Logical Dependence of Physical Determinism on Set-theoretic Metatheory

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0. Introduction: Are Set-Theoretic Foundations Irrelevant to Physics?

Baroque questions of set-theoretic foundations are widely assumed to be irrelevant to physics. In this article, we challenge this assumption. We show that even such fundamental questions as whether a theory is **deterministic** — whether it fixes a unique future given the present — depend on set-theoretic axiom candidates over which there is philosophical disagreement.

Suppose, as is customary (Earman 1986), that a deterministic theory is one whose mathematical formulation yields a unique solution to its governing equations. Then the question of whether a physical theory is deterministic becomes the question of whether there exists a unique solution to its mathematical model — typically a system of differential equations.

In this article, we show, first, that for some mathematically definable physical systems, reasonable set-theoretic assumptions (e.g. ZFC + large cardinals vs. ZFC + V = L) disagree about whether a solution with a particular property exists — such as a solution that's projectively definable. Second, and more dramatically, we show that they disagree about whether any solution to a PDE exists at all. Therefore, contrary to the methodology in physics and its philosophy, determinism is entangled with foundational disputes in set theory.

1. Theoretical Background

Let ZFC denote Zermelo–Fraenkel set theory with the Axiom of Choice. It provides the standard foundation for most mathematics.

Let LC denote a large cardinal axiom strong enough to imply projective determinacy (PD) — for instance, the existence of infinitely many Woodin cardinals (cf. Kanamori 2009).

Let V = L denote Gödel's axiom of constructibility, which postulates that every set is constructible in a certain precise sense. It leads to a minimalistic universe of sets and contradicts strong large cardinal axioms.

Key consequences:

- ZFC + LC proves certain regularity properties of definable sets of reals, including uniformization theorems — such as the Σ^1_3 uniformization results under PD (Moschovakis 1980, 6C).
- ZFC + V = L fails to prove those same properties, and often proves their negation (Jech 2003, Ch. 25; Koellner 2014).

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2. Definitions of Key Concepts

The **projective hierarchy** classifies definable subsets of $\mathbb{R}$ (or $\mathbb{R}^2$, etc.) according to how they can be described using quantifiers over reals and integers. A set $X \subseteq \mathbb{R}^2$ is $\mathbf{\Sigma^1_3}$ if there exists a formula of the form:

$\exists f: \mathbb{N} \to \mathbb{N}\ \forall n\ \exists m\ R(f, x, y, n, m)$

where R is computable (arithmetical). These sets involve quantifiers over functions and integers, with alternating complexity levels.

A set $X \subseteq \mathbb{R}^2$ has a **uniformizing function** if there exists a function $f: \mathbb{R} \to \mathbb{R}$ such that for every x such that there exists y with $(x, y) \in X$, we have $(x, f(x)) \in X$. That is, f picks out exactly one such y for each such x. This does not assume uniqueness of y in X — only the existence of one.

Global solutions and coding: For our purposes, we use standard techniques from recursion theory to encode mathematical objects as real numbers:

- **PDE encoding**: Each real number x can encode a partial differential equation via a standard recursive enumeration of syntactic expressions
- **Solution encoding**: Each real number y can encode a solution function via standard representations (e.g., coefficients of power series, or values on a dense rational subset) (Simpson 2009, §§II.6–7)
- **Global solution function**: A function $u: \mathbb{R} \to \mathbb{R}$ such that for each x encoding a PDE, u(x) encodes a solution to that PDE

3. Determinism and Solution Uniqueness

Determinism is standardly understood as the claim that the laws of nature and the initial conditions jointly determine a unique future evolution (Earman 1986). In the mathematical setting, this translates to the uniqueness of solutions to the differential equations describing a system.

We define determinism for a physical theory via the classical notion of well-posedness, originally formulated by Hadamard (1923):

- **Existence**: At least one solution exists for each initial condition.
- **Uniqueness**: At most one solution corresponds to each initial condition.
- **Well-posedness**: Small changes in initial conditions yield small changes in the solution.

