From Mollusk to Swarms of Observers:

An Observer-Based Operational Framework for General Relativity

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Abstract

We present a thought experiment extending Einstein's "reference mollusk" to explore operational realizations of coordinate systems in general relativity. Observers carrying four independently programmable clock-like devices assign numerical labels to events, constructing coordinate charts without the structural constraints of mollusk-type frames. This approach decouples coordinate assignment from both motion and geometry, enabling the representation of arbitrary charts.

The framework offers pedagogical clarity: coordinate freedom appears as the capacity to freely program clocks, subject only to the (pre-deployment) programming condition that their collective readings fulfill the minimum mathematical properties required of all charts (in particular smoothness). Following the logical structure of differential geometry, charts precede metric determination, which arises from comparing displayed values with measurements in local inertial frames.

We emphasize two key aspects. First, any smooth observer congruence suffices to construct every valid coordinate chart, generalizing Einstein's construction. Second, and more strongly, a single congruence suffices to simultaneously also represent *any* other chart: by corresponding programming of displays, observers following fixed timelike worldlines can display multiple charts (if more than four numbers are displayed), and emulate also any smooth coordinate transformation, including displaying charts with null coordinate directions in the spacetime in which the swarm is deployed. The timelike motion of the observers is only required to fulfill the task of covering a spacetime region, but does not restrict the freedom of programming.

Keywords: operational reference frames, programmable clocks, observer swarm, general covariance, diffeomorphism invariance, spacetime individuation, gauge symmetry, scalar field deparametrization

I. INTRODUCTION

The general theory of relativity permits arbitrary smooth coordinate systems, reflecting its diffeomorphism invariance and the absence of fixed background geometric structures. This mathematical freedom raises conceptual questions: which coordinate systems possess empirical significance, and how can their physical meaning be understood? The *hole argument*, originally formulated by Einstein in 1913, initially led him to reject general covariance due to concerns about underdetermination—the worry that the field equations might not uniquely specify the metric even given all physical boundary conditions. Only later did he recognize that diffeomorphically related models represent the same physical situation, thereby resolving this apparent failure of unique determination[1,2]. On this understanding, coordinate values lack intrinsic physical meaning; what matters empirically are coincidences—intersections of worldlines, measurement events, and localized interactions.

General covariance allows the use of any smooth coordinate chart, and practical modeling in relativity exploits this freedom. Yet the relationship between mathematical coordinate freedom and physical reference frames remains a source of conceptual tension. While physicists routinely employ coordinate systems for calculations and predictions, the operational meaning of such systems continues to generate philosophical discussion.

This paper presents a thought experiment that may help clarify these issues by exploring how arbitrary coordinate charts could, in principle, be operationally instantiated. Following Einstein's tradition of the "reference mollusk"[3]—itself an idealized construction meant to illustrate coordinate generality—we consider reference frames constructed from swarms of local observers, each equipped with four independently programmable clock-like devices. These devices assign numerical labels to locally encountered events, enabling the swarm to realize any smooth coordinate chart within its coverage region.[60]

This construction extends Einstein's mollusk metaphor while sharing its status as a conceptual tool rather than a practical proposal. Just as the mollusk clarified how non-rigid reference frames could be conceived without requiring actual mollusks, and just as dust-based scalar field models illuminate gauge-fixing without requiring actual dust distributions, our framework aims to clarify the operational meaning of coordinate freedom without proposing literal implementation.

The thought experiment offers several conceptual insights. It illustrates how coordinate charts could be understood as arising from programmable labeling conventions rather than geometric structures. It provides a concrete visualization of how general covariance might be interpreted operationally: coordinate freedom becomes the conceptual capacity to program clocks, while diffeomorphism invariance is mirrored by the ability to reprogram them equivalently. This perspective may offer pedagogical value in teaching general relativity and could inform discussions about the interpretation of gauge redundancy.

Additionally, since programmable clocks can, in principle, emulate arbitrary scalar field profiles, the framework suggests viewing scalar fields used in deparametrization schemes as pragmatic modeling tools rather than fundamental structures. This perspective complements existing approaches while highlighting their conventional aspects.

Like other thought experiments in the foundations of physics, this construction is not intended as a blueprint for actual implementation. Rather, it serves to clarify conceptual relationships and provide an intuitive framework for understanding the interplay between mathematical formalism and operational thinking in general relativity.

Outline of the paper

The paper is organized as follows. Section II reviews previous approaches to understanding reference frames in general relativity, including mollusk-like constructions, scalar-field-based methods, and observer-based frameworks, while noting connections to philosophical and pedagogical discussions. Section III briefly recalls operational reference frames in special relativity as context for the general framework. Section IV A revisits Einstein's reference mollusk, which motivates the thought experiment presented in Section IV B: a conceptual framework where observers carry four programmable clock-like devices to assign coordinate values along their worldlines. This construction explores how arbitrary coordinate charts might be operationally understood without requiring specific foliations or coordinate splittings. The framework is developed to include coordinate transformations and considerations for metric determination (Section IV C). Section IV D discusses conceptual issues including the meaning of programmability in deterministic theories. The appendices examine specific aspects of the construction. Appendix A analyzes mollusk-type reference frames and their limitations. Appendix B provides illustrative examples, showing how the proposed four programmable clocks per observer in an observer congruence can avoid such limitations and instantiate every smooth chart.

II. BACKGROUND AND PRIOR WORK

A. Mollusk-like reference frames

Einstein's mollusk provides a pedagogical metaphor for understanding deformable coordinate systems in curved spacetime. However, mollusk-adapted coordinates rely on a slicing into spatial hypersurfaces, with comoving observers carrying fixed spatial labels and a single arbitrarily advancing clock. Similar constructions have appeared throughout the literature, though often without explicit reference to Einstein's original conception.

Norton[2] surveys physical reference frames including a space-filling family of clocks that carry "three smoothly assigned indices (functioning as spatial coordinates), while requiring that the clocks tick smoothly—though not necessarily in proper time—and that time readings vary smoothly across neighboring clocks".

Rovelli[5] presents a similar model, introducing a "cloud" of particles labeled by a three-dimensional continuum index $\vec{y}$, with one clock per particle. As he notes, "Having matter elements distinguished by 'names' is, in a sense, the peculiar property of any reference system: think, for instance, of a rod and its ticks with numbers."

These constructions share a common feature: while each observer carries a time parameter, spatial coordinates remain tied to fixed labels that identify entire worldlines rather than individual spacetime points. The present work explores what happens when these fixed spatial labels are replaced by programmable values that can vary along each worldline. By equipping each observer with four programmable clock-like devices instead of one, we obtain a thought experiment that can, in principle, instantiate arbitrary coordinate values at each point along the worldlines. This appears to be a natural extension that has not been previously explored in the literature.

B. Fields defining reference frames

The use of four programmable devices per observer differs from approaches that derive coordinate systems from dynamical scalar fields. Understanding this distinction may help clarify both approaches.

Bergmann and Komar[7,8] introduced curvature-invariant scalars to define spacetime points intrinsically, with subsequent canonical formulations by others[9,10]. These constructions face technical challenges: ensuring that curvature-based scalars yield smooth, invertible coordinate charts requires addressing potential degeneracies and singularities.

Other approaches introduce dynamically independent scalar fields. Brown and Kuchař's dust models[14] employ one timelike and three spacelike scalar fields tied to pressureless dust, later extended to null dust[15]. Giesel and Thiemann[16] develop sophisticated deparametrization schemes using such models to construct gauge-invariant observables in canonical gravity. Tambornino[17] reviews these approaches comprehensively.

Additional examples include Klein-Gordon fields for reference frames[11], macroscopic scalar field coincidences[12], and explicitly operational frameworks[13].

A technical point worth noting: while scalar fields can define local charts when gradients are linearly independent, ensuring injectivity over finite regions requires additional conditions rarely verified in practice. The literature typically relies on local Jacobian conditions without addressing global invertibility, leaving the validity of scalar-field charts over extended domains as an open technical question.

The programmable clock approach sidesteps these issues by design. Since clock values are freely assignable rather than dynamically determined, questions of injectivity or critical points do not arise in the same way. Moreover, programmed values can emulate any scalar field behavior if desired, potentially suggesting simplified scalar field constructions tailored for specific technical purposes. This flexibility might be particularly relevant for deparametrization schemes in canonical quantum gravity, where scalar fields are often introduced primarily for their technical utility in addressing the problem of time.

C. Other Observer-Based Approaches

Several recent developments share an observer-centric perspective, each with distinct methodologies and goals.

Geometric Observer Space: Gielen and Wise[18] reformulate general relativity using the seven-dimensional observer space (unit future-directed timelike tangent bundle), equipped with Cartan geometry. While mathematically elegant, this approach operates at a different conceptual level than the present thought experiment, which focuses on how coordinate values might be operationally assigned by physical devices.

Observer-Based Decompositions: Bini and colleagues[19] develop $1+3$ and $1+1+2$ splittings of Einstein's equations relative to timelike congruences. These methods, while valuable for calculations, presuppose a given metric structure. The present framework instead explores how coordinate systems might be established prior to metric determination, following the standard logical sequence of differential geometry.

Relativistic Positioning Systems: The RPS program by Coll and collaborators[20] assigns coordinates via proper times broadcast by moving emitters, proceeding independently of background metrics. Rovelli[21] employs similar GPS-inspired constructions for defining relational observables in canonical gravity. Real GPS systems[22] demonstrate practical implementation, while recent work explores metric reconstruction from signal data[23]. These signal-based approaches offer material economy but are optimized for receivers actively determining their position. The observer-swarm thought experiment instead imagines dense local recording of arbitrary events, providing conceptual clarity at the cost of requiring extensive (idealized) deployment.

D. Causal Reconstruction and the Role of Charts

Spacetime geometry can, in principle, be reconstructed from causal structure without coordinate charts[24–26,28,29]. However, this chart-free reconstruction does not extend to matter fields, which in current theoretical frameworks require coordinate representations for their description. The programmable clock framework offers one perspective on this situation: it provides a thought experiment for how arbitrary charts might be operationally realized, potentially useful when considering both geometric and matter field descriptions.

E. Gauge-Invariance and Reference Frames

The hole argument and related discussions[2,31–34] emphasize that coordinate values lack intrinsic physical meaning unless grounded in operational procedures. The programmable clock framework offers a concrete thought experiment for understanding this point: coordinate freedom can be visualized as the freedom to program clocks, while gauge equivalence corresponds to different programming choices yielding the same physical content.

This perspective does not resolve foundational debates but may provide a useful mental model for understanding the operational meaning of diffeomorphism invariance.

F. Relation to Spacetime Discussions

While this framework touches on issues discussed in spacetime philosophy[28,32,35,36], it makes no claims about spacetime ontology. Instead, it offers a thought experiment for how coordinate systems might be operationally understood, remaining neutral on deeper metaphysical questions. The framework simply explores one way to think about the relationship between mathematical coordinate freedom and potential physical realization, without claiming this relationship has ontological significance.

G. Reference frames as pedagogical tools

From a pedagogical perspective, this thought experiment follows the operational tradition of Bridgman and Reichenbach[37,38], who emphasized that physical concepts gain clarity through consideration of measurement procedures. Standard textbooks on general relativity typically begin by noting that global inertial frames cannot be extended to curved spacetimes, motivating the introduction of curvilinear coordinates and pseudo-Riemannian geometry[39–41]. However, such treatments rarely explore how general coordinate systems might be operationally understood, leaving students to work with abstract formalism without a concrete picture connecting coordinates to potential measurements.

