Quantum Systems as Indivisible Stochastic Processes

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Abstract

According to the stochastic-quantum correspondence, a quantum system can be understood as a stochastic process unfolding in an old-fashioned configuration space based on ordinary notions of probability and 'indivisible' stochastic laws, which are a non-Markovian generalization of the laws that describe a textbook stochastic process. The Hilbert spaces of quantum theory and their ingredients, including wave functions, can then be relegated to secondary roles as convenient mathematical appurtenances. In addition to providing an arguably more transparent way to understand and modify quantum theory, this indivisible-stochastic formulation may lead to new possible applications of the theory. This paper initiates a deeper investigation into the conceptual foundations and structure of the stochastic-quantum correspondence, with a particular focus on novel forms of gauge invariance, dynamical symmetries, and Hilbert-space dilations.

1 Introduction

'Indivisible' stochastic processes are a very new idea. They first appeared in the research literature in a 2021 review article, which introduced them in passing as a generalization of the sorts of stochastic processes described in textbooks (Milz, Modi 2021, Fig. 6). One can find rudimentary notions of indivisible dynamics in earlier work on open quantum systems, although those forms of indivisibility corresponded to quantum channels, not to stochastic processes (Wolf, Cirac 2008).[1]

Like a textbook stochastic process, an indivisible stochastic process is based on ordinary probability theory, and describes a system's trajectory unfolding in an old-fashioned configuration space according to probabilistic dynamical laws. However, as the present work will review, indivisible stochastic processes do not satisfy the Markov property, nor are they non-Markovian only in the familiar sense of requiring the specification of higher-order conditional probabilities. The key difference between an indivisible stochastic process and a textbook stochastic process is that for an

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[1] Indivisibility in the sense of a stochastic process should also not be confused with the older and altogether different notion of *infinite divisibility* for a static probability distribution.

indivisible stochastic process, the underlying laws are simpler, more general, and are not guaranteed to be decomposable or 'divisible' over time, to an even greater extent than is the case for a run-of-the-mill non-Markovian process that involves so-called memory effects (Barandes 2025).

Remarkably, indivisible stochastic processes generically exhibit all the hallmark empirical features of quantum systems, including interference, decoherence, entanglement, and noncommutative observables. These features agree with the observed behavior of quantum systems not only qualitatively, but quantitatively. Indeed, by a straightforward choice of dynamics, an indivisible stochastic process can replicate all the empirical predictions made using a given quantum system, while evading the famous *measurement problem* and the more recently named *category problem* (Ibid.). The measurement problem refers to the manifest ambiguity in the Dirac-von Neumann axioms for textbook quantum theory (Dirac 1930, von Neumann 1932) over precisely which sorts of processes count as measurements. The category problem refers to the separate problem of accounting for the larger category of non-measurement phenomena that seem to be happening in the world around us.[2]

Based on this reasoning, one can arguably view any quantum system as an indivisible stochastic process in disguise, leading to what one could naturally call the *indivisible interpretation of quantum theory*, or just *indivisible quantum theory* for short. Going in the other logical direction, one can also represent indivisible stochastic processes as quantum systems in the usual Hilbert-space formalism, which may open up new applications for Hilbert-space methods.

As a consequence, there exists a *stochastic-quantum correspondence* between indivisible stochastic systems and quantum systems (Barandes 2025). This correspondence has the potential to deflate and demystify many of the exotic features that are usually associated with quantum theory, with superpositions no longer describing a literal smearing of physical configurations, and with measurements demoted to just an ordinary kind of stochastic interaction. Moreover, the correspondence gives a first-principles way to understand not only why the time evolution of closed quantum systems is linear, but why it is unitary.[3]

Finally, by replacing the Dirac-von Neumann axioms of textbook quantum theory—and their internal ambiguities—with a simpler, more internally consistent, and more physically transparent set of axioms based on configuration spaces and stochastic laws, one opens up the possibility of new applications and generalizations of quantum theory. This paper will conclude by describing some of these downstream implications, as well as questions surrounding causation, locality, and Bell's theorem.

[2]Despite what one reads in some textbooks on quantum mechanics (for example, Shankar 1994, Chapter 6), one cannot get around the category problem by appealing to expectation values, because the only expectation values provided by the Dirac-von Neumann axioms are averages of numerical *measurement outcomes* statistically weighted by *measurement-outcome* probabilities. In the absence of a rigorous argument to the contrary, these expectation values are categorically narrower than ensemble averages or time averages of phenomena simply *happening* in the world around us.

[3]The conditions of linearity and probability conservation alone are not sufficient to pick out unitary time evolution. Completely positive trace-preserving (CPTP) maps, also called quantum channels, need not be unitary, but are linear and conserve probability. Even adding on the condition of logical invertibility is not enough to imply unitarity, because there exist non-unitary quantum channels that are invertible, such as for a quantum system with a 2×2 density matrix ρ evolving according to the bit-flip channel $\rho \mapsto (1-p)\rho + p\sigma_x \rho \sigma_x$, where p is a probability not equal to $1/2$ and where σ_x is the first Pauli matrix (Chuang 2014).

The goals of this paper are to review the theory of indivisible stochastic processes, the stochastic-quantum correspondence, and indivisible quantum theory, as well as to describe the ramifications of the stochastic-quantum correspondence for dynamical symmetries and for formal enlargements or dilations of a system's Hilbert space, including several new results. Along the way, this paper will identify two general forms of gauge invariance, the first of which has not yet been described in the research literature.

Attempts to reformulate or reconstruct quantum theory in terms of stochastic processes, without a fundamental physical role for wave functions or Hilbert spaces, are far from new. As Everett pointed out in the unpublished 1956 version of his dissertation, which eventually led to the 'many worlds' interpretation, Bopp was already developing a "stochastic process interpretation" of quantum theory starting in the 1940s (Everett 1956, pp. 114–115; Bopp 1947, 1952, 1953). Fenyes worked on a similar interpretation in the 1950s (Fenyes 1952), but the most well-known stochastic-process formulation was due to Nelson's work in the 1960s through the 1980s, and is known today as Nelsonian stochastic mechanics (Nelson 1966, 1985). One can find later examples in the research literature as well (Aaronson 2005; Strocchi 2008, Chapter 6; Frasca 2012; de Oliveira 2025). These stochastic approaches were all based on adaptations of methods like Brownian motion, and so they all assumed Markovian dynamical laws. This constraint limited the ability of these approaches to capture the full behavior of quantum systems without becoming very complicated, and also made it difficult to generalize them beyond systems of fixed numbers of finitely many non-relativistic particles.

These stochastic approaches are not to be confused with spontaneous-collapse theories, which treat the wave function or density matrix as a primary physical ingredient that reduces or collapses probabilistically with time (Ghirardi, Remini, Weber 1986; Bassi, Ghirardi 2003). Frameworks forming an entirely separate class, known as general(ized) probabilistic theories (GPTs), take a thoroughly instrumentalist perspective, and are phrased in terms of formal, external agents acting on probabilistic systems (Plavala 2023).