A theory is deterministic if for any admissible initial state, the corresponding equations have a unique solution. If this claim is sensitive to metatheoretic background (such as ZFC, ZF + V=L, etc.), then determinism is too.

4. Logical Encodings of Physical Systems

We now provide an explicit construction showing how physical systems can be encoded in logical language suitable for foundational analysis.

4.1 Encoding Scheme for PDEs

Step 1: PDE Syntax Encoding Using standard recursive techniques, we encode PDE systems as real numbers:

Each real number x encodes a complete PDE system by storing different pieces of information in different parts of its decimal expansion. Think of x as a "filing system" where each component of the PDE is stored in a designated location:

- **Spatial domain**: The interval [0, n] where n is the integer part of x
- **Boundary conditions**: Information extracted from the first few decimal places
- **Coefficient functions**: Rational approximations stored in subsequent decimal places
- **Initial conditions**: Initial data encoded in the remaining digits

Concrete example: Consider x = 3.14159265...

- **Spatial domain**: [0, 3] (from the integer part 3)
- **Boundary conditions**: u(t,0) = u(t,3) = 0 (encoded in digits 1,4)
- **Coefficient functions**: Heat equation $\partial u/\partial t = \nabla^2 u + V(z)u$ where the potential V(z) is determined by the pattern in digits 1,5,9,2,6,5...
- **Initial conditions**: $u(0,z) = \sin(\pi z/3)$ (encoded in the remaining decimal structure)

Step 2: Solution Encoding Just as we encode PDEs as real numbers, we also encode their solutions. For any real number y, we can extract a complete solution function u(t,z) by treating y as a storage device for the solution's essential information:

- **Fourier coefficients**: The decimal expansion stores the coefficients needed to reconstruct the solution as a sum of sine and cosine waves
- **Sample values**: Key values of the solution function at a carefully chosen grid of points
- **Function reconstruction**: Standard mathematical techniques allow us to rebuild the complete solution from this stored data

(Think of y = 2.71828... as encoding a solution where the digits 2,7,1,8,2,8... specify how much of each "wave component" (sine, cosine functions) to include in building up the complete solution function u(t,z).)

4.2 The Solution Relation

Define the binary relation $\mathscr{R}(x, y)$ that holds iff:

y codes a function u(t,z) that satisfies the PDE system encoded by x

This gives rise to the set:

$X = \{(x, y) \in \mathbb{R}^2 \mid \mathscr{R}(x, y)\}$

Relevant properties:

Now that we can encode both PDEs and their solutions as real numbers, we can ask three basic questions about our encoded system:

- **Existence**: Does every PDE (encoded as x) have at least one solution (encoded as some y)? In symbols: $\exists y\ \mathscr{R}(x, y)$
- **Uniqueness**: Does every PDE have exactly one solution? That is, no PDE has multiple different solutions. In symbols: $\exists! y\ \mathscr{R}(x, y)$
- **Uniformizability**: Is there a "master function" f that can systematically pick out a solution for every solvable PDE? This function would take any PDE code x as input and output a solution code f(x). In symbols: $\exists f\ \forall x\ (\exists y\ \mathscr{R}(x, y) \rightarrow \mathscr{R}(x, f(x)))$

Physical interpretation: Existence and uniqueness are the standard requirements for determinism. Uniformizability concerns whether systematic solution methods exist for entire families.

4.3 Detour: Logical Complexity

Before we can establish our main technical result, we need to understand how logicians classify the complexity of different types of statements. This classification system, called the **projective hierarchy**, works like a ladder of increasing complexity:

Basic level (Arithmetic): Simple statements about integers that can be checked by computation. Example: "n is prime."

First level (Π^1_1, Σ^1_1): Statements involving quantification over integers but about real numbers. Example: "x is rational" can be written as $\exists n,m\ (x = n/m)$.