The textbook *Gravitation*[42] motivates coordinate systems through their capacity to label events in orderly fashion. While conceptually sound, this approach may not distinguish between mathematical labeling conventions and physical measurement procedures. The programmable clock framework offers one way to visualize this distinction: coordinate labels could be understood as arising from programmable device readings, while physical relations like proper time intervals and causal connections remain independently measurable.

Actual experimental reference systems involve technical complexities unsuitable for introductory exposition. The observer-swarm thought experiment provides a middle ground: it is simple enough to visualize, general enough to illustrate arbitrary coordinate systems, and transparent enough to clarify the conceptual relationship between observer readings and coordinate assignments.

III. SPECIAL RELATIVITY'S REFERENCE FRAMES

To prepare for general relativity, we begin by revisiting and slightly reframing the construction of physical reference frames in special relativity (SR). The foundational principle—shared by all relativistic theories—is that only local coincidences of events can be assigned physically unique meaning. This *locality principle* therefore also serves as the operational starting point for the formulation of reference frames.

Consequently, even in special relativity, an "observer" must not be understood as a single pointlike abstraction, but rather as a *swarm of local observers*, each equipped with measuring devices and capable of recording only those events that occur in their immediate vicinity. A spacetime diagram, then, represents an operational construct: the aggregate record of all localized measurements gathered by these assistant observers, who conceptually fill the spacetime region of interest. Each local device assigns coordinate values—its own clock reading and spatial position—to the events it registers. The diagram emerges only by conceptually collecting and assembling these local records into a coherent global account.

Mathematically, these coordinates are typically denoted by x^μ with $\mu = 0, 1, 2, 3$, so that a spacetime event is labeled by $x = (x^0, x^1, x^2, x^3)$. The temporal coordinate is given by $x^0 \equiv c\bar{t}$, where c is the speed of light and $\bar{t}$ is the local clock reading. The spatial coordinates are written as $(x^1, x^2, x^3) \equiv \bar{\mathbf{x}}$.

Some introductory texts visually depict such a reference frame as a regular three-dimensional lattice, composed of idealized measuring rods and regularly placed idealized and synchronized clocks[43,44]. These constructions are designed to illustrate the operational assumptions of inertial frames in SR—such as the Euclidean geometry of space, idealized rods and clocks, global clock synchronization, and the idealization of zero spacetime separation between the event and its recording observer.

An equivalent and more locality-focused depiction is shown in Fig. 1. Here, spacetime events are assigned coordinates by a distributed network of observers, each using their locally maintained synchronized clocks and pre-established positions relative to others. When needed, spatial alignment across the network can be re-calibrated by activating orthogonal laser pulses, which form a transient coordinate grid. This operational picture emphasizes that it is the observers—and not the coordinate mesh—that serve as the physical basis of the reference frame.

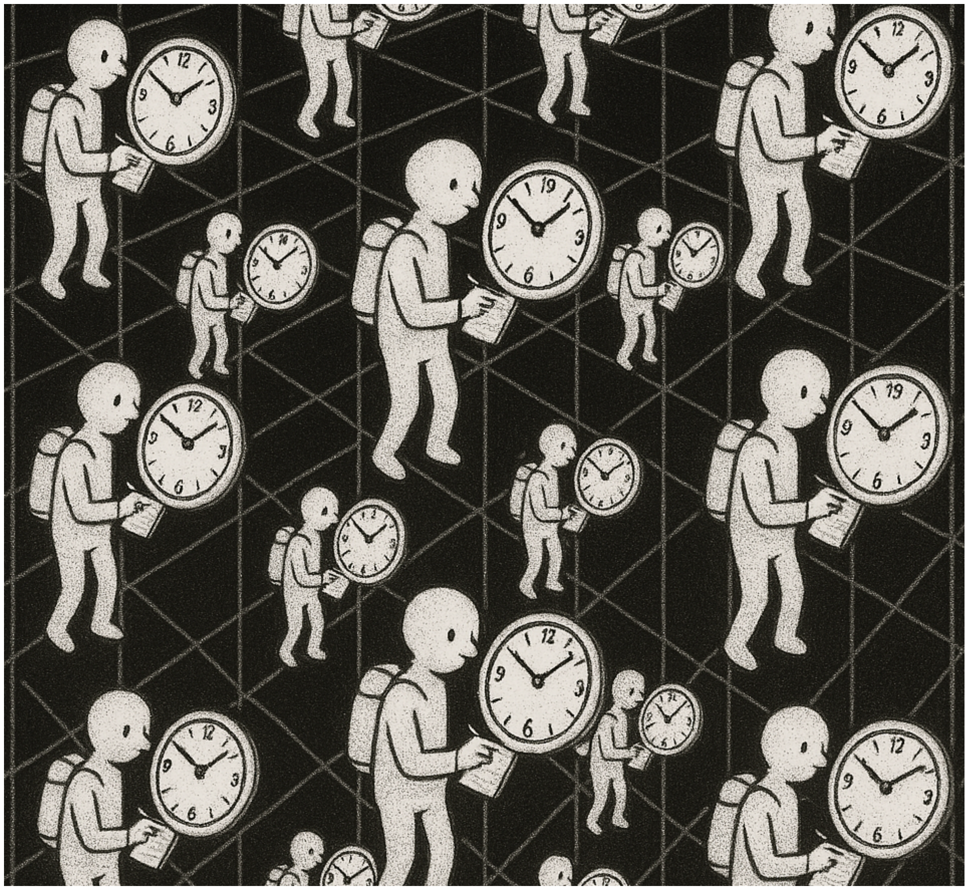

FIG. 1. A special relativistic reference frame depicted from an operational perspective.

Once established, a reference frame enables the measurement of spatial distances, time intervals, and kinematic quantities by combining purely local observations of events, even when those events are spatially and/or temporally separated. For instance, measuring the length of a small rod—whether it is at rest or moving uniformly relative to the frame—requires identifying two events that are simultaneous within the frame but occur at different locations: one recorded by the observer who happens to be positioned at one end of the rod at an agreed time, and the other by a second observer located at the opposite end at that same agreed time. Both observers rely on their synchronized clocks to ensure simultaneity within the frame. The length of the rod is then computed as the Euclidean distance between the spatial positions of the two observers at that moment.

The configuration of free-floating observers shown in Fig. 1 naturally generalizes to multiple overlapping inertial reference frames, such as frame A (Alice) and frame B (Bob), covering the same region of spacetime. Each frame is constructed independently, using only its own network of observers, synchronized clocks, and internal measurement protocols.

As a result, any given spacetime event can be assigned two sets of coordinates—x and $\tilde{x}$—depending on which frame's observers perform the local assignment. Because the two networks are coextensive, observers in one frame can also access the measurements made by their local counterparts in the other. This operational comparison allows one to empirically determine a transformation function:

$$\tilde{x} = \Lambda_{A \to B}(x). \tag{1}$$

A striking empirical result would be obtained when performing this experiment: when two inertial frames A and B are constructed independently—each relying solely on its own local observers and synchronization protocol—the form of the transformation $\Lambda_{A \to B}$ turns out not to be Galilean, as would be expected in Newtonian mechanics. Instead, it corresponds to a Lorentz transformation (up to translations), in accordance with the Poincaré symmetry that preserves the form of Maxwell's equations. This result is theoretically equivalent to the constancy of the speed of light as measured in any inertial reference frame — the principle originally postulated by Einstein and used to derive the Lorentz transformation. In modern terms, this invariance is understood as the *Poincaré invariance* of special relativity: the spacetime interval between any two events remains the same in all inertial frames.

Expressing the Poincaré transformation in component form in Einstein notation as

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu + o^\mu, \tag{2}$$

where the constant four-vector o^μ accounts for arbitrary (but physically irrelevant) offsets in spacetime origins between frames, and introducing the invariant Minkowski metric tensor

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1), \tag{3}$$

the invariance of the infinitesimal spacetime interval ds^2 is expressed as

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = \eta_{\mu\nu}\, d\tilde{x}^\mu d\tilde{x}^\nu. \tag{4}$$

These equations involve only the differences in spacetime coordinates between events, and as discussed above, such differences can be determined through purely local measurements within each reference frame, using synchronized clocks and known spatial positions. Consequently, the value of the invariant spacetime interval is directly computable from physically measurable quantities in either frame. The presented construction of SR reference frames is conceptual, but also approximately realized by particle detectors in collider experiments.

IV. GENERAL RELATIVITY'S REFERENCE FRAMES

Introductory treatments of general relativity typically emphasize that globally inertial reference frames, as defined in special relativity, cannot be extended to generic curved spacetimes. This motivates the transition to curvilinear coordinates, the adoption of general covariance, and the use of the formalism of pseudo-Riemannian differential geometry. Yet, during this transition, the analogy between physically realizable reference systems and their mathematical counterparts often remains conceptually incomplete.

Only part of this analogy is typically developed. The equivalence principle establishes a local correspondence between a freely falling physical reference frame—realized, for instance, by idealized observers equipped with mutually orthogonal measuring rods and synchronized clocks for temporal measurement—and the mathematical notion of a locally flat infinitesimal part of a curved manifold $\mathcal{M}$, formalized as the tangent space $T_p\mathcal{M}$ at a point $p \in \mathcal{M}$. However, no corresponding operational framework is typically provided for constructing general finite coordinate charts on the manifold. The extension from infinitesimal inertial frames to arbitrary curvilinear coordinates is usually carried out purely in the abstract, without offering concrete procedures that could be implemented by individual observers using only local measurement processes. Pedagogical analogies—when offered—frequently appeal to cartographic projections, such as mapping Earth's surface, but these remain disconnected from the question of how general reference frames might be physically realized in spacetime. The step needed to complete the analogy—constructing arbitrary reference frames based solely on the data physically accessible to observers operating independently—is often omitted, or substituted with technically elaborate formalisms.

As a result, the connection between arbitrary coordinate systems defined over finite regions of curved spacetimes and their operational construction remains underexplained and continues to call for more accessible and physically grounded introductions.

Even a textbook as conceptually thorough as *Gravitation*[42] exemplifies this broader trend. It begins its treatment of coordinates (pp. 5–10) with a vivid physical description of events as intersections of worldlines and physical interactions. The introduction of coordinates is explicitly delayed until events have been described physically. However, the transition to coordinate systems is then carried out in purely mathematical terms: Coordinates are introduced as abstract ordering devices applied *post hoc* to an idealized event structure:

Nothing is more distressing on first contact with the idea of "curved spacetime" than the fear that every simple means of measurement has lost its power in this unfamiliar context. [...] No numbers. No coordinate system. No coordinates. [...] To order events, introduce coordinates! [...] Coordinates are four indexed numbers per event in spacetime. [...] In christening events with coordinates, one demands smoothness but foregoes every thought of mensuration.

For some students, such remarks may be reassuring enough to allay the acknowledged "fear that every simple means of measurement has lost its power." Still, from an operational perspective, the lack of an explicit physical construction linking coordinate labels to measurable quantities may obscure the logical continuity between SR and GR. Since Einstein's mollusk (and in particular its generalization introduced below) offers a direct and conceptually smooth transition from operational SR frames to their GR counterparts, for others the absence of this construction might amount to a missed opportunity for conceptual clarity.

In what follows, we demonstrate how the extension from SR to GR reference frames can proceed continuously and transparently, requiring only a few well-motivated generalizations.

