Tatiana Ehrenfest-Afanassjewa's Contributions to Dimensional Analysis

Mahmoud Jalloh *†

St. John's College, Santa Fe

July 25, 2025

Abstract

Tatiana Ehrenfest-Afanassjewa was an important physicist, mathematician, and educator in 20^{th} century Europe. While some of her work has recently undergone reevaluation, little has been said regarding her groundbreaking work on dimensional analysis. This, in part, reflects an unfortunate dismissal of her interventions in such foundational debates by her contemporaries. In spite of this, her work on the generalized theory of homogeneous equations provides a mathematically sound foundation for dimensional analysis and has found some appreciation and development. It remains to provide a historical account of Ehrenfest-Afanassjewa's use of the theory of homogeneous functions to ground (and limit) dimensional analysis. We take as a central focus Ehrenfest-Afanassjewa's contributions to a debate on the foundations of dimensional analysis started by physicist Richard Tolman in 1914. I go on to suggest an interpretation of the more thoroughgoing intervention Ehrenfest-Afanassjewa makes in 1926 based on this earlier context, especially her limited rehabilitation of a "theory of similitude" in contradistinction to dimensional analysis. It is shown that Ehrenfest-Afanassjewa has made foundational contributions to the mathematical foundations and methodology of dimensional analysis, our conception of the relation between constants and laws, and our understanding of the quantitative nature of physics, which remain of value.

Keywords: dimensional analysis, constants, laws of nature, women in science, history of physics, history of mathematics

*Thanks to Porter Williams, Jeff Russell, Alexis Wellwood, and Katherine Brading for comments on earlier iterations of this material and encouragement in completing the work. Thanks also to the St. Andrews-Bonn Gravitational Constant Network for an opportunity to present some of this material and for feedback.

†mahmoudtimbojalloh@gmail.com

Contents

1 Introduction **2**

2 Early Contributions **4**

3 The Tolman Dispute **15**

4 After *Dimensional Analysis*: A Tripartite Dispute **22**

5 Conclusion **31**

References **33**

1 Introduction

Tatiana Ehrenfest-Afanassjewa (1876-1964) was an important physicist, mathematician, and educator in 20th century Europe.[1] Born in Kiev, Ehrenfest-Afanassjewa is well known as the wife of physicist Paul Ehrenfest, whom she met while studying in Göttingen, in its mathematical golden age. Famously, the couple wrote an influential encyclopedia article on statistical mechanics, at the instigation of Felix Klein.[2] Ehrenfest-Afanassjewa would later write a book on the foundations of thermodynamics (Ehrenfest-Afanassjewa 1956). Ehrenfest-Afanassjewa was also greatly involved with the didactics of mathematics, both in Russia and in the Netherlands. While her work in physics and mathematics—and especially in statistical mechanics and thermodynamics—have recently undergone reevaluation, little has been said regarding her groundbreaking work on dimensional analysis. This, in part, reflects an unfortunate dismissal of her intervention into debates in the 1910s and 1920s as "merely mathematical" or "irrelevant" by her contemporaries and some later commentators (Campbell 1926; Bridgman 1926; c.f. Bridgman 1931; Walter 1990). In spite of this, her work on the generalized theory of homogeneous equations—which provides a mathematically sound foundation for dimensional analysis, so central to physics and engineering—has found some appreciation and development (e.g. San Juan 1947; Palacios 1964; Johnson 2018). It remains to provide an account of Ehrenfest-Afanassjewa's initial use of the generalized theory of homogeneous functions to ground dimensional analysis in the context of disputes regarding the foundations of dimensional analysis. This paper introduces the debate, started by Richard Tolman, of whether Fourier's principle of dimensional homogeneity provides sufficient grounds for dimensional analysis. Tolman argued that a principle of similitude, inspired by relativity, provides a better basis.[3] Ehrenfest-Afanassjewa's refutation of Tolman's principle, especially

[1]For book length considerations of her life and work, see Uffink et al. (2021) and van der Heijden (2021a). For shorter biographical sketches, see Ogilvie and Harvey (2000) and van der Heijden (2024).

[2]Ehrenfest and Ehrenfest (1911). Translated into English as Ehrenfest and Ehrenfest (1959). It is also worth noting that Ehrenfest-Afanassjewa was one of a number of female mathematicians that Klein supported and mentored, see Tobies (2020).

[3]Tolman (1914). See Jalloh (2024b) and Walter (1990) for fuller accounts of the debate.

with regards to its consequences for the gravitational constant, will set the stage for an interpretation of the more thoroughgoing intervention Ehrenfest-Afanassjewa makes in 1926, especially her rehabilitation of a distinct "theory of similitude".

An outline of what is to follow: First, I will describe what I know regarding Ehrenfest-Afanassjewa's contributions relevant to the foundations of dimensional analysis prior to the Tolman dispute. Second, I will introduce the Tolman dispute surrounding his proposed principle of similitude, primarily taking place in *Physical Review*. While I will not track the full debate, some special attention will be paid to Buckingham's response, providing context and comparison for Ehrenfest-Afanassjewa's own response to Tolman. Notably, I will not directly discuss Bridgman's classic *Dimensional Analysis* (which coined the term "dimensional analysis") in any detail.[4] Third, I will pick up the conversation with the interventions of Campbell and Ehrenfest-Afanassjewa in the *Philosophical Magazine*, which broke the silence following what Bridgman intended to be the final word on the matter. The value of Ehrenfest-Afanassjewa's attempt to provide mathematical foundations for a theory of similitude (as well as her distinguishing her approach from dimensional analysis) were questioned by Campbell and Bridgman in turn. The continued influence of Ehrenfest-Afanassjewa's later intervention will be discussed in the conclusion.

I should add as a preliminary caveat: Ehrenfest-Afanassjewa was a trenchant critic of dimensional analysis and her primary contribution is to provide its mathematical foundations so as to demonstrate its limits. As will emerge, she more and more strongly argued that dimensional analysis was to be distinguished from the theory of similitude (i.e. "model theory" or "Ähnlichkeitstheorie"). I will argue (in §4) that her distinction between the two methods is erased by a more liberal conception of dimensional analysis, that is not necessarily committed to the traditional dimensional system of physics. Such flexibility had already appeared in early works known to her by Bridgman and Buckingham, but her recognition of them appears to be incomplete. That said, relinquishing this overly narrow conception of dimensional

[4]Bridgman (1931). While I will not be discussing Bridgman's book in detail (originally published in 1922), we will get some sense of his essential point of view. A similar (partial) omission is Campbell (1920).

analysis allows Ehrenfest-Afanassjewa's works to be seen even more clearly as the fundamental contributions that they are.[5]

2 Early Contributions

The earliest known contribution of Ehrenfest-Afanassjewa to what would come to be called dimensional analysis, comes in an August 1905 paper published in the *Mathematisch-Naturwissenschafftliche Blätter*, "Ueber die Willkürlichkeit bei der Dimensionerung phiskalischer Größen" ("On the Arbitrariness of the Dimensions of Physical Quantities").[6] There apparently exist some other early Russian articles, from around 1911, which, from their titles, seem relevant to dimensional analysis.[7] However, these articles are hard to find and this article will be comprehensive enough, though not complete, on their exclusion (to my knowledge they are not substantively cited in any of the proceeding literature).

In his introduction to the republication of the 1905 paper, Oliver Schauldt describes a central thread of the paper as following:

> The nature of these [dimensional] parameters is already central to the article reproduced here. In the much more strictly formalized approach of 1916, the author will characterize them as *formal variables.* There, she will also introduce the concept of dimension in group-theoretical terms[...] But already in the article reproduced here, she rigorously discusses the meaning and dependence of constants of nature on the choice of basic units. Her most important tool is the *thought experiment*, which allows her to check the consequences of alternative paths of development. The goal is the one that has always been a basic feature of

[5]This is not to say that there is not a distinction between dimensional analysis and the theory of similtude to be drawn, usage varies. For example, Sedov (1993), a very canonical, influential, and important book, seperates the methods in the title and throughout *linguistically*, but they are not clearly distinguished conceptually. I prefer "dimenisonal analysis" as a title for the broad set of connected methods and issues.

[6]Ehrenfest-Afanassjewa (1905). Republished in Schlaudt (2009).

[7]Citations in van der Heijden (2021b) lead one to believe that the papers are from 1911 and 1912. Contrarily Ehrenfest-Afanassjewa (1926), 258, cites two 1911 papers.