In the existing research literature on quantum theory, non-Markovianity is almost exclusively studied as a phenomenological approximation to the observable features of *open* quantum systems, for which deviations from Markovian time evolution arises from interactions with the environment and associated feedback effects. As an important exception, Glick and Adami showed that a form of non-Markovianity is inherent and experimentally accessible in the dynamics of *closed* quantum systems as well, and is deeply connected with entanglement (Glick, Adami 2020). Glick and Adami therefore established that a form of non-Markovianity is a fundamental feature of closed quantum systems, whether one likes it or not. One can regard the present work as an argument for taking the intrinsic non-Markovianity of closed quantum systems seriously as a starting point for *reformulating* quantum theory, in part as a means toward understanding what quantum theory is really about. Although the specific definition of non-Markovianity employed by Glick and Adami does not coincide with *indivisibility*, Glick and Adami also introduced a notion of Markovianity that coincides with *divisibility*, which refers to a breakdown in indivisibility. As a result, indivisible quantum theory makes a concrete observational prediction that the violations of Markovianity identified by Glick and Adami should show up in suitably arranged experiments, although other interpretations of quantum theory may make the same prediction as well.

There are very few previous examples of non-Markovian stochastic processes being used to reformulate quantum theory (Gillespie 1994; Gillespie 1997; Skorobogatov, Svertilov 1998; Gillespie 2000; Gillespie, Alltop, Martin 2001; Dennis 2010). Skorobogatov and Svertilov, for example, studied simple, two-level systems, and showed that with appropriately chosen memory kernels, one could get adequate predictions, but the paper did not construct a general theory or framework. Gillespie's approach was more similar to the one presented here, but focused on defining specific non-Markovian *realizers*, in the terminology of the present work, rather than generalizing to indivisible stochastic processes, which had not been discovered by that point.

Considering the mathematical simplicity of the stochastic-quantum correspondence between indivisible stochastic processes and quantum systems, it is surprising that it has apparently not shown up in the research literature before. To the author's knowledge, the only previous example that bears a suggestive resemblance to the approach taken in this paper, at least at the level of some of its equations, is an unpublished draft by Wetterich (2009).[4] Although that reference argues that some stochastic processes can be modeled using a formalism similar to that of quantum theory, it does not establish that the resulting Hilbert-space representation is fully general. Nor does it attempt to show that the correspondence is bidirectional, so that quantum systems can be modeled by stochastic processes in configuration spaces.

Section 2 will review the theory of indivisible stochastic processes. Section 3 will lay out the stochastic-quantum correspondence, including the dictionary that ultimately connects indivisible stochastic processes with quantum systems, as well as identify an important new class of gauge transformations that have not yet been described in the research literature. Section 4 will describe further implications of the stochastic-quantum correspondence, focusing on dynamical symmetries and Hilbert-space dilations. Section 5 will conclude with a summary and a discussion of future work.

2 Stochastic Processes

2.1 Textbook Stochastic Processes

According to widely used textbooks (Rosenblatt 1962, Parzen 1962, Doob 1990, Ross 1995), the most general kind of *stochastic process* requires only a *sample space*, an initial *probability distribution*, and one or more time-dependent *random variables*, meaning time-indexed families of functions from the sample space to the real numbers. However, stochastic processes defined in this narrow way lack an ingredient that plays the role of a dynamical law. This paper will be concerned with a slightly modified construction that allows the probability distribution itself to vary in time, and that also includes the notion of a dynamical law.

As an important case, one can naturally model classical physical systems as stochastic processes,

[4] The author thanks Logan McCarty for finding this reference.

ultimately with dynamical laws that may be probabilistic in only a trivial sense. In defining a stochastic process to serve as such a model, one can take the sample space to be the system's *configuration space* $\mathcal{C}$,[5] which is a *fixed* ingredient of the model, meaning that it remains the same for every physical run or instantiation of the model. Examples would include the discrete configurations of a system of finitely many digital bits, or the continuous set of possible arrangements of a collection of particles in three-dimensional physical space.

For each physical instantiation of the physical system to be modeled, one then has a *standalone probability distribution* $p(i,t)$, where i denotes a configuration of the system, t is the time, and $p(i,t)$ is the probability or probability density for the system to be in its ith configuration at the time t. The system's standalone probability distribution is a *contingent* feature of the model, meaning that it can be different each time the model is run or instantiated. As a probability distribution, $p(i,t)$ satisfies the usual conditions

$$0 \leq p(i,t) \leq 1, \quad \sum_i p(i,t) = 1, \tag{1}$$

where the discrete summation would be replaced with an integration in the case of a system with a continuous configuration space.

Time-dependent maps $A(t)$ from the system's configuration space $\mathcal{C}$ to the real numbers are then *random variables*, with individual values or magnitudes $a(i,t)$ that depend both on the system's configuration i as well as explicitly on the time t. Random variables can also inherit an implicit time-dependence from the contingent standalone probability distribution $p(i,t)$, which probabilistically determines the system's configuration i and therefore the random variable's value $a(i,t)$. The statistical average or expectation value of a random variable $A(t)$ is then defined as the probability-weighted sum

$$\langle A(t) \rangle \equiv \sum_i a(i,t)p(i,t), \tag{2}$$

where the two forms of time-dependence are manifest on the right-hand side—explicit in $a(i,t)$ and implicit in $p(i,t)$—and, again, the discrete summation would be replaced with an integration if the system has a continuous configuration space.

As explained already, the configuration space $\mathcal{C}$ is one of the *fixed* ingredients of the model, and provides the model with its kinematics, meaning its elementary physical or 'ontological' content. Meanwhile, the standalone probability distribution $p(i,t)$ is a *contingent* ingredient, and provides the model with its informational or 'epistemic' content.

A textbook stochastic process is generally also assumed to have a very intricate set of elementary dynamical laws, or 'nomological' content, consisting of an infinite hierarchy or *Kolmogorov tower* of first-order, second-order, third-order, and higher-order conditional probabilities, conditioned on

[5]For Newtonian systems, one could instead take the sample space to be the system's *phase space*, provided that one writes the dynamical laws in a different way.

a sequence of times $t_1, t_2, t_3, \ldots$ increasingly into the past, and satisfying $t > t_1 > t_2 > t_3 > \cdots$:[6]

$$\left.\begin{array}{rl} p(i,t|j_1,t_1) & \text{[first-order]}, \\ p(i,t|j_1,t_1;j_2,t_2) & \text{[second-order]}, \\ p(i,t|j_1,t_1;j_2,t_2;j_3,t_3), & \text{[third-order]}, \\ \vdots & \end{array}\right\} \tag{3}$$

Here vertical separators $|$ should be read as "given" and semicolons ; should be read as the "and" operator. In each of these conditional probabilities, the single time t to the left of the vertical separator will be called a *target time*, and the times $t_1, t_2, t_3, \ldots$ to the right of the vertical separator will be called *conditioning times*.

Essentially, at the conditioning times, one specifies 'initial conditions' that ultimately lead to a probabilistic prediction for the system's configuration at the target time. To make clear this important dynamical role played by the Kolmogorov tower (3), these conditional probabilities will also be called *transition probabilities*. It is somewhat more intuitive to think of them as aleatory ("objective chance") probabilities, rather than as epistemic ("subjective credence") probabilities.

Unlike the *contingent* standalone probability distribution $p(i,t)$, the transition probabilities (3) are *fixed* rules built into the laws of the model, and are not supposed to change in value for each run or instantiation of the model. However, by combining them with the contingent standalone probability distribution, one can construct contingent *joint* probabilities of greater and greater complexity:

$$\left.\begin{array}{rl} p(i,t;j_1,t_1) \equiv p(i,t|j_1,t_1)p(j_1,t_1) & \text{[two-way]}, \\ p(i,t;j_1,t_1;j_2,t_2) \equiv p(i,t|j_1,t_1;j_2,t_2)p(j_1,t_1;j_2,t_2) & \text{[three-way]}, \\ \vdots & \end{array}\right\} \tag{4}$$

Said somewhat more precisely, these joint probabilities mix together contingent-epistemic and fixed-nomological data—they are not fixed and purely nomological, in contrast with the transition probabilities (3). The joint probabilities are, in turn, required to satisfy an intricate set of marginalization consistency conditions known as Chapman-Kolmogorov equations:

$$\sum_{j_k} p(\ldots;j_k,t_k;\ldots) = p(\ldots[\text{no } j_k]\ldots). \tag{5}$$

[6]This nomological structure is emphasized, for example, in the work of Gillespie and collaborators (Gillespie 1996, Section IV; Gillespie 1998; Gillespie, Alltop, Martin 1999; Gillespie 2000) .