A. The Einstein Mollusk

As previously mentioned, the first popular exposition of general relativity[3] was written by Einstein himself, who did not shy away from introducing the vivid operational image of a physical "reference mollusk":

What does it mean to assign to an event the particular co-ordinates x_1, x_2, x_3, x_4, if in themselves these co-ordinates have no significance? More careful consideration shows, however, that this anxiety is unfounded [...] For this reason non-rigid reference-bodies are used, which are as a whole not only moving in any way whatsoever, but which also suffer alterations in form ad lib. during their motion. Clocks, for which the law of motion is of any kind, however irregular, serve for the definition of time. We have to imagine each of these clocks fixed at a point on the non-rigid reference-body. These clocks satisfy only the one condition, that the "readings" which are observed simultaneously on adjacent clocks (in space) differ from each other by an indefinitely small amount. This non-rigid reference-body, which might appropriately be termed a "reference-mollusk", is in the main *equivalent to a Gaussian four-dimensional co-ordinate system chosen arbitrarily. That which gives the "mollusk" a certain comprehensibility as compared with the Gauss co-ordinate system is the (really unjustified) formal retention of the separate existence of the space co-ordinates as opposed to the time co-ordinate. Every point on the mollusk is treated as a space-point, and every material point which is at rest relatively to it as at rest, so long as the mollusk is considered as reference-body. The general principle of relativity requires that all these mollusks can be used as reference-bodies with equal right and equal success in the formulation of the general laws of nature; the laws themselves must be quite independent of the choice of mollusk.*

A visual representation of this idea is shown in Fig. 2 which attempts to capture the mollusk metaphor using a nested-surface visualization, or the top-surface of a dough, but the physical framework is not tied to any particular visualization of physical three-dimensional space. The mollusk and its associated clock-carrying observers cover a (possibly infinite) spatial (sub-)set of $\mathbb{R}^3$. This suffices for the operational definition of the spatial labels[63].

The mollusk generalizes the spatial lattice of special relativity to a flexible reference structure. Local observers, each equipped with a clock that may run non-uniformly, are distributed across spatial coordinates that are allowed to warp arbitrarily. These coordinates define a continuous network of physically distinguishable events without relying on global synchronization or idealized rulers. The mollusk as a whole may also evolve in time, with the shape of each surface changing and clocks ticking at locally arbitrary rates — it is, metaphorically speaking, "alive and kicking."[64]

It is worth emphasizing how fundamentally Einstein's account anchors coordinate meaning in physical reference systems. In his explanation, even a complete specification of events does not yield coordinate values unless those events are situated relative to some other physical entity acting as a reference frame. This relational view may seem trivial, but it is easily overlooked when no attempt is made to operationalize general relativistic coordinates. The risk is particularly pronounced in coordinate-free formulations, where the absence of explicit charting procedures can obscure the empirically indispensable link between mathematical structure and observable quantities.

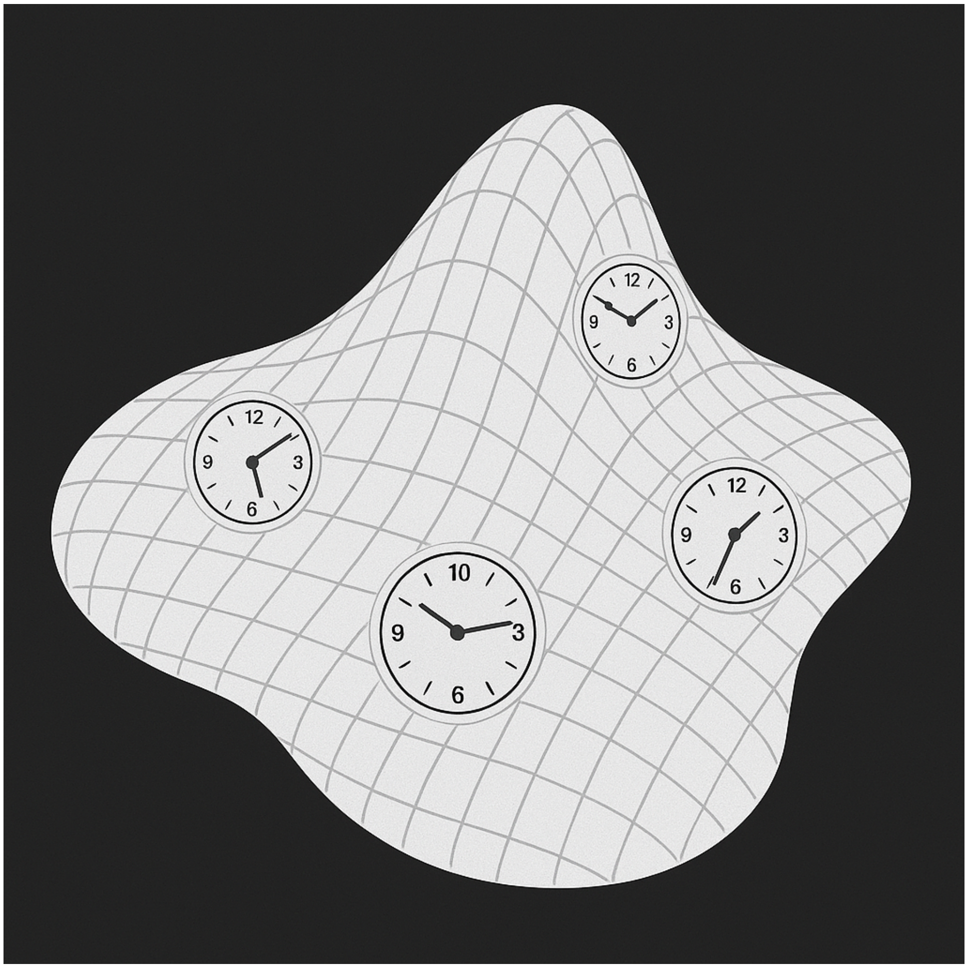

FIG. 2. One spatial surface of an Einstein mollusk, representing a local physical reference frame. Observers with arbitrarily varying clocks are positioned along nonlinearly deformed spatial coordinates. The surface is one of many spatial layers that, together, define a deformable three-dimensional reference structure. The entire mollusk may change its shape over time.

Since we are interested in physically realizable reference frames, and because the entire spacetime manifold $\mathcal{M}$ can be covered by a countable collection of overlapping (open) sets $U \subset \mathcal{M}$, we may approximately represent each of those regions in which spacetime is probed by a distinct physical mollusk — that is, by a distinct deformable reference frame constructed from comoving observers equipped with local clocks and measuring devices[65]. If mollusks are defined on multiple overlapping regions, their respective coordinate labels must be defined so as to ensure mutual consistency.

A mollusk-adapted coordinate system within U is then given by the coordinate functions

$$(x^0, x^1, x^2, x^3) := (\lambda(\tau, x^1, x^2, x^3), x^1, x^2, x^3), \tag{5}$$

where each co-moving observer remains at fixed spatial coordinates x^i, with $i = 1, 2, 3$, relative to the arbitrarily moving mollusk, while the local clock shows λ, which increases monotonically with the observer's proper time τ. These functions define a smooth local coordinate chart on spacetime, i.e., a local diffeomorphism from $\mathbb{R}^4$ to U.

However, it is important to distinguish between the mathematical existence of arbitrary local charts—guaranteed by the manifold structure of spacetime—and the construction of such charts via physical procedures. Material reference frames are constrained by their material nature and have to be erected in actual spacetime geometries. These factors may affect whether they can reproduce all mathematically admissible charts. The mollusk can only realize mollusk-adapted coordinates given the actual Lorentzian spacetime in which it happens to find itself. It is important to recognize that a mollusk is thus not merely a physical realization of an abstract coordinate chart, but a realization that is structurally constrained by its mode of implementation in a spacetime. A mollusk-adapted chart is more than a smooth diffeomorphism from $\mathbb{R}^4$ to a spacetime region U: it is a chart with built-in metric constraints. Specifically, the coordinate direction associated with $x^0 = \lambda(\tau, x^i)$ must be timelike, while the spatial directions associated with x^i must be spacelike within the mollusk's domain. This restriction is inherited from the mollusk's realization by a congruence of comoving timelike observers, each with a single clock and fixed spatial label.

As a consequence, the mollusk does not enjoy general covariance, as revealed by asking: can the mollusk instantiate *any* admissible mathematical chart? Importantly, we are not asking whether a mollusk-adapted chart can be transformed *into* any other chart via coordinate change; rather, we ask whether it can *directly represent* arbitrary charts through its physical construction. The answer is negative: irrespective of the geometry of the specific spacetime in which the mollusk is physically realized, the mollusk can never instantiate coordinate systems that require null directions, as found in (double-)null charts. In such cases, there is no way to associate the required coordinate structure with the mollusk's single timelike direction and its fixed spatial labeling (for details, and the mollusk's relationship to the ADM decomposition, see Appendix A).

This limitation reveals a deeper insight: physically implemented coordinate charts typically carry not only a smooth labeling of events, but also structural constraints inherited from the physical systems that realize them.

B. The reference swarm of unconstrained observers

To overcome this, and to locally realize *any* chart—regardless of its causal or foliation structure—we require a physical system in which the coordinate labels are decoupled from both spacetime geometry and observer motion. This is precisely what the generalized observer swarm provides: each observer carries four freely and independently programmable clock readings, which serve purely as dynamic numerical labels. These can be chosen to reflect any smooth chart on spacetime, without implying anything about the causal or geometric structure of the observer's worldline[66]. The role of the observers' timelike worldlines is merely to create a congruence which ensures that spacetime is locally covered by measurement devices; once this coverage is achieved, the freely programmable clocks can assign arbitrary coordinate values to any point in spacetime. Each observer's clocks can be freely programmed to advance according to any smooth function of their proper time or local measurements[67]. In this way, the observer swarm serves both as a conceptual tool and a physically realizable construction (at least in principle): a congruence of observers locally provides any chart and is free from geometric constraints.

We may arrive at the same insight from a slightly different point of view. Building on our earlier shift in special relativity—from an idealized lattice to a distributed network of local observers (as depicted in Fig. 1)—we now complete this progression by shifting attention away from coordinate labels derived from spatial slices. Instead, we regard the observers themselves—and, in particular, the information they locally assign—as the primary carriers of coordinate structure. Specifically, we propose eliminating all ties to fixed spatial coordinate values by implementing a general physical reference frame in general relativity as a *swarm of local observers in motion*, each equipped with *four* independent and arbitrarily evolving clocks (replacing any constant readings or fixed spatial positions). The readings of these clocks provide smooth, unique, but otherwise arbitrary numerical labels to events observed in the neighborhood of each observer (i.e. conceptually infinitesimal neighborhood in which the assumed chart smoothness allows the observers to distinguish which neighbors are closer: they show more similar numerical values). Together, they define a general local coordinate system—unrestricted by any geometric structure or synchronization constraints. This contrasts with, for example, ADM-like formulations based on global spacelike foliations and with tetrad-based approaches tied to orthonormal bases.

