A} \wedge \mathcal{E}] \boxright \mathcal{E})) \rightarrow ([\mathcal{A} \wedge \mathcal{E}] \boxright \Phi_A)$. By the fact that for any sentence φ in Γ_A, $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$ proves $\ulcorner(\mathcal{A} \wedge \mathcal{E}) \boxright \psi\urcorner$, where ψ is φ's nominalization$_M$, and $\wedge$ introduction: $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$ proves $\ulcorner([\mathcal{A} \wedge \mathcal{E}] \boxright \chi_1) \wedge ... \wedge ([\mathcal{A} \wedge \mathcal{E}] \boxright \chi_n)\urcorner$. By the fact that $\vdash (\mathcal{A} \wedge \mathcal{E}) \boxright \mathcal{E}$, trivially, $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$ proves $\ulcorner(\mathcal{A} \wedge \mathcal{E}) \boxright \mathcal{E}\urcorner$. By the last two conclusions and $\wedge$ introduction, $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$ proves $\ulcorner([\mathcal{A} \wedge \mathcal{E}] \boxright \chi_1) \wedge ... \wedge ([\mathcal{A} \wedge \mathcal{E}] \boxright \chi_n) \wedge ([\mathcal{A} \wedge \mathcal{E}] \boxright \mathcal{E})\urcorner$. Then by the last conclusion, the conclusion reached with CLOSURE, and modus ponens, $\mathcal{MS}(\Gamma_A, \mathcal{L}_A)$ proves $\ulcorner(\mathcal{A} \wedge \mathcal{E}) \boxright \Phi_A\urcorner$.

Now consider R_{Γ_F}. Let φ be an arbitrary mixed sentence in Γ_F and let φ_R be its modal-structuralist replacement. It is *a priori* that if platonism$_M$ is true, then $\ulcorner \varphi \leftrightarrow \varphi_R \urcorner$ is true. To see this, let us assume platonism$_M$ is true for conditional proof and establish the embedded biconditional on that assumption. φ_R effectively states that if $\mathcal{A} \wedge \mathcal{E}$ were true, then some X would stand in the same structural relations to each other and to the parts of α that mathematical objects stand in to each other and the parts of α according to φ: some f is playing the role of $\in$, and X the role of sets. (Effectively states because $\mathcal{E}$ guarantees that the consequents wouldn't be vacuous.) Given that α would be just how it actually is were $\mathcal{A} \wedge \mathcal{E}$ to obtain, it is obvious that φ iff φ_R on our assumption. It is the structures that mathematical objects reside in that explain the connections between them and concreta. φ and the consequent of φ_R describe the exact same structures and describe them as being occupied in exactly the same positions, and the antecedent of φ_R ensures that in the relevant worlds some objects will generate the needed structure and occupants. To be sure, φ and φ_R characterize the occupants of the structures very differently: according to φ, they are abstract, which φ_R does not require. But this detail is irrelevant. What matters is that various positions in the structure are occupied, not what their occupants are like. So on the assumption of platonism$_M$, we can see that the biconditional is true. Discharging our assumption, we arrive at the original conditional. And since this rationale is obviously *a priori*—one need not know anything particular about α to see that the conditional is true—the conditional itself is *a priori*. Thus if φ_R is true, φ is correct.

What we need, however, is the converse: if φ is correct, then φ_R is true. To show that, we need to show that any nominalistic$_M$ sentence ψ must have the same truth-value as φ_R if it is *a priori* that if platonism$_M$ is true, then $\ulcorner \varphi \leftrightarrow \psi \urcorner$ is true. This falls out of platonism$_M$ being consistent with the nominalistic$_M$ facts. Suppose that it is *a priori* that if platonism$_M$ is true, then $\ulcorner \varphi \leftrightarrow \psi \urcorner$ is true. If we assume for conditional proof that platonism$_M$ is true, we can conclude from the *a priori* conditionals pertaining to ψ and φ_R that $\ulcorner \varphi \leftrightarrow \psi \urcorner$ is true and $\ulcorner \varphi \leftrightarrow \varphi_R \urcorner$ is true. The truth of these two sentences entails that $\ulcorner \psi \leftrightarrow \varphi_R \urcorner$ is true. Discharging our assumption, we conclude that if platonism$_M$ is true, then $\ulcorner \psi \leftrightarrow \varphi_R \urcorner$ is true.