That is,

$$\left.\begin{aligned}\sum_i p(i,t;j_1,t_1) &= p(j_1,t_1),\\ \sum_{j_1} p(i,t;j_1,t_1) &= p(i,t),\\ \sum_i p(i,t;j_1,t_1;j_2,t_2) &= p(j_1,t_1;j_2,t_2),\\ \sum_{j_1} p(i,t;j_1,t_1;j_2,t_2) &= p(i,t;j_2,t_2),\end{aligned}\right\} \tag{6}$$

and so forth. There are many other consistency conditions that must be satisfied as well (Gillespie, Alltop, Martin 1999).

As mentioned earlier, the framework described here is general enough to accommodate classical, deterministic systems. For example, a Newtonian system based on deterministic, second-order equations of motion is equivalent to a stochastic process whose nomological content consists of only second-order conditional probabilities, with values equal to 1 or 0 (or, more properly delta functions), and with times t, t_1, t_2 separated by vanishingly small durations.

A stochastic process is said to be *Markovian* if all its second- and higher-order transition probabilities $p(i,t|j_1,t_1;j_2,t_2,\dots)$ exist and are equal in value to the *first-order* transition probabilities $p(i,t|j_1,t_1)$ for the time t_1 closest to t. That is, for a Markov process, the whole Kolmogorov tower (3) of transition probabilities is included in the model, but the latest conditioning time t_1 always 'screens off' all the earlier conditioning times $t_2, \dots < t_1$, thereby trivializing all second- and higher-order transitions probabilities. Crucially, all those second- and higher-order transition probabilities still *exist* in the laws—the point is just that their numerical values are all determined by first-order transition probabilities. It follows immediately from the Chapman-Kolmogorov equations (5) that the first-order transition probabilities concatenate or compose over time, in the sense that

$$p(i,t) = \sum_{j_1,j_2,\dots} p(i,t|j_1,t_1)p(j_1,t_1|j_2,t_2)\cdots p(j_{n-1},t_{n-1}|j_n,t_n)p(j_n,t_n). \tag{7}$$

Notice, in particular, that the time-evolution laws from t_n to t decompose or *'divide'* into time-evolution laws for all the intermediate time steps.

One can introduce some helpful notation to make this divisibility more manifest. Observe that each first-order transition probability appearing in (7) is labeled by a configuration i at a target time t, and by another configuration j at a conditioning time t'. One can therefore organize these first-order transition probabilities into a *transition matrix* $\Gamma(t \leftarrow t')$ with individual entries defined according to

$$\Gamma_{ij}\big(t \leftarrow t'\big) \equiv p\big(i,t|j,t'\big). \tag{8}$$

Because the entries of $\Gamma(t \leftarrow t')$ are all conditional probabilities, they are all non-negative, and each column sums to 1, meaning that $\Gamma(t \leftarrow t')$ is a *(column) stochastic matrix*:

$$\Gamma_{ij}\big(t \leftarrow t'\big) \geq 0, \quad \sum_i \Gamma_{ij}\big(t \leftarrow t'\big) = 1. \tag{9}$$

Introducing a column vector $p(t)$ with individual entries given by

$$p_i(t) \equiv p(i,t), \tag{10}$$

one can write the composition equation (7) as

$$p_i(t) = \sum_{j_1,j_2,\ldots} \Gamma_{ij_1}(t \leftarrow t_1)\Gamma_{j_1 j_2}(t_1 \leftarrow t_2)\cdots\Gamma_{j_{n-1}j_n}(t_{n-1} \leftarrow t_n)p_{j_n}(t_n), \tag{11}$$

or, more compactly, as

$$p(t) = \Gamma(t \leftarrow t_1)\Gamma(t_1 \leftarrow t_2)\cdots\Gamma(t_{n-1} \leftarrow t_n)p(t_n). \tag{12}$$

In other words, the transition matrix $\Gamma(t \leftarrow t_n)$ from t_n all the way to t *divides* into a product of transition matrices for all the subintervals:

$$\Gamma(t \leftarrow t_n) = \Gamma(t \leftarrow t_1)\Gamma(t_1 \leftarrow t_2)\cdots\Gamma(t_{n-1} \leftarrow t_n). \tag{13}$$

In particular, given any two times t and t_0, with $t > t_0$, then for any intermediate time t' between them, the system's transition matrices satisfy the *divisibility condition*

$$\Gamma(t \leftarrow t_0) = \Gamma\big(t \leftarrow t'\big)\Gamma\big(t' \leftarrow t_0\big). \tag{14}$$

The simplest example of a Markov process is a *discrete-time Markov chain* based on a finite time scale δt, for which the time-dependent transition matrix (8) at any integer number $n \geq 1$ of steps of duration δt from an initial time 0 can be expressed as n powers of a stochastic transition matrix Γ that is constant in time:

$$\Gamma(n\,\delta t \leftarrow 0) = \Gamma^n. \tag{15}$$

A textbook stochastic process that fails to be Markovian is said to be *non-Markovian*, and, in some cases, can be regarded as exhibiting *memory* effects. Memory here refers to the fact that conditioning on more and more previous times changes the values of the conditional probabilities:

$$\left.\begin{aligned} p(i,t|j_1,t_1) &\neq p(i,t|j_1,t_1;j_2,t_2) \\ &\neq p(i,t|j_1,t_1;j_2,t_2;j_3,t_3) \\ &\cdots . \end{aligned}\right\} \tag{16}$$

The Chapman-Kolmogorov equations (5) then yield equations that are more complicated than the

Markovian case (7), such as

$$p(i,t) = \sum_{j_1,j_2,j_3,\dots} p(i,t|j_1,t_1;j_2,t_2;j_3,t_3)p(j_1,t_1|j_2,t_2;j_3,t_3)p(j_2,t_2|j_3,t_3)p(j_3,t_3). \tag{17}$$

The simplest kind of non-Markovian stochastic process has an infinite Kolmogorov tower (3) that trivializes after n orders, for some finite integer n. Such a stochastic process is said to be *nth-order non-Markovian.* A Markov process corresponds to the case $n = 1$. Newtonian mechanics, viewed as a deterministic version of a stochastic process, is second-order non-Markovian, because Newton's second law is a second-order differential equation, so it requires specifying positions at the present time and at a time infinitesimally in the past—although, in practice, one instead specifies positions and velocities at just the present time.