> measurement theory since Reid, namely to separate *conventional* and *empirical* moments in physical theories. Of course, such thought experiments must be approached with caution, and the rigorous basic attitude that dissolves much into convention which appeared to be justified in fact has provoked criticism from Julius Wallot, for example. In this context, he raised the question of what the variables in physical equations actually stand for – quantities or measured values? These are the kinds of questions that need to be critically posed to Ehrenfest-Afanassjewa's thought experiments. But these are also fundamental questions of the philosophy of science, and it must therefore be surprising that such a fruitful debate about the theory of dimensions has received so little attention.[8]

It is also worth noting that Schlaudt, like others, contextualizes the early 20$^{\text{th}}$ century concerns with constants, units, and conventionality—most evident in the debate concerning the foundations of dimensional analysis—as a result of Einstein's radical rethinking of the fundamental quantities of physics in the special theory of relativity.[9]

In her short 1905 paper, Ehrenfest-Afanassjewa begins by addressing four foundational questions she believes will be useful in making the "limits and arbitrary character" (die Grenzen und der Charakter de Willkür) of dimensional reasoning more explicit.[10] These can be paraphrased so:

1. What determines number of basic dimensions, nature or historical and practical matters?
2. Where do the universal constants get their dimensions from?
3. What determines that a quantity (e.g. angle) is dimensionless?
4. Given a quantity, is its dimension determined by nature or arbitrary?

We will deal with her simpler answers to 2 and 3 first. The constants are given dimensions

[8]Schlaudt (2009), 233, his emphasis. Unless otherwise noted, translations from German are mine. Apologies for all mistakes.

[9]Compare Walter (1990) and Jalloh (2024b). A somewhat different, though not contradictory, contextualization can be found in Sterrett (2020).

[10]Schlaudt (2009), 234.

by the laws in which they figure and the constraints formed by the already given dimensions of the other quantities involved. For example, by Newton's second law, force is already given the dimensions $[F] = \mathrm{MLT}^{-2}$. Given that the gravitational force is to be proportional to the product of two masses and inversely proportional to the square of their distance, i.e., $F_G \propto \frac{m_1 m_2}{r^2}$, then the proportionality constant, G, can only have one set of dimensions: $[G] = [\mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2}]$. Following this general principle, dimensionless quantities come about in two ways: either as a special case of the general definition of dimension of a new quantity, where the proportionality factor is a proportionality of two quantities of the same dimension (e.g. the radian definition of angle as a proportion of an arc length to the radius of the corresponding circle), or, in the case in which more than one quantity is introduced at once, the two quantities may have dimensions divided between them in any way (including having some of the quantities be dimensionless) as long as the overall coherence (i.e. homogeneity, see below) of the equation is preserved.

Regarding 1 and 4, it may be best to discuss her systematic reflections which follow and then see how it is that answers fall out of them. Ehrenfest-Afanassjewa's systematic remarks begin by making two claims: (1) each physical (or geometrical) variable is a quantity of its own and can only be measured by units of the same kind, though the choice of unit is otherwise unconstrained; (2) equations are between *numbers*, either *variables* that are the ratio of quantities to units or *parameters* which are constant for all the cases covered by the equations in which they figure.[11] From these foundations comes a problem: both variables and constants will vary with choices of units.[12] There are two solutions to this problem that Ehrenfest-Afanassjewa mentions. The first is merely to fix the choice of units, i.e. never vary from C.G.S. or some other suitably complete set of units. The second is to fix the proportions of all the terms in an equation, regardless of units. This is done by having the variables and the constant parameters vary with the variation of some set of base variables

[11]This latter claim puts Ehrenfest-Afanassjewa on one-side of a debate concerning the interpretation of equations (which is criticized by Wallot, mentioned already). On the debate, see de Courtenay (2015).

[12]For a contemporary discussion of the issue of the unit-invariance of equations, see Grozier (2020).

by set proportions—this is done by giving them (derivative) dimensions. How we give them dimensions is in accordance with the answers given to questions (2) and (3), by a set of *defining* equations. As these equations are taken to *define* the dimensions of some quantities with respect to *some* chosen set of units, these dimensions are determined conventionally (answering 4). Since the basic units are chosen on the basis of practicality and history, they are similarly arbitrary (answering 1). Ehrenfest-Afanassjewa goes on to show that the sort of evident dimensional arbitrariness that appears in electricity and magnetism—the distinct dimensional systems that correspond to the electrostatic and electromagnetic unit systems—can appear even in geometry, with the conclusion:

> After all these considerations, one must admit that the dimensioning of quantities is purely a matter of convention and that one can dimension any quantity as desired, because in every equation that connects quantities of different types, there is always a proportionality factor present, through whose suitable dimensioning one can always ensure the homogeneity of the equation.[13]

The issue of the "homogeneity", or unit-invariance, of an equation will become more central as we go on.

One more issue remains to be discussed, which will figure majorly into Ehrenfest-Afanassjewa's later work in dimensional analysis. She found the idea of changing the values of the constant parameters troubling, almost a contradiction.[14] The end of the paper is an attempt to specify the conditions under which the constant parameters are invariant under unit transformations (which also preserve homogeneity). For a closed set of n' variables and n equations (i.e. setting aside the introduction or reduction of equations or the discovery that some constant is in fact a variable, etc.), Ehrenfest-Afnassjewa sets down two conditions:

1. The number of basic variables (i.e. basic dimensions or fundamental units) is equal to the difference between the number of variables and the number of equations, $p = n' - n$.

[13] Schlaudt (2009), 238.
[14] Schlaudt (2009), 236

2. There must not be an subset of n_1 equations such that $p_1 > n'_1 - n_1$.

She does not say much more about this, though similar ideas will recur in her later work. She shows, as an example, that the basic equations of Newtonian mechanics violate this stricture, with a redundancy of two ($p = 2$). Accordingly we can expect the parameters of Newtonian mechanics (e.g. the gravitational constant) to change under some changes of units.

Though it postdates Tolman's opening salvo in 1914, Ehrenfest-Afanssjewa's "Der Dimensionsbegriff und der analytische Bau physikalischer Gleichungen" (drafted in March of 1915, published in June of 1916) predates her entry into the Tolman dispute and makes no reference to Tolman or Buckingham, and so it will be discussed here.[15] Compared to the 1905 paper, the 1916 paper is much more formalized and thoroughgoing, though it draws on the 1905 paper at several points. I will here describe only its essential features.

Ehrenfest-Afanassjewa presents her purpose in this work as "extracting and systematizing" the "pure mathematical content" of the informal discussion of dimensions and the homogeneity of equations. This is done with an eye towards clarifying and correcting some mistakes regarding some of the analytical results of dimensional reasoning. She begins by setting the terminology in order.

Similarly to the 1905 paper she distinguishes quantities, units, and quantity numbers, which are given by the ratio of a quantity to a unit (or is the result of *measuring* a quantity by a unit).[16] A quantity number can accordingly change either due to a change in the magnitude (Spezialwerte) of the quantity or in the unit measuring the quantity. The former are *material changes* and the latter are *formal changes*. Besides these quantity numbers, usually represented by variables, physical equations include numbers which are constant under material changes and can be changed to preserve the validity of an equation under an arbitrary change of units—these are formal variables. The ratio between two values of a given quantity number, $\frac{x_1}{x_2} = \xi$, she calls a transition (or equivalently in later work, transformation) factor,

[15] Ehrenfest-Afanassjewa (1916a).

[16] This latter termiology of "measuring" goes all the way back to book 5 of Euclid's *Elements*.

distinguishing material and formal factors by whether the transition factor is determined by a material or formal change.[17] We can coordinate the formal transformations of different kinds of quantities by setting up sets of conditioning equations (or more narrowly, dimensional equations, see below):[18]

$$\begin{aligned}\xi_{k+1} &= \xi_1^{\alpha_{11}} \xi_2^{\alpha_{12}} \dots \xi_k^{\alpha_{1k}},\\ &\vdots\\ \xi_{k+r} &= \xi_1^{\alpha_{r1}} \xi_2^{\alpha_{r2}} \dots \xi_k^{\alpha_{rk}}.\end{aligned} \tag{1}$$

ξ_1 to ξ_k are the transition factors of the basic variables x_1 to x_k (Grundvariablen) and ξ_{k+1} to ξ_{k+r} are the transition factors of the derived variables x_{k+1} to x_{k+r}. The exponents α^{ih} give the dimension of the corresponding derived variable x_i with respect to (or "in") basic variable x_h. In a standard mechanical dimensional system, mass, length, and time are the basic variables, while velocity, force, density, and so on are derived variables.

From this definition of dimensional equations, Ehrenfest-Afanassjewa was able to make an important tripartite distinction between unit choice, unit system, and dimensional system. A unit choice is a complete, simultaneous choice of units for all relevant quantity types. A unit system is a choice of units and rules to convert to other units (i.e. a choice of dimensional system). A dimensional system is the totality of dimensional equations, which determine the rules for unit conversion, but does not determine a choice of units. Ehrenfest-Afanassjewa's distinction between a unit system and a dimensional system is important and useful but unfortunately did not suffuse the literature as it ought to have (likely due to Maxwell's

[17] Ehrenfest-Afanassjewa uses the same word, Spezialwerte, for the value of a quantity number and the magnitude of a quantity. As these two uses are clearly different, I have rendered them as "value" and "magnitude", in line with current practice. Magnitude is the unit independent "size" of the quantity, while value is the size relative to a unit.

