Noncommutative Geometry and Chronogeometry in Quantum Gravity

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Abstract

Chronogeometry is often conceived as a necessary condition for spatiotemporality, yet many theories of quantum gravity (QG) seem to challenge it. Applications of noncommutative geometry (NCG) to QG propose that spacetime exhibits noncommutative features at or beyond the Planck scale, thereby replacing relativistic symmetries with their deformations, known as quantum groups. This leads to an algebraic formulation of noncommutative structure that postulates a minimal length scale and deforms relativistic (commutative) physics, raising questions about whether noncommutative theories preserve spatiotemporal content, and specifically, chronogeometry. I argue that noncommutative approaches can satisfy an appropriate definition of chronogeometry, thus attaining physical significance within QG. In particular, I contend that noncommutativity is compatible with chronogeometricity, using $\boldsymbol{\kappa}$-Minkowski spacetime as case study in NCG. In this algebraic setting, physical interpretation hinges on two crucial elements: a representation of the noncommutative algebra and a corresponding set of observers. I show how this framework enables the algebra to encode localisation procedures for events in noncommutative spacetime, relative to a *noncommutative reference frame*, with frame transformations governed by the quantum group structure. By enriching the theory with noncommutative reference frames, NCG can satisfy the necessary representational principles to support chronogeometric content.

Keywords: Noncommutative geometry, Chronogeometry, Noncommutative reference frames, Interpretation, Quantum gravity

1 Introduction

Quantum gravity (QG) offers a new perspective on numerous key aspects of physical reality. In an attempt to describe physics at or beyond the Planck scale, candidate theories of QG postulate new fundamental degrees of freedom that challenge the physical picture described by quantum and relativistic theories. A significant number of these proposals, including loop quantum gravity, group field theory, and causal set theory, conjecture that the spacetime geometry, as described successfully by general relativity (GR), breaks down at extremely high energy due to quantum effects. Consequently, a new non-spatiotemporal description of this regime is warranted.

Noncommutative geometry (NCG) instantiates this novel approach to fundamental physics by conjecturing that spacetime has a non-classical, noncommutative structure at the Planck scale. This new structure, termed *noncommutative spacetime* (NCST), challenges key relativistic structures and properties, including the ability to sharply localise events, the distinction between timelike and spacelike distances, and the locality of fields. While NCG emerges from different theories of QG, it has often been neglected by philosophers of physics interested in the epistemology and metaphysics of QG.

In this context, Huggett, Lizzi, and Menon [1] argue that the noncommutative parameter, introduced as a fundamental scale of the new theory, interacts with the localisability of events in interesting ways. Specifically, they contend that operational criteria of localisation become meaningless due to severe limitations in the definition of a point-based spacetime structure. Similarly, many geometrical notions must be reformulated due to the impossibility of reconstructing a differential geometric picture out of the algebraic description of NCST models. These features raise crucial concerns regarding the spatiotemporality of said models.

Much of the philosophical literature has argued that, alongside arbitrary localisability, *chronogeometricity* is a necessary condition for spatiotemporality (see, e.g., [2–5]). Chronogeometricity refers to a geometry's capacity to have chronogeometric content, that is, to represent temporal and spatial distances. It has been extensively argued that only a chronogeometric structure can meaningfully make contact with empirical contexts, thereby enabling an appropriate physical interpretation of the mathematical structure. In this paper, the phrase "physical interpretation" refers to the identification of a semantic framework for the mathematical formalism, one that facilitates understanding of the theory's content in potential applications to physical contexts. Correspondingly, the term "physical geometry" denotes a mathematically defined geometric structure that is endowed with a physical interpretation, as informed by our best physical theories, and that is suitable for representing spacetime structure in specific investigative contexts.

Nevertheless, the chronogeometricity of mathematical structures raises important conceptual challenges, concerning the status of spatiotemporal probes, the interplay between special relativity (SR) and GR, and the relationship between spacetime and matter fields. In light of this, the absence of chronogeometricity can indicate disappearance of spacetime. In the context of NCST theories, one must therefore demonstrate that these possess chronogeometric structure. As I will argue, however, there is no universal consensus about which conditions a theory must satisfy to quality as chronogeometric.

This paper has three main goals. First, to discuss the problem of chronogeometry in QG through a case study: κ-Minkowski spacetime. In particular, the paper examines the viability of operational versus representational chronogeometry within the context of quantum gravitational phenomena.

Second, the paper aims to offer an overview of the quantum group approach to NCST. Not only this approach introduces a precise implementation of noncommutativity within standard algebraic structures, thereby modifying the perspective offered by relativistic physics. More in general, it also indicates a new perspective on the spatiotemporal interpretation of algebraic physical theories.

Finally, the paper aims to discuss the overall problem of the interaction between algebra, physical interpretation, and spatiotemporality. Specifically, it indicates the necessary representation relations and conditions that an algebraic theory should satisfy in order to be spatiotemporal within a given domain.

I argue that, contrary to expectations, noncommutativity is perfectly compatible with chronogeometry: the two features are not contradictory. Specifically, I indicate κ-Minkowski spacetime as a suitable model that can satisfy appropriate chronogeometric criteria, and thereby provide a chronogeometric noncommutative theory. To this end, a necessary condition is the definition of an appropriate interpretation for the relevant algebraic structures, one that includes *noncommutative reference frames*.

In this regard, this paper is structured as follows. In Section 2, I introduce the notion of chronogeometry in relativistic theories and illustrate how QG theories challenge it. In particular, I argue that, while QG theories cannot satisfy operational chronogeometry, they can still satisfy representational chronogeometry upon provision of suitable representational principles.

In Section 3, I introduce the algebraic approach to NCST theories outlining their connection with the underlying quantum group structures. κ-Minkowski spacetime serves as a key case study, with particular emphasis placed on the role of representations in defining concrete algebraic models.[1]

In Section 4, I formulate the problem of chronogeometry within the context of algebraic QG theories. I specify the criteria that a NCST theory must meet to be considered chronogeometric, and underscore the need for a physical interpretation of it to evaluate this property.

In Section 5, I provide a physical interpretation of κ-Minkowski spacetime, arguing that it must incorporate the concept of a noncommutative reference frame. Here, I demonstrate how this concept is essential in ensuring compatibility among different noncommutative structures derived from the abstract NCST theory: they must all implement noncommutative structures in a consistent manner, governed by the same noncommutative parameter.[2]

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[1] A *concrete algebraic model* of an algebraic theory is a pair $(H, \mathrm{End}(H))$, where H is an appropriate space and $\mathrm{End}(H)$ is a family of operators on H that realise the elements of the theory as transformations of H. This model arises from the theory via an algebraic representation, and satisfies the theory's axioms by virtue of homomorphism conditions that preserve the algebraic structure.

[2] More precisely, this requires the adoption of a *noncommutativisation protocol*: a quantisation procedure that deforms the commutative structures by introducing a noncommutativity parameter. Such a protocol

Finally, in Section 6, I demonstrate that κ-Minkowski spacetime, when interpreted through the notion of noncommutative reference frame, satisfies the appropriate criteria for chronogeometricity.

A final note of clarification: throughout this paper, I refer to the geometry described by these theories as NCST. This usage is primarily historical: many of these theories were originally intended to describe spatiotemproal structure in terms of NCG at high-energy scales. I maintain that this terminology does not introduce confusion regarding the evaluation of chronogeometric features. Specifically, "NCST" is employed here as a label without any presupposed spatiotemporal interpretation, until and unless chronogeometry is established.

2 The Problem of Chronogeometry in QG

Spatiotemporal models are often exhibit *chronogeometry*. This term refers to a geometry's capacity to represent spatial and temporal distances. It has been claimed (e.g., in [2–5]) that not only chronogeometricity is a necessary condition for spatiotemporality; also, that only chronogeometric structures can meaningfully make contact with empirical contexts, thereby enabling an appropriate physical interpretation of relevant mathematical structures.

SR and GR are both expected to be chronogeometric theories. Most of the debate between dynamicists and geometricists has focused on identifying a suitable justification for the claim that, in both cases, the metric field receives chronogeometric content. The former contend that chronogeometry derives from the symmetries of the laws of matter fields, which are encoded by the Lorentz-invariant structure of the metric. Conversely, the latter contend that spacetime chronogeometricity is prior to the identification of rods and clocks: matter fields are Lorentz-invariant because of the underlying metric structure, and so are insufficient to explain its chronogeometricity.

Several theories of QG, including loop quantum gravity, causal set theory, group field theory, and NCG, challenge the strong interrelationship between chronogeometry and spatiotemporality. The standard implication drawn from the extant philosophical literature is that a theory of QG that lacks chronogeometric structure also fails to be properly spatiotemporal (e.g., [3]). While chronogeometry is often acknowledged as a problematic or absent feature in many QG approaches, due to disappearance of spacetime, it is also important to note that the presence of chronogeometric structure alone is insufficient to ensure that a theory is spatiotemporal.

A key motivation for examining chronogeometric properties within QG theories stems from the following consideration: if a given QG theory can be shown to possess chronogeometric models, provided a specific definition of chronogeometry, then the common argument from the absence of chronogeometric structure to non-spatiotemporality can be blocked. Any further claim that such a model is non-spatiotemporal would then require a different line of argumentation, thus shifting the burden of proof to the adversary.

specifies how the new structures depend on this parameter and provides a method for calculating the commutative limit. Compatibility among structures, in this context, means that their respective noncommutative formulations arise from the same noncommutativisation protocol.

In order to identify the specific notion of chronogeometry involved in relevant QG theories, in Section 2.1, I discuss two possible characterisations, namely, operational and representational chronogeometry, and examine their viability in the context of QG. Then, in Section 2.2, I illustrate the metatheoretic function of representational principles for representational chronogeometry, and identify specific conditions that a QG theory, if chronogeometric, must satisfy.

2.1 Spatiotemporality and Chronogeometry

Recent trends in the philosophy of space and time advocate caution in accepting novel mathematical structures as legitimate representations of possible spacetime geometries (see, e.g., [6]). This cautious stance arises from two sources: (i) the increasing proliferation of alternative theories of gravity and spacetime, many of which employ highly abstract mathematical frameworks; and (ii) renewed interest in the historical development of SR and, in particular, GR as spatiotemporal theories. These developments have emphasised the need to clarify what conditions a mathematical structure must meet to count as "spatiotemporal." While this latter notion often includes an array of structures and properties that the models of the theory must satisfy, the present section focuses more narrowly on a specific aspect of spatiotemporality, namely, chronogeometricity.

Broadly speaking, chronogeometricity refers to the capacity of a geometric structure to measure distances, temporal durations, and angles. In a weaker sense, it entails its ability, within a physical theory, to *account* for, or *represent*, such measurements. This property is commonly taken as a demarcation criterion between genuinely spatiotemporal theories and mere mathematical spaces. For instance, Brown [2] repeatedly emphasises the importance of deriving the chronogeometric content of the metric tensor in GR by locally reducing it to the structure of SR; this is motivated by the ability to define sets of measuring rods and clocks in the latter. Similarly, Knox [3] stresses the need for the operational significance of chronogeometry to be grounded in the derivability of inertial frames from the underlying geometry, under suitable approximation conditions. Furthermore, Einstein's aim of reformulating SR as a constructive theory has often been interpreted (particularly by dynamicists like Brown) as rediscovering the central role of operations with rods and clocks in the foundations of relativity on an axiomatic basis.

Chronogeometricity is primarily a property of the models of the concerned theory. If the theory under investigation admits chronogeometric models, then it is chronogeometric. Two distinct notions of chronogeometricity can be identified. The first, which I call *operational chronogeometricity*, requires that the model include, in its domain, identifiable dynamical fields that serve as clocks and rulers, along with appropriate measurement protocols for distances, durations, and angles (see, e.g., [2, 3, 5]). In this view, a model of physical geometry is chronogeometric if and only if it describes the behaviour of material clocks and rods.[3] Specifically, these are characterised by protocols for the measurement of distances and durations that can be implemented

[3]Constructive approaches to spacetime physics may propose alternative targets for chronogeometric structure, including light rays and test particles. The problem of chronogeometry within these approaches, and specifically its relationship with the clock hypothesis, has been examined in [7].

in practical situations, up to idealisations and approximations. The latter typically include, for instance, conventions for synchronising clocks associated to different reference frames. The identification of these protocols is a demanding requirement, rooted in the dynamicist thesis that spacetime geometry is simply a codification of the dynamics and symmetries of matter fields: a "glorious non-entity" [5]. On this view, dynamical matter fields instantiate physical geometry, which is thereby termed "spatiotemporal:" they do not survey any prior underlying structure.[4]

According to this operational definition, many QG theories face significant obstacles in demonstrating the spatiotemporality of the postulated structures. Gravitational fluctuations, for instance, may preclude the identification of suitable dynamical fields. It may be impossible to define reliable clocks for measuring the lengths of timelike worldlines, assuming the theory even preserves a sharp distinction between timelike and spacelike directions (see Section 5.3). Violations of the clock hypothesis, or the failure of other standard constructions of physical clocks (such as those described in [8, ch. 3]), can thus be leveraged as arguments against the spatiotemporality of such theories (see, e.g., [9]).