2.2 Indivisible Stochastic Processes

An *indivisible stochastic process* generalizes a textbook stochastic process in a very simple way (Barandes 2025). Rather than laws consisting of an infinite Kolmogorov tower of conditional probabilities of arbitrarily high order, the laws of an indivisible stochastic process contain only first-order transition probabilities connecting target times t with conditioning times t_0, making up a transition matrix $\Gamma(t \leftarrow t_0)$ with individual entries

$$\Gamma_{ij}(t \leftarrow t_0) \equiv p(i,t|j,t_0). \tag{18}$$

The only available Chapman-Kolmogorov equations (5) take the simple form

$$p(i,t) = \sum_j p(i,t|j,t_0)p(j,t_0), \tag{19}$$

which is just the law of total probability. Equivalently, in matrix notation,

$$p(t) = \Gamma(t \leftarrow t_0)p(t_0). \tag{20}$$

For continuity, the transition matrix $\Gamma(t \leftarrow t_0)$ will typically be assumed to approach the identity matrix $\mathbb{1}$ in the limit $t \to t_0$:

$$\lim_{t\to t_0} \Gamma(t \leftarrow t_0) = \mathbb{1}. \tag{21}$$

Importantly, notice that the law of total probability (19) is *linear*, in the sense that it establishes a linear relationship between the system's standalone probabilities at t_0 and the system's standalone probabilities at t. This linearity will turn out to underwrite the linearity of time evolution for closed quantum systems.

Note that no assumption is made here that the transition probabilities $p(i,t|j,t_0)$ exist as part of the laws for all real-valued choices of t_0. Allowed conditioning times t_0 are called *division events* for the given system, and, without any real loss of generality, are assumed to include an 'initial' time 0.

Division events are not global properties of the whole universe, but are system-centric, just like various other kinds of spontaneous time-translation-breaking in physics. In practice, a system may have multiple exact division events, or they may be generated to an extremely good approximation through interactions with other systems, after marginalizing over those other systems (Barandes 2025).

The target time t, by contrast, can be treated as a free variable. In particular, no assumption is made that $t > t_0$. One can choose $t < t_0$ as well. An indivisible stochastic process does not, therefore, need to violate logical time-reversal invariance in any fundamental way.

Crucially, an indivisible stochastic process, as befits its name, will not generally obey a divisibility condition like (7) or (14). Given transition matrices $\Gamma(t \leftarrow t_0)$ and $\Gamma(t' \leftarrow t_0)$, where t' lies in the interval between t_0 and t, it might seem reasonable to try to define an intermediate transition matrix $\tilde{\Gamma}(t \leftarrow t')$ from t' to t according to

$$\tilde{\Gamma}\big(t \leftarrow t'\big) \equiv \Gamma(t \leftarrow t_0)\Gamma^{-1}\big(t' \leftarrow t_0\big), \tag{22}$$

at least if $\Gamma(t' \leftarrow t_0)$ is invertible. By construction, it would then seem to follow that the system obeys a divisibility condition akin to (14):

$$\Gamma(t \leftarrow t_0) = \tilde{\Gamma}\big(t \leftarrow t'\big)\Gamma\big(t' \leftarrow t_0\big). \tag{23}$$

However, it turns out that a matrix $\tilde{\Gamma}(t \leftarrow t')$ defined according to (22) will generically fail to be a column stochastic matrix, and, indeed, will typically have negative entries, and so will form a so-called *pseudo-stochastic matrix* (Chruściński, Man'ko, Marmo, Ventriglia 2015). The reason is that the inverse of a stochastic matrix can only itself be a stochastic matrix if both matrices are *permutation matrices*, meaning that they consist solely of 0s and 1s, and therefore do not contain nontrivial probabilities.[7]

Given just the minimalist ingredients that define a given indivisible stochastic process, there will generically exist a large or infinite number of ways of choosing a complete Kolmogorov tower (3) consistent with those ingredients. Each such choice of Kolmogorov tower is called a non-Markovian *realizer*.[8] A given indivisible stochastic process will therefore encompass a whole equivalence class of non-Markovian realizers. Because only the minimalist ingredients specified by the indivisible stochastic process will end up being connected to the empirical predictions of quantum theory, the

[7]Proof: Let X and Y be $N \times N$ matrices with only non-negative entries and with $Y = X^{-1}$, so that $XY = \mathbb{1}$. Then, in particular, the first row of X must be orthogonal to the second-through-Nth columns of Y. Because, by assumption, Y is invertible, the columns of Y must all be linearly independent, so the first row of X must be orthogonal to the $(N-1)$-dimensional subspace spanned by the second-through-Nth columns of Y. Because the entries of X and Y are all non-negative by assumption, the only way that this orthogonality condition can hold is if precisely one of the entries in the first row of X is nonzero, with a 0 in the corresponding entry in each of the second-through-Nth columns of Y. Repeating this argument for the other rows of X, one sees that X can only have a single nonzero entry in each row. If X is a stochastic matrix, then each of these nonzero entries must be the number 1, so X must be a permutation matrix. Because the inverse of a permutation matrix is again a permutation matrix, it follows that Y must likewise be a permutation matrix. QED

[8]The author thanks Alexander Meehan for suggesting this terminology.

specific non-Markovian realizer is potentially unknowable, and perhaps meaningless. Indeed, from a more fundamental perspective, the system ultimately takes only one trajectory, and if one knew that trajectory in detail, then probabilities would not strictly be needed in the first place.

Rather surprisingly, one can model quantum systems as indivisible stochastic processes for suitable choices of the first-order conditional probabilities (18) that make up their laws, as will be explained in the next section.

3 Review of the Stochastic-Quantum Correspondence

3.1 The Time-Evolution Operator

To see how this construction works in the finite-dimensional case, consider an indivisible stochastic process with N total configurations $i = 1, \ldots, N$ making up the system's configuration space $\mathcal{C}$. Letting 0 denote a suitable choice of initial division event, define the system's time-dependent, $N \times N$ transition matrix as usual according to (18):

$$\Gamma_{ij}(t \leftarrow 0) \equiv p(i, t|j, 0). \tag{24}$$

Then solve the inequality $\Gamma_{ij}(t \leftarrow 0) \geq 0$ appearing in (9) by introducing a "potential" consisting of complex-valued matrix elements $\Theta_{ij}(t \leftarrow 0)$ related to $\Gamma_{ij}(t \leftarrow 0)$ according to the modulus-squaring operation:

$$\Gamma_{ij}(t \leftarrow 0) = |\Theta_{ij}(t \leftarrow 0)|^2. \tag{25}$$

Note that this formula is not a postulate, but an identity, and that the potential matrix $\Theta(t \leftarrow 0)$, which will be called a *time-evolution operator* in what follows, is not unique.[9] It follows from the other formula $\sum_i \Gamma_{ij}(t \leftarrow 0) = 1$ in (9) that the time-evolution operator $\Theta(t \leftarrow 0)$ satisfies the summation condition

$$\sum_{i=1}^{N} |\Theta_{ij}(t \leftarrow 0)|^2 = 1. \tag{26}$$

There are several helpful ways to re-express the identity (25). To begin, one can introduce the *Schur-Hadamard product* $\odot$, which is defined for arbitrary $N \times N$ matrices X and Y as entry-wise multiplication (Schur 1911, Halmos 1958, Horn 1990):

$$(X \odot Y)_{ij} \equiv X_{ij} Y_{ij} \quad \text{[no sum on repeated indices]}. \tag{27}$$

One can then regard (25) as expressing the transition matrix $\Gamma(t \leftarrow 0)$ as a Schur-Hadamard factorization of the complex-conjugated potential matrix $\overline{\Theta(t \leftarrow 0)}$ with $\Theta(t \leftarrow 0)$ itself:

$$\Gamma(t \leftarrow 0) = \overline{\Theta(t \leftarrow 0)} \odot \Theta(t \leftarrow 0). \tag{28}$$

[9] No assumption is made at this point that the time-evolution operator $\Theta(t \leftarrow 0)$ is unitary. Also, the terms "matrix" and "operator" will be used synonymously in this paper.

Schur-Hadamard products are not widely used in linear algebra, in part because they are basis-dependent. For the purposes of analyzing a given indivisible stochastic process, however, this basis-dependence is unimportant, because the system's configuration space $\mathcal{C}$ naturally singles out a specific basis, to be defined momentarily.