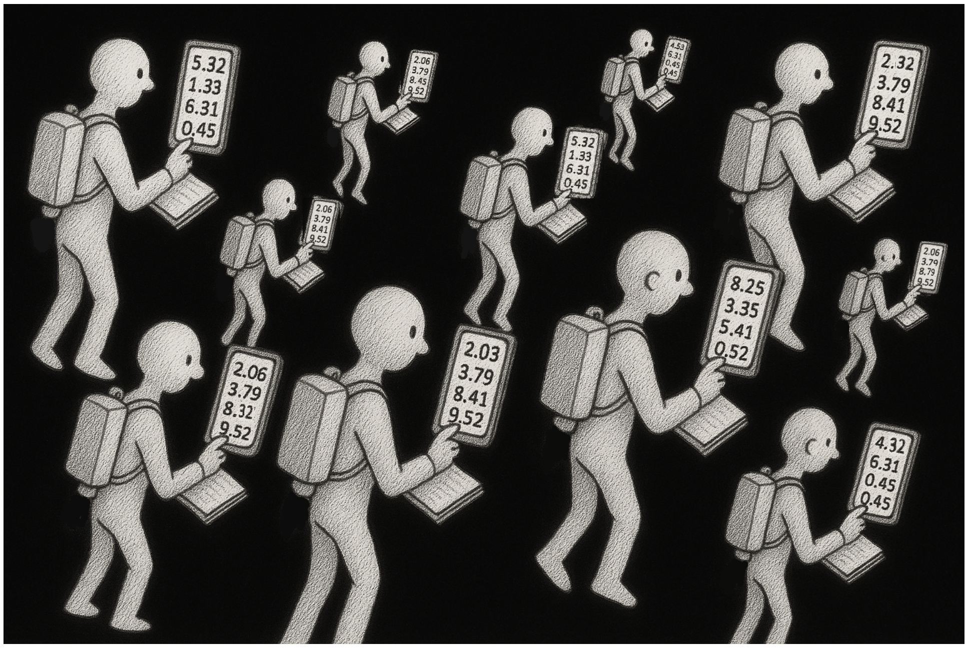

FIG. 3. A swarm, flock or cloud of observers in motion fills spacetime without relying on any foliation. Each observer locally records observed events in a notebook and uses four arbitrarily evolving numerical labels to index them — for instance, as displayed on a single device. These values define general local coordinates in a covariant, slicing-independent framework (smooth chart).

While we summarize the construction as involving four arbitrary clock values per observer, in practice a single physical clock—such as an atomic timekeeping device—suffices. The four coordinate values can then be algorithmically generated as functions of this single internal clock reading, and/or additionally or exclusively depending also on local measurements of physical fields. The reference to arbitrary clocks should emphasize the dynamic nature of the displayed values, not imply four different time-keeping mechanisms. Regardless of the underlying mechanism[67], the key requirement is that the displayed coordinate values evolve freely and independently of external physical dynamics, ensuring the strict decoupling between the possibilities for coordinate assignment and physical behavior in an ambient spacetime. For pedagogical concreteness, we may imagine that the four continuously updated clock readings are displayed on a single device—such as a smartphone screen—carried by each observer, as depicted in Fig. 3. The proposed generalization thus yields an operational realization of a reference frame that respects general covariance. It introduces no additional structure beyond the smooth invertibility of the observer-assigned labels and makes no assumptions about slicing, symmetries, or background geometry. Indeed, it remains agnostic even to spacetime dimensionality (if more or fewer than four numbers are displayed).

Moreover, since the numerical displays of the observers are entirely unconstrained — evolving arbitrarily in both space and time — the framework naturally accommodates the mathematical structure of an arbitrary atlas on a smooth manifold. The local continuity of the four numbers ensures that overlapping neighborhoods can collectively form a smooth cover of spacetime in which chart transitions are realized algorithmically through the consistent labeling of events in intersecting regions.

Let $\mathcal{M}$ be a smooth 4-dimensional Lorentzian manifold, and let $U \subset \mathcal{M}$ be a neighborhood in which we implement our physical reference frame. We define a local congruence of timelike worldlines via a smooth, future-directed, unit timelike vector field $u^\mu(x)$. The integral curves of u^μ then define the worldlines of the observers. We keep U sufficiently small, and the density of observers sufficiently low, so as to neglect back-reaction on the ambient geometry and to exclude both self-gravitational effects and caustics.

Let now $x(\tau)$ denote the worldline of one observer in U, parametrized by a scalar parameter τ denoting the proper time of the observer, as measured by a (fifth) clock that is no longer arbitrary (e.g., an atomic clock). Associated with the congruence of observer worldlines, we introduce four functions $X^\mu(x)$, interpreted as the numerical values assigned to events by local observers—that is, the arbitrary clock readings or algorithmically defined numbers displayed on each observer's screen. This setup is analogous to the situation in fluid mechanics: the observer worldlines correspond to the Lagrangian viewpoint, while the quadruples $X^\mu(x)$ reflect an Eulerian description.

For the following, the Eulerian description $X^\mu(x)$ is preferred because the mapping

$$X : x \mapsto \big(X^0(x), X^1(x), X^2(x), X^3(x)\big) \tag{6}$$

can define a physically realized coordinate chart on the region $U \subset \mathcal{M}$, mapping events to coordinate tuples in $\mathbb{R}^4$, provided the Jacobian matrix $\partial X^\mu/\partial x^\nu$ is invertible. It is important to stress that the functions $X^\mu(x)$ are smooth, real-valued assignments made by observers. They are neither scalar fields nor components of a vector field in the tensorial sense. They do not transform covariantly under diffeomorphisms and carry no intrinsic geometric meaning beyond their operational role in providing general arbitrary labels for events. We intentionally use Greek superscripts for both the abstract manifold coordinates

x^μ and the arbitrary observer-assigned functions X^μ, with $\mu = 0, 1, 2, 3$, to highlight their respective roles as abstract and observable coordinates. The use of a capital X emphasizes the operational origin — a reference frame realized physically by measurement procedures. This distinction, while useful here, may be safely omitted in what follows, since any mathematical coordinate chart x defined on a region $U \subset \mathcal{M}$—for instance, one used in solving the Einstein field equations—can now be trivially realized on the observers' screens by choosing the observable coordinates $X^\mu(x)$ to coincide with the abstract coordinates:

$$X^\mu(x) := x^\mu.$$

We are allowed to use any assignment and choose $X^\mu(x) := x^\mu$ without any loss in generality since already our choice of chart coordinate values x^μ is completely unconstrained. Therefore this construction reflects a central feature of the observer swarm: the coordinate values carried by each observer need not correspond to physically intrinsic quantities. Rather, these values can be freely assigned and algorithmically updated, enabling observers to carry abstract coordinate labels that encode structures not directly tied to their own local state and the realization of coordinate systems not adapted to the underlying timelike congruence—including null foliations or even coordinate charts with formal singularities. The roles of the observers are thus cleanly separated: they must be present throughout the region U in order to record everything that happens, but the coordinate values they carry are not restricted by their motion. In this respect, the observers are best understood as carriers of programmable labels which implement the chosen chart, which can be conceptually realized irrespective of the details of the local dynamics of the observers themselves.

Even after selecting chart coordinates x^μ and identifying $X^\mu(x) := x^\mu$, we retain freedom in specifying the observer motion $u^\mu(x)$ through the choice of initial conditions.

Of course, once these design choices have been fixed, the resulting quantities must satisfy certain consistency conditions. In particular, since the clock readings $X^\mu(x)$ are defined operationally by clocks carried by moving observers, their evolution along each worldline must satisfy the mathematical identity $\frac{dX^\mu}{d\tau} = u^\nu \partial_\nu X^\mu$. Our identification $X^\mu := x^\mu$ yields $\partial_\nu X^\mu = \delta^\mu_\nu$, and the expression therefore simplifies to

$$\frac{dX^\mu}{d\tau} = u^\mu.$$

This relation is not a physical constraint. It reflects the bookkeeping requirement underlying the programming of each observer's clocks if they should keep their readings in synchronicity with freely assigned coordinate values x^μ as the swarm moves through spacetime. The construction is illustrated in a few concrete examples given in Appendix B.

As for examples, any timelike congruence in the GR literature can be interpreted as an observer swarm by viewing the coordinate evolution $u^\mu(x)$ along worldlines as programmed labels rather than dynamically determined values. For instance, the standard congruence of radially infalling observers in Schwarzschild spacetime, which naturally implements Painlevé-Gullstrand coordinates, becomes an example of our framework when the coordinate values are understood as programmable clock displays (cf. Appendix B). This reinterpretation applies to any coordinate system realized by observer congruences in the literature: Lemaître coordinates, Gaussian normal coordinates, or even exotic constructions—all become examples of programmable clock displays.

Every diffeomorphism between two coordinate systems, $x \mapsto \tilde{x}(x)$ now corresponds in the operational setting to a transformation between two physically realized coordinate systems:

$$\tilde{X}^\mu(X) = \tilde{x}^\mu\big(x(X)\big), \tag{7}$$

where $x(X)$ is the inverse of the physically realized chart $X : x \mapsto X(x)$, and $\tilde{X}^\mu$ denotes the new observer-assigned coordinates. This shows that diffeomorphisms appear as transformations between observable coordinate assignments, and that any mathematical chart or coordinate transformation can be physically realized by a suitably constructed swarm of observers and their associated quadruples of clock readings.

Crucially, a single observer congruence suffices to implement any coordinate chart in a region. Given one reference congruence with coordinates X^μ, observers can display any transformed coordinates $\tilde{X}^\mu(X)$ through appropriate programming. This includes charts with null coordinate directions: for instance, radially infalling observers in Schwarzschild can display Kruskal-Szekeres coordinates U,V despite these being null. The observers themselves follow timelike paths, but their programmable displays can represent any smooth coordinate system: because of their coordinated pre-deployment programming they are capable of displaying in combination even coordinate systems having no timelike directions (cf. again Appendix B). Expressed in a somewhat catchy manner: the observer swarm achieves maximum generality with minimum machinery.

In particular, it must be noted that none of the four displayed numbers would be known to directly correspond to time or space directions as such. At this stage of the operationalization, these numbers are only four smoothly varying labels, not implying more information than any purely mathematical chart would imply before the manifold has been equipped with a metric. In this sense the construction follows the logic of differential geometry even though the observers must be deployed in a real spacetime: the freedom of programming has decoupled their coordinate values from any constraints potentially arising through their timelike motion.

Once the observer-assigned labels X^μ are identified with a chosen mathematical chart x^μ, it becomes convenient—though conceptually distinct—to adopt a unified notation. Since both sets of functions share the same mathematical properties as coordinate functions on a manifold, their distinction lies purely in their operational meaning: one stems from an idealized chart, the other from observer-based measurement procedures. In the formalism itself, however, no confusion arises if both the abstract coordinates and their physically realized counterparts are denoted using lowercase symbols x^μ and $\tilde{x}^\mu(x)$.

Operationally, coordinate transformations can be realized in different equivalent ways. For example, we may again introduce two interspersed swarms of observers — defining two independent reference frames, A (for Alice) and B (for Bob) — each covering the same region U of spacetime. Alternatively, we may choose to assign multiple distinct sets of four numerical values to each observer within a single swarm. In this case, each observer carries several coordinate quadruples simultaneously, which may be displayed on two screens per observer (or a shared screen for both coordinate systems). In all of these scenarios, a general coordinate transformation between frames is implemented physically by having each observer record both their own set of coordinate labels x and those of the nearest observer from the other frame $\tilde{x}$. The transformation $\tilde{x}^\mu = \tilde{x}^\mu(x)$ in all of U is then inferred from all local comparisons. Moreover, since observers can observe a (conceptually infinitesimal) neighborhood around them, the smoothness of charts allows them to order their (infinitesimally close) neighbors according to proximity (the closer neighbors have more similar coordinate values). Accordingly, an infinitesimal coordinate displacement $d\tilde{x}$ in the reference frame A of the Alice observers can be established operationally, and it is related to the corresponding difference dx in the reference frame B of the Bob observers via the Jacobian matrix (differentials remaining differentials in every frame due to the smoothness of chart transforms):

$$d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\, dx^\nu. \tag{8}$$

All quantities appearing in this vectorial transformation law eq. (8) correspond to observable differences: both $d\tilde{x}^\mu$ and dx^ν represent measurable differences between the coordinates of two nearby events, as recorded in two distinct reference frames by the respective swarms of Alice and Bob observers. Note again that "nearby" is operationally definable without knowledge of any metric only using the smoothness of charts (closer events have more similar coordinate values). The partial derivatives in the Jacobian matrix can be obtained directly from such measured values using basic linear algebra, or by numerically differentiating the transformation relation given in Eq. (7). Each term thus admits a concrete operational interpretation and reflects actual measurements followed by computational analysis.