A second, alternative notion, which I term *representational chronogeometricity*, requires only that the model of the theory *represent* distances, durations, and angles (see, e.g., [10]). Unlike its operational counterpart, this weaker notion does not demand the model to supply concrete implementations or measurement protocols. Instead, it requires that certain components of the model be interpretable in empirical terms. That is, chronogeometricity is achieved when representational links between model and empirical context can be established. In this sense, the emphasis shifts from experimental realisability to the modelling itself.

Notably, this distinction is often blurred in the literature. For example, Fletcher's [10, 7–12] critique of the dynamicist account highlights the latter's conflation of the two notions. He argues that operational chronogeometry pertains to metrology, the science of actual measuring instruments, and should be clearly separated from theoretical considerations about spatiotemporal structure.

This distinction does not, however, preclude the construction of idealised models of clocks and rods. The familiar picture of an ideal clock, such as a photon bouncing between two mirrors (a light clock), remains a legitimate part of certain spacetime models (e.g., in the case of Minkowski spacetime). Yet, the ability to define such an idealised clock does not, in itself, justify the chronogeometricity of the model. Rather, it is a consequence of it satisfying deeper *representational principles* that mediate between the model and empirical, albeit idealised, practices.[5] These principles not only authorise the model's use in representing spatiotemporal features of a target system,

[4]In particular, this use of dynamical fields is guaranteed by the local coincidence of their groups of dynamical symmetries with the group of spacetime symmetries. Consequently, the dynamicist argues, the structure of spacetime is explained by the symmetries of the equations of motion of matter fields. I also emphasise that operational chronogeometry can only be exhibited by those spacetime models that ensure the causal isolation of local measurements. If the selected dynamical fields are nonlocal, then the outcomes of measurement protocols will be influenced by physics outside the scope of the measurement, thereby making any procedure unreliable.

[5]For instance, in the case of GR, Fletcher indicates the existence of test matter, the geodesic principle, the association of photon propagation with null geodesics, and the clock hypothesis as representational principles that secure representational chronogeometricity of the relevant models.

but also justify the construction of idealised probes, such as light clocks, as models of real-world instruments and procedures.

In contrast to operational chronogeometry, representational chronogeometry requires a precise articulation of the representational relations that link the model to its target. As mentioned above, these relations are codified in a set of *representational principles* that the model must satisfy in order to qualify as chronogeometric. This model-based perspective shifts the focus of the problem of assessing chronogeometricity and opens the possibility for QG theories to meet the necessary conditions. This is particularly important, given the lack of direct empirical access to the high-energy regimes they describe. Consequently, such theories may plausibly satisfy representational, though not necessarily operational, chronogeometricity.

2.2 Representational Principles

The explication of the function of representational principles in the context of chronogeometry requires the definition of a suitable conceptual framework. Specifically, it requires the identification of the precise relationship between these principles and the representation relation discussed in the philosophy of scientific modelling (see, e.g., [11]).

A representation relation, or *model representation*, is a relation between the domain of a theory and the intended target. This relation is a *concrete interpretation* of a theory when it provides a semantics for the formal theory.[6]

Interpretations of individual elements of the theory are rarely made in isolation. Typically, the interpretation of a designated structure depends on the presence and interpretation of other elements within the theory, forming what I call a *representational system*. This system provides the necessary background to assign empirical significance to the structure, and may include both interpreted structures and mathematically meaningful elements that lack direct physical targets (e.g., a symplectic geometrical structure). Without such a background, isolated structures may yield ambiguous, conflicting, or incorrect interpretations.

A crucial feature of model representations is that they must be *defeasible*. That is, interpretations should remain open to revision in light of new theoretical developments or conflicting intuitions (or, outside the scope of current QG, conflicting evidence). Following [10, 11–12], a successful interpretation should support a wide range of predictions, accommodate representations from empirically successful prior theories, and furnish new paradigmatic cases. In the context of specific theories, this entails: (i) offering an intelligible theoretical framework (e.g., in the sense of [13]); (ii) maintaining compatibility with interpretations of other theories (e.g., with relativistic theories in the case of QG); and (iii) producing physically meaningful case studies as

[6] In philosophy of science there is a vast debate concerning the specification of this representation relation, often with heterogeneous answers. In this work, I remain neutral with respect to specific accounts. As discussed below, the analysis of NCST approaches only requires a local definition of this relation, in the sense of being intended to provide a background for an appropriate explication of chronogeometry in this specific context. It thus has no ambition of generality. Nevertheless, it is important to note that the model representation relation is expected to be extremely complicated: naive accounts of the representation relation as direct designation of the target may be inappropriate. Indeed, theories of QG are still far from being "used in practice" like other standard theories. Furthermore, we can expect the target to be a model system itself (in the sense, e.g., of [12]), due to both idealisation from known physics and impossibility to experimentally access the relevant regimes.

applications. Moreover, model representations must remain responsive to theoretical refinement, conceptual change, and evolution of *in fieri* theories.

The interpretation of elements within a representational system can be guided and constrained by *representational principles*. These are metatheoretical principles that connect abstract structures of a theory to empirical content. In standard physics, such principles include: identifying (q, p) as position and momentum in classical mechanics; assigning inertial frames to freely falling systems; interpreting the length of timelike worldlines as proper time; and applying the Born rule to compute measurement probabilities in quantum theory. Each metatheoretical property of the models of a theory is thus governed by an associated set of representational principles, and failure to satisfy these principles indicates that the model lacks that property.

In the context of this paper, the focus is specifically representational chronogeometry. Since this is a property of models (and, by extension, of the corresponding theories), it must be associated with a specific set of representational principles. These principles serve to identify, within the domain of each model, those elements that can be meaningfully interpreted as representing durations, distances, or angles in appropriate representational systems. Accordingly, a model is *chronogeometric* if it contains elements that satisfy (or "saturate") chronogeometric representational principles. A theory, in turn, is chronogeometric if it admits at least one such model.

To illustrate, consider the following representational principle in the context of GR:

(CH) γ is a timelike curve if and only if $|\gamma|$, i.e., the length of γ, represents the duration of the events in $\gamma[I]$, for $I \subseteq \mathbb{R}$, i.e., the proper time elapsed between two such events. [10, 3]

From an operationalist perspective, (CH) can be substantiated by indicating certain configurations of dynamical fields, used as reliable clocks for measuring intervals of proper time along worldlines. In other words, satisfaction of the so-called *clock hypothesis* guarantees that (CH) serve as a chronogeometric principle. The specific realisation of this clock, however, is debated (see, e.g., [14]).

On the other hand, representational chronogeometry can introduce (CH) as a suitable representational principle without specifying any further justification for it. In other words, (CH) can indicate a constraint on timelike curves that brings them in contact with empirical contexts. At the same time, satisfaction of (CH) requires for the concerned model to possess sufficient structures that can be interpreted according to the appropriate representational system, including timelike-ness, worldlines, length, and events. In other words, (CH), if adopted as a chronogeometric principle, can only be satisfied by those models that contain suitable structures to be interpreted as timelike worldlines, lengths of these worldlines, and events. Absence of such structures implies that the model cannot be chronogeometric.

3 NCST from Quantum Groups

NCST theories are always formulated in algebraic terms. This primacy of an algebraic over a geometric formulation is often motivated by their construction process. To illustrate, consider an algebraic description of a relativistic structure. Its noncommutative deformation prevents any classical differential geometric structure from encoding the same algebraic content. Instead, NCST theories raise the issue of identifying new kinds of geometric structures that can serve as duals to NCST algebras. This difficulty arises because commutative dualities between algebraic and geometric model (including Gelfand duality), central to classical scenarios, break down due to the introduction of noncommutativity, and so new duality relations must be identified.

Algebraic theories raise important questions when they are intended to describe the quantum gravitational regime. Certain approaches to QG incorporate algebraic methods through combinatorial reasoning. For instance, groups field theory relies on the identification of simplexes as fundamental building blocks in the constitution (or emergence) of spacetime structure at the appropriate scale. Each simplex is described by a state in a Hilbert space, and gluing of simplexes corresponds to tensoring the associated Hilbert spaces to describe the collective state of these atoms of spacetime.

In contrast, NCST theories directly associate an algebra to the quantum gravitational structure they aim to describe. These theories are not combinatorial in nature. The algebra itself encodes all the relevant relativistic structure (affine, differential, projective, and so on) and, in the commutative limit, enables the reconstruction of smooth differentiable manifolds. In the noncommutative regime, however, the manifold structure is replaced by more complicated geometries. As a result, the connection between the mathematical geometry described by a NCST theory and the standard spatiotemporal picture becomes subtle. This complexity brings forth a central question: in what sense can an algebraic theory be considered spatiotemporal?

This section frames the question within the context of NCST theories. Specifically, in Section 3.1, I introduce the notion of strict algebraicism, which characterises the fundamental stance underlying NCST approaches. In Section 3.2, I present the κ-Poincaré quantum group as a paradigmatic case study of strict algebraicism. In Section 3.3, I outline how the corresponding NCST structure is reconstructed from this algebraic structure. Finally, in Section 3.4, I argue that quantum groups can be interpreted as algebras of (deformed) symmetries of the underlying NCST geometry. Emphasis is placed on the importance of algebraic representations, which are essential for assigning representational content to the abstract algebraic theory.

3.1 Strict Algebraicism and Representations

Of course, algebraic approaches are not used exclusively in QG (see, e.g., [15, 16]). Varied instances of this practice share a core insight: algebraic methods can offer significant conceptual and practical advantages over more conventional frameworks. Not only do they preserve many of the essential features typically expressed in differential geometric terms (e.g., using tensor fields); they can also provide alternative research strategies that are equally, if not more, effective.

The viability of an algebraic approach hinges on whether physicists can address each of the following three questions [17]:

1. Which structures are relevant to the formulation of the physical theory?

2. Which algebraic structures can be defined within this theory, that is, which structures are supported by the underlying theory?
3. Among all possible algebraic structures, which ones are physically meaningful, and which are purely mathematical?

To illustrate, consider a field theory formulated in the language of differential geometry. In response to question 1., one might list a number of relevant features, such as having the structure of a topological manifold or an affine connection. Question 2. then concerns identifying algebraic structures that are well-defined on these geometric foundations. For instance, every open region of the manifold supports a C^*-algebra of complex-valued smooth functions. Similarly, the affine structure permits the definition of an affine connection as a derivation operator acting on the algebra.

Finally, question 3. requires singling out those algerbraic structures that bear physical interpretation. Local C^*-algebras of smooth functions can be interpreted as local fields, that is, observables, while the affine connection operator defines geodesic motion, representing curves along which generating vector fields are parallel transported. By contrast, inner automorphisms of the C^*-algebra may be considered formal redundancies rather than physical structures.

A more radical position, which I term *strict algebraicism* or *algebra-first approach*, asserts that algebraic methods are not only advantageous but fundamentally necessary: they are "the only game in town" [1, 4697]. This is primarily a methodological claim: certain problems can only be adequately addressed through algebraic means, either because algebraic tools are uniquely effective or because no viable alternatives exist.[7] Consequently, strict algebraicism does not necessarily deny the definability of non-algebraic frameworks; rather, it prioritises the algebraic approach due to either practical convenience (*pragmatic strict algebraicism*) or conceptual necessity (*theoretical strict algebraicism*).

Pragmatic strict algebraicism is provisional: new non-algebraic methods may eventually match or surpass the utility of current algebraic ones. In contrast, theoretical strict algebraicism demands a principled argument that no alternative non-algebraic framework is possible. In other words, it indicates a foundational limitation.