Since all the reasoning involved was *a priori* and we began with two *a priori* conditionals, the derived conditional is *a priori* as well. Thus if ψ and φ_R have different truth-values, there is a nominalistic$_M$ fact with which platonism$_M$ is *a priori* incompatible: the latter's falsehood follows from the former, modus tollens, and an *a priori* truth. The only way a nominalistic$_M$ fact could be *a priori* incompatible with platonism$_M$ is if that fact and platonism$_M$ are inconsistent. While many complaints can be made against platonism$_M$, being inconsistent with the nominalistic$_M$ facts is surely not one of them. Thus we can conclude that if φ is correct, then φ_R is true. And what goes for φ and φ_R goes for all the mixed sentences of Γ_F and their modal-structuralist replacements.

Lastly, for the members of B_F, correctness is truth. So (I) is true. (II) is true because the only members of $\mathcal{MS}(\Gamma_F, \mathcal{L}_F)$ are the members of B_F, which are base sentences, and some nominalistic$_M$ counterfactuals. And with (I) and (II) established, nominalistic safety is demonstrated.

8. More Mathematics: Pure and Possible Applied

So goes the safety result. But what of pure mathematics? The modal-structuralist framework can be adapted provide an account of it as well. For any sentence φ of ZFC, the modal-structuralist substitute for φ is $\Box(\mathcal{E} \mathrel{\Box\!\!\rightarrow} \psi)$, where ψ is φ's nominalization$_M$. In the context of pure mathematics, the nominalization$_M$ of sentence mentions ZFC rather than ZFCU^+ because urelements are irrelevant. Modal-structuralist substitutes can given for sentences of $\mathsf{ZF}^2\mathsf{C}$ in the same manner: as noted in §3, the modal-structuralist framework can express second-order set theory. A platonistic interpretation of the claims of classical mathematics might be false, but their counterfactual, plural-mereological substitutes have the right features to do justice to them. First, they have the right truth-conditions. By the reasoning given in the proof of preservation, the nominalization$_M$ of any mathematical theorem is provable from $\mathcal{E}$.[48] If $\mathcal{E}$ proves ψ, it is of course true that $\Box(\mathcal{E} \mathrel{\Box\!\!\rightarrow} \psi)$. Likewise, if a mathematical claim is not a theorem, its nominalization$_M$ is not provable from $\mathcal{E}$, and we therefore have

[48]From here until the second point, I am speaking of first-order mathematics, setting aside what is beyond the reach of first-order ZFC.

no reason to affirm the corresponding modal-structuralist substitute. Thus, we can only be confident that the modal-structuralist substitute for a mathematical statement is true if that statement is a mathematical theorem. Second, the substitutes capture the thought that mathematical truths are not contingent. They are, after all, assertions of the necessity of certain facts: it is not an accidental fact that if $\mathcal{E}$ were to obtain, such and such would be true, but part of the essential fabric of reality. Third, because the modal-structuralist framework can express $\mathsf{ZF^2C}$, it can express set theory in a powerful enough way to exclude, e.g., countable models.[49] This matters because one might wonder whether a set theory that has models that are too weak to reflect the intuitive picture of the set hierarchy can really do justice to mathematics. Thankfully, even if the answer is no, this theory is unaffected.[50]

A further question arises as to how the sentences of pure mathematics should be seen as being related to their modal-structuralist substitutes. One option is that the substitutes express the claims of pure mathematics in a logically perspicuous way. Lewis (1993) thought that mathematics might have been structuralist all along (p. 17), so that his megethology does not revise mathematics so much as lay bare what its real commitments are. If Lewis' claim is plausible, it is not less plausible that mathematics might be the study not of structures, but of possible structures—an insight also developed by Hellman (1989), pp. 6-8. On the other hand, one might hold that the claims of pure mathematics are false and think of the substitutes as alternatives. If so, however, the claims of pure mathematics at least reliably correlate with certain necessary structural truths. The conservative impulse is still to some degree satisfied. The same question can be asked of applied mathematical sentences and their modal-structuralist replacements, and parallel answers can be given to it.

[49] As is well known, the Löwenheim–Skolem theorem entails first-order ZFC has countable models. For details on models of $\mathsf{ZF^2}$, see Gabriel Uzquiano (1999).

[50] The modal-structuralist framework can even be modified to include large cardinal axioms, so that even the study of very large structures can be done justice with small changes. Burgess (2015) points out that a plural-mereological version of Paul Bernay's reflection principle guarantees the existence of large cardinals and most of the axioms of set theory on its own (pp. 467-469). The idea here is to modify $\mathcal{E}$ to incorporate this nominalistic reflection principle in modal-structuralist replacements / substitutes and counterparts, then derive $\mathsf{ZF^2CU^+}$ and a large cardinal axiom. Working out the details is left as a task for another time. (In connection with Lemma, note that Burgess' proposal was intended to strengthen Lewis' megethology.)