Although physical laws must be expressed in locally covariant form to ensure consistency under coordinate transformations, non-covariant expressions—such as coordinate values or Christoffel symbols—retain operational significance *within a fixed reference frame*. Their empirical relevance derives from the fact that they correspond to directly measurable quantities, once the reference frame is specified. For example, coordinate values correspond to actual numbers displayed on screens in Fig. 3, reflecting measurable quantities within a physically realized reference frame. These values can be freely chosen, but once assigned, they constitute real outputs of local measurements.

Such distinctions are easily overlooked when physical constructions of reference frames are not explicitly considered. Yet they are both pedagogically and conceptually important. For example, in Rovelli's terms[34], coordinate values and non-tensorial quantities like the Christoffel symbols qualify as *partial observables*—quantities that are directly measurable, even though they are not tensorial.

All operations required to verify Eqs. (7)–(8) can be carried out locally by the observers in the frame. Any quantity computed from coordinates—whether tensorial or not—can thus be viewed as the result of data processing based on numerical results obtained from physical operations. Properties like tensoriality can, in principle, be operationally tested.

C. Measuring the Metric in GR

In special relativity (SR), the flatness of spacetime[68] is reflected in the ability to choose globally rectilinear coordinates in which the metric tensor assumes its canonical Minkowski form η. Its global constancy ensures that coordinate differences dx^μ correspond directly to physically meaningful intervals, as shown in Eq. (4). In curvilinear coordinates, coordinate differences dx^μ alone are not physically meaningful. To obtain, for example, proper distances or durations, one must also know the locally varying metric tensor $g_{\mu\nu}(x)$, which allows to extend beyond SR by encoding spacetime curvature. This section shows how this additional structure can be gently introduced using the operational framework developed so far.

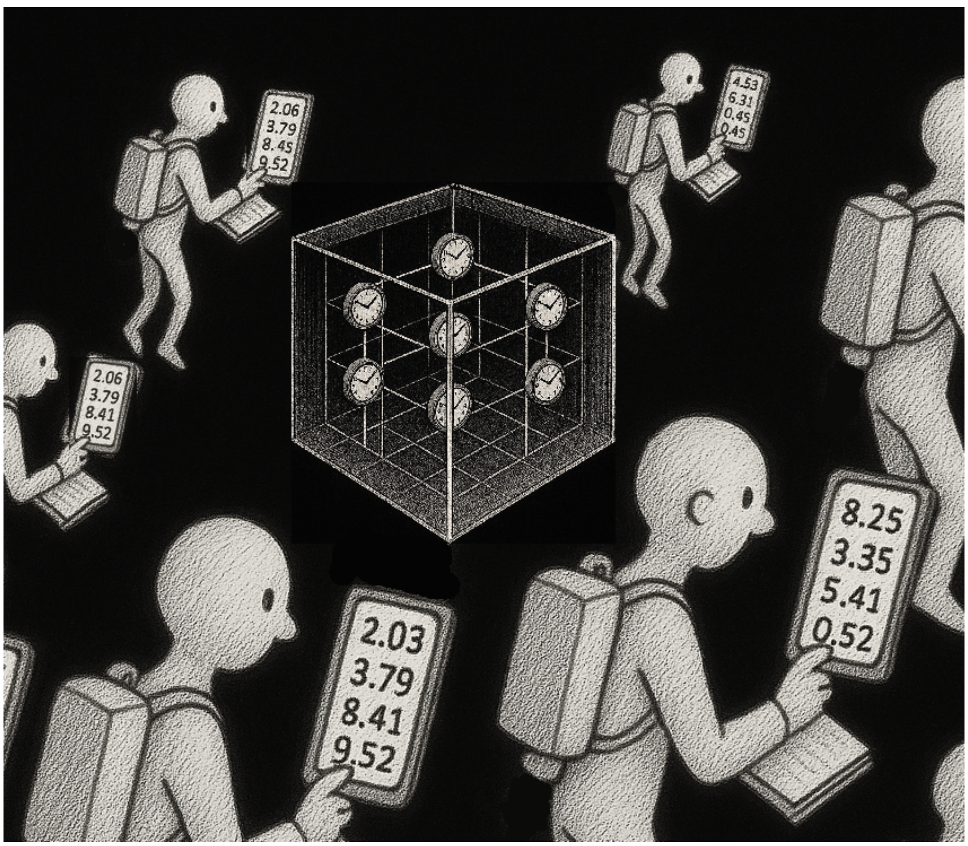

FIG. 4. A GR reference frame consists of a swarm of unconstrained observers traveling arbitrarily in spacetime, measuring whatever happens in their immediate vicinity. A second reference frame, in the form of an Einstein elevator, covers an infinitesimal region in space and time in which the laws of SR hold and can be expressed in the elevator's special relativistic coordinate system.

Einstein was led to general relativity (GR) by a second stroke of genius: he recognized the true reason why all bodies fall with the same acceleration in Galileo's famous Leaning Tower of Pisa experiment. Gravitational effects can be locally eliminated by using the coordinate values of a freely falling reference frame. More figuratively, the interior of an infinitesimally small, freely falling elevator constitutes a local inertial frame in which the laws of special relativity (SR) hold—without curvature or gravitational effects—for a very short duration and within a very small spatial region. This is the content of the equivalence principle.

This empirical insight can be applied to the analysis of any event p with coordinate values x^μ in an arbitrary GR reference frame. In the infinitesimal neighborhood of p, one can always construct a freely falling Einstein elevator passing through the event, as illustrated in Fig. 4. The condition of free fall can be operationally verified by ensuring zero readings from an (infinitesimal) accelerometer. Since SR holds locally inside this tiny, transient frame, an orthonormal local SR coordinate system $\tilde{x}$ can also be constructed—a latticework as in Fig. 1—but now centered on p and valid only within an infinitesimally small region of spacetime.

For ease of visualization, we assume that the Einstein elevator has no physical walls, allowing the observers from both reference frames—the general relativistic (GR) frame and the local inertial special relativistic (SR) frame—to be interspersed throughout the same region, including inside the elevator. Observers from both systems can thus record the same events occurring within the elevator. The GR observers assign four coordinate values x^μ, while the SR observers use locally valid coordinates $\tilde{x}^\mu$. These coordinate systems are related by a smooth transformation $\tilde{x}^\mu = \tilde{x}^\mu(x)$, which is valid only in a small neighborhood around the event p, but is otherwise no different from a conventional operationalization of coordinate transforms.

Operationally, this transformation can be physically realized in the usual manner: each observer in one reference frame simply records the coordinate values displayed on the device of their neighboring counterpart in the other frame. Since the $\tilde{x}^\mu$ coordinates belong to a local SR reference frame, spacetime intervals expressed in these coordinates have immediate physical significance; in particular, the invariant expression of the interval $ds^2 = \eta_{\mu\nu}\, d\tilde{x}^\mu d\tilde{x}^\nu$ remains valid throughout the short lifetime and spatial extent of the Einstein elevator.

By applying the coordinate transformation $\tilde{x}^\mu = \tilde{x}^\mu(x)$ and using Eq. (8), the spacetime interval can be rewritten in terms of the general coordinates x^μ, which alone do not carry immediate physical meaning:

$$ds^2 = \eta_{\mu\nu} \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} dx^\alpha dx^\beta \equiv g_{\alpha\beta}(x)\, dx^\alpha dx^\beta, \tag{9}$$

where we have defined the general relativistic metric tensor by

$$g_{\alpha\beta}(x) = \eta_{\mu\nu} \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta}. \tag{10}$$

Once the functional form of the coordinate transformation $\tilde{x}^\mu(x)$ is determined—through local comparison of observer readings over sufficiently small neighborhoods—its derivatives yield the metric components via Eq. (10), completing the reconstruction from local SR measurements.

Alternatively, one may reconstruct the components of $g_{\mu\nu}(x)$ directly from physically measured finite but sufficiently small displacements $d\tilde{x}^\mu$ in the elevator frame and the corresponding coordinate differences dx^μ in the general frame, using the defining relation $g_{\alpha\beta} dx^\alpha dx^\beta \equiv \eta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu$ and standard linear algebra, as first presented by Einstein.[45]

If we switch to a different GR reference frame with coordinates x'^μ, and construct a corresponding local SR frame $\tilde{x}'^\mu$, we again define the metric via local measurements as

$$ds^2 = \eta_{\alpha\beta}\, d\tilde{x}'^\alpha d\tilde{x}'^\beta = g'_{\mu\nu}(x')\, dx'^\mu dx'^\nu. \tag{11}$$

Since both the original and the new GR frames use the same underlying local SR measurements (up to Lorentz or Poincaré transformations, which preserve ds^2), and since both expressions yield the same invariant interval, we have

$$g_{\mu\nu}(x)\, dx^\mu dx^\nu = g'_{\alpha\beta}(x')\, dx'^\alpha dx'^\beta. \tag{12}$$

But the coordinate displacements are related via the Jacobian matrix in eq. (8).

Substituting into the left-hand side of eq. (12), we get (since the infinitesimal displacements were arbitrary):

$$g'_{\alpha\beta}(x') = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}(x), \tag{13}$$

which is the transformation law of a (0,2)-tensor. Thus, the metric's tensorial character follows from the invariance of ds^2 which is operationally defined through local measurements in overlapping reference frames.

Since the local GR metric $g_{\alpha\beta}(x)$ is operationally defined via a transformation from a local SR coordinate system, it can also be used to analyze causal relationships in the GR coordinates. For any event p with assigned coordinates x^μ in a general reference frame, one can construct the local light cone by first identifying the standard SR light cone at p in the local inertial frame, defined by the null condition

$$ds^2 = \eta_{\mu\nu}\, d\tilde{x}^\mu d\tilde{x}^\nu = 0, \tag{14}$$

and then applying the local coordinate transformation to express these directions in terms of the GR coordinates. This yields the GR light cone at x, defined by

$$ds^2 = g_{\alpha\beta}(x)\, dx^\alpha dx^\beta = 0, \tag{15}$$

whose causal structure thus arises directly from the mapping of the local SR structure into the general frame.

This provides a useful practical insight, especially in spacetimes with potentially misleading coordinate labels (such as the Schwarzschild interior). It allows one to determine unambiguously which coordinate displacements dx^α are timelike, spacelike, or null by evaluating the associated invariant:

$$ds^2 \begin{cases} < 0 & \text{timelike,} \\ = 0 & \text{null,} \\ > 0 & \text{spacelike.} \end{cases} \tag{16}$$

Regardless of the coordinate names or conventions, the causal character of any direction can thus be determined locally from the metric.