Higher levels (Σ^1_3, etc.): More complex statements involving quantification over functions and sets. These have the form:

$\exists \varphi: \mathbb{N} \rightarrow \mathbb{N}\ \forall n\ \exists m$ [some basic condition involving φ, n, m]

The key insight is that **Σ^1_3** statements are complex enough to express sophisticated mathematical relationships, but simple enough that we have good tools for analyzing them. This is the "sweet spot" where foundational assumptions like V=L and large cardinals can disagree about whether certain objects exist.

4.4 Making Our PDE Relation Σ^1_3

Now we can state a crucial technical result:

Lemma 4.1: The solution relation $\mathscr{R}(x,y)$ can be made Σ^1_3-definable.

Proof idea: The statement "y codes a solution to the PDE encoded by x" has exactly the right logical structure. Here's why:

To verify that y codes a genuine solution, we need to check that it satisfies the differential equation. Since we can't verify this exactly (differential equations involve continuous functions), we instead verify it approximately:

$\exists\varphi: \mathbb{N} \to \mathbb{N}\ \forall n\ \exists m$ [approximation_condition(φ, x, y, n, m)]

Breaking this down:

- **$\exists\varphi$**: There exists an approximation scheme φ (this function tells us how to approximate the solution)
- **$\forall$n**: For every level of precision n we might want
- **$\exists$m**: There exists a computational bound m such that
- **approximation_condition**: The approximation φ makes y satisfy the PDE to within precision n using at most m computational steps

This pattern—quantifying over functions ($\exists\varphi$) followed by alternating quantifiers over integers ($\forall$n $\exists$m)—is exactly what makes a statement Σ^1_3.

Because our PDE relation has Σ^1_3 complexity, it falls into the class of mathematical statements where V=L and large cardinals can disagree. This is what allows us to construct our results.

5. Determinism, Quasi-Functionality, and Uniformization

Definition: A relation $\mathcal{R} \subseteq \mathbb{R}^2$ is **quasi-functional** if $\forall x\ \exists{\leq}1\ y\ \mathcal{R}(x, y)$ — that is, each x is related to at most one y.

Lemma 5.1: If we encode only deterministic PDE families, then $\mathcal{R}$ is quasi-functional by construction.

Proof: By definition of deterministic systems, each initial condition x can have at most one solution y. Therefore, our encoding naturally yields quasi-functional relations.

Key insight: For quasi-functional relations, uniformizability (as defined in Section 2) becomes equivalent to what working physicists tend to care about: having a systematic method to solve entire families of deterministic systems.

The connection works because the mathematical concepts align with physical intuitions in deterministic systems (quasi-functional relations):

- **Uniformization** provides a "master function" f that can systematically pick out a solution for any solvable system in our family
- **Determinism** guarantees that each individual system has at most one solution, so when f picks a solution, it's picking *the* unique solution

Having a uniformizing function for a deterministic family means we can solve the entire family systematically. This is exactly what we mean by the family being "globally solvable".

The upshot is that for deterministic systems, uniformization corresponds to global solvability. So, disagreements about uniformization translate into disagreements about whether physics admits systematic solution methods – and, hence, into disagreements about properties of those solutions.

6. Concrete Construction: Heat Equations with Projective Potentials

We now provide a concrete example showing how foundational disagreements about projective sets translate into disagreements about systematic solution methods for physical systems.

6.1 The Construction

Consider the family of heat equations:

$\partial u/\partial t = \nabla^2 u + V\_x(z)u$, $(t,z) \in [0,\infty) \times [0,1]$
$u(t,0) = u(t,1) = 0$ (boundary conditions)
$u(0,z) = \sin(\pi z)$ (initial condition)

where the potential V_x(z) is defined via a specific projective set construction.