Support for strict algebraicism can be found in algebraic geometry. It is commonly accepted that all the information contained in a differentiable manifold can be encoded in a $*$-algebra $C^\infty(M)$ of smooth, complex-valued functions on the manifold M with pointwise commutative multiplication. Conversely, any commutative C^*-algebra can be realised as the algebra of smooth, complex-valued functions on some smooth manifold. This correspondence is formalised by Gelfand duality. Geroch's [16] proposal to model physical fields directly via the elements of these algebras (modulo inner automorphisms) reflects this insight: the algebraic fields admit direct representation as physical fields.

To illustrate, one can encode the full structure of Minkowski spacetime in a suitably defined *Minkowski algebra* $\mathcal{M}$. It suffices to associate self-adjoint operators X^μ to

[7] The emphasis on methodology is pivotal. Indeed, strict algebraicism has no direct bearing on the question of the status of the algebraic structures it uses. One is free to take them as real (algebraic substantivalism) or as proxies for solving problems (algebraic anti-realism). These metaphysical claims are independent of the usage of algebraic methods in scientific and mathematical practice.

physical events in Minkowski spacetime.[8] These operators, acting on a suitable Hilbert space (e.g., $L^2(\mathbb{R}^3)$, the space of square-integrable functions on $\mathbb{R}^3$), can be interpreted as localisation procedures relative to a chosen reference frame. The action of spacetime coordinate operators on states in $L^2(\mathbb{R}^3)$ determines the spatiotemporal location of the corresponding physical events. The mutual independence of spacetime coordinates is expressed via the commutation relations $[\mathrm{X}^\mu, \mathrm{X}^\nu] = 0$, ensuring that the uncertainty in measuring coordinates vanishes:

$$\Delta \mathrm{X}^\mu \Delta \mathrm{X}^\nu = \frac{1}{2} |\langle [\mathrm{X}^\mu, \mathrm{X}^\nu] \rangle| = 0. \tag{1}$$

Therefore, events can in principle be sharply localised in spacetime.

Both cases of a differentiable manifold and Minkowski spacetime indicate how algebraic elements can be realised on specific structures: in one case as functions on the manifold, in the other as operators on a Hilbert space. However, these concrete realisations potentially limit the generality sought by strict algebraicism. Indeed, a key strength of the algebraic approach lies in its abstraction. Algebraic elements possess mathematical meaning independently of the structures that realise them. They are not reducible to their action on a particular space or support structure. For this reason, the strict algebraicist must clearly distinguish the abstract algebra and its various representations, such as sets of transformations acting on a manifold.

For instance, Menon [19, 5] characterises an algebraic field as a "*non-manifold-based* representation of matter" (my emphasis). He writes:

> If the abstract algebra out of which the dynamically possible algebraic fields are constructed admits a realisation as a set of material scalar fields, then all of the information about the underlying manifold–its topological and smooth structure–is already contained in the abstract algebra. (*ivi*)

Consequently, the algebraic and manifold-based formulations are not only formally equivalent; the algebraic description may be considered more fundamental, or privileged, dispensing with the manifold entirely: "[o]n the algebraic field view, all we are doing is taking the field to be primitive."

While the abstract algebra may encode geometric information, this is only evident when a representation on a manifold is available: strict algebraicism depends on representations. This is not necessarily problematic, unless the algebraicist seeks to eliminate the manifold altogether.[9] In doing so, she risks undermining the interpretability of an algebra in contexts where no such manifold can be defined: the algebra may lack a clear physical interpretation.

The issue is especially pressing in NCG. Here, the algebraic approach has been advocated as not only useful but indispensable, a claim that aligns with both pragmatic

[8] Although the definition of a time operator is complicated: see, e.g., [18].

[9] Compare with the case of algebraic structuralism: see [20]. In that case, no ontological commitment should be attached to the concrete representations of a given algebra; rather, the representational content of the algebra is entirely associated to the underlying abstract structure that is common to all different representations. Moreover, it is interesting to note that the same issues arise outside the context of QG. It could be argued that strict algebraicism also applies to quantum theories in general, particularly to algebraic QFT. However, as discussed below, NCG differs from these other theories in that it only admits an algebraic formulation. Whether strict algebraicism in NCG is merely pragmatic or also theoretic remains an open issue.

and theoretical strict algebraicism (see, e.g., [1]). NCG thus provides a privileged case study for evaluating the viability of strict algebraicism in the absence of manifold-based representations. To assess this, one must first define NCST models in properly algebraic terms. This task requires a preliminary analysis of the underlying quantum group structure.

3.2 Quantum Groups

Quantum groups lie at the core of the eponymous approach to NCG. In this section, I present a technical treatment of their construction, whereas for an informal overview of their main features and historical reconstruction of their development, see [21]. Such a reconstruction is motivated by the aim of examining the algebraic structures underlying NCST models and clarifying how these structures are related to one another. To maintain independence from specific spatiotemporal realisations, I first define the relevant algebraic framework in an abstract form, postponing interpretational issues to Section 3.3.

To begin, recall that an algebra $\mathcal{A}$ over a field $\mathbb{K}$ is a vector space equipped with an appropriate product. Dually, a *coalgebra* $\mathcal{C}$ over a field $\mathbb{K}$ is a vector space with a *coproduct* $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and a *counit* $\epsilon : \mathcal{C} \to \mathbb{K}$, satisfying coassociativity and counit conditions. The coproduct is essential for defining representations of the coalgebra. In fact, it allows to define a representation of $\mathcal{C}$ on a tensor of vector spaces, while ensuring that the representation is linear, coassociative, and homomorphic. A coproduct is said to be *cocommutative* if $\Delta(a) = a^1 \otimes a^2 = a^2 \otimes a^1$.

$\mathcal{A}$ and $\mathcal{C}$ form a categorical-dual pair: their structures can be expressed using commutative diagrams where the morphisms (arrows) are reversed between the two. This duality implies that the algebraic structure of a coalgebra $\mathcal{C}$ can be derived from a corresponding algebra $\mathcal{A}$, and vice versa, without loss of information. Accordingly, any deformation applied to one structure induces a corresponding deformation in the other. This leads to the concept of a *bialgebra*, a structure $(\mathcal{B}, \mu, \eta, \Delta, \epsilon)$ where $\mathcal{B}$ is simultaneously an algebra, with product μ and unit η, and a coalgebra, with coproduct Δ and counit ϵ.[10] The coproduct and counit of a bialgebra must satisfy suitable homomorphisms compatibility conditions.

A *Hopf algebra* over $\mathbb{K}$ is a bialgebra $(\mathcal{H}, \mu, \eta, \Delta, \epsilon)$ equipped with an additional map, called the *antipode* $S : \mathcal{H} \to \mathcal{H}$, with appropriate axioms. The antipode acts as an anti-homomorphism on the elements of the algebra. Furthermore, the composition of product, antipode, and coproduct must satisfy the *Hopf identity*, which codifies the relationship between group inverse and identity and implies that the antipode generalises the notion of group inverse with respect to Δ.

Many familiar algebraic structures can be trivially promoted to Hopf algebras by appropriately enriching their structure. For instance, the space $\mathcal{C}(G)$ of continuous functions on a finite group G becomes a Hopf algebra by defining product and unit in a natural way, coproduct and counit as actions of the functions on elements of G, and the antipode as the action of functions on inverse group elements.[11] In this

[10] By construction, μ is the opposite map of Δ, and η the opposite of ϵ. This inversion of the arrows of the commutative diagrams ensures the compatibility of the two structures due to the category-theoretic duality.

[11] Similarly, the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ can be extended to a Hopf algebra $U(\mathfrak{g})$. Here, Δ defines the actions of elements of $U(\mathfrak{g})$ on tensor products and codifies the Leibniz rule. If $\mathfrak{g}$ is a

construction, Δ induces a group multiplication law, ϵ defines the identify element of G, and S defines the group inversion.

Given this framework, a *quantum group* is defined as a deformation of Hopf algebras.[12] This means that, given a Hopf algebra $\mathcal{H}$, the elements of the quantum group $\mathcal{H}_q$ are required to reduce to elements of $\mathcal{H}$ modulo q (i.e., $x \sim y \bmod q$ if and only if $x - y \propto q^n$ for some n). Similarly, the product on $\mathcal{H}_q$ is a deformation of the product on $\mathcal{H}$ that trivialises in the *quasiclassical limit*, i.e., as $q \to 0$.

3.3 From Deformed Symmetries to Spacetime Models

The commutative case indicates a clear and well-established relationship between groups and manifolds. The construction of a manifold as the support for a group of symmetries relies on the decomposition of the latter into a semi-direct product of subgroups. Here, the manifold is reconstructed as the quotient of the group with one of its isotropy subgroups.

A classical example of this construction is the derivation of the Minkowski algebra from the Poincaré group $\mathcal{P}_4$. In this case, $\mathcal{P}_4$ can be written as a semi-direct product of the translation group T_4 (a normal subgroup) and the Lorentz group $SO(1,3)$, which is the isotropy group at the origin: $\mathcal{P}_4 = SO(1,3) \ltimes T_4$. The resulting homogeneous space, Minkowski spacetime, is then identified with the quotient: $\mathcal{M}$ is isomorphic to $T_4 = \mathcal{P}_4/SO(1,3)$.

A natural question arises: can this construction be extended to quantum groups? The answer is yes. To show this, first define a *right-action* of a Hopf algebra $\mathcal{H}$ on an algebra $\mathcal{A}$. This is a linear map $\triangleleft : \mathcal{A} \otimes \mathcal{H} \to \mathcal{A}$, $a \otimes x \mapsto a \triangleleft x$, such that $a \triangleleft xy = (a \triangleleft y) \triangleleft x$ and $a \triangleleft 1 = a$ for $x, y \in \mathcal{H}$, $a \in \mathcal{A}$. It codifies the effect of applying elements of $\mathcal{H}$ on the elements of $\mathcal{A}$. This action is *covariant* if it preserves the structure of $\mathcal{A}$, i.e., if it is an algebra homomorphism. Similarly, a *left-coaction* is a linear map $\mathcal{A} \to \mathcal{C} \otimes \mathcal{A}$, where $\mathcal{C}$ is a coalgebra, specifying how elements of the algebra transform under the coalgebraic sector.

A *bicrossproduct algebra* $\mathcal{Q} \bowtie \mathcal{A}$ is constructed as a tensor product of Hopf algebras, $\mathcal{Q} \otimes \mathcal{A}$, equipped with compatible left-coaction and right-action. In such a structure, both the algebraic and the coalgebraic sectors are allowed to act on each other under appropriate compatibility conditions. This bicrossproduct structure is the base for reconstructing an underlying space. To illustrate this, the κ-Poincaré algebra serves as a paradigmatic example of derivation of a NCST model, namely the κ-Minkowski algebra $\mathcal{M}_\kappa$:[13]

$$[\mathrm{X}^0, \mathrm{X}^j] = \frac{i}{\kappa}\mathrm{X}^j \quad , \quad [\mathrm{X}^i, \mathrm{X}^j] = 0 \tag{2}$$

Lie algebra and G its corresponding Lie group, then the Hopf algebras $U(\mathfrak{g})$ and $\mathcal{C}(G)$ are dual. Note that any Lie algebra can be extended to a Lie bialgebra, whose structure is characterised by the introduction of a tensor, called the *classical r-matrix* r. Moreover, a Lie bialgebra can be further equipped with a Poisson structure. For instance, the smooth functions of a Poisson-Lie group define a Poisson-Hopf algebra that serves as base for the extension of classical symmetry groups to the quantum group framework: see Section 3.3.

[12] For each classical case, there are typically two Hopf algebras that can be quantised. One can either quantise the Hopf algebra of smooth functions on a Poisson group, or the universal enveloping of the Poisson algebra. These are two possible ways to characterise the resulting quantum group.

[13] Roman indices indicate spatial dimensions: $i, j = 1, 2, 3$.