Slightly more advanced approaches may start from the idea that local light cone structures could anyway be determined directly through the observation of light rays, without explicitly constructing any local inertial system. This provides an independent observational route to determining the local metric structure[26,42,48]. Since null displacements satisfy eq. (15), knowledge of enough linearly independent null directions dx^α, obtained for instance by tracing the trajectories of different light rays through the same event p having coordinates x, determines the metric $g_{\alpha\beta}(x)$ up to a conformal factor. The remaining scale ambiguity can be fixed, for example, via a (fifth) clock which is no longer arbitrary but rather has to record proper time along the timelike worldline of the observer passing through p. While this approach may be more directly practical, it tends to obscure the conceptual link to the equivalence principle and the analogy with locally flat coordinate patches in differential geometry.

Regardless of which operational route is chosen—local SR-based construction or direct causal probing—the same procedure can be repeated (at least conceptually) at each event $p, q, r, \ldots$ in the reference frame. In this way, one reconstructs the spacetime-dependent metric tensor $g_{\alpha\beta}(x)$ over the domain of interest, over the domain of interest, subject to the practical constraints of measurement density and accuracy. Once the metric tensor is known throughout spacetime, all geometrical quantities—such as the connection and curvature tensors—become, at least in principle, derivable from observable data. The foundational structure of general relativity is therefore rooted in conceptually transparent measurement procedures, based entirely on coordinate readings and local observations[24].

While conceptually not necessary, it is of practical value to simplify any measured metric tensor by checking whether it can (in our operational setup only afterwards) be fit to a model geometry, and to establish the coordinate transform to such model-adapted coordinate systems.

We may also note that the physical framework developed so far already suffices to describe non-gravitational physical processes in GR coordinate frames. The procedure is straightforward: such processes are first analyzed in the local inertial SR frame, where the known laws of special relativity apply for short durations and over small spatial regions. These results are then translated into GR coordinates using the local coordinate transformation and the metric tensor defined in Eq. (10), as needed.

In this way, locally force-free motion becomes geodesic motion in spacetime, ordinary derivatives in SR translate to covariant derivatives in GR coordinates, and so forth. Moreover, the notion of a local inertial frame can be formalized *a posteriori* through the construction of Riemann normal coordinates $\tilde{x}$ centered at a point p, in which the metric reduces to the Minkowski form $\eta_{\mu\nu}$ and its first derivatives vanish[49]. This completes the conceptual circle: the observer swarm itself is composed of local observers, each of whom experiences a momentarily special relativistic description of spacetime.

It is worth emphasizing, however, that we have so far only scratched the surface of the mathematical structure of differential geometry. The reason further mathematical development becomes useful is not abstract preference, but physical necessity: the local flatness implied by the equivalence principle is conceptually analogous to the flatness of small patches on curved surfaces. In this sense, physics has led us naturally to the tools of differential geometry, not the other way around. This conceptual flow can be obscured in treatments that begin with a few physical images (such as local SR latticeworks) but then make a rapid leap to abstract mathematics (e.g., tangent spaces to curved manifolds), without developing the underlying physical motivation.

Moreover, since we were forced to engage in local metric reconstruction precisely because gravity prohibits the existence of a global SR frame, it should come as no surprise that local variations in the metric tensor $g_{\alpha\beta}(x)$ will also play a central role in the gravitational field equations themselves. It is instructive to contrast this with the case of curvilinear coordinates in special relativity, which also yield a locally varying metric, but one with a vanishing Riemann curvature tensor[49]. Hence, also the operational framework developed here makes it clear from the outset that any genuine generalization of special relativity to include gravitation must involve field equations that yield nonzero spacetime curvature.

D. Interpretational and Conceptual Remarks

Before concluding, it is worth highlighting several important qualifications of the physical picture developed here, in which moving timelike observers serve as carriers of a general reference frame via their freely programmable four coordinate readings.

1. Practice versus Gedanken-Experiments

In typical experimental practice, the coordinate chart is often treated as a mathematically declared structure rather than an operationally instantiated one. This is not an oversight but a pragmatic and methodologically sound approach: the operational content resides in the measurements themselves—ticks of clocks, detection events, frequency shifts—which are then related to theoretical quantities such as scalars, distances, and intervals using the postulated metric *expressed in the assumed chart*. The chart itself frequently remains a formal background structure, implicitly assumed rather than constructed from first operational principles. Indeed, this pragmatic methodical choice may have contributed to the view that charts are not themselves objects requiring or even capable of unrestricted operationalization. This approach, however, implies that a whole chain of assumptions is tested by observations. While this is the regular research practice, it may interfere with the step-by-step pedagogical development of concepts desirable at an early stage of contact with relativity. Thus, the proposed framework actually serves a different purpose: it primarily acts as a *conceptual and pedagogical scaffold* and a *mathematically and logically rigorous operationalization* of coordinate freedom.

2. Determinism vs. Freely Programmable Arbitrary Clocks

A subtle but important tension arises between the mathematical freedom to define arbitrary coordinate charts and the physical determinism of classical field theories like general relativity. In our construction, each observer is equipped with freely programmable clocks, allowing arbitrary numerical labels to be assigned to events within a region $U \subset \mathcal{M}$. This operationalizes chart freedom: any smooth, invertible map $X^{\mu}(x)$ can be implemented through suitable clock programming. From a deterministic perspective, however, such freedom is physically illusory. If all field values—including the observers' internal states—are fixed by initial data and deterministic evolution, then the clock readings themselves are also determined. In this view, coordinate labels become a derived consequence of physical evolution, not a freely chosen input. This is a reflection of a general issue: operational interpretations of mathematical freedoms in deterministic theories must confront this constraint. The swarm makes this tension explicit. Just as scalar fields or Einstein's mollusk evolve deterministically once initialized, so too do the clock values.

The only genuine freedom lies in the choice of initial conditions during pre-deployment - and this freedom is restricted by the necessity to ensure that the motion of the observers will cover the region U and that their collectively coordinated programming of clocks will ensure displaying of a smooth chart during deployment; "arbitrary" coordinate assignments mean that these conditions can lead to any smooth coordinate chart via deterministic evolution (see also again the examples in the Appendix B).

V. CONCLUSION

This paper has introduced a general, physically realizable framework for constructing reference frames in general relativity using swarms of local observers, each equipped with four independently programmable clocks. The construction provides a transparent operational implementation of coordinate systems, avoiding reliance on abstract geometric structures introduced *a priori*, while respecting the diffeomorphism freedom of the theory.

Moreover, this approach allows one to represent general curvilinear coordinate systems—including those defined by null surfaces or irregular foliations—without introducing artificial structural assumptions such as global slicings or synchronized emitter networks. It also utilizes the programmability of observers to allow one single congruence (for example radially in-falling observers in a Schwarzschild spacetime) to display any other coordinate system (even, for example, Kruskal–Szekeres).

By foregrounding the material realization of coordinate systems, the framework offers a concrete model for interpreting general covariance as an operational capacity: the freedom to assign arbitrary smooth coordinates is here instantiated as the freedom to program observers' clocks. This provides a new vantage point on the physical interpretation of gauge redundancy and the role of observables.

The same structure also supports pedagogical and conceptual aims. It generalizes Einstein's "reference mollusk" into a covariant setting, extending familiar images such as the special-relativistic spacetime lattice into the regime of curved spacetime. Because coordinate labels arise from programmable clocks, the model offers an accessible yet rigorous way to connect local measurement procedures to the abstract formalism of general relativity—useful both in teaching and in conceptual analysis. This operational construction thus directly realizes—to our knowledge for the first time without conceptual gaps—the textbook analogy between differential geometry's local flat patches and curvilinear coordinates. What is usually visualized through cartographic projections (flat map versus curved Earth) here becomes physically realized by observer swarms: local Minkowski frames are the' 'flat patches" while programmable displays provide arbitrary "curvilinear" labels.

Because the assigned coordinate values are generated entirely by local, freely programmable devices, the framework mirrors—to our knowledge for the first time—in a transparent operational manner the logical architecture of differential geometry: charts, transitions, and manifolds are not presupposed but enacted through physical procedures. Metric and other structures are measured and follow only after the coordinate frame is established as such.

The observer-swarm interpretation may complement existing scalar-field-based constructions of reference frames. Since freely programmable clock values can emulate any scalar field behavior, the framework provides a possibility to conceptually reinterpret every scalar field as algorithmically chosen.

While elementary in formulation, the framework touches on several foundational and interpretive themes in general relativity. It does not aim to resolve longstanding debates but to offer a concrete structure in which the operational content of coordinate freedom, measurement, and reference systems can be examined without undue abstraction. It is hoped that this perspective proves helpful in both research and pedagogy, and in clarifying the role of material systems in the realization of spacetime structure.

Appendix A: Mollusk-like Charts: Scope and Limitations

This appendix examines cases which illustrate that mollusk-adapted coordinate systems are remarkably general but still fail to realize the general covariance permitted by general relativity. The examples highlight that, while mathematically any smooth coordinate chart is admissible, only a subset can be physically realized by mollusk observers who are by definition subject to built-in causal and metric constraints. By contrast, the observer swarm can assign arbitrary programmable numerical values to cover such charts, since the labels are decoupled from the observers' physical motion. This situation may be summarized in a somewhat catchy manner as: The mollusk realizes only a constrained subset of charts; the observer swarm realizes them all.

The Einstein mollusk is a chart realized by a physical congruence of timelike worldlines, each labeled by fixed spatial coordinates and equipped with a clock defining a monotonically increasing time coordinate. This construction imposes additional geometric constraints beyond the mathematical smoothness of arbitrary charts.

In mollusk-adapted (co-moving) coordinates, the chart necessarily takes the form:

$$(x^0, x^1, x^2, x^3) := (\lambda(\tau, x^1, x^2, x^3), x^1, x^2, x^3),$$

where λ is a monotonically increasing function of the proper time τ along each worldline, and observers are located at fixed values of (x^1, x^2, x^3). The coordinate differential dx^0 must be timelike. The spatial directions dx^i span the directions of infinitesimal displacements between neighboring observers and must be spacelike. These causal types are physical constraints imposed by the reference body's construction.

To clarify this distinction, we first state the *necessary conditions* that a coordinate chart $x^\mu = (\lambda, x^1, x^2, x^3)$ must satisfy to be interpreted as mollusk-adapted:

1. **Timelike time direction:** The coordinate vector field associated with λ must be everywhere timelike. In terms of the metric,

$$g_{00} < 0. \tag{A1}$$

2. **Spacelike spatial grid:** The subspace spanned by $\{dx_i\}$, for $i = 1, 2, 3$, must be spacelike. That is, the induced spatial 3-metric g_{ij} must be positive definite:

$$\text{For any nonzero spatial vextor } \xi^i, \quad g_{ij}\xi^i\xi^j > 0. \tag{A2}$$

3. **Causal realizability:** The worldlines defined by holding $x^i = \text{const}$ and increasing λ must form a smooth timelike congruence. This means observers sit at fixed spatial labels and experience λ as a valid arbitrary clock reading.

4. **Lorentzian signature:** The spacetime metric must preserve the Lorentzian signature to ensure a well-defined causal structure.