[18] This schematic set of dimensional equations are labeled both (1) and (1*) in Ehrenfest-Afanassjewa's paper. Further theorems of hers refer to the closer restatement of these equations, (1*). I will modify her later statements to refer to (1) for simplicity. (1) and (1*) are identical in her paper, so I assume the asterisk is just to indicate the second printing of it.

influential usage of the term "fundamental unit" which confused the issue).[19]

I will here state and remark upon Ehrenfest-Afanassjewa's eight theorems (Satzen) before discussing some of the more general contributions of the paper.

> (Theorem I)[20] Dimensional equations can be grounded in the requirement that the substitutions that correspond to unit changes form a group.[21]

This theorem is notable for two reasons: (1) It establishes a foundation for dimensions and dimensional analysis that is *not* dimensional homogeneity but more fundamental than it. (2) It is the earliest (that I can recall) group-theoretic account of the theory of dimensions.[22]

> (Theorem II)[23] In the case where all transition factors $\xi_1, \ldots, \xi_n$ are independent of each other,
>
> $$H = kP,$$
>
> where k is independent of the variables $x_1, \ldots, x_n$ and P is a power product $P = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$.

To understand this and following theorems it is essential to know that H is a homogeneous function defined by Ehrenfest-Afanassjewa's equation (3):

$$H(x_1\xi_1, \ldots x_n\xi_n) = \varphi(\xi_1, \ldots, \xi_n)H(x_1, \ldots, x_n) \tag{3}$$

which can simplify to have the form $H = 0$. More directly, this means that this function is invariant under arbitrary changes in the basic variables by transition factors.

[19] A recent exception to this is Jalloh (2024b), who explicitly makes this distinction, with reference to Ehrenfest-Afanassjewa. For more on Maxwellian dimensional analysis, see Mitchell (2017).

[20] Ehrenfest-Afanassjewa (1916a), 262.

[21] As synedoche for showing that they form a group, she shows that units changes are closed under multiplication.

[22] Other early group-theoretic accounts include San Juan (1947) (explicitly influenced by Ehrenfest-Afanassjewa) and Whitney (1968).

[23] Ehrenfest-Afanassjewa (1916a), 263.

> (Theorem III)[24] In the case where relationships exist between the transition factors $\xi_1, \ldots, \xi_n$, it is necessary, in order for equation (3) [i.e. homogeneity] to be satisfied, that these relationships can be reduced to the form [of equations (1)].

This theorem has come to be known as Bridgman's theorem or lemma, though evidently Ehrenfest-Afanassjewa (among others) came to it earlier.[25] The result requires that for homogeneity to be preserved when derivative transformation factors (derivative dimensions) are introduced, they can only take the form of products of powers (or "power products") of some basic transition factors (basic dimensions).

> (Theorem IV)[26] If between the transition factors equations of form (1) exist and equation (3) is satisfied [i.e. homogeneity], then the function $H(x_1, \ldots, x_n)$ has the form:
>
> $$H(x_1, \ldots, x_n) = kP(x_1, \ldots, x_n)\Phi(\frac{x_{k+1}}{x_1^{\alpha_{11}}, \ldots x_k^{\alpha_{1k}}}, \ldots, \frac{x_{k+r}}{x_1^{\alpha_{r1}}, \ldots, x_k^{\alpha_{rk}}}) \tag{4}$$
>
> where k is a coefficient independent of x, $P(x_1, \ldots, x_n)$ is a power product, and Φ is an arbitrary function of its arguments.

This—in combination with her earlier statement of what a homogeneous function, H, is and theorem III—amounts to a statement of what, following Buckingham (see discussion in §3), has been called the Π-theorem, a fundamental theorem of dimensional analysis.[27] From this, Ehrenfest-Afanassjewa draws the conclusion that the φ function in equation (3) must have the same form as the P function in equation (4)—but with ξs as arguments rather than xs.

[24] Ehrenfest-Afanassjewa (1916a), 263.

[25] For its attribution to Bridgman see Berberan-Santos and Pogliani (1999) and Jalloh (2025). This condition appears as early as Vaschy (1892). Ehrenfest-Afanassjewa attributes the result to Federman (1911). See my remark on theorem IV and note 27 below for more information on priority issues.

[26] Ehrenfest-Afanassjewa (1916a), 263.

[27] Both theorems III and IV cite Federman (1911). For at least a couple of reasons I cannot deal with Federman here, but see Görtler (1975) for a defense of his fundamental role in the develpoment of the Π-theorem and other important historical notes aside (many not mentioned elsewhere in the anglo-literature). See also Pobedrya and Georgievskii (2006) and Sterrett (2017).

(Theorem V)[28] An equation of the form

$$f(x_1, \ldots, x_n) = 0$$

is invariant under the substitution

$$x_1' = \xi_1 x_1,$$

$$x_2' = \xi_2 x_2,$$

in which the ξ_i have relationships, such that it can always be reduced to a conditionally homogeneous function.

For Ehrenfest-Afanassjewa, a conditionally homogeneous function is a function that is homogeneous (i.e. invariant) once the dimensional equations (1), or "conditioning equations", are taken into account.[29] Ehrenfest-Afanassjewa further describes a corollary to this theorem which states that that if this substitution holds for any equation, the transition factors can only have a relation of the form given by the dimensional equations (1). This theorem and corollary show that the invariance of equations (of a certain form) under arbitrary transformations of quantity numbers and that the transformation factors form a dimensional system imply each other. Ehrenfest-Afanassjewa remarks that theorems III and V together produce the same dimensional relation between the formal factors as does theorem I, but the establishment of the dimensional system by the later theorems is more general because the dimensional exponents α_{ih} are left free.

(Theorem VI)[30] If an equation or a system of equations is conditionally homogeneous, let

[28]Ehrenfest-Afanassjewa (1916a), 264.

[29]This point is made to show that a mathematical notion of "homogeneity" (used largely in a different context) is a special case of this conditional homogeneity.

[30]Ehrenfest-Afanassjewa (1916a), 265.

$$x_r = \frac{d^n x_i}{dx_h^n},$$

> and if ξ_i, ξ_h, ξ_r are the transition factors of x_i, x_h, x_r respectively, then among the conditioning equations the following must necessarily be included:

$$\xi_r = \frac{\xi_i}{\xi_h^n}.$$

This theorem can be easily understood by an example: acceleration is the second derivative of length with respect to time. Therefore the dimensional equation (or "conditioning equation") for acceleration must be:

$$\xi_a = \frac{\xi_l}{\xi_t^2}.$$

> (Theorem VII)[31] If an equation is a consequence of a specific system of equations and all its constants are completely defined by this system of equations, then it is conditionally homogeneous with respect to the same conditioning equations as the given system of equations.

Ehrenfest-Afanassjewa goes on to distinguish two kinds of homogeneity depending on the sources of the conditioning equations. If the conditioning equations are determined by material changes, they are "model equations" and constrain *materially* homogeneous functions or equations. If the conditioning equations are determined by formal changes, they are "dimensional equations" and constrain *formally* homogeneous functions or equations. Generally the dimensional equations may outrun the model equations, reducing the determinancy of their derivations. Constraining the dimensional equations to the modeling equations may aid in making dimensional analysis more accurate and useful, but there are possible hang-ups preventing such a "specialization" of the dimensional equations—these issues are central to Ehrenfest-Afanassjewa's later criticism of Tolman.

> (Theorem VIII) Every coefficient k_i, which accompanies any power product $P_i(x)$ as a factor, can be represented as a power product of values [Spezialwerten] of the quantity numbers entering into the equation.

[31] Ehrenfest-Afanassjewa (1916a), 267.

The formal variables, k_i, through being defined out of the other quantity numbers in the equation, can be given dimensions such that any equation can be made formally (or dimensionally) homogeneous for any given dimensional system. These are dimensional constants.

Now the mathematical foundations are set. In the rest of the paper, Ehrenfest-Afanassjewa moves to address the actual traditional dimensional system of physics. Her criticism targets a particular conviction:

> There is a general conviction that all equations in physics cannot be anything but homogeneous and that the dimension of a quantity is closely related to its essence and its functional dependence on other quantities.[32]

I believe her criticism of this conviction is best understood as a dilemma: one can either force every physical equation to be homogeneous through the introduction of formal variables (and give up that the dimension of a quantity has anything to do with its "nature") or one can require that all quantities have some material grounding (and give up the necessity of homogeneity). One of the example she uses to demonstrate this dilemma draws on her 1905 paper: Consider how the dimensions of the gravitational constant, $[G] = [\mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2}]$, were determined by the constraints of the formal or dimensional homogeneity of the gravitational law $F = G\dfrac{m_1 m_2}{r^2}$ and the predetermined dimensions of force, length, and mass. If these constrain both the dimension and the *magnitude* of the gravitational constant, then it is a formal variable, in line with theorem VIII. Oddly she express a further conclusion that the gravitational law is not homogeneous, and more generally, following other examples, that not all physical equations, in the traditional dimensional system, are homogeneous.[33] This is extended in her ninth conclusion as: "No system of units is possible in which all equations of physics would be free from formal variables." (Ehrenfest-Afanassjewa 1916a, 276.). This may seem to contradict how Ehrenfest-Afanassjewa set up the introduction of formal variables, as *saving* homogeneity. One clue is that in discussion of this very issue, in her third conclusion,

[32] Ehrenfest-Afanassjewa (1916a), 271.
[33] Ehrenfest-Afanassjewa (1916a), 272.

she states:

> If one adjoins these formal variables to the system of quantity numbers, then the requirement of formal homogeneity can no longer impose any restrictions on the form of physical equations.[34]

If one attempts to preserve the "physical meaning" of the constants (as she'll later put it in her objection to Tolman), one cannot accept them as formal variables. The first horn of the dilemma is assumed to be untenable—so then the dimensional analyst cannot accept dimensional constants, otherwise their "derivations" are reduced to triviality. So the dimensional analyst is forced into accepting the necessary existence of inhomogeneous equations. This issue is dealt with more in the next two sections.