$\mathcal{M}_\kappa$ can be characterised as a structure that supports the κ-Poincaré Hopf algebra as its symmetry quantum group. This implies identifying κ-Poincaré as a Hopf algebra of continuous functions on the Poincaré group $\mathcal{P}_4$, write $\mathcal{C}_\kappa(\mathcal{P}_4)$, with the condition that the quantum group preserves the κ-Minkowski commutation relations (2). Here, κ refers to the noncommutative parameter that governs the deformation of the commutative structures.[14]

Let this quantum group be generated by the self-adjoint operators Λ^μ_ν and a^ρ acting on a suitable Hilbert space, e.g., $L^2(SO(1,3)\times\mathbb{R}^3)$. These operators indicate, respectively, the generators of Lorentz transformations and translations. The problem of describing the κ-Poincaré algebra reduces to that of studying the conditions under which a map:

$$\mathcal{M}_\kappa \to \mathcal{C}_\kappa(\mathcal{P}_4)\otimes\mathcal{M}_\kappa \quad , \quad \mathrm{X}^\mu \mapsto \mathrm{X}'^\mu = \Lambda^\mu_\nu\otimes\mathrm{X}^\nu + a^\mu\otimes 1, \tag{3}$$

on the generators X^μ of the κ-Minkowski algebra, leaves the commutation relations invariant.

This occurs if the Lorentz sector remains undeformed,

$$[\Lambda^\mu_\nu, \Lambda^\alpha_\beta] = 0, \tag{4}$$

whereas both translation sector and cross-relations are deformed:

$$\begin{aligned} [a^\mu, a^\nu] &= -\frac{i}{\kappa}(a^\mu\delta^\nu_0 - a^\nu\delta^\mu_0) \\ [\Lambda^\mu_\nu, a^\alpha] &= -\frac{i}{\kappa}((\Lambda^\mu_0-\delta^\mu_0)\Lambda^\alpha_\nu + (\Lambda_{0\nu}-g_{0\nu})g^{\mu\alpha}) \end{aligned} \tag{5}$$

In other words, while the noncommutativity is entirely encoded by the translation sector and its cross-relations with the Lorentz sector, rotations and boosts are classical.

The effects of $\mathcal{C}_\kappa(\mathcal{P}_4)$ on $\mathcal{M}_\kappa$ can be specified by an appropriate left-action. This requires the definition of the Hopf algebra structure of $\mathcal{C}_\kappa(\mathcal{P}_4)$.[15] In particular, one can define coproducts, counits, and antipodes as follows:

$$\Delta(a^\mu) = \Lambda^\mu_\nu\otimes a^\mu \qquad \epsilon(a^\mu) = 0 \qquad S(a^\mu) = -a^\nu(\Lambda^{-1})^\mu_\nu$$

[14] Alternatively, the κ-Poincaré algebra can be defined as the deformation of the Hopf universal enveloping algebra $U(\mathfrak{p})$ of the Poincaré algebra. This construction quantises the universal enveloping of the de Sitter algebra, and then derives $U(\mathfrak{p})$ by contraction, i.e., by sending the de Sitter radius $R\to\infty$ while $iR\log q\to\kappa^{-1}$, The resulting algebra differs from $\mathcal{C}_\kappa(\mathcal{P}_4)$: translation and rotation sectors are undeformed, whereas boosts and boost-momentum cross-relations are highly deformed. Similarly, the coproducts of spatial translations and boosts exhibit noncommutative features that introduce non-classical correlations when acting on pairs of states. This algebra is not in a nice bicrossproduct basis, as the Lorentz sector does not form a subalgebra. The derivation of $\mathcal{M}_\kappa$ can only be obtained by a nonlinear transformation into the Maid-Ruegg basis. Here, the Lorentz sector is not deformed, whereas the cross-relations between translations and Lorentz transformations are. The resulting bicrossproduct Hopf algebra is $U_\kappa(\mathfrak{p}) = U(\mathfrak{so}(1,3)) \bowtie T_4$, where $\mathfrak{so}(1,3)$ is the Lorentz algebra and $U_\kappa(\mathfrak{p})$ is dual to $\mathcal{C}_\kappa(\mathcal{P}_4)$. The quantum group is the semidirect product of an undeformed Lorentz algebra, with a deformed action on translations as algebraic sector, and the translation algebra, with the coaction of the Lorentz algebra as coalgebraic sector. The standard coset procedure then derives $\mathcal{M}_\kappa$ as the algebra on which $U_\kappa(\mathfrak{p})$ acts covariantly.

[15] Note that $\mathcal{C}(\mathcal{P}_4)$ is a Poisson-Lie algebra, and so the group laws must be compatible with the Poisson structure as well as the undeformed Leibniz rule. These constrain the possible quantum group laws, as expressed by coproduct and antipode.

$$\Delta(\Lambda^{\mu}_{\nu}) = \Lambda^{\mu}_{\alpha} \otimes \Lambda^{\alpha}_{\nu} \qquad \epsilon(\Lambda^{\mu}_{\nu}) = \delta^{\mu}_{\nu} \qquad S(\Lambda^{\mu}_{\nu}) = (\Lambda^{-q})^{\mu}_{\nu}. \tag{6}$$

The structure (6) completes the *κ-Poincaré algebra*.[16]

In an appropriate basis (the so-called Majid-Ruegg basis), $\mathcal{C}_{\kappa}(\mathcal{P}_4)$ can be written as a bicrossproduct of two algebras:

$$\mathcal{C}_{\kappa}(\mathcal{P}_4) = T_4^* \bowtie \mathcal{C}(SO(1,3)). \tag{7}$$

$\mathcal{C}(SO(1,3))$ is the commutative algebra of continuous functions on $SO(1,3)$, whereas T_4^* is the dual of the noncommutative translation sector. By means of the classical construction, one obtains that $\mathcal{M}_4 \sim T_4^*$ is the coset of $\mathcal{C}_{\kappa}(\mathcal{P}_4)$ in this basis.

In conclusion, the correspondence between the noncommutative structure of the NCST algebra and the quantum group is mediated by the cross-actions defined by an appropriate choice of bicrossproduct basis. Specifically, the construction of $\mathcal{M}_{\kappa}$ as a coset of the κ-Poincaré algebra supports the interpretation of the latter as the algebra of quantum symmetries associated with this NCST model. These quantum symmetries preserve the commutation relations of the spacetime coordinate operators by construction. This invariance is ensured by the covariance condition imposed on both right-action and left-coaction, as encoded in the bicrossproduct structure.

3.4 Quantum Groups and NCST: What relation?

The characterisation of NCST models as cosets of the corresponding quantum groups in a suitable basis is crucial for the algebra-first approach. However, this characterisation remains purely algebraic. This raises a crucial question: can such a NCST algebra exhibit any spatiotemporal feature at all? If not, its inability to satisfy even some weak notion of spatiotemporality would represent a serious limitation to the strict algebraic approach. The issue is further compounded by the fact that, in the current formulation of NCST theories via quantum groups, only algebraic methods are available.[17]

One possible resolution is to extend the construction beyond its purely algebraic setting. As discussed in Section 3.1, representation theory plays a pivotal role in applying algebraic structures. More precisely, the introduction of a representation theory allows one to build concrete models of the concerned theory, being essential to guarantee the success of strict algebraicism. In the case of NCST, the algebraic theory is initially defined via commutators between the generators of the coordinate algebra. To obtain a concrete model, the theory must be equipped with a suitable representation of algebras, typically in the form of a triple $(\mathcal{A}, \rho, H)$, where ρ is a representation of the algebra $\mathcal{A}$ as operators on a support space H. In particular, H is the Hilbert space

[16] Alternatively, I will refer to this as the κ-Poincaré group, with the understanding that it is a *quantum* group.

[17] Notably, alternative approaches to NCG, such as Connes' spectral triple approach, also employ geometric methods ranging from algebraic geometry to functional analysis. This is because spectral triples, the fundamental structures used to characterise different models, comprise a Hilbert space along with a representation of the relevant algebra as a family of operators acting upon it. Further geometric information is encoded by the Dirac operator, an unbounded differential operator that acts on the algebra and enables the definition of a topological metric. For more details, see [22, ch. 1, §10].

containing the states on which the representations of elements of $\mathcal{A}$ act. The homomorphism condition ensures that the representation of the algebraic theory preserve the invariant algebraic structures and carry them over into the NCST models.

Importantly, this representation is generally non-unique, so specifying it is necessary for constructing a specific algebraic model.[18] It is this model, not the abstract algebraic theory, that represents a NCST structure. While the commutation relations define the core structural features of NCST, a concrete model (potentially) with spatiotemporal features requires well-defined *states* for events. Crucially, these states encode the salient properties of events, including, for example, their localisation properties, but cannot be extracted from the bare abstract theory alone.

A specific example of this construction is provided by the representation of κ-Minkowski and its associated quantum symmetries. For simplicity, the following presentation focuses on the kinematical aspects of the model, leaving out momentum space considerations.

Recall that the κ-Minkowski algebra $\mathcal{M}_\kappa$ has the following commutator structure:

$$[\mathrm{X}^0, \mathrm{X}^j] = \frac{i}{\kappa}\mathrm{X}^j \quad , \quad [\mathrm{X}^i, \mathrm{X}^j] = 0, \tag{8}$$

where κ is the noncommutative parameter. $\mathcal{M}_\kappa$ has a Lie-algebra-type noncommutativity between the time coordinate operator and the spatial coordinate operators. By a suitable substitution of the generators with a polar basis of self-adjoint operators $(\mathrm{R}, \cos(\theta), \mathrm{e}^{i\phi})$, one obtains:

$$[\mathrm{X}^0, \cos(\theta)] = [\mathrm{X}^0, \mathrm{e}^{i\phi}] = 0 \quad , \quad [\mathrm{X}^0, \mathrm{R}] = \frac{i}{\kappa}\mathrm{R}. \tag{9}$$

This indicates that $\mathcal{M}_\kappa$ is equivalently characterised by noncommutativity between the time coordinate operator and the radial coordinate operator R.

Drawing an analogy with quantum mechanics, we can represent the algebra $\mathcal{M}_\kappa$ as a set of self-adjoint operators acting on a Hilbert space. One such representation assigns:

$$\begin{aligned} \rho(\mathrm{X}^i)\psi(x) &= x^i\psi(x) \\ \rho(\mathrm{X}^0)\psi(x) = \frac{i}{\kappa}\Big(x^i\partial_i + \frac{3}{2}\Big)\psi(x) &= \frac{i}{\kappa}\Big(r\partial_r + \frac{3}{2}\Big)\psi(r), \end{aligned} \tag{10}$$

acting on normalised wavefunctions $\psi \in L^2(\mathbb{R}^3_x)$.[19] When the action of $\rho(\mathrm{X}^0)$ is expressed in polar coordinates, it can be shown that the time operator commutes with all spherical harmonics, but not with functions depending on the radial coordinate r.

[18] This is the case unless proved otherwise. For instance, the definition of the GNS representation of a finite-dimensional C^*-algebra offers a case of unitary equivalence between all irreducible representations. In the more general context, such specification modifies the concrete model. This has been emphasised not only in the case of field theories from a more philosophical perspective (see, e.g., [23]), but also in the specific case of κ-Poincaré: see, e.g., [24].

[19] The subscript x indicates the basis for the integral representation of the wavefunctions. Note that the term $3/2$ is added for symmetrisation reasons.

The operators must form a complete basis. In the case at hand, there are two possibilities: $(\mathrm{R}, \cos(\theta), \mathrm{e}^{i\phi})$ or $(\mathrm{X}^0, \cos(\theta), \mathrm{e}^{i\phi})$. Here, (improper) states can be represented in either basis and related via integral transforms, ensuring no information is lost when translating between them.[20]

Given a basis, localised states can be constructed as limits of normalised vectors in $L^2(\mathbb{R}^3_x)$ by saturating the generalised uncertainty relations implied by $\mathcal{M}_\kappa$. In particular, it is possible to achieve sharp localisation of a state in both space and time by using a family of approximating functions, the *log-Gaussians*, provided that they are either centered at the origin, or peaked at another point with null variance. Consequently, the model allows the complete localisation of a state at the origin; this is allowed by the construction of this state as the limit of normalised elements of the Hilbert space. However, localisation at the origin cannot be extended to other arbitrary states.

Furthermore, the time evolution of such a localised state at the origin can then be precisely tracked, for instance via a one-parameter family of states representing its evolution through time. However, sharp localisation is restricted for states away from the origin: in such cases, spatial localisation necessarily introduces temporal uncertainty. This limitation is imposed by the intrinsic noncommutative structure of the spacetime model.