3.2 Schur-Hadamard Gauge Transformations

To make the nonuniqueness of the potential matrix $\Theta(t \leftarrow 0)$ more manifest, it will be helpful to introduce an analogy with the Maxwell theory of classical electromagnetism.[10]

In classical electromagnetism, the electric and magnetic fields are physically meaningful quantities, but it is often very convenient to work instead in terms of scalar and vector potentials, which are not uniquely defined. All choices for the potentials that yield the same electric and magnetic fields are said to be related by gauge transformations. Any one such choice for the potentials is called a gauge choice, and the scalar and vector potentials themselves are called gauge potentials or gauge variables.

Making a suitable gauge choice can greatly simplify many calculations, such as using Lorenz gauge to compute the electric and magnetic fields for delayed boundary conditions. Ultimately, however, the theory does not treat any gauge choice as fundamentally more correct than any other gauge choice, and all calculations of physical predictions in classical electromagnetism must conclude with expressions that are written in terms of gauge-invariant quantities.

To set up the claimed analogy with electromagnetic gauge transformations, one starts by observing from the basic relationship $\Gamma_{ij}(t \leftarrow 0) = |\Theta_{ij}(t \leftarrow 0)|^2$ in (25) that the Schur-Hadamard product (27) of the time-evolution operator $\Theta(t \leftarrow 0)$ and a matrix of arbitrary, time-dependent phases $\exp(i\theta_{ij}(t))$ is a transformation of $\Theta(t \leftarrow 0)$ with no physical effects, and therefore corresponds to a genuine form of gauge invariance:

$$\Theta(t \leftarrow 0) \mapsto \Theta(t \leftarrow 0) \odot \begin{pmatrix} e^{i\theta_{11}(t)} & e^{i\theta_{12}(t)} & \\ e^{i\theta_{21}(t)} & \ddots & \\ & & e^{i\theta_{NN}(t)} \end{pmatrix}. \tag{29}$$

This gauge transformation can be written equivalently at the level of individual matrix entries as

$$\Theta_{ij}(t \leftarrow 0) \mapsto \Theta_{ij}(t \leftarrow 0) e^{i\theta_{ij}(t)}. \tag{30}$$

Although it may seem surprising that these time-dependent changes of phase have no physical effects, keep in mind the *crucial* fact that this gauge transformation will end up having downstream effects on the definitions of various Hilbert-space ingredients ahead, in just such a way that all empirical predictions will remain unchanged.[11]

[10] For pedagogical treatments of classical electromagnetism, see Griffiths (2023), Zangwill (2012), or Jackson (1998).

[11] In particular, this form of gauge invariance is *not* equivalent to changing merely the relative phases of state vectors *alone*.

This kind of gauge transformation will be called a *Schur-Hadamard gauge transformation*, because it involves the sort of entry-wise multiplication that characterizes Schur or Hadamard products. To the author's knowledge, this kind of gauge invariance has not yet been described in the research literature, and is therefore a new result. It will turn out to play a key role in the analysis of dynamical symmetries that will be presented in Subsection 4.1, and will be extended in an interesting way in the context of Hilbert-space dilations in Subsection 4.2. A different but equally general form of gauge invariance will be discussed later on.

3.3 The Dictionary

As an important alternative approach to writing the identity $\Gamma_{ij}(t \leftarrow 0) = |\Theta_{ij}(t \leftarrow 0)|^2$ in (25), one starts by introducing a set of $N \times 1$ column vectors $e_1, \ldots, e_N$, where e_i has a 1 in its ith component and 0s in all its other components:

$$e_1 \equiv \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_N \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{31}$$

That is, e_i has components

$$e_{i,j} = \delta_{ij}. \tag{32}$$

It follows that the column vectors $e_1, \ldots, e_N$ form an orthonormal basis for the vector space of all $N \times 1$ column vectors, and $e_1, \ldots, e_N$ will be called the system's *configuration basis*.

In particular,

$$e_i^\dagger e_j = \delta_{ij}, \quad e_i e_i^\dagger = P_i, \tag{33}$$

where δ_{ij} is the usual Kronecker delta,

$$\delta_{ij} \equiv \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases} \tag{34}$$

and where $P_1, \ldots, P_N$ are an N-member collection of $N \times N$ constant, diagonal projection matrices, which will be called *configuration projectors*. For each $i = 1, \ldots, N$, the configuration projector P_i consists of a single 1 in its ith row, ith column, and 0s in all its other entries. That is, P_i is defined as

$$P_i \equiv \operatorname{diag}(0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{th entry}}}{1}, 0, \ldots, 0) = e_i e_i^\dagger, \tag{35}$$

with individual entries

$$P_{i,jk} = \delta_{ij}\delta_{ik}. \tag{36}$$

It follows immediately that the configuration projectors satisfy the conditions of *mutual exclusivity*,

$$P_i P_j = \delta_{ij} P_i, \tag{37}$$

and *completeness*,

$$\sum_{i=1}^{N} P_i = \mathbb{1}, \tag{38}$$

where again $\mathbb{1}$ is the $N \times N$ identity matrix. These two conditions are the defining features of a projection-valued measure (PVM) (Mackey 1952, 1957), so the configuration projectors $P_1, \dots, P_N$ naturally constitute a PVM.

Letting $\mathrm{tr}(\cdots)$ denote the usual trace, one can then recast the identity $\Gamma_{ij}(t \leftarrow 0) = |\Theta_{ij}(t \leftarrow 0)|^2$ from (25) as

$$\boxed{\Gamma_{ij}(t \leftarrow 0) = \mathrm{tr}(\Theta^\dagger(t \leftarrow 0) P_i \Theta(t \leftarrow 0) P_j).} \tag{39}$$

This formula provides the *dictionary* that translates between indivisible stochastic processes, as represented by the left-hand side, and the formalism of quantum-theoretic Hilbert spaces, as represented by the right-hand side. This dictionary is the basic ingredient of the *stochastic-quantum correspondence*.

3.4 The Hilbert-Space Representation

Given an initial standalone probability distribution,

$$p_j(0) \equiv p(j, 0), \tag{40}$$

perhaps capturing epistemic ("subjective credence") uncertainty about the system's configuration j at the initial time 0, one can define an initial *density matrix*

$$\rho(0) \equiv \mathrm{diag}(\dots, p_j(0), \dots) = \sum_{j=1}^{N} p_j(0) P_j, \tag{41}$$

which is a diagonal matrix whose diagonal entries are just the initial probabilities $p_j(0)$, where P_j are the configuration projectors defined in (35). One can then define a time-dependent density matrix by a congruence transformation

$$\rho(t) \equiv \Theta(t \leftarrow 0) \rho(0) \Theta^\dagger(t \leftarrow 0), \tag{42}$$

which is not generically diagonal, but is self-adjoint, positive-semidefinite, and has trace equal to 1, due to the summation condition (26) satisfied by the potential matrix $\Theta(t \leftarrow 0)$.