Step 1: Let $A \subseteq \mathbb{R}^2$ be a Σ^1_3 set satisfying the independence conditions established by Steel and Woodin:

- ZFC + V = L ⊢ "A has no projective uniformizing function"
- ZFC + LC ⊢ "A has a projective uniformizing function" (Steel 1984; Woodin 2011)

Step 2: For each $x \in \mathbb{R}$, define:

$$V\_x(z) = \begin{cases} \delta(z - \text{decode\_point}(x)) & \text{if } \exists y\ (x,y) \in A \\ 0 & \text{if } \nexists y\ (x,y) \in A \end{cases}$$

where δ is the Dirac delta function and decode_point extracts a point in [0,1] from x.

Step 3: Define our solution relation:

$\mathscr{R}(x,y) \Longleftrightarrow$ y codes the unique solution to the heat equation with potential V_x

6.2 The Independence Result

Theorem 6.1: The family {Heat equation with V_x}_{x∈ℝ} exhibits foundational dependence as claimed above.

Proof sketch:

Step 1: By standard results in descriptive set theory (Steel 1984), there exists a Σ^1_3 set A such that:

- ZFC + V = L ⊢ "A has no projective uniformizing function"

- ZFC + LC ⊢ "A has a projective uniformizing function"

Step 2: Our construction in Section 6.1 defines V_x in terms of membership in A. The key observation is that determining the potential V_x requires deciding whether $\exists y\ (x,y) \in A$.

Step 3:

- **In ZFC + V = L**: Without a uniformizing function for A, there's no definable way to systematically determine which x satisfy $\exists y\ (x,y) \in A$, and, thus, no definable global method to construct the potentials V_x.
- **In ZFC + LC**: The uniformizing function f for A provides such a method: we can definably compute V_x by checking whether $(x, f(x)) \in A$.

Step 4: Each individual heat equation has a unique solution by standard PDE theory (assuming appropriate regularity conditions). The foundational dependence concerns global definable solvability, not individual solution existence. Both ZF + V = L and ZFC + LC agree that each individual PDE in the family has a unique solution when its coefficients are well-defined. They disagree about whether a global solution method can be definably constructed for the family.

See Appendix A.1 for the technical proof.

7. From Abstract Uniformization to Concrete PDEs

Let us take stock. The connection between abstract uniformization and concrete determinism works as follows:

Abstract: We have a Σ^1_3 set A whose uniformizability is independent of ZFC.

Concrete: We construct a PDE family whose "global solvability" depends on uniformizing A.

Bridge: The existence of definable global solution methods depends on the uniformization properties of definable sets. This is not yet to say that determinism per se depends on this. Let us turn to that now.

8. Generalizing: Disagreement About Solution Existence

It might be thought that global solution methods are of interest mainly to the practicing physicist whose stock and trade is the computation of solutions. The philosopher of physics, by contrast, is concerned with the nature of determinism itself—not how to solve equations, but what it means for a physical system to be deterministic in the first place. However, we now establish that even this more rudimentary question—what counts as a deterministic physical system—depends on contested foundational assumptions over which V=L and large cardinals disagree.

8.1 Construction via Coefficient Regularity

Consider the PDE system:

$\partial u/\partial t = a(x,t)\ \partial u/\partial x + b(x,t)\ u$

$u(0,x) = \varphi(x)$

This is a first-order linear PDE—a type commonly used in physics for transport equations, wave propagation, and fluid dynamics. The key point is that we can construct the coefficient functions a(x,t) and b(x,t) using projective sets in such a way that:

- **The coefficient functions are well-defined** in both foundational contexts
- **Their regularity properties differ dramatically** depending on whether we assume V = L or large cardinals
- **Standard existence theorems depend on these regularity properties**. So, different foundations yield different conclusions about whether solutions exist

We will build coefficients that "look smooth" under large cardinals but "look pathological" under V = L, causing the same PDE to be solvable in one foundational context but unsolvable in another.

8.2 Specific Construction

We now construct a PDE whose solvability depends on which foundational axioms we accept.

Step 1: Choose a foundationally sensitive property Many regularity properties of projective sets depend on foundational assumptions. For concreteness, we use measurability, but similar constructions work with continuity properties, boundedness, differentiability, etc.