Up to this point, the construction of an algebraic model has relied solely on a representation of $\mathcal{M}_\kappa$. However, any viable model for a description of NCST must also exhibit invariance under the relevant group of quantum symmetries. This requirement arises not only from the abstract definition definition of $\mathcal{M}_\kappa$ as a coset of the κ-Poincaré group, but also from the need for consistency at the level of concrete models.

A realisation of the κ-Poincaré algebra is a representation of its generators on a suitable Hilbert space. As illustrated in Section 3.3, the algebra $\mathcal{C}_\kappa(\mathcal{P}_4)$ contains an undeformed Lorentz sector, whose representation theory is well understood. Specifically, the Lorentz sector can be realised by multiplication operators $\rho(\Lambda)$ acting on wavefunctions in $L^2(SO(1,3))$, whereas the representation of translation sector carries the noncommutative deformation over to the state space.

To obtain a faithful representation,[21] it is necessary to extend both the action of the translations and the Hilbert space to include a representation of $\mathcal{M}_\kappa$. In other words, the representation of $\mathcal{C}_\kappa(\mathcal{P}_4)$ must be constructed as a direct sum of two representations on $L^2(SO(1,3) \times \mathbb{R}^3 \times \mathbb{R}^3)$, subject to the condition that it also satisfies the nontrivial commutation relations characterising the translation sector.

This construction ensures that the representation of $\mathcal{C}_\kappa(\mathcal{P}_4)$ is sufficiently rich: any vector in $L^2(\mathbb{R}^3)$ (i.e., the support space of the representation of $\mathcal{M}_\kappa$) can be approximated as a limit of vectors in $L^2(SO(1,3) \times \mathbb{R}^3)$ (i.e., the partial support of the representation of $\mathcal{C}_\kappa(\mathcal{P}_4)$), under suitable conditions. Accordingly, the algebraic relationship between $\mathcal{C}_\kappa(\mathcal{P}_4)$ and $\mathcal{M}_\kappa$ is preserved in the concrete model through a judicious choice of representations. In other words, a concrete model of κ-Minkowski

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[20] Similarly, in quantum mechanics states can be expressed in either position or momentum basis via the Fourier transform. In the case of κ-Minkowski, [25] suggests the use of the Mellin transform in order to transform a basis of observables in the radial coordinate r to a basis of observables in the temporal coordinate τ.

[21] Faithfulness guarantees that distinct algebraic elements are represented as distinct linear maps.

can be recovered by selecting particular states in the representation of the κ-Poincaré algebra, provided these states are localised around the origin.

4 The Problem of Chronogeometry in NCG

As emphasised, an algebraic theory, such as $\mathcal{M}_\kappa$, can give rise to multiple inequivalent concrete models, depending on the chosen algebraic representation. Accordingly, inequivalent representations yield different families of concrete models. Within this context, one can ask whether at least one such family of concrete models can adequately represent a spatiotemporal structure.

Chronogeometry is often considered a necessary condition for spatiotemporality (Section 2), and so NCST approaches share with other QG approaches the challenge of demonstrating the chronogeometricity of their models. In addition, strict algebraicism requires the identification of a precise relationship between the algebraic structures and the elements of the domain of each concrete model, so that the algebraic representation of the former facilitates the ascription of empirical content.

The examination of the representational chronogeometric content of NCST models, therefore, requires a suitable conceptual framework that accounts for their algebraic foundations.[22] Here, the distinction between the *algebraic representation* of a theory, and the *model representation* relation (central to the philosophy of scientific modelling and discussed in Section 2.2) becomes crucial.

An algebraic representation provides a mathematical realisation of the elements of a structure: in this case, an algebra. These elements are typically represented as operators on a Hilbert space. This realisation concerns the formalism of the NCST theory: it does not, in itself, establish a direct link to any intended physical target, beyond the mathematical interpretation of the algebraic elements. Crucially, any finite-dimensional C^*-algebra is a direct sum of matrix algebras, and its algebraic representations are determined (up to unitary equivalence) by the multiplicities with which each irreducible component appears; changing the multiplicities thus results in inequivalent representations of the same algebra, some of which may have even different dimensions. If each of these algebraic representations were treated as a distinct model representation, the algebra would then designate different physical targets, contrary to expectations.

Nonetheless, an algebraic representation remains a necessary step for identifying a concrete model that can be compared with a physical system. Let us denote a concrete model of the algebraic theory T, under a chosen representation π_i, as Mod_{π_i}. Similarly, $\{\mathrm{Mod}_{\pi_i}\}_{i\in I}$ denotes a family of concrete models, each one associated with a different indexed representation. For each concrete model, the domain $\mathrm{dom}(\mathrm{Mod}_{\pi_i})$ consists of all transformations that realise the elements of T. Then, a model representation assigns a physical target to each element of this domain, under the constraints imposed by the algebraic structure of Mod_{π_i}: let this be a concrete interpretation ι of the model. Therefore, an interpretation of T is given by the pair (ι, π_i), where π_i is an algebraic representation of T, and ι is a model representation compatible with it.[23]

[22] Comparable analyses have been undertaken, for example, in the philosophy of QFT: see [23].
[23] Compare this definition with [26, ch. 1].

To illustrate, consider the concrete model introduced in equation (10). This model admits an interpretation in terms of localisation procedures. Each operator $\rho(\mathrm{X}^\mu)$ can be interpreted as a measurement on a state in a Hilbert space, yielding the spatiotemporal position of the corresponding event along the x^μ-axis. Due to the idealised nature of the model, this interpretation does not necessarily involve physical interaction with the localised system. However, the interpretation is incomplete without the specification of a reference frame and an observer situated at its origin: these constitute the representational system that supports the present interpretation. Representational adequacy further requires that the reference frame accommodate uncertainty in position when the system is localised along an axis x^ν, given that $[\rho(\mathrm{X}^\mu), \rho(\mathrm{X}^\nu)] \neq 0$.

While this interpretation is a necessary step towards spatiotemporality, it is still insufficient to ground chronogeometricity. Specifically, the peculiar localisation properties of the NCST model raise concerns regarding the definability of local measurements of distances and durations. It is expected that chronogeometric representations of NCST models will face limitations, particularly due to the presence of a noncommutativity parameter, which constrains their regime of applicability. This limitation also affects the identification of candidate physical targets. Moreover, given a specified family of chronogeometric representational principles, it is necessary to identify additional model representations to complete the representational system needed to satisfy them. These additional interpretations provide the background required to assign chronogeometric content to salient elements within the designated concrete model. In the following section, I explicitly formulate these model representations and outline the associated representational system.

5 Noncommutative Observers

The problem of providing a comprehensive interpretation of NCST algebras can be traced back to the 1940s. In those years, original works in noncommutative approaches defined the relevant structure of NCST models as an algebra of spacetime coordinate operations, naturally yielding an interpretation in which these operators represent possible localisation procedures for high-energy events in NCST. This interpretation, further substantiated by operationalist assumptions, accounted for both a minimal length scale characteristic to the NCST structure and an intrinsic uncertainty in localisation procedures [27, 28].

This operationalist interpretation found resonance in later applications of NCG to QG, particularly in interpreting the commutative limit as a coarse-graining over measurements of noncommutative spatiotemporal effects. However, it proves ultimately unsatisfactory for two key reasons. First, the representational system underpinning the interpretation of the noncommutative algebra as a set of spacetime coordinate operators is incomplete. While it offers some initial understanding of the formalism, it ultimately fails to attain empirical significance without presupposing a more refined model representation relation. Second, the operationalist conception clashes with applications to the quantum gravitational regime and, as such, ought to be bracketed.

In light of this shortcomings, in this section I propose a way to refine the original interpretation, showing how it allows NCST theories to exhibit representational chronogeometry. In doing so, I develop a complete representational system that supports the interpretation of NCST without committing to a strongly operationalist stance. A central role is played by the κ-Poincaré algebra: as suggested by Lizzi and collaborators [25], quantum symmetries should be interpreted through a suitably defined notion of "quantum observer."

The section is thus structured as follows. In Section 5.1, I develop the proposal of [25], articulating a representational system based on noncommutative observers. This includes characterising noncommutative observers and clarifying their role in interpreting NCST theories. In Section 5.2, I extend the interpretation by analysing the non-localisation effects induced by noncommutativity and their compatibility with the representational system. Specifically, I highlight the connection between noncommutative observers and spatiotemporal structure. Finally, in Section 5.3, I illustrate how this interpretation allows one to use the algebraic theory of NCST for reconstructing the notion of a timelike worldline.

5.1 Observers and Reference Frames for NCST

As discussed in Section 3.4, a concrete model of κ-Minkowski spacetime can be constructed through specific states arising from a representation of the κ-Poincaré algebra. Lizzi and collaborators motivate this result by introducing the notion of an "observer" associated with the origin of the Hilbert space that supports the representation. They point out that, for κ-Poincaré states to be perfectly localised at the origin, one must specifically indicate "the *observer* making the observation" ([25, 23], emphasis in the original). Accordingly, the operators X^μ are interpreted as localisation procedures *relative to a reference frame associated to a designated observer, located at its origin.* The action of elements of the κ-Poincaré group is then understood as a passive transformation between reference frames that respects the NCST structure. Lizzi and collaborators describe this transformation in the following terms:

> A spacetime event (*i.e.* the clicking of a particle detector) seen by Alice will be described by the expectation value of its coordinates $\langle x^\mu \rangle$, their variance $\langle (x^\mu - \langle x^\mu \rangle)^2 \rangle$, which measures how localized it is, the skewness $\langle (x^\mu - \langle x^\mu \rangle)^3 \rangle$ measuring how asymmetric it is around the expectation value, and all higher moments $\langle (x^\mu - \langle x^\mu \rangle)^n \rangle$ which describe in increasingly finer details the distribution of probability of where the event can be localized. The same event, seen by Bob, will be described by a tower of moments of the transformed coordinate operators: $\langle (x'^\mu - \langle x'^\mu \rangle)^n \rangle$, which are in general different from Alice's, unless the transformation that connects Alice and Bob is the identity [...]. (pp. 24-25)

Notably, Lizzi and collaborators introduce these "quantum observers" without fully characterising them. Here, I further develop their interpretation by specifying an appropriate definition of "observer" for the NCST context. While such observers differ from those defined in relativistic settings, I argue that they share important affinities with the recently developed notion of a *quantum reference frame* (QRF; see fn. 27).

In physics, the notion of observer is theory-dependent. In orthodox non-relativistic quantum mechanics, observers are typically macroscopic systems capable of interacting with a quantum system during a measurement, thereby inducing state collapse. These observers select measurement settings and record outcomes.

By contrast, in relativistic frameworks, observers are closely tied to reference frames: collections of physical degrees of freedom dynamically coupled to the observer's state. An observer may be idealised as a point-like particle affected by the spacetime geometry, but not contributing to the stress-energy tensor nor exerting gravitational attraction on nearby systems. Alternatively, one can define an observer as a timelike worldline within a reference frame equipped with a clock.[24]

In NCST approaches, an adequate notion of observer must incorporate both relativistic and quantum features. On the one hand, the relativistic association of an observer with a reference frame promises to account for the noncommutative effects of the underlying geometry. Moreover, it contributes to the definition of the X^{μ}'s as localisation operators. On the other hand, the noncommutative geometric structure can only be surveyed by a system that is, in some sense, quantum in nature. This necessity is supported by two main arguments.

First, several localisation arguments produced over the last century reveal a crucial incompatibility that transcends the specific context of NCG: general relativistic black holes and the infinite-mass limit of quantum probes, required to eliminate quantum gravitational uncertainties, cannot coexist.[25] This result rules out the viability of using classical probes, i.e., infinitely massive quantum probes, to minimise the uncertainties characterising quantum spacetime models, since such probes would induce black hole formation before yielding sharp results. In contrast, appropriate probes for detecting quantum spacetime structure must operate at specific quantum gravitational scales. This imposes stringent constraints on the choice of dynamical fields: they must be sufficiently sensitive to quantum gravitational effects, while also bounded by the minimal uncertainties predicted by the theory.

Second, the structure of the observer must be compatible with the representation theory of the underlying quantum group. This is because the observer must be represented by a certain structure within the concrete model under examination. In the case at hand, the translation sector exhibits nontrivial commutators, as shown in equation (5). The composite structure formed by the observer and reference frame must provide a suitable representational system for a physical interpretation of the underlying noncommutativity.