The following table summarizes the first two key invariant conditions on the coordinate differentials:

Direction	**Invariant requirement**	**Metric condition**
Time ($d\lambda$)	$d\lambda$ timelike	$g_{00} < 0$
Space (dx^i)	span of dx_i spacelike	$g_{ij}\xi^i\xi^j > 0 \; \forall \xi^i \neq 0$

No condition is imposed on the cross terms g_{0i}; they may vanish or not, depending on whether the spatial coordinate grid is orthogonal to the time flow. Also, no condition is imposed on the diagonal spatial components g_{ii}; these need not be positive individually, since the spatial coordinate differentials dx^i are not required to be mutually orthogonal. What matters is that the spatial metric g_{ij} is positive definite, ensuring that the spatial subspace is spacelike. However, conditions 1–4 must all be satisfied for a mollusk interpretation to be valid. Note also that permuting the coordinate labels merely permutes the corresponding metric indices and thus preserves the logical content of the constraints; the underlying geometry remains unchanged. What matters is that one coordinate direction corresponds to the worldlines of the timelike observers, while the remaining three coordinates label fixed spatial positions along those worldlines.

1. Failure to instantiate charts with null directions

First consider the Schwarzschild metric in standard coordinates t, r, θ, ϕ:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{A3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$. This chart and its associated metric satisfy the mollusk conditions outside the horizon ($r > 2M$): the mollusk can be used as a physical reference frame in this situation.

Now consider the Schwarzschild geometry expressed in Kruskal–Szekeres coordinates[42]. These are defined via null coordinates

$$U = -e^{-u/4M}, \quad V = e^{v/4M}, \tag{A4}$$

where $u = t - r^*$ and $v = t + r^*$ are the usual retarded and advanced times, and r^* is the tortoise coordinate,

$$r^* = r + 2M \ln \left| \frac{r}{2M} - 1 \right|. \tag{A5}$$

In these coordinates, the Schwarzschild metric takes the form:

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} dU dV + r^2 d\Omega^2, \tag{A6}$$

where r is implicitly a function of UV via the coordinate transformation.

The key feature of this chart is that both dU and dV correspond to null directions: the metric has $g_{UU} = g_{VV} = 0$, with only a cross-term $g_{UV} \neq 0$. This means that neither U nor V labels a direction that can be associated with a timelike vector field. Instead, both coordinate directions are null, and the others space-like.

This violates a basic requirement of mollusk-adapted charts: that there exist one timelike and three spacelike coordinate directions. A coordinate system based on null directions cannot be instantiated by a mollusk-style congruence of observers carrying proper-time clocks and fixed spatial labels. It is also important to stress that Kruskal–Szekeres coordinates are globally regular and extend across the Schwarzschild horizon, and yet they provide a clear example of a smooth, maximally extended chart for Schwarzschild spacetime that is incompatible with any mollusk realization. This highlights a clear shortcoming of the Einstein mollusk: even in cases in which it can model spacetime geometry in one coordinate chart, it may fail to realize all charts in which the same spacetime geometry is expressed, including charts providing highly regular metric expressions.

2. Canonical ADM charts can be instantiated by a single mollusk

The ADM decomposition[42] expresses the spacetime metric using a time coordinate t, a lapse function N, a shift vector N^i, and a spatial metric h_{ij}. The metric takes the form:

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{A7}$$

with components:

$$g_{tt} = -N^2 + h_{ij}N^iN^j, \tag{A8}$$

$$g_{ti} = h_{ij}N^j, \tag{A9}$$

$$g_{ij} = h_{ij}. \tag{A10}$$

To determine whether an ADM chart can be physically instantiated by a mollusk, we require that the coordinate chart (t, x^i) satisfy the mollusk metric conditions:

1. The coordinate time differential dt must define a timelike direction: $g_{tt} < 0$.
2. The spatial coordinate differentials dx^i must span a spacelike subspace: h_{ij} must be positive definite.
3. The worldlines at fixed x^i and varying t must be timelike curves that form a smooth congruence.

The second and third conditions are automatically satisfied by the structure of ADM: h_{ij} is Riemannian, and the coordinate grid defines smooth families of curves. The crucial constraint is the first: the metric component g_{tt} must be negative. Since

$$g_{tt} = -N^2 + h_{ij}N^iN^j, \tag{A11}$$

it follows that this condition is violated when the shift vector becomes sufficiently large relative to the lapse. In such regions, the coordinate time direction dt becomes null or spacelike. However, then the ADM formulation breaks down: the hypersurfaces Σ_t are no longer valid Cauchy surfaces, the evolution vector leaves the causal cone, and the initial-value problem becomes ill-posed. Such cases are considered pathological and are explicitly avoided in both canonical ADM theory and numerical relativity practice. ADM charts as employed in practice can therefore also be directly instantiated by mollusks since they also fulfill the condition $g_{tt} < 0$.

This example shows that the Einstein mollusk can represent a broad class of spacetimes, since material reference frames are typically constructed only on subsets $U \subset \mathcal{M}$ where local $3+1$ foliations exist. However, it cannot realize arbitrary coordinate charts and their metric forms in U and thus does not capture the mathematical freedom of general covariance.

Appendix B: Example Implementations of Observer Swarms

To concretely illustrate the operational model of reference frames developed in the main text, we will consider a few specific examples of observer swarms. For concreteness we will consider Schwarzschild spacetime, but assuming knowledge of the underlying spacetime metric is a pedagogical device, not a prerequisite: the logical flow in section IV leads from initial physical chart construction in section IV B to subsequent estimation of the metric in section IV C. If the swarm is deployed without knowledge of the metric it will still implement an allowable chart (chart-freedom). Most likely this chart will not be adapted to the actual metric of the spacetime in which the observer swarm is deployed. The measured metric tensor will thus typically not be of a simple form and contain many non-zero off-diagonal values. However, the free programmability of the devices can then be used to change to metric-adapted coordinates, if so desired.

The first example demonstrates in detail how radially in-falling observers can be programmed to assign chart values X^μ in a physically meaningful way. The second example will briefly discuss the idea of satellites as observers in an observer swarm.

Assigning Chart Values by Local Programming

We consider the Schwarzschild solution in ingoing Painlevé–Gullstrand (PG) or rain coordinates (T, r, θ, ϕ), where the line element is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dT^2 + 2\sqrt{\frac{2M}{r}}\, dT\, dr + dr^2 + r^2 d\Omega^2, \tag{B1}$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$.

As mentioned in the main text, any coordinate system may be locally modeled by the observer swarm. We choose Painlevé–Gullstrand coordinates as starting point only because they allow for a particularly simple mathematical chart from which to build a first operational chart, which can then be generalized to any arbitrary operationally defined chart.

In all of our examples, we consider any finite subset U excluding the singularity $r = 0$ and we cover U by a swarm of observers in motion, idealizing each observer as a test particle moving along a timelike curve parameterized by proper time τ, starting at $\tau = 0$. Their position in spacetime is therefore described by a curve $x^\mu(\tau)$, and the four-velocity is

$$u^\mu := \frac{dx^\mu}{d\tau}. \tag{B2}$$

We define the observer's displayed chart values X^μ operationally as the values of the coordinates along its worldline:

$$X^\mu(\tau) := x^\mu(\tau), \tag{B3}$$

and therefore obtain

$$\frac{dX^\mu}{d\tau} = u^\mu. \tag{B4}$$

1. Geodesic Congruences in Schwarzschild Spacetime

In the first example, each observer is a test particle in free fall moving along a timelike geodesic parameterized by proper time τ, starting at $\tau = 0$.

In other words, we choose an observer swarm whose four-velocity components should be:

$$u^\mu = \left(1, -\sqrt{\frac{2M}{r}}, 0, 0\right), \tag{B5}$$

which satisfies the normalization condition $g_{\mu\nu}u^\mu u^\nu = -1$ and the geodesic equation $u^\mu \nabla_\mu u^\nu = 0$. This congruence is irrotational, timelike, and the observers' worldlines are mutually non-intersecting until $r = 0$ and admit a smooth coordinate assignment. The corresponding evolution equations of the coordinate readings of our observers are:

$$\frac{dX^T}{d\tau} = 1, \tag{B6}$$

$$\frac{dX^r}{d\tau} = -\sqrt{\frac{2M}{X^r}}, \tag{B7}$$

$$\frac{dX^\theta}{d\tau} = 0, \quad \frac{dX^\phi}{d\tau} = 0. \tag{B8}$$

These equations determine the chart values to be displayed along each observer's trajectory. The angular coordinates remain fixed and can thus also be displayed directly. The time coordinate evolves as $X^T = \tau$ and can thus be directly displayed from the reading of the atomic clock carried by the observer. Note, however, that the PG time coordinate coincides with proper time only for the specific infalling geodesic congruence, not for arbitrary observers. The radial coordinate satisfies

$$X^r(\tau) = \left(X^r(0)^{3/2} - \frac{3}{2}\sqrt{2M}\,\tau\right)^{2/3}, \tag{B9}$$

which remains valid until the particle reaches the caustic limit $r = 0$.

Both X^T and X^r evolve along each single observer's worldline (Lagrangian viewpoint). The observer swarm may be conceptually imagined as being initialized at $T = 0$ to fill a corresponding spatial region such that the swarm of observers covers U as time progresses and each individual observer falls inward to smaller radii and therefore must display a different radial value $X^r(\tau)$. Importantly, the ensemble continues to span the spatial domain of our considered region U: at any fixed radial position r and at each time T inside U there will be a (different) in-falling observer at these coordinates who is displaying the correct values $X^T = T$ and $X^r = r$ (Eulerian viewpoint). In mathematical terms the Eulerian field description $X^\mu(x)$ is recovered by inverting the motion equations. For instance, given the radial trajectory $X^r(\tau)$ we solve for the proper time at which a freely falling observer starting at r_0 at $\tau = 0$ crosses a given radius:

$$\tau(r) = \frac{2}{3\sqrt{2M}} \left(r_0^{3/2} - r^{3/2} \right).$$

Using this, one can express $X^r(\tau(r)) = r$ and thereby show that the observers' programmed clock readings reproduce the chart $X^\mu(x) = x^\mu$. The operational field $X^\mu(x)$ thus recovers the standard Eulerian coordinates from the ensemble of programmed observers. This distinction between individual motion and ensemble structure reflects the standard picture of a fluid or dust: while individual elements move, the cloud they define remains spatially complete. Thus, the congruence as a whole forms a valid reference frame for each spacetime point outside the singularity, even though each observer will ultimately become affected by the black hole center, which lies outside the considered region U. Ultimately our congruence worldlines meet at the center in finite proper time, marking the caustic limit.