3 The Tolman Dispute

Motivated by the shocks to the foundations of physics initiated by relativity, Richard C. Tolman sought to ground the foundations of measurement on a relativity principle of his own: *relativity of size*. The simple idea that an observer cannot know the absolute scale of their universe is behind Tolman's principle of similitude.

In his seminal 1914 article "The Principle of Similitude", Tolman states the titular principle so:

> *The fundamental entities out of which the physical universe is constructed are of such a nature that from them a miniature universe could be constructed exactly similar in every respect to the present universe.*[35]

Tolman proceeds by way of a thought experiment: consider two observers O and O' whose measuring devices and respective unit systems stand in such a way that O' assigns the same

[34] Ehrenfest-Afanassjewa (1916a), 276.
[35] Tolman (1914) 244, his emphasis.

numerical values to the counterpart quantities in a miniature universe as O does to those quantities in the actual universe. Their length measurements will have the relation $l' = xl$. If x is one-half, then when O measures one foot, O' measures a "half-foot", etc. From this and the speed of light postulate, their temporal measurements given by clocks must also stand in the same relation: $t' = xt$. By inspection it is evident that: $v' = v$ and $a' = x^{-1}a$. Holding the electric charge as constant and as the unit of charge, i.e. $e' = e$, Tolman derives the mass similitude transformation from the preservation of Coulomb's law:

$$\text{(Coulomb's Law for } O\text{)}\ ma = \frac{e_1 e_2}{l^2}$$

$$\text{(Coulomb's Law for } O'\text{)}\ m'a' = \frac{e'_1 e'_2}{l'^2} \rightarrow m' = \frac{x e_1 e_2}{x^2 l^2 a}$$

$$\text{(Mass Similitude Transformation)}\ m' = x^{-1}m$$

With the fundamental similitude transformations determined, those of derived quantities, like force, energy and temperature are produced by dimensional analysis. From these results Tolman proposes to determine the functional form of several physical equations describing important physical phenomena: ideal gases, blackbody radiation, the electromagnetic field, and the electron (its mass-radius ratio and its radiation).

Quantity Kind	**Similitude Transformation**
Length	$l' = xl$
Time Duration	$t' = xt$
Velocity	$v' = v$
Acceleration	$a' = x^{-1}a$
Mass	$m' = x^{-1}m$
Force	$f' = x^{-2}f$
Energy	$U' = x^{-1}U$
Energy Density	$u' = x^{-4}u$
Electrical Charge	$e' = e$
Entropy	$S' = S$
Temperature	$T' = x^{-1}T$

Table 1: Induced transformations of quantities under similitude transformations. Adapted from Tolman (1914, 226).

The details of these derivations need not concern us here, with the exception of his treatment of gravity. Tolman argues that his principle of similitude yields a functional relation which shows the necessity of a new theory of gravity. This is because Newton's law of gravity is not consistent with the principle of similitude:

$$\text{(Newton's Gravitational Law for } O\text{)}\ f = G\frac{m_1 m_2}{l^2}$$
$$\text{(Newton's Gravitational Law for } O'\text{)}\ f' = G\frac{m_1' m_2'}{l'^2} \rightarrow f = \frac{l}{x^2} G\frac{m_1 m_2}{l^2}$$

The numerical value of the gravitational force between two masses is not preserved across the transformation. Tolman wants to hold on to the truth of the principle of similitude and presents two possible solutions: (1) Gravitational force is not proportional to mass but is only accidentally so, and the gravitational force is really proportional to some invariant quantity, like e.[36] (2) The gravitational force does not only depend on the masses but also on the properties of some mechanism which produces the force. Therefore, rather than falsifying the principle of similitude, this failed derivation shows that:

> The search for the true nature of gravitational action will now become an important problem of physics, and the principle of similitude will be a criterion for judging the correctness of proposed solutions.[37]

Any such correct law must meet the condition:

$$\text{(Invariant Gravitation)}\ f = F(A, B, C, \dots)\frac{m_1 m_2}{l^2} = F(x^a A, x^b B, x^c C, \dots)\frac{m_1 m_2}{x^2 l^2}$$

This bold conjecture, the general validity of the principle of similitude, the true standard for physical similarity, the possibility of miniature worlds, and Tolman's robustly metaphysical conception of quantity dimensions, were all to be questioned in the years to follow.

[36] This path is taken by Nordström (1915), who distinguishes the gravitational mass from the inertial mass and on this basis develops an early competitor to Einstein's general relativity.

[37] Tolman (1914), 254.

Edgar Buckingham's response in the same year is deeply significant and does many things, but I will focus narrowly on what is relevant for a contrast with Ehrenfest-Afanassjewa's response.[38] The main result is the proof of the Π-theorem on the basis of dimensional homogeneity. The principle of dimensional homogeneity is attributed to Fourier:[39]

> (Dimensional Homogeneity) Every complete physical equation is dimensionally homogeneous, i.e. all terms of a physical equation have the same dimension.[40]

This principle prohibits equations that, for example, equate a velocity and an acceleration, or involve adding a mass to a length. Due to this condition, every complete physical equation is expressible as the conservation of some function of dimensionless quantities, Π_i: $\psi(\Pi_1, \Pi_2, \ldots, \Pi_i) = 0$. This is the Π-theorem.[41]

Setting aside the details of the proof, the Π-theorem shows that complete dimensionless equations can be put into a form that is invariant under any arbitrary transformation, passive or active, of the fundamental units. Given the absolute generality of this result, it is unsurprising that Buckingham takes Tolman's principle to a mere special case of the physical similarity entailed by the Π-theorem. While Tolman more or less accepts that in many contexts his principle is a special case of standard dimensional analysis, he claims that in particular contexts his principle is applicable where the principle of dimensional homogeneity is not (or is not applicable to much effect).[42] Ehrenfest-Afanassjewa also presses this point to Tolman, that his principle is reducible to a special case of dimensional analysis.[43]

[38] Buckingham (1914).

[39] See De Clark (2017) for more on the significance of Fourier to dimensional analysis.

[40] "Complete" has a bit of a vexed history. We can read it as "empirically adequate" for our purposes here. See discussion in Jalloh (2024b, n. 16). Palacios (1964, 50) claims that completeness is equivalent to Ehrenfest-Afanassjewa's postulate of conditional homogeneity.

[41] Jalloh (2025) calls this the "Ur-Equation". Sterrett (2005) calls this "The Reduced Relation Equation of 1914". Here I stick with the more standard usage.

[42] Tolman (1915).

[43] As does Bridgman (1916). For more on Bridgman's contributions, see Walter (1990), Jalloh (2024b), and Okamoto (1998). Okamoto's paper is especially of interest as he has uncovered some archival records of interactions between Bridgman and Ehrenfest-Afanassjewa on dimensional analysis that were previously unknown to me and otherwise unknown to the broader literature.

The other significant objection to Tolman that we can glean from Buckingham is that his “miniature universe” fails to be mechanically similar to our own—unless he modifies the intensity of the gravitational force (i.e. the gravitational constant).

> [I]f we construct a miniature universe by multiplying all actual lengths by a, and if we change the densities in such a way that the mass of every volume element of the miniature universe is b times the mass of the corresponding volume element of the actual universe, then if the miniature universe is to be *mechanically similar* to the actual universe, *the gravitational forces in the miniature universe must bear to the corresponding gravitational forces in the actual universe a ratio fixed by the law of gravitation.* And if the speeds at which the gravitational phenomena occur in the miniature universe are to have the same numerical values as the corresponding speeds in the actual universe, the unit [of] time or speed can not be fixed arbitrarily but must have a particular relation to our actual unit.[44]

This objection dogged Tolman's principle and points to some deep issues regarding the nature of the constants and the laws, which are investigated more deeply by Ehrenfest-Afanassjewa.

Compared to Buckingham's response, Ehrenfest-Afanassjewa's response is at once more focused and more general. That Tolman's principle of similitude is to serve as an entry into a deeper discussion regarding the foundations of physics is evident from the first paragraph:

> An accurate analysis shows that Tolman's considerations possess at least a close connection with the reduction to a definite hypothesis of the conviction of the homogeneity of all the equations of physics, a conviction which is commonly used without any foundation. This is not the intention of the author, as appears from his third paper on the same subject, yet he really does nothing else but construct a system of dimensions of his own (indeed one that in some respects deviates from

[44]Buckingham (1914), 375, emphasis added.