We take a specific Σ^1_3 set C such that:

- **Under ZFC + V = L**: C is not Lebesgue measurable
- **Under ZFC + LC**: C is Lebesgue measurable (by projective determinacy) (Moschovakis 2009; Kanamori 2009)

Working with measurability is strategically convenient because non-measurable functions immediately break the standard frameworks (L^2, Sobolev spaces) that PDE theory relies on.

Step 2: Build coefficient functions that depend on this property

Consider the PDE system:

$\partial u/\partial t = \nabla^2 u + V(x,t)u$

$u(0,x) = \varphi(x)$

where the potential V(x,t) is defined as:

$$V(x,t) = \begin{cases} +\infty & \text{if } (x,t) \in C \\ 0 & \text{if } (x,t) \notin C \end{cases}$$

Step 3: Demonstrate foundational dependence

In the V = L universe:

- C is not measurable, so the characteristic function χ_C is not measurable
- Therefore V(x,t) is not a measurable function
- For any u in a standard function space (L^2, H^1, H^2, and other Sobolev spaces, etc.), the product V(x,t)u is not well-defined in measure theory
- Since the product V(x,t)u is undefined, the differential equation $\partial u/\partial t = \nabla^2 u + V(x,t)u$ becomes meaningless when u is substituted into it
- A function cannot satisfy an equation that becomes meaningless when that function is substituted into it
- Therefore, ZFC + V = L proves "no solutions exist in standard function spaces"

Note: Standard function spaces (L^2, Sobolev spaces H^k, etc.) are the mathematical frameworks that physicists routinely use for formulating and solving PDEs. These spaces require functions to be measurable and satisfy certain integrability conditions.

In the large cardinal universe:

- C is measurable, so χ_C is a measurable function
- Therefore V(x,t) is measurable (though unbounded on C)
- The PDE can be interpreted as having "infinite potential barriers" on the set C
- Such systems have unique solutions with u(x,t) = 0 whenever $(x,t) \in C$ (the infinite potential forces the solution to vanish)
- This gives a well-defined, unique solution

This construction works because we have chosen a regularity property (measurability) that is both foundationally sensitive and essential for PDE theory. Similar arguments work with other regularity properties. The important point is that projective sets can satisfy different regularity conditions depending on foundational assumptions.

The upshot is that **the same differential equation is mathematically meaningless in one 'reasonable' foundational context but has a unique solution in another**. So, if a deterministic theory is one whose mathematical formulation yields a unique solution to its governing equations, then **whether a theory is deterministic is relative to one's background metatheory.**

8.3 The Independence Result

Theorem 8.1: There exists a PDE system such that:

- ZFC + V = L $\vdash$ "The system has no solutions in standard function spaces"
- ZFC + LC $\vdash$ "The system has a unique solution in standard function spaces"

Proof: By the construction above, in V = L the potential V(x,t) is not measurable, which makes the PDE operator undefined on standard function spaces. Therefore V = L proves no solutions can exist in L^2, Sobolev spaces, etc. In contrast, under LC the potential is measurable and the system has a unique solution by standard PDE theory.

This is a considerably more dramatic consequence. It is not just that foundational disagreement translates into disagreement about properties of solutions or global solution methods. It translates into disagreement about whether individual PDE systems have solutions at all in the function spaces that physicists rely on.

9. Discrete Analogues: Foundational Sensitivity Without the Continuum

A natural objection to our results is that foundational sensitivity is an artifact of continuum mathematics. Discrete physical models such as cellular automata, finite difference equations, or lattice field theories might escape this foundational dependence. We now show this assumption is false.

9.1 Diophantine Equations and Discrete Systems

We exploit the connection between computation and Diophantine equations to construct discrete physical systems with foundational sensitivity.

By the MRDP theorem (Matiyasevich-Robinson-Davis-Putnam), every recursively enumerable set can be represented as the solution set of a Diophantine equation—a polynomial equation with integer coefficients where we seek integer solutions.