If the system's density matrix $\rho(t)$ happens to be rank-one, then it can be factorized as the outer product of an $N \times 1$ column vector $\Psi(t)$ with its complex-conjugate-transpose, or adjoint, $\Psi^\dagger(t)$:

$$\rho(t) = \Psi(t)\Psi^\dagger(t) \quad \text{[if rank-one]}. \tag{43}$$

The *state vector* or *wave function* $\Psi(t)$ then evolves in time according to

$$\Psi(t) = \Theta(t)\Psi(0), \tag{44}$$

and is guaranteed to have unit-norm, in the sense that

$$\sqrt{\Psi^\dagger(t)\Psi(t)} = 1. \tag{45}$$

However, $\Psi(t)$ is not unique, as it implicitly depends on the choice of potential matrix $\Theta(t \leftarrow 0)$, and can therefore be altered by Schur-Hadamard gauge transformations (29). Later on, this paper will review a different kind of gauge invariance under which $\Psi(t)$ is not unique. Even fixing both these kinds of gauge invariance, the state vector $\Psi(t)$ is still only defined up to an overall phase factor.

In terms of the system's density matrix $\rho(t)$, as defined in (42), one can express the law of total probability (19) as

$$p_i(t) = \mathrm{tr}(P_i\rho(t)), \tag{46}$$

which is just the *Born rule* for the configuration basis, as expressed in terms of the system's density matrix $\rho(t)$ and the configuration projector P_i, as defined in (35). The fact that this formula works out correctly, given the definition (42) of the system's density matrix, is ultimately due to the linearity of the law of total probability (19). For the case of a rank-one density matrix, for which a state vector $\Psi(t)$ is available, the Born rule (46) takes a form that is more commonly found in elementary textbooks,

$$p_i(t) = |\Psi_i(t)|^2, \tag{47}$$

where $\Psi_i(t)$ is the complex-valued ith entry of $\Psi(t)$.

For any random variable $A(t)$ with possible values $a(i,t)$, one can introduce a diagonal, self-adjoint matrix, which, by an re-appropriation of notation, will be denoted by the same symbol $A(t)$:

$$A(t) \equiv \mathrm{diag}(\ldots, a(i,t), \ldots) = \sum_{i=1}^{N} a(i,t)P_i. \tag{48}$$

From straightforward linear algebra, one can rewrite the expectation value (2) as

$$\langle A(t)\rangle = \mathrm{tr}(A(t)\rho(t)), \tag{49}$$

or, for the rank-one case, as

$$\langle A(t)\rangle = \Psi^\dagger(t)A(t)\Psi(t). \tag{50}$$

In keeping with Bell's terminology, random variables will now be called *be-ables*, or *beables* (Bell 1973, 1976). When the system is any given configuration i, each beable $A(t)$ has a definite, underlying value $a(i,t)$. Bell intended to distinguish beables from mere observables, which, by the Kochen-Specker theorem (Bell 1966; Kochen, Specker 1967), cannot all be beables.

Observables that are not beables are non-diagonal self-adjoint operators that correspond to emergent patterns that show up when systems interact with measuring devices, and will be called *emergeables* (Barandes 2025). The idea of emergeables goes back to Bohr (Bohr 1935, Bell 1971), and also figures prominently in Bohmian mechanics (Bell 1982, Daumer et al. 1996). A system's beables and its emergeables collectively comprise the system's overall non-commutative algebra of observables. When a measuring device is properly modeled as one additional part of a larger stochastic process, along the lines presented in the present paper, one can show that at the end of the measurement process, the measuring device will end up in one of its possible measurement-outcome configurations with a stochastic probability that coincides with the standard Born rule, whether the measuring device has been tuned to measure a beable or an emergeable (Barandes 2025).

The Hilbert-space formulas introduced so far are all expressed in what would conventionally be called the *Schrödinger picture*. It can also be useful to introduce the *Heisenberg picture*, which is defined according to

$$\left.\begin{aligned} \rho^H \equiv \rho(0), \quad \Psi^H \equiv \Psi(0), \\ A^H(t) \equiv \Theta^\dagger(t \leftarrow 0) A(t) \Theta(t \leftarrow 0), \end{aligned}\right\} \tag{51}$$

where $A^H(t)$, which is no longer generically diagonal, now includes both a possible explicit dependence on time through its magnitudes $a_i(t)$ as well as an implicit dependence on time through the time-evolution operator $\Theta(t \leftarrow 0)$. In the Heisenberg picture, the probability formula (46) becomes[12]

$$p_i(t) = \mathrm{tr}(P^H(t)\rho^H), \tag{52}$$

and the formula (49) for expectation values becomes

$$\langle A(t) \rangle = \mathrm{tr}(A^H(t)\rho^H). \tag{53}$$

With the Heisenberg picture available, one can construct a general example of an emergeable. Let A be a time-independent beable, meaning a random variable represented by a constant diagonal matrix (48), and consider the time derivative of its Heisenberg-picture counterpart

$$A^H(t) \equiv \Theta^\dagger(t \leftarrow 0) A(t) \Theta(t \leftarrow 0), \tag{54}$$

as defined for the time-evolution operator $\Theta(t \leftarrow 0)$, in the limit $t \to 0$:

$$\dot{A} \equiv \lim_{t \to 0} \frac{dA^H(t)}{dt} = \dot{A}^\dagger. \tag{55}$$

This self-adjoint $N \times N$ matrix will generically be non-diagonal and will therefore not generally commute with the original random variable A itself:

$$[A, \dot{A}] \neq 0. \tag{56}$$

[12]Note that for a generic time-evolution operator $\Theta(t \leftarrow 0)$, the Heisenberg-picture version $P_i^H(t) \equiv \Theta^\dagger(t \leftarrow 0) P_i \Theta(t \leftarrow 0)$ of a projector P_i will not necessarily still be a projector.

Nonetheless, the matrix $\dot{A}$ has physical uses. For example, one has

$$\mathrm{tr}(\dot{A}\rho(0)) = \lim_{t\to 0} \frac{d\langle A(t)\rangle}{dt}, \tag{57}$$

which is a perfectly meaningful physical quantity, even though the time derivative of an expectation value is not necessarily the expectation value of something physical.

The matrix $\dot{A}$ therefore resembles a random variable in some ways, but incorporates stochastic dynamics directly into its definition through the time-evolution operator $\Theta(t \leftarrow 0)$, and does not have a transparent interpretation at the level of the underlying configuration space $\mathcal{C}$ of the given indivisible stochastic process. Instead, $\dot{A}$ is an emergent amalgam of kinematical and dynamical ingredients—that is, it is an emergeable.

As a concrete example, consider a particle whose underlying *position* is regarded as a physical configuration, corresponding to some random variable A. If the particle's dynamics is stochastic, in the sense that the particle can be described as an indivisible stochastic process, then the particle's *velocity* (or, equivalently, the particle's *momentum*) will not generally have a well-defined value at all times, and will naturally be representable as an emergeable $\dot{A}$ along the lines described here.

3.5 Kraus Decompositions and Unitary Time Evolution

Given an $N \times N$ time-evolution operator $\Theta(t \leftarrow 0)$ for the system, one can define a set of N *Kraus operators* $K_\beta(t \leftarrow 0)$, for $\beta = 1, \ldots, N$, according to

$$K_\beta(t \leftarrow 0) \equiv \Theta(t \leftarrow 0)P_\beta. \tag{58}$$

Due to the summation condition $\sum_{i=1}^{N} |\Theta_{ij}(t \leftarrow 0)|^2 = 1$ from (26), these operators satisfy the Kraus identity

$$\sum_{\beta=1}^{N} K_\beta^\dagger(t \leftarrow 0)K_\beta(t \leftarrow 0) = \mathbb{1}, \tag{59}$$

as befits their name (Kraus 1971). In terms of these Kraus operators, one can write the stochastic-quantum dictionary (39) as the Kraus decomposition

$$\Gamma_{ij}(t \leftarrow 0) = \sum_{\beta=1}^{N} \mathrm{tr}(K_\beta^\dagger(t \leftarrow 0)P_i K_\beta(t \leftarrow 0)P_j). \tag{60}$$

Consequently, one can write the time-evolution formula (42) for the system's density matrix $\rho(t)$ as the Kraus decomposition

$$\rho(t) \equiv \sum_{\beta=1}^{N} K_\beta(t \leftarrow 0)\rho(0)K_\beta^\dagger(t \leftarrow 0). \tag{61}$$

Like all the other mathematical objects in the Hilbert-space formulation, the Kraus operators $K_1(t \leftarrow 0), \ldots, K_N(t \leftarrow 0)$ are not unique. Notice also that any number of $N \times N$ matrices satisfying the Kraus identity (59) and giving a suitable Kraus decomposition akin to (60) are guaranteed to yield a valid transition matrix $\Gamma(t \leftarrow 0)$. Said in another way, the preceding argument establishes the existence but not the uniqueness of Kraus operators for any given time-evolution operator $\Theta(t \leftarrow 0)$.