In this framework, I introduce a notion of *noncommutative reference frame* (hereafter, NCRF). A NCRF is defined as a family of physical degrees of freedom that are dynamically coupled to a *noncommutative* (or "quantum") *observer*. The observer is modelled as an idealised physical system with no gravitational effect and a negligible contribution to the stress-energy tensor. It possesses a state space that supports the action of a noncommutative algebra, compatible with the geometry of the NCST model

[24] See [29, 30] for this coordinate-free construction. Interestingly, they define a reference frame as a (1,1)-tensor field $\mathbf{R}$ of rank 1 on the space of events. Each frame is decomposed into a vector field $\mathbf{\Gamma}$ and a 1-form α, i.e., $\mathbf{R} = \alpha \otimes \mathbf{\Gamma}$. $\mathbf{\Gamma}$ generates a family of worldlines as integral curves, whereas α defines a family of three-dimensional hyperplanes that foliate the space of events transversally to the curves defined by $\mathbf{\Gamma}$. The condition $\mathbf{R}^2 = \mathbf{R}$ on the reference frame is equivalent to saying that $\alpha(\mathbf{\Gamma}) = 1$. In particular, a vector $\mathbf{v}_p$ at point p is timelike if $\alpha_p(\mathbf{v}_p) \neq 0$. The pair $(\mathbf{\Gamma}, \alpha)$ defines a family of observers that evolve along $\mathbf{\Gamma}$, each having a rest frame defined by $\alpha(\mathbf{\Gamma}) = 0$.

[25] See, e.g., [31–33]. For a discussion of the interrelationship between these arguments and the development of NCG as an approach to QG, see [21].

in which the observer is embedded. The NCRF is able to "detect the noncommutativity" of the underlying geometry: it has a characteristic energy scale that coincides with the domain of applicability of the NCST theory. The degrees of freedom of the reference frame are therefore coupled to, i.e., dependent on, the state of the observer.

The compatibility between the states of the noncommutative observer and the NCST structure warrants further elaboration. The κ-Poincaré algebra can be interpreted as the family of transformations between different observers. As such, the Hilbert space $L^2(SO(1,3)\times\mathbb{R}^3)$ supporting its representation is interpreted as the state space of the observer. This space carries an algebraic representation of the κ-Poincaré algebra interpreted as the family of transformations between NCRFs.

In the representation adopted in Section 3.4, only translations of the origin of the NCRF are deformed (subject to a minimal scale), whereas Lorentzian boosts and rotations remain undeformed. This is important because, in SR, the notion of an inertial reference frame is deeply linked to the structure of the Lorentz group. Indeed, the latter is the set of transformations between inertial reference frames. Similarly, in NCST, the preservation of the Lorentz sector allows to identify *inertial* NCRFs among all possible ones. In particular, we can refine the interpretation of the κ-Poincaré algebra as the family of transformations between such inertial NCRFs.[26]

At this juncture, one might argue that the introduction of reference frames is, at best, a "toy construction" within the context of QG. In particular, quantum gravitational effects might obstruct the identification of suitable degrees of freedom that realise a NCRF at the relevant energy scales. While such concerns are valid and depend on the specifics of each algebraic theory, the preservation of the Lorentz sector in the construction of the κ-Poincaré algebra used here alleviates this issue. The construction remains compatible with a (potentially enriched) notion of relativistic reference frame.

However, this preservation is not a general feature: it depends on the chosen basis of the quantum group. Therefore, a natural question arises: what is the extension of the family of NCST models that admit NCRFs? The limitation of the number of instances would not preclude the interpretation of successful cases, such as the one presented in this section. Rather, it suggests that one should look at the collection of all different instances and define local interpretations piecewise. A thorough investigation of such exceptional cases lies beyond the scope of this paper and is left for future work.

The concept of a NCRF becomes clearer when compared to the closely related idea of a QRF.[27] A QRF is a quantum system that is associated with a reference frame. The composite system, consisting of a quantum particle and a reference frame, is described by a collective quantum state and associated extended symmetries. The state of the target system is defined relative to the QRF, and the description of a multi-particle quantum system becomes relational: any particle can serve as a QRF, eliminating the need for absolute, external frames. As a result, quantum features, such

[26] It is important to remark that this interpretation crucially relies on the definability of inertial frames, and thus on the preservation of the Lorentz sector. If this sector were not preserved, it would be necessary to prove that such a notion is still well-defined.

[27] The literature on QRFs is extensive. Early proposals include [34–36], and, in quantum information theory, [37]. Applications to spacetime are discussed, e.g., in [38]. For a discussion of the generalised symmetries between QRFs, see, e.g., [39–42]. More philosophically oriented examinations have been conducted, e.g., in [43], building on [44].

as superposition and entanglement, become frame-dependent: they can, in principle, be removed via a suitable transformation of QRF.

NCRFs exhibit a similar relational structure: the state of a target system is defined in terms of the state of the NCRF. However, the relationship between QRFs and spacetime is inverted in comparison to that between NCRFs and NCST. In the case of QRFs, the reference frame is defined internally, via the relation between the frame and the quantum system; the spacetime structure is reconstructed from frame-dependent constructions and their mutual transformations. In contrast, NCST geometry is defined independently of any reference frame. Rather, NCRFs instantiate specific structures of this geometry through the κ-Poincaré symmetry: they realise, in physical terms, the algebraic transformations that characterise the NCST. In this way, NCRFs allow us to interpret the algebraic elements of the theory as physical transformations between the states of noncommutative observers.

Let us now consider an arbitrary system in NCST. Suppose we want to determine its position. Let $|o\rangle$ be the state of a noncommutative observer at the origin of its NCRF. A coordinate system is defined by choosing a parametrisation of the timelike and spacelike directions associated with this frame.[28] Let $|\psi\rangle$ represent the state of the system relative to the NCRF. Since $|\psi\rangle$ lies in the support of a representation of the κ-Minkowski algebra $\mathcal{M}_\kappa$, the operators X^μ can be interpreted as localisation operators, defining the position of the system with respect to the NCRF and the chosen coordinate system.[29] In other words, interpreting (X^μ) as NCST coordinate operators requires the specification of a NCRF as part of the representational system.

Moreover, one can consider a second NCRF with state $|o'\rangle$, and ask where the same target system is localised relative to this new frame. This implies a comparison between $\mathrm{X}^\mu |\psi\rangle$ and $\mathrm{X}'^\mu |\psi\rangle$, where (X'^μ) is the algebra of NCST coordinate operators defined by the new NCRF. Because the geometry is noncommutative, the comparison between these two localisations must also involve the frame states $|o\rangle$ and $|o'\rangle$. Indeed, we expect the localisability of the target system to depend on each frame state.

Formally, one must specify a transformation from the composite state $|o\rangle \otimes |\psi\rangle$ to $|o'\rangle \otimes |\psi\rangle$. Notably, $|o\rangle \otimes |\psi\rangle$ lies in the support of a faithful representation of the κ-Poincaré algebra. Consequently, the latter can be interpreted as a family of transformations of the state of an event, relative to the coordinate system (X^μ) of the first NCRF, to the state of that same event relative to the coordinates (X'^μ) of the second NCRF.

5.2 The Function of NCRFs

The introduction of NCRFs contributes to the interpretation of the generalised uncertainty relations that characterise the NCST algebraic theory. The first step in this analysis is to characterise the state $|o\rangle$ of a noncommutative observer in relation to the action of the κ-Poincaré algebra within concrete models.

[28] Since the framework of this paper is a flat, albeit noncommutative geometric structure, I will not be concerned with the locality of the coordinate chart. Instead, I will suppose that there is always a global coordinate chart for each NCRF. This assumption must be dropped in the context of noncommutative gravity: there, curved NCST models are accepted as solutions to the deformed Einstein field equations.

[29] For the sake of simplicity, I will omit explicit reference to the representation ρ in the notation throughout the rest of the paper. The distinction between abstract algebraic elements and their concrete representations will be evident from the context.

Let g be an arbitrary element of the κ-Poincaré algebra in a given representation. The observer's state $|o\rangle$ is characterised by the condition $\langle o| g |o\rangle = \epsilon(g)$, where ϵ is the counit map. This means that the expected effect of applying a transformation g to the observer's own state is trivial: it acts as the identify. In other words, an observer associated with a NCRF can perfectly localise itself at the origin of its own frame. A noncommutative observer has thus complete information about its own location in its own frame.

Uncertainty, however, arises when the observer considers an event displaced from the origin. Suppose this observer, call her Alice, is associated with the state $|o\rangle$. Alice can determine the expected position of a distant event by computing the expectation value $\langle \mathrm{X}^\mu\rangle$, where X^μ's are the coordinate operators. Higher statistical distributions refine this information: $\langle(\mathrm{X}^\mu - \langle \mathrm{X}^\mu\rangle)^2\rangle$ is interpreted as the variance of the event relative to Alice's NCRF and measures its position uncertainty; $\langle(\mathrm{X}^\mu - \langle \mathrm{X}^\mu\rangle)^3\rangle$ is interpreted as the skewness measured by Alice and reflects asymmetry in the probability distribution; and so on. The introduction of NCRFs in the representational system of the X^μ operators allows these statistical distributions to be interpreted relationally, i.e., in terms of the state of the observer.

The scenario becomes more intricate with the introduction of a second observer, Bob, associated with a distinct NCRF. Two key points follow. First, Bob can, just like Alice, compute and interpret the statistical distributions of the same event relative to his own frame. Second, if Bob is located at the origin of Alice's NCRF, then the transformation between their states is trivial. In this special case, both observers can perfectly localise one another, and their calculated statistical distributions coincide.

In general, however, Bob and Alice can occupy different positions, and the relation between their frames is nontrivial. In such cases, additional uncertainty arises.

To illustrate, as discussed in the previous section, the state of an event relative to Alice's NCRF can be written as $|o\rangle \otimes |\psi\rangle$, where $|o\rangle$ is Alice's state and $|\psi\rangle$ is the state of the event. This tensor product belongs to the support of a representation of the κ-Poincaré algebra. A transformation from Alice's frame to Bob's frame, associated with observer state $|o'\rangle$, maps this to $|o'\rangle \otimes |\psi\rangle$. Such transformations realise the elements of the κ-Poincaré algebra as operators acting on a suitable collective Hilbert space, e.g., $L^2(SO(1,3) \times \mathbb{R}^3 \times \mathbb{R}^3)$. While these transformations produce changes of reference frame for the state of the event, they also actively map the initial tensor state $|o\rangle \otimes |\psi\rangle$ to a new one.

A crucial observation is that κ-Poincaré transformations do not act "sharply" on NCRFs. Each element of the representation is accompanied by a corresponding uncertainty. The NCRF framework provides a natural interpretation of this effect: transformations on tensor states $|o\rangle \otimes |\psi\rangle$ result in uncertainties in localisation. For example, a translation with nonzero uncertainty implies that the origin of the NCRF is shifted to an indeterminate location. From the perspective of the original frame, the origin of the translated NCRF appears delocalised, reflecting the noncommutative structure of spacetime.

As an illustration, Bob cannot sharply localise Alice unless he is situated exactly at the origin of her NCRF. In that unique case, the translation is trivial. Otherwise, localising Bob from Alice's frame amounts to translating Alice with respect to Bob in

NCST, and this process introduces an additional uncertainty. In essence, the uncertainty in Bob's location relative to Alice's frame is equivalent to the uncertainty of translating Bob from the origin of Alice's NCRF.

The study of the localisation of events also requires a symmetrisation of the relation between event and NCRF. The symmetrisation allows to treat the event as a third noncommutative observer, call him Charles, with its own NCRF. As with Alice and Bob, Charles can self-localise precisely at the origin of his own frame. Similarly, Charles can sharply localise any event that coincides with the origin of his NCRF. However, from Charles's perspective, Alice is no longer sharply localised. The symmetry in the construction ensures that all observers and events can self-localise, but cannot perfectly localise others unless they coincide.

Within this framework, different sectors of the κ-Poincaré algebra act distinctly on the composite states $|o\rangle \otimes |\psi\rangle$.[30] First, the Lorentz sector is undeformed in the chosen representation. These transformations act on tensor states in the standard way, and they do not alter the expectation value of the event's position from Bob's point of view. This is due to the fact that the target event can be treated as a third noncommutative observer, with sharp localisation in its own NCRF: its contribution to the expectation value is zero. Higher statistical distributions (variance, skewness, etc.) also remain invariant under pure Lorentz transformations.