Consider a family of discrete dynamical systems parameterized by natural numbers:

For each $n \in \mathbb{N}$: Find integer solutions to $P_n(x_1, x_2, ..., x_k) = 0$

where P_n is a polynomial whose coefficients are determined by n.

We construct this family so that:

- **Each individual system** corresponds to solving a specific Diophantine equation
- **The existence of solutions** depends on membership in a Σ^1_3 set $X \subseteq \mathbb{N}^2$
- **Global solvability** depends on uniformizing X

9.2 Specific Construction

Step 1: Let $X \subseteq \mathbb{N}^2$ be a Σ^1_3 set such that uniformization of X is independent of ZFC (as established in our previous constructions).

Step 2: For each parameter $n \in \mathbb{N}$, define the Diophantine system:

$P_n(x_1, x_2, ..., x_k) = Q(x_1, x_2, ..., x_k) + R_n(x_1, x_2, ..., x_k)$

where:

- Q is a fixed polynomial with integer solutions
- R_n is constructed so that $P_n = 0$ has solutions iff $\exists m\ (n,m) \in X$

Step 3: The foundational dependence:

- **Individual solvability**: Each equation $P_n = 0$ either has integer solutions or doesn't
- **Global solvability**: A systematic method to solve the entire family exists iff X can be uniformized
- **Independence**: Since uniformization of X depends on V = L vs. LC, so does global solvability

9.3 Physical Interpretation

This construction applies to discrete physical models:

Lattice field theories: Discrete approximations to quantum field theories where field values are computed at lattice points according to discrete update rules.

Cellular automata: Systems where cell states evolve according to local rules, with global behavior emerging from local interactions.

Finite difference methods: Numerical approaches to solving PDEs by discretizing space and time.

In each case, we embed our Diophantine construction into the discrete update rules, making the global behavior of the discrete system depend on foundational assumptions about uniformization.

9.4 The Stronger Discrete Result: Basic Determinism

The constructions above show that global solution methods for discrete systems depend on foundational assumptions. But, as before, we can go further and show that even determinism —whether individual discrete systems have unique solutions at all—depends on contested axioms.

We construct discrete dynamical systems whose update rules depend on foundationally-sensitive regularity properties.

Consider the discrete dynamical system:

$x_{n+1} = F_A(x_n, n)$

$x_0 = \text{initial\_value}$

where F_A is defined using a foundationally-sensitive set $A \subseteq \mathbb{N}^2$:

$$F_A(x, n) = \begin{cases} g(x, n) & \text{if } (x, n) \in A \\ \text{undefined} & \text{if } (x, n) \notin A \end{cases}$$

Here $g(x, n)$ is a well-defined function, but the domain of F_A depends on the set A.

Step 1: Choose A to be a specific Σ^1_3 subset of $\mathbb{N}^2$ such that:

- **Under ZFC + V = L**: A is not computably enumerable in any uniform way

- **Under ZFC + LC**: A has sufficient regularity to be systematically decidable

Step 2: The discrete evolution depends on membership in A:

In ZFC + V = L:

- The set A lacks computational regularity
- For many values (x, n), we cannot effectively determine whether $(x, n) \in A$
- Therefore the update rule F_A(x, n) cannot be implemented as an algorithm
- **Result**: The discrete system is not computationally well-defined → No algorithmic solutions exist

In ZFC + LC:

- A has sufficient regularity for systematic membership testing
- The update rule F_A(x, n) can be implemented algorithmically for all inputs
- **Result**: The discrete system has a well-defined algorithmic evolution → Unique computable solution exists

Physical interpretation: This applies to:

- **Cellular automata** where update rules depend on foundationally-sensitive pattern recognition
- **Digital physics models** where computational processes depend on decidability properties
- **Algorithmic information theory** applications where compression depends on regularity assumptions

The key point is that the same discrete dynamical system cannot be implemented algorithmically in one foundational context but has a unique algorithmic implementation in another. Just as continuous PDEs can be foundationally sensitive in both global solvability and individual solution existence, discrete systems exhibit the same foundational dependence at both levels.