> the C.G.S. system), and he examines all equations with a view to homogeneity *as regards this system of dimensions.*[45]

Ehrenfest-Afanassjewa's analysis applies the mathematical foundation of modeling systems in miniature, developed in her other 1916 paper (discussed in §2), to Tolman's point of view. Mathematically her results are near-identical to the Π-theorem, which is unsurprising as the Π-theorem is a condensation of the commonly understood principles of dimensional analysis. However, she is sceptical of the universality of dimensional homogeneity. Indeed, her criticism of Tolman depends to the impossibility of the homogeneity of all of the equations of physics (in the sense described in §2).

Ehrenfest-Afanassjewa indicates a preference for the passive interpretation of this group of symmetry transformations: unit changes. She places conditions on Tolman's active interpretation of these transformations as indicating material changes in size (e.g. a miniature universe):

> (1) that a model universe in the sense defined above is possible,
>
> (2) that we possess all equations which are wanted for a full description of the whole universe,
>
> (3) that the latter condition is especially fulfilled by those equations which in the C.G.S. system serve to fix the dimensions of the different quantities.[46]

To these conditions Ehrenfest-Afanassjewa raises three remarks. First she remarks that the transformation coefficients for time, length, and mass are fixed independently of investigation into the possibility of such model universes. Second, the full description condition necessitates that the transformation coefficients of the other quantities are fixed by the transformation so that definitions of novel quantities are invariant under such transformations; that is to say that dimensional homogeneity is indeed universal. Third, there is no reason to think that

[45] Ehrenfest-Afanassjewa (1916b), 1, her emphasis.
[46] Ehrenfest-Afanassjewa (1916b), 4.

the current fundamental dimensions are sufficient to capture all of nature, and in the case of Tolman's reduced dimension set (where length is the only basic mechanical dimension) it is insufficient to capture Newtonian gravity. In order to understand these transformations actively, it must be shown that they can capture all of the quantity dimensions needed to model nature, and Tolman as given no reason to assume so (and has indeed given one major counter-reason).

Ehrenfest-Afanassjewa suggests two possible ways of saving the active interpretation of the dimensional symmetries: (a) artificially introduce coefficients, perhaps newly postulated constants of matter, to each dimensional term which makes the term dimensionless (i.e. into a Π-term), such that each term transforms so as to preserve invariance under the scale transformations; (b) refer to said coefficients as power products of the values or magnitudes of the variables involved such that the equations are homogeneous across the any transformation (i.e. introduce formal variables). In this latter case homogeneity across a model transformation is a triviality and the constants therefore lack physical meaning as their invariance is no longer "a criterion for distinguishing between equations which are 'physically allowable' and arbitrary equations".[47] For the similitude transformation to a scale model universe to have *meaning*, the constants must remain invariant. However, as Tolman himself showed, there is no such model universe that can preserve the gravitational constant and the gravitational equation, so: "the equations of physics are of such a nature that no model transformation exists in which all the universal constants have the same values as in our universe."[48]

Immediately following Ehrenfest-Afanassjewa's article was a response from Tolman. Tolman begins with agreements, he accepts Ehrenfest-Afanassjewa's presentation of the issue, with regards to the fact that his principle of similitude requires invariance under the arbitrary transformation of one quantity dimension while her (and Buckingham's) principle is more generally arbitrary—the principle of homogeneity is more general than the principle of similitude. He does object to her characterization of his principle as determining another

[47]Ehrenfest-Afanassjewa (1916b), 6.
[48]Ehrenfest-Afanassjewa (1916b), 7.

"system of dimensions" distinct from the that induced by the standard CGS system. In rejecting this characterization, Tolman puts forward a metaphysically robust notion of a dimension as "a shorthand statement of the definition of that kind of quantity in terms of certain fundamental kinds of quantity, and hence also as an expression of the essential physical nature of the quantity in question".[49] Note that this is precisely the sort of view which Ehrenfest-Afanassjewa explicitly rejects in her earlier 1916 paper (or at least shows it to be inconsistent with the universality of dimensional homogeneity, see §2). Tolman would go on to elaborate this view in a seminal 1917 paper, his last (published) word on dimensional analysis.[50] This is all to ground the universality of dimensional homogeneity in metaphysics rather than mathematics, as Ehrenfest-Afanassjewa showed it could not be, unless one accepts the physical possibility of worlds in which the constants have different magnitudes.[51] While Tolman was right to distinguish his metaphysically loaded conception of dimensions with Ehrenfest-Afanassjewa's formal conception of dimensions, the case that this move would save his principle of similitude is dubious at best, which he more or less recognized.[52]

4 After *Dimensional Analysis*: A Tripartite Dispute

Bridgman intended his *Dimensional Analysis* to settle once and for all the issues evident in the previously discussed dialogues and related disputes—in his preface to the first edition (1922) he states:

> There is [...] nowhere a systematic exposition of the principles of the method [of dimensional analysis.] Perhaps the reason for this lack is the feeling that the subject is so simple that any formal presentation is superfluous. There do,

[49]Tolman (1916), 9.
[50]See Tolman (1917). For his later thoughts on dimensional analysis, see Jalloh (2024a).
[51]This issue remains disputed, see below and the relevant citations. An overview can be found in Martens (2024).
[52]See Jalloh (2024b) and Jacobs (2024) on the metaphysics of dimension.

> nevertheless, exist important misconceptions as to the fundamental character of the method and the details of its use. These misconceptions are so widespread, and have so profoundly influenced the character of many speculations [...] that I have thought an attempt to remove the misconceptions well worth the effort.(Bridgman 1931, v)

Insofar as his aim was to clear up once and for all the misconceptions fueling debate regarding dimensional analysis and its proper use, the book was less than a complete success, though by most other standards it was a great success. He notes this in his preface to the revised edition (1931):

> Since the first printing of the book I have observed to my great surprise that in spite of what seemed to me a lucid and convincing exposition there are still differences in fundamental points of view, so that the subject cannot yet be regarded as entirely removed from the realm of controversy.(Bridgman 1931, vi)

Ehrenfest-Afanassjewa's final contribution to dimensional analysis in 1926 is cited among the more important articles that appeared in the intervening years (Ehrenfest-Afanassjewa 1926). This article was a response to the stirring polemic of Norman Campbell, which also managed to pull Bridgman back into an issue he thought he had settled.

Campbell reignited the debate regarding the fruitfulness of dimensional analysis, now on the pages of *Philosophical Magazine*. His essay is a polemic against highly mathematical approaches to "dimensional analysis" (an "imposing title") which obscure the physical assumptions at play for the dubious benefit of "unnecessary elaboration" of the logic. Campbell states three stages at which physical assumptions play a role in dimensional analysis:

> (1) When it is decided that the systems to which the argument is to be applied are physically similar, and have throughout the same no-dimensional [dimensionless] magnitudes [quantities].

> (2) When it is decided what are the magnitudes which determine the process discussed.
>
> (3) When definite dimensions are assigned to these magnitudes. (Campbell 1924, 482)

Campbell claims that only the second stage has gotten any significant attention and that the final stage is usually accounted for by simply lists of magnitudes and their dimensions, which is "if not actually erroneous, highly misleading". The assignment of dimensions to quantities only has truth-apt meaning in the context of some "numerical law", else the designation is arbitrary and of no physical significance. That quantities appear in multiple equations involving different functional relations requires us to stipulate the dimensions of those quantities contextually. The first question is: What laws figure in the problem I am dealing with? The proper dimensions will fall out of those.[53] In a quick disposal of the so-called Riabouchisky-Rayleigh paradox regarding the dimensions of temperature, Campbell claims that whether temperature is given the dimensions of energy is dependent on the laws that are relevant for the description of the phenomena at hand. There are no context-independent dimensions proper to temperature.[54]

Campbell's main point is that the assignment of dimensions to quantities and the application of dimensional analysis cannot be prior to knowledge of the laws that describe some specific phenomena—this is a direct attack on the assumed usefulness of dimensional analysis for the derivation of laws.[55] The dimensional analysts have gotten their epistemology backwards and prematurely exclude the assignment of a different set of dimensions to quantities in case of the discovery of a novel law which indicates a distinct assignment of dimensions. This makes the second stage trivial, as knowledge of the laws involved necessarily involves

[53] In a footnote, Campbell rejects homogeneity as a grounds for the dimensions of derivative quantities: "It cannot be justified by considering how the unit of the magnitude would have to be changed if that of other magnitudes were changed, for there is no compulsion whatever to change units so as to preserve the formal constants in law." Campbell (1924), 483.

[54] See Riabouchinsky (1915).

[55] The laws are similarly privileged by Palacios (1964).

knowledge of the quantities involved.