As shown in other work (Barandes 2025, 2023), and as will be explained in detail later in the present paper, the existence of these Kraus decompositions has an important implication. Specifically, after an application, if necessary, of the Stinespring dilation theorem (Stinespring 1955, Keyl 2002), which involves expanding the original N-dimensional Hilbert space to a 'dilated' Hilbert space of dimension no greater than N^3, one can always assume that the system's time-evolution operator $\Theta(t \leftarrow 0)$ is unitary, meaning that

$$\Theta(t \leftarrow 0) = U(t \leftarrow 0), \tag{62}$$

with

$$U^\dagger(t \leftarrow 0) = U^{-1}(t \leftarrow 0). \tag{63}$$

The preceding arguments therefore provide a first-principles motivation for *unitary time evolution*, or *unitarity*, in quantum theory.

It follows that the basic relationship (25) now takes the form

$$\Gamma_{ij}(t \leftarrow 0) = |U_{ij}(t \leftarrow 0)|^2, \tag{64}$$

and the dictionary (39) of the stochastic-quantum correspondence becomes

$$\Gamma_{ij}(t \leftarrow 0) = \mathrm{tr}(U^\dagger(t \leftarrow 0) P_i U(t \leftarrow 0) P_j). \tag{65}$$

The system's density matrix time-evolution rule (42) is now

$$\rho(t) \equiv U(t \leftarrow 0)\rho(0)U^\dagger(t \leftarrow 0), \tag{66}$$

and if the density matrix is rank-one, so that it factorizes as $\rho(t) = \Psi(t)\Psi^\dagger(t)$ as in (43) for a state vector $\Psi(t)$, then the state vector's time-evolution rule (44) is

$$\Psi(t) = U(t \leftarrow 0)\Psi(0). \tag{67}$$

The Heisenberg picture (51) is then

$$\left.\begin{aligned} \rho^H \equiv \rho(0), \quad \Psi^H \equiv \Psi(0), \\ A^H(t) \equiv U^\dagger(t \leftarrow 0)A(t)U(t \leftarrow 0). \end{aligned}\right\} \tag{68}$$

In general, an $N \times N$ matrix is called *unistochastic* if its individual entries are expressible as the modulus-squares of the corresponding entries of an $N \times N$ unitary matrix. It follows that (64) is just the statement that the system's transition matrix $\Gamma(t \leftarrow 0)$ can be taken to be unistochastic, and will therefore be said to describe a *unistochastic process.*

The preceding analysis implies that an indivisible stochastic process can be viewed either as a unistochastic process itself, or (if a nontrivial dilation was required) as a *subsystem* of a unistochastic process. This statement is called the *stochastic-quantum theorem* (Barandes 2023).

Note that a unitary time-evolution operator $U(t \leftarrow 0)$ will not generically remain unitary under arbitrary Schur-Hadamard gauge transformations (30). Hence, writing a unistochastic transition matrix $\Gamma(t \leftarrow 0)$ in terms of a unitary time-evolution operator $U(t \leftarrow 0)$ corresponds to making a gauge choice—or, somewhat more precisely, to a partial fixing of the gauge freedom (30).

If the unitary time-evolution operator $U(t \leftarrow 0)$ is an appropriately differentiable function of the target time t, then one can define a corresponding self-adjoint matrix $H(t)$, called the *Hamiltonian*, according to

$$H(t) \equiv i\hbar \frac{\partial U(t \leftarrow 0)}{\partial t} U^\dagger(t \leftarrow 0) = H^\dagger(t). \tag{69}$$

Here the factor of i ensures self-adjointness. The reduced Planck constant $\hbar$ is introduced to give $H(t)$ measurement units of energy, with an explicit value fixed by the definition of one's system of measurement units. It follows immediately that the system's density matrix satisfies the *von Neumann equation*,

$$i\hbar \frac{\partial \rho(t)}{\partial t} = [H(t), \rho(t)], \tag{70}$$

the system's state vector (which exists if the density matrix is rank-one) satisfies the *Schrödinger equation*,

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = H(t)\Psi(t), \tag{71}$$

expectation values satisfy the *Ehrenfest equation*,

$$\frac{d\langle A(t)\rangle}{dt} = \left\langle \frac{i}{\hbar}[H(t), A(t)] \right\rangle + \left\langle \frac{\partial A(t)}{\partial t} \right\rangle, \tag{72}$$

and Heisenberg-picture random variables $A^H(t)$ evolve according to the *Heisenberg equation of motion*,

$$\frac{dA^H(t)}{dt} = \frac{i}{\hbar}[H^H(t), A^H(t)] + \left(\frac{\partial A(t)}{\partial t} \right)^H, \tag{73}$$

where the matrix $H^H(t)$ appearing in the Heisenberg equation of motion (73) is the Hamiltonian expressed in the Heisenberg picture. Note that the brackets here are *commutators*, $[A, B] \equiv AB - BA$, not Poisson brackets, and that these equations are not mere analogies rooted in a classical Liouville picture.

This paper has reviewed how the stochastic-quantum correspondence allows one to write an indivisible stochastic process in a Hilbert-space form describing an associated quantum system. By reading the dictionary (65) in the other direction, a quantum system evolving unitarily can be understood as an indivisible stochastic process in disguise. The correspondence between indivisible stochastic processes and quantum theory is akin to the correspondence between classical Newtonian systems and Hamiltonian mechanics—the former member of each pair gives a clearer picture of the ontology, whereas the latter member of each pair provides powerful mathematical tools for constructing new dynamics and calculating predictions. In both cases, the correspondence is many-to-one in both directions.

3.6 Foldy-Wouthuysen Gauge Transformations

The Hilbert-space formulation has *another* form of gauge invariance, which appears to have first been written down by Foldy and Wouthuysen in a 1950 paper on the Dirac equation (Foldy, Wouthuysen).[13] Working with a generic time-evolution operator $\Theta(t \leftarrow 0)$ that may or may not be unitary, and letting $V(t)$ be a time-dependent unitary matrix, the following transformation is a gauge invariance of the entire Hilbert-space formulation, leaving all probabilities $p_i(t)$, expectation values $\langle A(t) \rangle$, and the transition matrix $\Gamma(t \leftarrow 0)$ as a whole unchanged:[14]

$$\left.\begin{aligned} \rho(t) &\mapsto \rho_V(t) \equiv V(t)\rho(t)V^\dagger(t), \\ \Psi(t) &\mapsto \Psi_V(t) \equiv V(t)\Psi(t), \\ A(t) &\mapsto A_V(t) \equiv V(t)A(t)V^\dagger(t), \\ \Theta(t \leftarrow 0) &\mapsto \Theta_V(t \leftarrow 0) \equiv V(t)\Theta(t \leftarrow 0)V^\dagger(0). \end{aligned}\right\} \tag{74}$$

If the unitary matrix $V(t)$ is taken to be time-*independent*, then the Foldy-Wouthuysen gauge transformation (74) is merely a simple change of orthonormal basis. However, if $V(t)$ depends nontrivially on time, and if one regards the system's Hilbert space at each moment in time as a fiber over a one-dimensional base manifold parameterized by the time coordinate t, then $V(t)$ represents a local-in-time, unitary transformation of each individual Hilbert-space fiber. In particular, any given time-dependent state vector $\Psi(t)$, regarded as a trajectory through the system's Hilbert space, can be mapped to any other trajectory by a suitable choice of time-dependent unitary matrix $V(t)$, so trajectories in the Hilbert space do not describe gauge-invariant facts.