9.5 The Broader Point

Foundational sensitivity is not an artifact of continuum mathematics but reflects a deeper entanglement between physical concepts and mathematical foundations. Even when we retreat to the most basic discrete models—integer arithmetic, finite systems, computational processes—we cannot escape foundational dependence. The sensitivity appears whenever we have:

1. **Families of systems** (rather than isolated cases)
2. **Questions about systematic solvability** (rather than individual solutions)
3. **Definable mathematical structure** (rather than purely empirical data)

Moreover, we've now seen that foundational sensitivity affects **both levels** of physical determinism:

- **Global solution methods**: Whether systematic procedures exist to solve entire families

- **Individual solution existence**: Whether basic physical processes are even well-defined

This suggests that foundational relativity is a systematic feature of mathematical physics, not a curiosity that only appears in exotic continuous systems. The discrete analogues show that the phenomenon persists across the spectrum of physics, from continuous field theories to discrete computational models.

10. Philosophical Upshot

Philosophers and physicists often assume that foundational disagreements about logic or set theory — such as whether $V = L$, or whether large cardinals exist — have no bearing on physical inquiry. This paper shows otherwise. Our examples demonstrate that the properties of deterministic physical theories, and, more radically, the very fact that they are deterministic, can depend on one's background metatheory. Disagreements between ZFC + LC and ZFC + $V = L$ are not mathematical curiosities: they can be encoded into the structure of definable physical laws.

It might be objected that our constructions are too artificial to matter for real physics, or even its philosophy. But this objection misses the point. We are not claiming that working physicists should worry whether their equations "really" have solutions. We are showing that the **concept of determinism** is not metatheoretically stable. It depends on debates in the foundations of set theory. Whether foundational sensitivity appears "naturally" in physics is a separate question.

Quantum gravity theorists sometimes remark that we may need "new math" to formulate a final theory. We have shown that this "new math" may go deeper than anticipated — not just new tools within a familiar framework, but a new framework. Future progress in the foundations of physics may therefore depend on novel interaction physics and the foundations of mathematics.

Appendix A: Technical Proofs

A.1 Proof of Theorem 6.1 (Heat Equation Independence)

Theorem: The family {Heat equation with V_x}_{x∈ℝ} exhibits foundational dependence.

Proof:

Preliminaries: Let $A \subseteq \mathbb{R}^2$ be the specific Σ^1_3 set from Steel (1984) satisfying:

- $\mathrm{ZFC} + V = L \vdash \neg\exists f$ [f uniformizes A projectively]
- $\mathrm{ZFC} + \mathrm{LC} \vdash \exists f$ [f uniformizes A projectively]

Recall our construction: V_x(z) = δ(z - decode_point(x)) if $\exists y\ (x,y) \in A$, and V_x(z) = 0 otherwise.

Part I: Individual solution existence For any fixed $x \in \mathbb{R}$, the heat equation $\partial u/\partial t = \nabla^2 u + V_x(z)u$ with appropriate initial/boundary conditions has a unique solution by standard PDE theory. This holds in both ZFC + $V = L$ and ZFC + LC, since:

1. V_x is either zero or a delta function (both well-defined distributions)
2. Standard existence/uniqueness theorems apply
3. The question is purely about classical PDE theory, independent of set-theoretic assumptions

Part II: Global definable solvability

Case 1 (ZFC + V = L):

- By assumption, A has no projective uniformizing function
- To construct a global solution method, we would need to definably determine V_x for each x
- This requires deciding the predicate $P(x) := $ "$\exists y\ (x,y) \in A$"
- But deciding P(x) for all x would yield a uniformizing function for A (map $x \mapsto$ witness y if P(x) holds)
- Since no such uniformizing function exists, no definable global solution method exists

Case 2 (ZFC + LC):