Ensuring coverage of the region $U \subset \mathcal{M}$

As a technical subtlety note that we will have to account for observers leaving and entering the spatial domain represented by U during the observation period. This may be implemented by allowing for an observer swarm in a slightly larger region $V \supset U$ such that the observers in the spatial domain of V at $T = 0$ will ensure that U is densely covered. For a concrete example, let U be a finite spacetime region defined as follows: we consider a fixed spatial domain bounded between Schwarzschild radii r_a and r_b, with $r_a < r_b$, and an observation period from coordinate time $T = 0$ to $T = T_1$, all defined within Painlevé–Gullstrand coordinates. The region $U \subset \mathcal{M}$ is then given by:

$$U := \{(T, r, \theta, \phi) \mid 0 \leq T \leq T_1,\ r_a \leq r \leq r_b\}\,. \tag{B10}$$

To ensure that chart values are assigned by a continuous observer swarm throughout U, we must take into account that freely falling observers may enter or exit the spatial domain $[r_a, r_b]$ during the observation interval. Therefore, we define a larger region $V \supset U$ which must be covered by the swarm at $T = 0$ to ensure complete coverage of U at all times $T \in [0, T_1]$. Let an observer begin free fall from rest at radius r_c at time $T = 0$, following the geodesic equations for Painlevé–Gullstrand coordinates. To determine the maximal initial radius r_c such that the corresponding observer reaches r_b precisely at time $T = T_1$, we start from

$$r_b = \left(r_c^{3/2} - \frac{3}{2}\sqrt{2M}\,T_1\right)^{2/3}\,, \tag{B11}$$

and solve for r_c, obtaining

$$r_c = \left(r_b^{3/2} + \frac{3}{2}\sqrt{2M}\,T_1\right)^{2/3}\,. \tag{B12}$$

Hence, to cover the spacetime region U with infalling observers whose worldlines start at $T = 0$, the spatial domain of the initial swarm must extend from $r = r_a$ to $r = r_c$, where:

$$V := \{(T, r, \theta, \phi) \mid 0 \leq T \leq T_1,\ r_a \leq r \leq r_c\}\,. \tag{B13}$$

Caution. While the regions U and V are geometrically well-defined and coordinate-independent as subsets of spacetime, their temporal boundaries are expressed in Painlevé–Gullstrand time T and therefore do not correspond to constant ranges of Schwarzschild time; only the radial boundaries $r_a \leq r \leq r_b$ are shared across both coordinate systems.

Any chart by transforming $\tilde{X}^\mu(X)$

While we selected a chart corresponding to our Painlevé–Gullstrand coordinates for Schwarzschild spacetime, we could of course also display any other coordinate chart $\tilde{X}^\mu = \tilde{x}^\mu$ simply by implementing the corresponding coordinate transform $\tilde{X}^\mu(X)$. For example, we may choose to display Kruskal–Szekeres coordinates as discussed in Appendix A.

To achieve this, we need to specify the exact coordinate transformation from Painlevé–Gullstrand (PG) coordinates (T, r) to Kruskal–Szekeres (KS) coordinates (U, V). Since both systems share the angular part (θ, ϕ), we restrict ourselves to the $(T, r) \leftrightarrow (U, V)$ sector.

First, we recall the Schwarzschild radial tortoise coordinate r^*, defined by

$$r^* = r + 2M \ln \left| \frac{r}{2M} - 1 \right| . \tag{B14}$$

We then define the Schwarzschild null coordinates $u = t - r^*$, $v = t + r^*$, where t is Schwarzschild coordinate time.

Kruskal–Szekeres coordinates (U, V) are given in terms of these null coordinates by:

$$U = -\exp\left(-\frac{u}{4M}\right), \quad V = \exp\left(\frac{v}{4M}\right). \tag{B15}$$

This yields a transformation from Schwarzschild (t, r) to (U, V).

To express this in terms of Painlevé–Gullstrand coordinates (T, r), we use the relation between Schwarzschild time t and PG time T:

$$t = T - 2\sqrt{2Mr} + 2M \ln \left| \frac{\sqrt{r} + \sqrt{2M}}{\sqrt{r} - \sqrt{2M}} \right| . \tag{B16}$$

Hence, the transformation from PG coordinates (T, r) to Kruskal–Szekeres coordinates (U, V) becomes:

$$r^* = r + 2M \ln \left| \frac{r}{2M} - 1 \right| , \tag{B17}$$

$$t = T - 2\sqrt{2Mr} + 2M \ln \left| \frac{\sqrt{r} + \sqrt{2M}}{\sqrt{r} - \sqrt{2M}} \right| , \tag{B18}$$

$$u = t - r^*, \quad v = t + r^*, \tag{B19}$$

$$U = -\exp\left(-\frac{u}{4M}\right), \quad V = \exp\left(\frac{v}{4M}\right). \tag{B20}$$

These expressions determine (U, V) as explicit functions of (T, r), and hence define the coordinate transformation $\tilde{X}^\mu(X)$. Each observer in the swarm can now be programmed to display $\tilde{X}^\mu(X^\nu(\tau))$, where $X^\nu(\tau) = x^\nu(\tau)$ follows from its geodesic motion. The field $\tilde{X}^\mu(x)$ is then operationally realized as before.

This demonstrates how the very same observer swarm can be used to instantiate arbitrary coordinate charts, even those like the Kruskal–Szekeres system that globally extend the spacetime manifold across the horizon. The operational framework is thus completely general with respect to the choice of coordinates: only the transformation function $\tilde{X}^\mu(X)$ needs to be specified.

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Endnotes:

[60] We adopt the simplifying assumption that observer worldlines follow geodesic motion. This is not a conceptual limitation: the coordinate construction relies on freely programmable clocks and thus allows to display any values, provided the observers are able to cover the probed region. A fundamental constraint arises instead from the distinction between mathematical congruences, which assume smooth, gapless coverage, and any physically realizable swarm with inherent resolution limits. This limitation is shared with all other dust-like models that conceptually assume dense coverage but can practically only achieve an approximation.

[61] Dense constructions based on light-signal exchanges between observers typically presuppose that each observer is already equipped with an individual clock—potentially nonlinear but locally monotonic in proper time. Such signal exchanges are inherently local and should not be conflated with global synchronization procedures, as used for inertial frames in special relativity. Geroch[24] presents a locally consistent operational method in which idealized clocks, combined with light-signal reflections, define simultaneity hypersurfaces relative to an observer's worldline. This allows for the local measurement of spacetime intervals, including proper lengths defined along spacelike geodesics within those hypersurfaces. Conceptually similar are constructions of physical light-clocks—devices where photons bounce between mirrors whose separation is assumed to be locally fixed and known. While light-clocks are sometimes invoked to ground time measurement in the causal structure itself, their logical consistency in curved spacetime is nontrivial since it depends on presupposed spatial distance definitions. In all these constructions each worldline is ultimately associated with a single clock, and distinct worldlines require additional (fixed) labels for disambiguation. This naturally results in mollusk-like reference frames, where the spatial labeling is fixed and not freely programmable as in the observer swarm framework developed here.

[62] For pedagogical clarity, we also adopt a differential-geometric language in which expressions like $\mathrm{d}x$ denote coordinate displacements along the direction x, rather than employing the more abstract identification of partial derivatives $\partial/\partial x$ with tangent vectors in $T_p\mathcal{M}$, the tangent space to the Lorentzian manifold $\mathcal{M}$ at a point p. In standard differential geometry, each coordinate function x in a local chart induces two associated structures at each point $p \in \mathcal{M}$: the coordinate displacement $\mathrm{d}x$, which is a differential one-form belonging to the cotangent space $T_p^*\mathcal{M}$, and the partial derivative operator $\partial/\partial x$, which is a basis vector in the tangent space $T_p\mathcal{M}$. The formal duality between these arises because the tangent space $T_p\mathcal{M}$ consists of directional derivative operators acting on smooth functions, while the cotangent space $T_p^*\mathcal{M}$ consists of linear functionals acting on tangent vectors. In this paper, we intentionally adopt the older physicists' convention in which $\mathrm{d}x$ is read as a differential or an infinitesimal coordinate displacement, rather than invoking the more abstract identification of $\partial/\partial x$ as a tangent vector at p. This choice avoids requiring the reader to work within the modern coordinate-free formulation of differential geometry. It is therefore intended to make the presentation accessible to a wider audience, including readers with only rudimentary familiarity with general relativ-

ity and without formal training in modern differential geometry. It also aligns with the logical structure of the main text, where elementary physical operationalizations naturally lead to the mathematical apparatus of differential geometry.

63 Various visualization metaphors may help to convey the structure of such a 3D region. For example, one may imagine it as built from stacked, warped paper-like layers (suggesting a topology corresponding to $\mathbb{R} \times \mathbb{R}^2$), or as composed of warped, nested surfaces (each homeomorphic to a topological two-sphere, suggesting $\mathbb{R} \times S^2$). A third, perhaps even more flexible, metaphor would be a deformable blob of dough, without any preferred foliation at all. These visualizations are not topologically restrictive; each describes the same underlying mathematical object: a three-dimensional region with standard topology, continuously populated by local observers and their clocks. In the nested-surfaces analogy, the surface shown in Fig. 2 would be read as representing a two-dimensional, warped spatial slice. Multiple such surfaces would be imagined to be stacked spatially to fill a three-dimensional region (possibly extending to infinity in some or all of the three-spatial directions), like the layers of an onion or the nested shells of a matryoshka doll.

64 This formulation should not be misunderstood as implying that the mollusk reference frame is situated in any other higher-dimensional frame. The most general (co-moving) reference frame of mollusk-type is instead given by eq. (5). The implications and limitations inherent due to this formulation are overcome by the subsequently proposed generalization: a congruence of observers in which each observer carries four clock values, replacing fixed spatial coordinates with four coordinate values that can vary along the timeline of each observer.

65 To remain consistent with the geometric and dynamical constraints of general relativity, each mollusk must be of sufficiently low density to avoid backreactions on spacetime geometry (and keep also gravitational radiation negligible), and restricted to a region U small enough to neglect its own self-gravitation and to exclude the formation of collapsing matter. This may involve non-trivial trade-offs and constraints on the scaling of the observer number density: the mollusk must be sufficiently dense to support smooth interpolation of its coordinate functions but sufficiently dilute to avoid generating significant gravitational backreaction. See also the corresponding discussion below.[66]

66 This generalization permits the physical realization of any smooth local chart, including those with null coordinate differentials or non-spacelike level surfaces. Regarding the viability of material reference systems that produce null coordinates: this relies entirely on the free programma-

bility of clocks and the assumed pre-deployment of a swarm across the entire measurement region. In particular, material observers need neither signal nor travel along the null directions themselves; rather, they are simply deployed in such a manner that their collectively displayed coordinate values instantiate null coordinates. Indeed, the only physical requirement is sufficient swarm coverage of the region $U \subset \mathcal{M}$. This physical requirement introduces exactly the same physical viability constraints as for any other cloud/swarm/Mollusk/dust-based model in the literature, notably avoidance of caustics (restricting the regions U), idealized zero back-reaction on the geometry (affecting i.a. achievable density, maximum mass and motion-types of observers). However, once the physical coverage of U is guaranteed, the programming achieves smoothness and local invertibility, and the displayed labels define a valid coordinate chart—without the need for synchronization conventions, background symmetries, or orthonormal frames. This approach avoids the structural assumptions built into other approaches, for example, ADM-type formulations or tetrad-based methods and enables the construction of any chart in any spacetime geometry.

[67] If the freely programmable clock displays are made to depend on local measurements—either instead of, or in addition to, an observer's internal (e.g., atomic) clock—then the observer swarm may be realized as a hybrid construction. In such cases, the displayed coordinate values can reflect quantities used in other approaches, such as physical fields (e.g., scalar or tensor fields) permeating spacetime, or signals locally received from external sources, including other observers. The observer swarm may then help to mitigate some of the potential limitations of these alternative approaches, by providing additional freedom through the independently programmable clocks (and the potential availability of a local proper time clock).

[68] Flatness corresponds to the vanishing of the Riemann curvature tensor[49]. In such cases, the metric can always be brought locally — and globally if the manifold is topologically trivial — into the canonical Minkowski form η via a coordinate transformation. Thus, any coordinate system defined in a flat spacetime, even if curved, describes special relativity when properly interpreted. For pedagogical reasons, we assume rectilinear coordinates and the canonical form η for SR in freely falling local inertial frames, cf. also Fig. 4, such that the correspondence between coordinate differences and physical intervals becomes most transparent.