Second, pure temporal translations also leave expectation values invariant. A temporal shift of Bob with respect to Alice (i.e., a static displacement along her time direction) introduces no additional uncertainty.

Finally, arbitrary translations behave differently. When Bob is translated along both spatial and temporal directions relative to Alice, the uncertainty in the localisation of an event (say, Charles) with respect to Bob increases compared to Alice's measurement. This increase is due to the nontrivial uncertainty introduced by the κ-deformed translation acting on Bob's reference frame. Only when the transformation is trivial (i.e., an identity map) does the localisation remain invariant.

To summarise, the introduction of NCRFs provides a representational system for the action of concrete representations of the κ-Poincaré algebra. These transformations act on composite states involving both the noncommutative observer and the event. The noncommutative structure of the κ-Poincaré algebra introduces additional, frame-dependent uncertainties. Specifically, while κ-deformed translations contribute to delocalisation, pure Lorentz transformations and pure temporal translations leave the expectation values invariant. This framework also explains why an observer can only perfectly localise themselves: other observers can only be sharply localised if they are located at the origin of the reference frame in question.

5.3 NCRFs and Worldlines

NCRFs are essential for defining NCST models as chronogeometric structures, yet their specification alone is still insufficient to establish representational chronogeometricity. What is missing is an appropriate family of worldlines. This request is motivated by the standard treatment of chronogeometry in the SR literature: see Section 2.2, and specifically (CH). There, a structure exhibits chronogeometricity only if notions of proper

[30] The action of the identify has already been discussed in the case of self-localisation.

time and length can be defined for timelike and spacelike worldlines, respectively.[31] Consequently, any chronogeometric theory must preliminarily allow the definition of a suitable structure of worldlines in order to saturate the required representational principles.

Similarly, in the context of NCST theories, we must ensure that worldlines are suitably defined. Such a definition is proper if it (i) allows a distinction between timelike and spacelike curves, and (ii) is compatible with the underlying noncommutative structure. As discussed in the preceding sections, this also requires specifying the relationship between worldlines in NCST and a chosen NCRF.

In the standard framework, events are regarded as point-like, sharply localised coincidences of worldlines. However, as previously discussed, in κ-Minkowski spacetime, the localisation of events is nontrivial. The variance of the location depends on the specification of the NCRF. We expect that worldlines should also be compatible with this result and somehow depend on the choice of observer, up to a κ-Poincaré transformation.

To illustrate the structure of worldlines, Ballesteros, Gutierrez-Sagredo, and Herranz [45] have proposed defining them as elements of a *space of worldlines*. In the classical case, this construction employs the group properties of $\mathcal{P}_4$ to identify worldlines by their invariant features. The isotropy group H of timelike worldlines is generated by temporal translations and spatial rotations. By suitably parametrising $\mathcal{P}_4$ and H, one can form the quotient $\mathcal{P}_4/H$, which corresponds to the space of timelike worldlines in Minkowski spacetime. The elements of this quotient are equivalence classes representing timelike curves, and their position in the space of worldlines coincides with their position in Minkowski spacetime only when the associated system is at rest. In this construction, $\mathcal{P}_4$ acts nonlinearly on the space of timelike worldlines.

A similar approach can be used to define the space of noncommutative timelike worldlines in κ-Minkowski spacetime.[32] The resulting elements are parametrised by two coordinates: the position y^a and the rapidity η^a. While the rapidities commute with each other, both the position-position and position-rapidity brackets are nonzero. In particular, the brackets between positions and rapidities depend explicitly on the latter.

As Ballesteros, Gutierrez-Sagredo, and Herranz observe,

> the noncommutative spaces of worldlines seem to provide a privileged arena in order to explore the physical role of the κ-deformation. In particular, once the canonical coordinates have been found, noncommutativity in the space of worldlines could be rephrased in more physical terms as the impossibility of determining simultaneously and with infinite precision the six (q, p) coordinates of a given worldline.[33] (p. 180)

The construction of timelike worldlines as invariant structures of a subalgebra of $\mathcal{C}_\kappa(\mathcal{P}_4)$ implies that the space of worldlines and $\mathcal{M}_\kappa$ are different realisations of the same noncommutative structure, as reconstructed by a NCRF. In this sense, their noncommutative structures are said to be *compatible*.

[31] This is implied by the satisfaction of the clock hypothesis in SR.

[32] Here H is replaced by the subalgebra $\mathfrak{h}$ of $\mathcal{C}_\kappa(\mathcal{P}_4)$. One can show that $\mathcal{C}_\kappa(\mathcal{P}_4)$ in a suitable bialgebra basis is coisotropic with respect to $\mathfrak{h}$. This guarantees that the space of timelike worldlines inherits a homogeneous Poisson structure from $\mathcal{C}_\kappa(\mathcal{P}_4)$ by a suitable canonical projection.

[33] Where (q, p) are obtained from position and rapidity operators (Y, H) by a suitable change of basis.

Once this compatibility is established, we can ask how noncommutativity affects worldlines. As previously discussed, distant events appear fuzzy to a noncommutative observer, i.e., they exhibit non-zero position uncertainty. Consequently, worldlines that intersect such events are also expected to be fuzzy.

The construction of the timelike worldlines can be extended to define a space $\mathcal{W}_\kappa$ of arbitrary noncommutative worldlines [46]. Here, timelike worldlines are those with speeds less than one, and spacelike worldlines those with speeds greater than one. This distinction arises not from a conventional light-cone structure, which may be deformed by noncommutative effects, but rather from the characterisation of worldlines via their invariant properties under NCST symmetries.

Moreover, the algebra $\mathcal{C}_\kappa(\mathcal{P}_4)$ naturally endows $\mathcal{W}_\kappa$ with a noncommutative structure governed by κ^{-1}. As discussed in Section 3.4, we can represent $\mathcal{W}_\kappa$ on a suitable Hilbert space $L^2(\mathbb{R}^3)$. For example:

$$Q^i\psi(p) = \frac{i}{\kappa}\frac{\partial\psi(p)}{\partial p^i} \quad , \quad P^i\psi(p) = p^i\psi(p). \tag{11}$$

Here, (Q^i, P^i) are generators of $\mathcal{W}_\kappa$ and define a Heisenberg-Weyl algebra with noncommutative parameter κ^{-1}. The standard coordinates (y^i, η^i) can be recovered through a suitable change of basis. In general, optimal localisation of worldlines is achieved via a special family of states (known as *squeezed states*) that minimise the uncertainty relations of $\mathcal{W}_\kappa$. Perfect localisation is only attained in the limit of such states. Notably, the origin w_0 of the space of worldlines is sharply localisable and can be interpreted as the worldline of a particle at rest at the origin of its own NCRF.

The noncommutativity of $\mathcal{W}_\kappa$ becomes evident when we associate a probability distribution to each worldline. The physical interpretation of this distribution remains somewhat unclear; nevertheless, it is mathematically well-defined and understood. This distribution smears the location of the worldline, preventing sharp localisation. We thus obtain a bundle of possible worldlines centered around a given q^i in $\mathcal{W}_\kappa$, with the spread growing proportionally to the distance from the NCRF origin. This uncertainty or *fuzziness*, governed by κ^{-1}, is compatible with expectations that sequences of distant events (respectively, worldlines) become highly delocalised due to noncommutativity. The compatibility between noncommutative structures guarantees that the spread in worldline distributions within a region be consistent with the variance of the associated events.

To further illustrate these effects, note that the probability distributions associated with noncommutative worldlines can have non-overlapping tails. For any pair of worldlines, an *impact factor* β measures their minimal spatial separation at equal coordinate time. This is defined only with respect to a reference frame that provides such a coordinate chart. If $\beta = 0$, the worldlines intersect with certainty. Due to fuzziness, however, there is a nonzero probability that both $\beta = 0$ and $\beta \neq 0$. Specifically, to an observer at the origin of the NCRF, distant worldlines appear increasingly blurred, with larger variance: the observer assigns a nonzero probability that a given worldline intersects its own.

In conclusion, the space of noncommutative worldlines inherits a noncommutative structure compatible with that of $\mathcal{M}_\kappa$. The fuzziness of worldlines mirrors the position uncertainty assigned by a noncommutative observer to distant events. The only sharply localised worldline is that of the observer at the NCRF origin. Consequently, the structure of NCRFs extends the interpretation of the NCST theory from single events to worldlines in NCST.

6 Representational Chronogeometry in NCST Approaches

In Section 3, I examined the relationship between quantum groups and NCST theories. The resulting algebraic theory lacks a direct physical interpretation unless a suitable representational system is introduced. In Section 5, I argued that this system must include a notion of NCRF. This representational system thus enables the interpretation of concrete NCST models as families of localisation operations for events, each relative to a corresponding observer. Moreover, it allows one to interpret the algebraic representations of the quantum group as transformations between these NCRFs. Finally, the structure of worldlines in NCST is shown to be compatible with the NCRF framework: it shares the same noncommutative deformation and exhibits uncertainty effects proportional to those assigned by the observer to distant events.

At this juncture, a crucial question arises: can this construction and interpretation satisfy the representational principles for representational chronogeometricity, as indicated in Section 2?

On the one hand, satisfaction of these principles is expected to depend on the specific NCST model under consideration. While κ-Minkowski spacetime appears promising in this regard, the same may not hold for other flat NCST models. For example, ρ-Minkowski spacetime, i.e., a different NCST model with polar noncommutativity, shares sufficient structural similarities with κ-Minkowski spacetime to allow interpretation in terms of NCRFs: the polar noncommutativity introduced by the deformation of the Minkowski algebra produces a similar Lie-algebra-type NCG.[34] In contrast, θ-Minkowski spacetime presents additional interpretational challenges: the introduction of a deformation, mediated, in this case, by the use of a Drinfel'd twist, diverges from the κ-Minkowski construction in significant ways.

Nonetheless, the need to specify one NCST model among many does not undermine the viability of the interpretation. Ultimately, only one model is expected to be empirically adequate. If even one NCST model can be shown to exhibit chronogeometric features, then this is sufficient to counter arguments against NCST approaches in this context: this provides motivation for further exploring these approaches, rather than serving as a reason to dismiss the possibility that they may be chronogeometric.

On the other hand, demanding strict adherence to standard representational principles may be inappropriate in the context of QG. It is indisputable that relativistic principles will not fully apply here: QG describes structures and phenomena essentially different from those in classical relativistic theories. Hence, I do not take the trivial and unsurprising failure of QG to conform to standard chronogeometric principles as evidence of its non-chronogeometricity. In fact, the opposite would be more surprising.

[34] However, also note the differences: see, e.g., [47].

Instead, I submit that in the context of QG, the core problem of chronogeometricity should be the formulation of new representational principles that are compatible with the distinctive features of the quantum gravitational target domain. For instance, if a theory quantises gravitational phenomena, the corresponding criteria of chronogeometricity must account for quantum effects on distance and duration measurements.[35] Similarly, a NCST theory should be evaluated against criteria that reflect the presence of uncertainty effects in the measurement of spatiotemporal quantities.

I contend that representational chronogeometricity in NCST theories requires three conditions. First, it requires a suitably developed algebraic structure. The algebra must support additional structures (e.g., affine, projective, etc.) and admit a geometric counterpart. NCST satisfies these requirements in virtue of generalised algebraic-geometric dualities that replace the standard Gelfand duality (see, e.g., [50–52]).

Notably, NCST theories can also define a metric structure by introducing a differential calculus that is compatible with the underlying quantum symmetries. This calculus is often non-unique due to noncommutativity; nonetheless, the definability of this structure in a purely algebraic setting implies that the NCST theory can yield all the structures necessary to saturate chronogeometric principles. Moreover, the metric is compatible with the NCRF framework: it measures distances between events as localised by the noncommutative observer positioned at the origin of the frame, while being affected by noncommutative effects compatible with the underlying quantum group structure.

Second, representational chronogeometricity requires a spatiotemporal interpretation. As argued above, such interpretation is only possible when appropriate notions of noncommutative observers and NCRFs are introduced. These arise naturally from the representation theory of the algebra and its symmetries. They complete the representational system and enable the interpretation of the NCST theory as a framework for localising events. They also allow to understand the action of quantum group representations as transformations between NCRFs, thereby accounting for the observer-dependent uncertainty of events.