According to Campbell, the first stage, the determination of physical similarity, is unproblematic in simple cases that barely go beyond our direct knowledge of geometrical similarity, e.g. the pendulum. However for more complicated cases of dynamical similarity, where forces must be assumed to be similar across models, like modeling a propeller or the movement of a ship, our assumptions can only be justified by reference to the elementary laws of nature. Dimensional analysis is only to be applied in those situations in which the laws of motion do not yield to analytical or simple solutions, owing to the complexity of the system in question. That the laws are prior to dimensional analysis is supposed to show that the mathematical apparatus invoked to independently justify dimensional analysis, like Buckingham's Π-theorem, are useless. Campbell argues that all that is needed of the formalities of dimensional analysis is knowledge of the meaning of similarity and the invariance of dimensionless quantities for similar systems.[56]

Ehrenfest-Afanassjewa replies with another attempt to provide a mathematical and foundational account of the use of dimensional reasoning, which she calls the "method of similitude" in contrast to the less useful "dimensional analysis". Her aim is to reconcile much of what has been said before and to investigate further into the fundamentals:

> Indeed, in his introduction Campbell quotes my paper, but he does not express his agreement with my conception. However, I hope by means of the following analysis to further the desired agreement. Yet the note here presented is not a mere review of my earlier ones, but it embodies deeper insight into dimensional and similitude analysis in two respects: first, concerning the role of the constants of integration, and second, concerning the view upon similitude-transformations of different quantities having the same dimension. In addition to this, the difference of opinion between Lord Rayleigh and Riabouchinsky will also be

[56] Buckingham (1924) responds with a pragmatic defense of both dimensional analysis and formal investigations of it, especially in "technical" or "engineering" physics.

cleared up.(Ehrenfest-Afanassjewa 1926, 258)

There is a great deal to comment upon in this paper, but let me focus on the contrastive elements between her method of similitude and dimensional analysis.[57]

Rehearsing some of her earlier results, Ehrenfest-Afanassjewa comes to the most general form of an equation which admits the transformations of the constituent quantities by the "conditional equations" (equivalent to the "conditioning equations" of earlier work):

$$F(\frac{x_{r+1}}{x_1^{a_{11}} \dots x_r^{a_{1r}}} \dots \frac{x_{r+l}}{x_1^{a_{l1}} \dots x_r^{a_{lr}}}) = 0,$$

where F is any arbitrary function, the x's are the quantities which figure in the fundamental equations that describe a system, the a's are their powers, r is the number of transformation quantities for each transformation coefficient ξ_i and l is the number of conditional equations that define the relationships between the ξ_i's, thus defining the symmetry transformations of the fundamental equations taken together ($l + r = n$, the number of quantities altogether). This is again a version of the Π-theorem.[58] For Ehrenfest-Afanassjewa, the theory of similitude and dimensional analysis are formally similar, both are expressible by the Π-theorem. The difference between them comes in their usage (especially the determination of the conditioning equations) and their conception of the constants.

Before discussing further the issues raised by Ehrenfest-Afanassjewa's attempt to distinguish the theory of similitude and dimensional analysis, let's consider the response to her restatement of their shared mathematical foundations.

Campbell reprised his criticism of the unnecessary sophistication of the mathematics used in this literature, now aimed at Ehrenfest-Afanassjewa, though they agree regarding inadequacy of dimensional analysis in most contexts. Though his criticism was mere repetition of his earlier points and, due to a severe lack of mathematical engagement, rather weak, his admonishment of the *possibility* of a mathematical foundation to dimensional argument was

[57]In particular, I will not deal with her remarks on constants of integration.

[58]Ehrenfest-Afanassjewa again cites Federman (1911). See note 27 above for more on priority issues.

put quite forcefully:

> On the other hand, she apparently agrees with the dimensional analysts in thinking that the argument can be expressed mathematically; here I differ completely. The only distinctive mathematical proposition that it employs (namely, that multiplication is distributive and associative) is so elementary that it is assumed and not proved in her analysis. The introduction of elaborate symbolism is not merely superfluous, but positively misleading. It concentrates attention on 'elegant irrelevancies' and distracts it from essentials. It suggests that the circumstances in the argument are applicable are far wider than they actually are, and thus prepares the way for all the charlatanry with which the word dimensions has unfortunately become associated.[59]

Given the current state of play, it is easy to dismiss Campbell's rejection of the mathematization of dimensional analysis. The level of mathematical rigor and usefulness of the method has only increased since Ehrenfest-Afanassjewa began giving it its foundations.[60]

On the other side, Bridgman's short response was dismissive of both the formalist treatment and what he perceived as its lurking metaphysics.[61] The criticism of the former aspect amounts to a complaint that Ehrenfest-Afanassjewa's dimensional analyst is a strawman and her results indistinguishable from those of his own. Insofar as the results are the same, her approach is far less applicable for the working physicist and requires more of them: that they know the fundamental equations involved rather than the general physical character of the system at hand. Bridgman's criticism of Ehrenfest-Afanassjewa's *strict* distinction between the theory of similitude and dimensional analysis has been born out in the literature. Further, Bridgman's charge that Ehrenfest-Afanassjewa's dimensional analyst is something of a strawman holds. One may simply look at the supposed assumption of a mechanical dimensional basis of

[59] Campbell (1926), 1146.

[60] Consider just two influential and highly technical books downstream of Ehrenfest-Afanassjewa's influence (supported by citations): Birkhoff (1960); Barenblatt (1996).

[61] Bridgman (1926).

mass, length, and time as *essential* to the method as an overstatement—I elaborate on this below. On the other hand, Ehrenfest-Afannasjewa's application of mathematical method to dimensional analysis has proved clarifying and useful to those who have attempted to understand it. A foundational project need not be grounded on immediate practical grounds in order to be worthwhile. Bridgman was also wrong not to see Ehrenfest-Afanassjewa as an ally against metaphysical conceptions of dimensional analysis.

Regarding use, as opposed to mathematical foundation, Ehrenfest-Afanassjewa notes two rules required of the similitude method, which traditional applications of dimensional analysis, and Tolman's application of *his* principle of similitude, violate:

> (Rule I) All the fundamental equations have to be considered, of which the one desired is an analytical consequence.
>
> (Rule II) No conditional equations may be introduced, except those which are imposed on the transition-factors (transition coefficients, ξ_i) by the fundamental equations of the problem. (Ehrenfest-Afanassjewa 1926, 267.)

Violations of these rules by the consideration of irrelevant equations as fundamental, like the kinetic equivalent of heat in the problem that raises the Riabouchinsky-Rayleigh paradox, introduce illusory indeterminateness in the equation sought. On the other hand, illusory definiteness can be generated by ignoring the independence of particular quantities, like failing to distinguish different length quantities.

Ehrenfest-Afanassjewa claims that dimensional analysis necessarily violates both of these rules. It does so due to the the two further assumptions that distinguish it from the theory of similitude: (1) that the set of basic dimensions (i.e. the independent transition factors) are fixed as mass, length, and time and (2) that the derivative dimensions (i.e. the exponents of dependent transition factors) are fixed by the equations which introduced these quantities historically. However, it is simply not the case that "dimensional analysis" as a set of practices was ever so limited. Besides Tolman's experiment in reducing the basis of the mechanical dimensions, Rayleigh (who Ehrenfest-Afanassjewa goes on to discuss) introduces an independent dimension for temperature. Both Bridgman and Buckingham deal with some issues regarding the conventionality of which dimensions are treated as basic and how derivative dimensions are defined. If this is the only basis for distinguishing the theory of similitude and the dimensional analysis, then there is no difference beyond terminology.

This brings us to the murkiest issue in all of dimensional analysis to this day. This may ultimately provide some way of delineating the theory of similitude from dimensional analysis, though even if it does not, Ehrenfest-Afanassjewa deserves credit for being one of the earliest to be sensitive to the issues involved. The issue is the nature of the physical constants. First, we might start with Bridgman's criticism of Ehrenfest-Afanassjewa in this matter and work backwards:

> Thus she begins by committing the dimensional analysist to the thesis that all equations expressing physical relations are dimenisonally homogeneous, which is supposed to be equivalent to the statement that dimensional constants cannot occur. She then allows him to talk the most meaningless nonsense about other possible universes in which the gravitational constant might be different from ours [. . .][62]

Starting with the second characterization first: Bridgman is right that Ehrenfest-Afanassjewa claims the dimensional analyst leaves open the possibility of different values of the universal constants. However, like Bridgman, she takes this to be a pathology of dimensional analysis and eliminates the possibility from her theory of similitude. Further, as not every dimensional analyst had the modal strictures that Bridgman had, Ehrenfest-Afanassjewa's characterization is in this respect correct. Buckingham, in his response to Tolman, showed that for similitude transformations grounded in length transformations, the value of the gravitational constant would have to change.[63] This surely counts as talk about other possible universes, though I am

[62]Bridgman (1926), 1265.
[63]Buckingham (1914). See discussion in §3.