As for the contingency of applied mathematics, no matter how the concrete world had turned out, the residents of a given spacetime would be able to do applied mathematics using the framework I've developed. Suppose some reasoners live in a relationalist spacetime with a finite number of occupants, or suppose they live in a spacetime with k occupants for some transfinite cardinal $k > \beth_1$. They could repeat the proof of safety for themselves and avail themselves of as rich a mathematics as they might need to describe their world while knowing that their theories will have sound modal-structuralist counterparts. This is because in their language, '@' has a different interpretation, serving to pick out the reality that they inhabit. Despite the variation in meaning, their proof will look exactly the same.

9. Conclusion

I have argued that there is a nominalistic account of the success of applied mathematics in natural science that does not face the limitations of extant accounts. Moreover, it also yields a nominalistic account of set theory. This goes a long way toward defending the intellectual satisfactoriness of nominalism about mathematics. Whatever the merits of mathematical platonism might be, mathematical objects are not needed to account for the success of applied mathematics in science or to interpret pure mathematics as being something more significant than a mere formal game.

10. Acknowledgments

I especially thank Geoffrey Hellman for his generous feedback on the manuscript during the review process. I also thank my anonymous reviewers, Geoffrey Hall, and Benjamin Middleton for their helpful comments on the manuscript. Finally, I am grateful to David Mark Kovacs and the Israel Science Foundation for their support through a postdoctoral fellowship, during which I completed most of this work.

References

Berry, S. (2022). *A Logical Foundation for Potentialist Set Theory*. Cambridge, UK: Cambridge University Press.

Burgess, J. P., Hazen, A. P., and Lewis, D. (1991). Appendix to *Parts of Classes*, pp. 121-149. Cambridge, MA: Basil Blackwell Ltd.

Burgess, J. P. (2015). Lewis on Mereology and Set Theory. In B. Loewer and J. Schaffer (Eds.), A Companion to David Lewis (pp. 459-469). West Sussex, UK: Wiley-Blackwell.

Chihara, C. S. (1990). *Constructibility and Mathematical Existence*. Oxford, UK: Clarendon Press.

Colyvan, M. (2010). There Is No Easy Road to Nominalism. *Mind, 119(474)*: 285-306.

Craig, W. (1956). Replacement of Auxiliary Expressions. *Philosophical Review, 65*: 38-55.

Dorr, C. (2008). There Are No Abstract Objects. In J. Hawthorne, T. Sider, and D. W. Zimmerman (Eds.), *Contemporary Debates in Metaphysics* (pp. 32-63). Malden, MA: Blackwell Publishing.

Feferman, S. (1998). *In the Light of Logic*. Oxford University Press: New York, NY.

Field, H. (1989). *Realism, Mathematics, and Modality*. Oxford, UK: Blackwell.

Field, H. (1992). A Nominalistic Proof of the Conservativeness of Set Theory. *Journal of Philosophical Logic, 21(2)*: 111-123.

Field, H. (2016). *Science Without Numbers: A Defense of Nominalism* (2nd Ed.). Oxford, UK: Oxford University Press. (Original work published 1980.)

Hellman, G. (1989). *Mathematics Without Numbers*. New York, NY: Oxford University Press.

Hellman, G. (1996). Structuralism Without Structures. *Philosophia Mathematica, 4(3)*: 100-123.

Hellman, G. (2003). Does Category Theory Provide a Framework for Mathematical Structuralism? *Philosophia Mathematica, 11(3)*: 129-157.

Leng, M. (2017). Reasoning Under a Presupposition and the Export Problem: The Case of Applied Mathematics. *Australasian Philosophical Review, 1(2)*: 133-142.

Lewis, D. (1991). *Parts of Classes*. Cambridge, MA: Basil Blackwell Ltd.

Lewis, D. (1993). Mathematics is Megethology. *Philosophia Mathematica, 1(3)*: 3-23.

Malament, D. (1982). Review of *Science without Numbers: A Defence of Nominalism*. *The Journal of Philosophy, 79(9)*: 523-534.

Nolan, D. (1996). Recombination Unbound. *Philosophical Studies, 84*: 239-262.

Putnam, H. (1969). On Properties. In Rescher Nicholas (Ed.), *Essays in Honor of Carl G. Hempel: A Tribute on the Occasion of his Sixty-Fifth Birthday* (pp. 235-254). Dordrecht, NL: Springer Science.

Simpson, S.G. (1999). *Subsystems of Second Order Arithmetic*. Cambridge University Press: New York, NY.

Skiba, L. (2019). Fictionalism, the Safety Result and Counterpossibles. *Analysis, 79(4)*: 647-658.

Uzquiano, G. (1999). Models of Second-Order Zermelo Set Theory. *Bulletin of Symbolic Logic, 5(3)*: 289-302.

Woodward, R. (2010). Fictionalism and Inferential Safety. *Analysis, 70(3):* 409-417.