Finally, representational chronogeometricity requires the compatibility between the algebraic structures and their interpretation. Differently put, the noncommutative structure of a NCRF must align with the noncommutativity found in other structures derived from the NCST theory. This includes uncertainties and quantum symmetries, as well as the structure of noncommutative worldlines and the aforementioned metric field. By construction, these are already compatible with the definition of NCRFs.

These three requirements are necessary for representational chronogeometricity. To demonstrate this, suppose that a theory is chronogeometric. By definition, it must enable the representation of durations and distances. Thus, it must possess a complete representational system to facilitate it. Conversely, the absence or incompleteness of such a representational system implies that the theory cannot be interpreted in terms of measurable durations and distances. For instance, Maxwell's electromagnetic theory is not chronogeometric: its entities represent fields and sources, not spatiotemporal measurements.

[35] To illustrate, see the discussion about the measurement of time in quantum regimes, e.g., in [48, 49].

Similarly, if a theory is chronogeometric, then it must possess a suitable geometric structure to be interpreted. This geometry is typically reconstructed via measurements of durations and distances, as in constructive approaches. The converse situation, namely an ill-defined geometry, trivially results in the failure of chronogeometricity..

Finally, formalism and interpretation must be mutually compatible: the latter must facilitate understanding of the specific geometry under consideration. Without such compatibility, measurements could not correspond to the underlying geometry, and the theory would fail to be chronogeometric. Indeed, in this case, a representational system (e.g., for observers or apparatuses) would exist, but the results it yields would not be grounded in the theory's geometry. An example of such a pathological scenario would be discrepancies between observed quantum uncertainties and those predicted by the theory's geometric structure, signaling incompatibility.

To illustrate the link between representational chronogeometry and these three requirements, consider the following principle, which extends (CH):

(NC-CH) A curve γ is a timelike worldline in NCST if there exists a NCRF such that $|\gamma|$ represents the duration of the events in $\gamma[I]$, for some $I \subseteq \mathbb{R}$, as measured by the associated noncommutative observer.

As discussed, κ-Minkowski spacetime supports the definition of NCRFs. These frames encode the noncommutativity of $\mathcal{M}_\kappa$, inheriting it from the associated quantum group. In this setting, $\mathcal{M}_\kappa$ represents the geometry reconstructed by the noncommutative observer. Furthermore, the noncommutative features of a worldline align with the uncertainties attributed to it by the observer. A NCRF also provides a characterisation of timelike worldlines through their invariant properties, and allows the definition of a differential calculus and, subsequently, of a noncommutative metric.

Therefore, (NC-CH), if introduced, provides chronogeometric meaning by establishing a link between compatible mathematical structures and interpretations. This link is expressed by a conditional statement. κ-Minkowski algebraic theory possesses both the formal and interpretative tools required to satisfy this principle. In other words, it is compatible with the introduction of (NC-CH) in order to endow the theory of chronogeometric meaning.

An objection might be raised concerning the definition of $|\gamma|$, which denotes an integral of the metric and is interpreted, according to (NC-GR), as the proper time measured along γ. As discussed in Section 2.2, in relativistic theories this relies on the application of the clock hypothesis to Minkowski spacetime. In κ-Minkowski, instead, the metric need not be identical to that of Minkowski spacetime: it is in fact non-unique. Therefore, one might argue that $|\gamma|$ lacks meaning without an explicitly defined metric field.

In response, I emphasise that a suitable differential calculus can be chosen, and with it, a metric field that defines proper time for timelike worldlines. A good choice of calculus must be covariant, ensuring that the resulting noncommutative metric remains compatible with both the underlying geometry and the worldline's intrinsic fuzziness. Put simply, the computation of $|\gamma|$ based on a quantum metric yields results compatible with the observer-relative indeterminacy of γ.

A second objection might assert that duration cannot be represented due to the lack of a proper clock in the noncommutative context. The cause would be the minimal uncertainty that affects duration, as well as position, measurements. Yet, by definition, a noncommutative observer carries a clock structure that is special relativistic at the origin of the NCRF. Its dynamics can be extended across the NCRF via an active κ-Poincaré transformation, incorporating quantum effects into the clock's operations. In this sense, the complications posed by NCG are not greater than those encountered in SR.

Finally, one might object that even if noncommutative observers were equipped with appropriate clocks, representational chronogeometry alone would still be insufficient for the NCST theory to be genuinely chronogeometric. In other words, an opponent might insist that, for the NCST theory to be properly chronogeometric and spatiotemporal, one must also specify a convention for synchronising clocks between NCRFs, thereby requiring full operational chronogeometricity.

In this regard, two points deserve emphasis. First, representational chronogeometry, as presented here, is not incompatible with operational chronogeometry. Rather than standing in opposition, the former constitutes a weaker or more preliminary condition relative to the latter. Consequently, the identification of a chronogeometric model representation does not preclude the possibility of developing a richer, operational interpretation of the NCST theory, despite being a demanding task (see Section 2.1).

Second, following Fletcher [10, 9], I consider the formulation of such operational protocols to be a distinct endeavour from the interpretation of the theory's concrete models. Specifically, the former requires making explicit an operational interpretation of the NCST theory, which need not coincide with its physical interpretation. Here, I understand a physical interpretation of the theory as any ascription of model representational principles that enables a coherent understanding (both informal and technical) of the theory's content; that is, how the theory describes its target domain. Any proposed operational implementation goes beyond the minimal criteria of physical interpretability and will necessarily be, at best, controversial, owing to the inaccessibility of the relevant domain of applicability.

For these reasons, while I simpathise with the concerns raised by proponents of operational chronogeometry, I maintain that the issue they highlight is, strictly speaking, distinct from the one addressed in this paper. Nevertheless, I concede that candidate operational implementations, albeit controversial, may be proposed. One can reasonably expect these implementations to diverge from standard relativistic protocols, due to the need to incorporate characteristic noncommutative effects within the proposed synchronisation procedures.[36]

[36] To illustrate, one possibility for an operational implementation involves shifting the focus from rulers and clocks to measurements of energy and momentum. The latter quantities are arguably more appropriate for probing Planckian and sub-Planckian regimes. Moreover, standard chronogeometric approaches implicitly assume that measurements with rulers and clocks are reliable because such instruments are causally isolated from their surroundings: a premise ultimately supported by the particular geometry of the associated momentum space. NCG, in contrast, aligns naturally with an energy-momentum-based approach to operational chronogeometry, yet challenges standard constructions in specific contexts: see, for example, discussions of momentum space curvature and relative locality in [53]. A detailed examination of these issues is left to future work.

In conclusion, κ-Minkowski spacetime, equipped with NCRFs, is sufficient to saturate (NC-CH): the theory possesses the formal and interpretational resources needed to satisfy it. The introduction of NCRFs imposes no privileged structure: all are related by κ-Poincaré transformations. Each NCRF reconstructs its own model of the NCST geometry, constrained only by the requirement of mutual compatibility with others.

7 Conclusion

Chronogeometry is frequently discussed as a necessary condition for a mathematical geometry to be considered physical. Standard definitions are typically formulated within the framework of classical relativistic theories and identify core criteria that a theory should satisfy to be deemed chronogeometric. However, these definitions often exclude many theories of QG from fulfilling the specified conditions. As a result, such theories are said to exhibit disappearance of spacetime due to their lack of chronogeometricity.

NCST theories may fall into this category, given the complicated relationship between operational assumptions and the absence of localisability of events within arbitrarily small regions. As such, they raise the challenge of indicating an appropriate definition of chronogeometry that, if satisfied by these theories, would guarantee the spatiotemporality of the postulated noncommutative structures.

In this paper, I argued that noncommutativity, in general, does not constitute a direct obstacle to chronogeometry: the two are not incompatible. This is illustrated through the example of κ-Minkowski spacetime. I argued that κ-Minkowski serves as a successful case of noncommutative geometric theory that remains compatible with appropriate chronogeometric conditions. This implies that, with the right interpretation of its mathematical structure, the geometry can indeed be chronogeometric and thus capable of representing the intended physical target, namely, physics at or near the Planck scale.

According to strict algebraicism, the theory of κ-Minkowski spacetime is identified with an abstract algebra intended to describe the underlying NCST geometry. The foundamental algebraic structure is provided by the associated quantum group, κ-Poincaré. The NCST algebra emerges as a coset of the κ-Poincaré algebra and can be realised as a concrete NCST model via an appropriate algebraic representation. This relationship underpins the interpretation of a quantum group as encoding the symmetries of a NCST structure.

However, strict algebraicism may raise concerns regarding the chronogeometricity of these concrete models. Accordingly, it becomes necessary to specify a set of conditions that a theory (especially an algebraic one) must satisfy in order to support representational chronogeometricity. While QG theories may be incompatible with operational chronogeometricity, they may still satisfy representational principles, provided that the models can support the interpretation of geometrical quantities as physically meaningful spatiotemporal structures.

To this end, I argued that a viable interpretation of the κ-Minkowski algebra requires the introduction of new structures within the designated representational system. These structures, termed noncommutative reference frames, are inspired by the literature on quantum reference frames. They enable an interpretation of both κ-Minkowski and κ-Poincaré algebras: the former describes the space of events from the perspective of a noncommutative observer, while the latter encodes the transformations between distinct NCRFs. It is demonstrated that NCRFs accommodate the uncertainties in event localisation induced by noncommutativity, as well as the "fuzziness" of worldlines in κ-Minkowski spacetime. Moreover, the delineated framework outlines a concrete interpretation for κ-Minkowski spacetime theories, as well as similar NCST theories.

Finally, I argued that concrete models of κ-Minkowski spacetime can indeed saturate representational chronogeometric principles when equipped with the NCRF interpretation. More generally, a QG theory can be interpreted as chronogeometric only if the following three conditions are met: (i) it possesses an appropriate mathematical geometry; (ii) it includes a representational system that accounts for observers or experimental contexts; (iii) the structures of the representational system exhibit behaviour consistent with the predictions derived from the mathematical geometry. In the κ-Minkowski case, chronogeometricity is illustrated by the satisfaction of a standard representational principle for flat NCST theories: a noncommutative analogue to the clock hypothesis.

In conclusion, while the key aim of this paper has been to pave the way to the exploration of chronogeometry in algebraic theories of QG, the specific case of NCST also opens new issues. First, the identification of suitable rods and clocks, or structures that can represent distances and durations within the concrete models of the theory, would corroborate the argument for chronogeometry not only on a representational basis, but also on operationalist grounds; nevertheless, it proves insufficient unless one can demonstrate that said rods and clocks are *reliable*. A suitable notion of reliability would amount to their causal isolation from neighbouring fields: the result of any such measurement cannot be affected by spacelike separated measurements.

Second, SR is supposed to be recovered in the commutative limit of any NCST theory, that is, when the noncommutative effects are neglected. The chronogeometricity of both theories at different energy scales requires us to tell a story about how the two chronogeometric structures connect in the limit. Specifically, a deep philosophical question can be raised as to how the reducibility of the NCST theory to SR in the commutative limit can inform the chronogeometric interpretation of the noncommutative structures, based on their relativistic counterparts.

Third, the full extent of the analogy between NCRFs and QRFs remains unexplored. While NCRFs exhibit non-classical features arising from their underlying noncommutative structures, it is still unclear whether they can also exhibit the full spectrum of quantum behaviours captured by QRFs, such as superposition and entanglement. In this regard, the analogy serves as a valuable heuristic tool for guiding further investigations into NCRFs, but it is equally important to delineate its precise limits.

Finally, a pivotal step for establishing the chronogeometricity of NCST theories is to assess the scope of the interpretation presented in this paper. While κ-Minkowski is a fortuitous case, nothing proves that other significant NCST theories can saturate relevant chronogeometric principles. This is especially interesting when considering theories of curved NCST. However, the examination of these proposals is currently hindered by the difficulty of building an entire field theory on curved NCST. This theory is still in the making, and thus any analysis of its chronogeometricity would ultimately be premature.

Acknowledgements. This paper was written during a visiting period at the Radboud Center for Natural Philosophy (Nijmegen). I am deeply grateful to all the participants of the conference *Foundations 2025* (Gdańsk), where this work was presented. Special thanks go to Elena Castellani, Nick Huggett, and Fedele Lizzi for insightful discussions and for commenting on earlier drafts of this manuscript.

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