unsure that it is "the most meaningful nonsense". Ehrenfest-Afanassjewa makes a carveout for the dimensional analyst: If he wishes his discussion of other conceivable worlds—if he wishes to consider active similitude transformations as opposed to mere passive unit changes—to be "controllable" (that is to say, "physically meaningful"), he must leave his universal constants invariant, which requires non-homogeneous fundamental equations.[64]

But why does preserving the invariance of the constants require inhomogeneity? I have already indicated that there is something odd about Ehrenfest-Afanassjewa's conception of the *principle* of dimensional homogeneity (see §2). A conception which Bridgman rightly characterizes as rejecting the possibility of dimensional constants.[65] In this work, Ehrenfest-Afanassjewa merely assumes this conception of dimensional homogeneity. In order to make sense of this we must recall some of her earlier work. As early as the 1905 paper, Ehrenfest-Afanassjewa characterized the transformations induced on constants with derivative dimensions as merely formal and in the first 1916 paper, named such constants as "formal variables". On the other hand, material changes, like changes in constants of matter between samples, would indicate changes according to "modeling equations" as opposed to "dimensional equations", representing *physical* changes. These changes would violate dimensional (formal) homogeneity, unless the dimensional equations were restricted to those of the modeling equations, reducing dimensional analysis to her method of similitude. Alternatively, if the constants are held to make equations *dimensionally* homogeneous, then it must be that under material transformations of other quantities, they vary, so as to preserve the equations in which they figure, but this appears to undermine the constants having any *physical* grounding. Again, she sets up a dilemma between the role of the constants in preserving dimensional homogeneity (qua formal variables) and their physical meaning.

Ehrenfest-Afanassjewa's final work on dimensional analysis failed to realize the consensus she had aimed for. However, it did greatly enrich our understanding of dimensional analysis and its limits. She has not only provided it with explicit and rigorous foundations, but has

[64]Like the classical gravitation equation. See Ehrenfest-Afanassjewa (1926), 260, first note.
[65]Compare Ehrenfest-Afanassjewa (1926), 258.

presented problems that remain to be solved, particularly regarding the nature of the constants. In the conclusion I will present some direct consequences of Ehrenfest-Afanassjewa's influence and will suggest where her influence still ought to be felt.

5 Conclusion

Though the discussion here is nonexhaustive, I hope to have shed light on most of the contributions of Tatiana Ehrenfest-Afanassjewa to our understanding of dimensional analysis, complementing recent work done to reevaluate her contributions to statistical mechanics, thermodynamics, and didactics. In this conclusion I just want to specify some of the areas of research in which Ehrenfest-Afanassjewa's work on dimensional analysis has been influential and areas in which it ought to have some (further) influence.

(Mathematical Foundations of Dimensional Analysis) Direct followers of Ehrenfest-Afanassjewa's group theoretic grounding of dimensional analysis and the theory of generalized homogeneous functions include San Juan and Birkhoff.[66] Though they haven't always made reference to Ehrenfest-Afanassjewa, other algebraic or even geometrical accounts of dimensional analysis continue to be developed.[67]

(Methodological Foundations of Dimensional Analysis) Ehrenfest-Afanassjewa's foregrounding of the role of physical laws (along with Campbell) and account of illusory indeterminateness and definiteness were further developed by Palacios (who followed San Juan with respect to mathematical foundations). This distinction for Palacios becomes a distinction between "insufficient" and "superabundant" bases for a dimensional system. In a "well-tuned" system or what we might call a "goldilocks" system, the derivations of dimensional analysis are at maximum definiteness without falling into error.[68]

[66]See San Juan (1947) and Birkhoff (1960).

[67]See, e.g., Tao (2012); Raposo (2018); Raposo (2019).

[68]Palacios (1964). For further discussion see González de Posada and González Redondo (2002); Johnson

(Metaphysics of Constants and Quantities) While Ehrenfest-Afanassjewa might have had qualms about metaphysics as a discipline, her work brought to fore a major metaphysical question regarding the nature of the constants which has only recently become a regular topic of discussion: whether their magnitudes are contingent or necessary, according to the laws of physics.[69] This revival is part of a broader concern with the metaphysics of quantity which has become topic of concern in the philosophy of science, the methodology of psychology, and general metaphysics.[70]

Those working in these and related areas ignore the contributions of Tatiana Ehrenfest-Afanassjewa at their own peril.

(2018); Jalloh (2024b).

[69]Recent work on this issue, some of which cites Ehrenfest-Afanassjewa, includes: Johnson (2018); Jacobs (2022); Jalloh (2025); Martens (2024).

[70]Some of the most important recent work on the metaphysics of quantities include: Dasgupta (2013); Baker (2020); Sider (2020); Wolff (2020); Mari, Wilson, and Maul (2021). Those with more of a focus on psychology include: Michell (1999) and Briggs (2022).

References

Baker, David John. 2020. "Some Consequences of Physics for the Comparative Metaphysics of Quantity." In *Oxford Studies in Metaphysics Volume 12*, edited by Karen Bennett and Dean W. Zimmerman, 75–112. Oxford University Press. https://doi.org/10.1093/oso/9780192893314.003.0004.

Barenblatt, G. I. 1996. *Scaling, Self-similarity, and Intermediate Asymptotics: Dimensional Analysis and Intermediate Asymptotics.* Cambridge Texts in Applied Mathematics. Cambridge University Press.

Berberan-Santos, Mario N, and Lionello Pogliani. 1999. "Two Alternative Derivations of Bridgman's Theorem." *Journal of Mathematical Chemistry* 26: 255–61. https://doi.org/10.1023/A:1019102415633.

Birkhoff, Garrett. 1960. *Hydrodynamics, A Study in Logic, Fact, and Similitude.* Revised. Princeton: Princeton University Press.

Bridgman, Percy W. 1916. "Tolman's Principle of Similitude." *Physical Review* 8 (4): 423–31. https://doi.org/10.1103/PhysRev.8.423.

———. 1926. "CXX. Dimensional Analysis Again." *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 2 (12): 1263–66. https://doi.org/10.1080/14786442608564160.

———. 1931. *Dimensional Analysis.* Revised. New Haven: Yale University Press.

Briggs, Derek C. 2022. *Historical and Conceptual Foundations of Measurement in the Human Sciences: Credos and Controversies.* New York, NY: Routledge.

Buckingham, Edgar. 1914. "On Physically Similar Systems; Illustrations of the Use of Dimensional Equations." *Physical Review* 4 (4): 345–76. https://doi.org/10.1103/PhysRev.4.345.

———. 1924. "XV. Dimensional Analysis." *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 48 (283): 141–45. https://doi.org/10.1080/14786442408634474.

Campbell, Norman Robert. 1920. *Physics: The Elements*. Cambridge: Cambridge University Press.

———. 1924. "XL. Dimensional Analysis." *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 47 (279): 481–94. https://doi.org/10.1080/14786442408634384.

———. 1926. "CIII. Dimensional Analysis." *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 1 (5): 1145–51. https://doi.org/10.1080/14786442608633715.

Dasgupta, Shamik. 2013. "Absolutism Vs Comparativism About Quantity." *Oxford Studies in Metaphysics* 8: 105–50.

De Clark, Sybil Gertrude. 2017. "Qualitative Vs Quantitative Conceptions of Homogeneity in Nineteenth Century Dimensional Analysis." *Annals of Science* 74 (4): 299–325. https://doi.org/10.1080/00033790.2017.1372519.

de Courtenay, Nadine. 2015. "The Double Interpretation of the Equations of Physics and the Quest for Common Meanings." In *Standardization in Measurement: Philosophical, Historical and Sociological Issues*, edited by Oliver Schlaudt and Lara Huber, 53–68. London; Brookfield: Pickering & Chatto.

Ehrenfest, Paul, and Tatiana Ehrenfest. 1911. "Begriffliche Grundlagen Der Statistischen Auffassung in Der Mechanik." In *Encyklopädie Der Mathematischen Wissenschaften*, edited by Felix Klein. Vol. 4. Leipzig: Teubner.

———. 1959. *The Conceptual Foundations of the Statistical Approach in Mechanics*. Translated by M. J. Moravcsik. Ithaca: Cornell University Press.

Ehrenfest-Afanassjewa, Tatiana. 1905. "Ueber Die Willkürlichkeit Bei Der Dimensionierung Physikalischer Größen." *Mathematisch-Naturwissenschaftliche Blätter* 2 (8).

———. 1916a. "Der Dimensionsbegriff und der analytische Bau physikalischer Gleichungen." *Mathematische Annalen* 77 (2): 259–76. https://doi.org/10.1007/BF01456903.

———. 1916b. "On Mr. R. C. Tolman's 'Principle of Similitude.'" *Physical Review* 8 (1):

1–7. https://doi.org/10.1103/PhysRev.8.1.

———. 1926. “XVII. Dimensional Analysis Viewed from the Standpoint of the Theory of Similitudes.” *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 1 (1): 257–72. https://doi.org/10.1080/14786442608633625.

———. 1956. *Die Grundlagen Der Thermodynamik*. Leiden: Brill.