- By assumption, A has a projective uniformizing function f
- Define the global solution method as follows:
  - For each x, compute f(x)
  - Check if $(x, f(x)) \in A$
  - If yes, set $V_x(z) = \delta(z - \text{decode\_point}(x))$; if no, set $V_x(z) = 0$
  - Solve the resulting heat equation using standard methods
- This provides a definable, systematic method for solving the entire family

Part III: The independence The same mathematical objects (the PDE family) satisfy different global properties in different foundational contexts:

- ZFC + V = L: No definable global solution method
- ZFC + LC: Definable global solution method exists

This completes the proof. □

A.2 Tree-Based Construction

Following Harrington's construction (Harrington 1978), let W be the set of Gödel numbers of well-founded recursive trees on ω. Define:

$R(x, y) \iff$ "x codes a well-founded recursive tree T_x, and

$y = \text{rank}(T_x)$ in the Kleene-Brouwer ordering"

Then define:

$\mathscr{R}(x, y) \iff R(x, y) \text{ and } \forall y' < y\ \neg R(x, y')$

This gives a quasi-functional Σ^1_3 relation $\mathscr{R} \subseteq \mathbb{R}^2$.

A.3 PDE Interpretation

Method 1: Direct Translation

Construction: For each real number x, we extract both tree data and PDE data using standard coding techniques:

- **Tree component**: decode_tree(x) extracts a recursive tree T_x from the first part of x's decimal expansion
- **PDE component**: decode_pde(x) extracts initial conditions and domain information from the remaining digits

Solution encoding: For each real number y, we encode both mathematical objects:

- **Tree rank**: If T_x is well-founded, rank(T_x) encodes the ordinal rank in the Kleene-Brouwer ordering
- **PDE solution**: u_x(t,z) encodes the solution function to the PDE specified by x

The connection:

- **Solution existence**: The PDE encoded by x has a solution iff the tree T_x is well-founded (only well-founded trees have ranks)
- **Global solvability**: A uniform method to solve all PDEs in the family exists iff there's a uniformizing function for the tree-rank relation $\mathscr{R}(x,y) \Longleftrightarrow$ "T_x is well-founded and y = rank(T_x)"

Independence result: Since tree-rank uniformization is independent of ZFC (provable under LC, refutable under V = L), so is global solvability of this PDE family.

Method 2: Coefficient Construction

Construction: Use tree ranks to build coefficient functions with foundationally-dependent regularity:

- For each x, let T_x = decode_tree(x) be the associated recursive tree
- Define the coefficient function: a_x(t,z) = g(rank(T_x), t, z) where g is a fixed smooth function
- If T_x is not well-founded, set a_x(t,z) = ∞ (pathological coefficient)

PDE family:

$\partial u/\partial t = a_x(t,z)\, \nabla^2 u + b(t,z)\, u$

$u(0,z) = \varphi(z)$

Regularity dependence:

- **Well-founded case**: If T_x is well-founded, then rank(T_x) exists and a_x(t,z) is well-defined and smooth

- **Ill-founded case**: If T_x is not well-founded, then a_x(t,z) = ∞, making the PDE unsolvable

Global regularity:

- **Under V = L**: Some trees appear well-founded locally but have no global rank assignment (no uniformizing function exists)
- **Under LC**: All well-founded trees have definable ranks (uniformizing function exists), ensuring systematic regularity

The independence: Whether the entire family admits systematic solution methods depends on whether tree ranks can be uniformly assigned, which is independent of ZFC.

Connecting to Main Results:

Both methods show how the abstract uniformization independence translates into concrete PDE independence:

- **Method 1** gives existence dependence: individual solution existence tied to tree well-foundedness
- **Method 2** gives regularity dependence: coefficient regularity tied to uniform rank assignment

In both cases, the same foundational assumptions that affect uniformization of abstract tree-rank relations also affect solvability of concrete PDE families. This demonstrates that the connection between abstract set theory and physical determinism is not accidental but systematic.

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