Federman, A. K. 1911. “On Some General Methods of Integration of First-Order Partial Differential Equations.” *Proc. St.-Petersburg Polytech. Inst. Sect. Tech. Nat. Sci. Math* 16: 97–155.

González de Posada, Francisco, and Francisco A. González Redondo. 2002. “Génesis Histórica, Enunciado y Evolución Del análisis Dimensional de Julio Palacios.” *Llull: Revista de La Sociedad Española de Historia de Las Ciencias y de Las Técnicas* 25 (53): 399–424.

Görtler, H. 1975. “Zur Geschichte des Π-Theorems.” *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik* 55 (1): 3–8. https://doi.org/10.1002/zamm.19750550103.

Grozier, Jim. 2020. “Should Physical Laws Be Unit-Invariant?” *Studies in History and Philosophy of Science Part A* 80 (April): 9–18. https://doi.org/10.1016/j.shpsa.2018.12.009.

Jacobs, Caspar. 2022. “The Nature of a Constant of Nature: The Case of G.” *Philosophy of Science*, 1–20. https://doi.org/doi:10.1017/psa.2022.96.

———. 2024. “In Defence of Dimensions.” *British Journal for the Philosophy of Science*. https://doi.org/10.1086/729749.

Jalloh, Mahmoud. 2024a. “The Bridgman-Tolman-Warburton Correspondence on Dimensional Analysis, 1934.” *Hopos: The Journal of the International Society for the History of Philosophy of Science* 14 (2): Supplement. https://www.journals.uchicago.edu/doi/suppl/10.1086/731678.

———. 2024b. “Metaphysics and Convention in Dimensional Analysis, 1914–1917.” *HOPOS: The Journal of the International Society for the History of Philosophy of Science* 14 (2):

275–322. https://doi.org/10.1086/731678.

———. 2025. “The Π-Theorem as a Guide to Quantity Symmetries and the Argument Against Absolutism.” In *Oxford Studies in Metaphysics*, edited by Karen Bennett and Dean W. Zimmerman. Vol. 14. Oxford: Oxford University Press.

Johnson, Peter. 2018. *The Constants of Nature: A Realist Account*. New York, Oxford: Routledge.

Mari, Luca, Mark Wilson, and Andrew Maul. 2021. *Measurement Across the Sciences: Developing a Shared Concept System for Measurement*. Springer Series in Measurement Science and Technology. Cham: Springer International Publishing. https://doi.org/10.1007/978-3-030-65558-7.

Martens, Niels C. M. 2024. *Philosophy of Physical Magnitudes*. Cambridge: Cambridge University Press.

Michell, Joel. 1999. *Measurement in Psychology: Critical History of a Methodological Concept*. Ideas in Context. New York: Cambridge University Press.

Mitchell, Daniel Jon. 2017. “Making Sense of Absolute Measurement: James Clerk Maxwell, William Thomson, Fleeming Jenkin, and the Invention of the Dimensional Formula.” *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* 58 (May): 63–79. https://doi.org/10/gr6p4c.

Nordström, Gunnar. 1915. “R. C. Tolmans "Prinzip Der Ähnlichkeit" Und Die Gravitation.” *Öfversigt Af Finska Vetenskaps-Societetens Förhandlingar* 57: 1–10.

Ogilvie, Marilyn Bailey, and Joy Dorothy Harvey, eds. 2000. “Ehrenfest-Afanassjewa, Tatyana Alexeyevna (1876–1964).” In *The Biographical Dictionary of Women in Science: Pioneering Lives from Ancient Times to the Mid-20th Century*, 823–25. New York: Routledge.

Okamoto, Takuji. 1998. “Demystifying Dimensional Analysis: The Beginning of Bridgman’s Scrutiny of Theoretical Physics.” *Historia Scientiarum. Second Series: International Journal of the History of Science Society of Japan* 8 (1): 21–52.

Palacios, Julio. 1964. *Dimensional Analysis*. Translated by Peter Lee and L. Roth. London; New York: Macmillan; St. Martin's Press.

Pobedrya, B. E., and D. V. Georgievskii. 2006. "On the Proof of the Π-theorem in Dimension Theory." *Russian Journal of Mathematical Physics* 13 (4): 431–37. https://doi.org/10.1134/S1061920806040066.

Raposo, Álvaro P. 2018. "The Algebraic Structure of Quantity Calculus." *Measurement Science Review* 18 (4): 147–57. https://doi.org/10.1515/msr-2017-0021.

———. 2019. "The Algebraic Structure of Quantity Calculus II: Dimensional Analysis and Differential and Integral Calculus." *Measurement Science Review* 19 (2): 70–78. https://doi.org/10.2478/msr-2019-0012.

Riabouchinsky, D. 1915. "The Principle of Similitude." *Nature* 95 (2387): 591–91. https://doi.org/10.1038/095591c0.

San Juan, Ricardo. 1947. *Teoría de las magnitudes físicas y sus fundamentos algebraicos*. Madrid: Real Academia de Ciencias.

Schlaudt, Oliver. 2009. *Die Quantifizierung der Natur: klassische Texte der Messtheorie von 1696 bis 1999*. Paderborn: Mentis.

Sedov, Leonid Ivanovič. 1993. *Similarity and Dimensional Methods in Mechanics*. Translated by A. G. Volkovets. 10th ed. Boca Raton, Fla: CRC Press.

Sider, Theodore. 2020. *The Tools of Metaphysics and the Metaphysics of Science*. Oxford: Oxford University Press.

Sterrett, Susan G. 2005. *Wittgenstein Flies a Kite: A Story of Models of Wings and Models of the World*. Penguin/Pi Press.

———. 2017. "Physically Similar Systems: A History of the Concept." In *Springer Handbook of Model-Based Science*, edited by Lorenzo Magnani and Tommaso Wayne Bertolotti, 377–412. Dordrecht, Heidelberg, London, New York: Springer.

———. 2020. "The Physics of Miniature Worlds." In *WITTGENSTEINIAN (Adj.)*, edited by Shyam Wuppuluri and Newton Da Costa, 289–339. Cham: Springer International

Publishing. https://doi.org/10.1007/978-3-030-27569-3_17.

Tao, Terrence. 2012. "A Mathematical Formalisation of Dimensional Analysis." https://terrytao.wordpress.com/2012/12/29/a-mathematical-formalisation-of-dimensional-analysis/.

Tobies, Renate. 2020. "Internationality: Women in Felix Klein's Courses at the University of Göttingen (1893–1920)." In *Against All Odds: Women's Ways to Mathematical Research Since 1800*, edited by Eva Kaufholz-Soldat and Nicola M. R. Oswald, 9–38. Cham: Springer International Publishing. https://doi.org/10.1007/978-3-030-47610-6_1.

Tolman, Richard C. 1914. "The Principle of Similitude." *Physical Review* 3 (4): 244–55. https://doi.org/10.1103/PhysRev.3.244.

———. 1915. "The Principle of Similitude and the Principle of Dimensional Homogeneity." *Physical Review* 6 (3): 219–33.

———. 1916. "Note on the Homogeneity of Physical Equations." *Physical Review* 8–11 (1): 8. https://doi.org/10.1103/PhysRev.8.8.

———. 1917. "The Measurable Quantities of Physics." *Physical Review* 9 (3): 237–53.

Uffink, Jos, Giovanni Valente, Charlotte Werndl, and Lena Zuchowski, eds. 2021. *The Legacy of Tatjana Afanassjewa: Philosophical Insights from the Work of an Original Physicist and Mathematician*. Women in the History of Philosophy and Sciences. Springer International Publishing. https://doi.org/10.1007/978-3-030-47971-8.

van der Heijden, Margriet. 2021a. *Denken Is Verrukkelijk: Het Leven van Tatiana Afanassjewa En Paul Ehrenfest*. Amsterdam: Prometheus.

———. 2021b. "Tatiana Ehrenfest-Afanassjewa: No Talent for Subservience." In *The Legacy of Tatjana Afanassjewa: Philosophical Insights from the Work of an Original Physicist and Mathematician*, edited by Jos Uffink, Giovanni Valente, Charlotte Werndl, and Lena Zuchowski, 3–28. Women in the History of Philosophy and Sciences. Cham: Springer International Publishing. https://doi.org/10.1007/978-3-030-47971-8_1.

———. 2024. "More Is Known about Him Than about Her: Tatiana Ehrenfest-Afanassjewa." *Physics Today* 77 (1): 40–47. https://doi.org/10.1063/PT.3.5381.

Vaschy, Aimé. 1892. "Sur Les Lois de Similitude En Physique." *Annales Telegraphiques* 19: 25–28.

Walter, Maila L. 1990. *Science and Cultural Crisis: An Intellectual Biography of Percy Williams Bridgman (1882-1961)*. Stanford: Stanford University Press.

Whitney, Hassler. 1968. "The Mathematics of Physical Quantities: Part II: Quantity Structures and Dimensional Analysis." *The American Mathematical Monthly* 75 (3): 227–56. https://doi.org/10.2307/2314953.

Wolff, J. E. 2020. *The Metaphysics of Quantities*. Oxford: Oxford University Press.