These formulas make manifest that the Hilbert-space formulation of an indivisible stochastic process is ultimately a collection of gauge-dependent quantities, or gauge variables. In any physical theory, one does not typically try to assign gauge variables an ontological meaning. The familiar ingredients of the Hilbert-space formulation are highly gauge-dependent according to both kinds of gauge transformations described in the present paper—Schur-Hadamard gauge transformations (29) and Foldy-Wouthuysen gauge transformations (74). Hence, although a Hilbert-space formulation may be extremely useful for constructing stochastic dynamics or for carrying out calculations, one might rightly be skeptical about trying to assign direct physical meanings to its mathematical ingredients, or suspicious of any interpretation of quantum theory based on reifying, say, wave

[13]Foldy and Wouthuysen originally used the term *canonical transformation* rather than *gauge transformation*, and wrote the operator V instead as $\exp(iS)$, where $S = S(t)$ is a time-dependent, Hermitian operator. These transformations were later described in the textbook on quantum mechanics by Messiah (1958, Volume 1, Chapter XX, Section 33), and also in the textbook on relativistic quantum mechanics by Bjorken and Drell (1964, Chapter 4). They were a particular focus of a paper by Goldman (1977), and were apparently discovered independently by Brown (1999), inspired by his study of transformations of the Schrödinger equation between inertial and non-inertial reference frames.

[14]One should be mindful of the appearance of the initial time 0 in $V^\dagger(0)$ in the transformation rule for $\Theta(t \leftarrow 0)$.

functions or density matrices as parts of the ontological furniture of reality.

Intriguingly, if the system's time-evolution operator $\Theta(t \leftarrow 0) = U(t \leftarrow 0)$ can be taken to be unitary, then under Foldy-Wouthuysen gauge transformation defined by (74), the Hamiltonian $H(t)$ defined in (69) transforms precisely as a non-Abelian gauge potential:[15]

$$\left.\begin{aligned} H(t) &\mapsto H_V(t) \\ &= V(t)H(t)V^\dagger(t) - i\hbar V(t)\frac{\partial V^\dagger(t)}{\partial t}. \end{aligned}\right\} \tag{75}$$

This transformation behavior makes clear that a Hamiltonian is not a gauge-invariant observable, even though it may happen to coincide with various observables according to particular gauge choices. For example, for a simple system of non-relativistic point particles, there is generically a gauge choice in which the Hamiltonian is equal to the sum of the observables representing the system's kinetic and potential energies, but the Hamiltonian may look different according to other gauge choices.

Observe that one can rewrite the Schrödinger equation (71) as

$$\mathcal{D}(t)\Psi(t) = 0. \tag{76}$$

Here $\mathcal{D}(t)$ is a gauge-covariant derivative defined according to

$$\mathcal{D}(t) \equiv \mathbb{1}\frac{\partial}{\partial t} + \frac{i}{\hbar}H(t), \tag{77}$$

which maintains its form under Foldy-Wouthuysen gauge transformations (74), in the sense that

$$V(t)\left[\mathbb{1}\frac{\partial}{\partial t} + \frac{i}{\hbar}H(t)\right](\cdots) = \left[\mathbb{1}\frac{\partial}{\partial t} + \frac{i}{\hbar}H_V(t)\right][V(t)(\cdots)]. \tag{78}$$

Notice that if one picks the Foldy-Wouthuysen gauge-transformation matrix $V(t)$ to be the adjoint of the unistochastic process's time-evolution operator $U(t \leftarrow 0)$,

$$V(t) \equiv U^\dagger(t \leftarrow 0), \tag{79}$$

then the Hamiltonian precisely vanishes:

$$H_V(t) = 0. \tag{80}$$

This choice of gauge is nothing other than the definition (68) of the Heisenberg picture. Foldy-Wouthuysen gauge transformations (74) can therefore be viewed as generalized changes of time-evolution picture.[16]

[15] For pedagogical treatments of non-Abelian gauge theories, see Peskin, Schroeder (1999) or Weinberg (1996).

[16] The fact that one can set $H_V(t) = 0$ for all t is a manifestation of the fact that the fiber bundle in this case, consisting of copies of the system's Hilbert space fibered over a one-dimensional base manifold parameterized by the time t, has vanishing curvature.

4 Further Implications

4.1 Dynamical Symmetries

The stochastic-quantum correspondence developed in this paper provides new ways to think about *dynamical symmetries* in quantum theory, meaning transformations that leave the dynamics invariant. Going in the other direction, the stochastic-quantum correspondence also makes it more straightforward to impose dynamical symmetries systematically as constraints in the construction of a given indivisible stochastic process.

In the textbook approach to quantum theory, based on the Dirac-von Neumann axioms, the only clear ontology is associated with observers, measuring devices, and their measurement results. Symmetry transformations are then most naturally phrased in *passive* terms, as changes in the perspective of an observer or measuring device.[17]

From this standpoint, it is not clear what it would mean to carry out an *active* symmetry transformation on the quantum system itself. From the standpoint of indivisible quantum theory, by contrast, active symmetry transformations have essentially the same meaning as they do in classical physics—as changes to the given system's physical states or configurations.

As an example, an invertible transformation of a system's configurations $i = 1, \dots, N$ could take the form of a permutation transformation $\sigma : \{1, \dots, N\} \to \{1, \dots, N\}$ of the configuration projectors (35):

$$\left.\begin{aligned} &P_i \mapsto P_{\sigma(i)}, \\ &\quad \text{with } \{\sigma(1), \dots, \sigma(N)\} = \{1, \dots, N\}. \end{aligned}\right\} \tag{81}$$

More generally, an invertible transformation that alters the system's configurations in some more fundamental way should still allow for a Hilbert-space description in which the system's new configurations are again represented by a PVM. If P_i represented the ith configuration of the system before the transformation, then let $\tilde{P}_i$ represent the corresponding configuration after the transformation. An invertible transformation between two PVMs $P_1, \dots, P_N$ and $\tilde{P}_1, \dots, \tilde{P}_N$ is always a similarity transformation of the form

$$P_i \mapsto \tilde{P}_i \equiv V^\dagger P_i V, \tag{82}$$

where V is some unitary operator.[18] This similarity transformation reduces to the simple transformation (81) if and only if V is a permutation matrix.

The next step is to determine what condition ensures that the more general transformation (82) is a dynamical symmetry, meaning that it leaves the transition matrix—and thus the stochastic

[17] Weinberg, for instance, writes that "A symmetry transformation is a change in our point of view that does not change the results of possible experiments." (Weinberg 1996, Section 2.2, pp. 50–51).

[18] Proof: Let $e_1, \dots, e_N$ be the orthonormal configuration basis (31), with $e_i^\dagger e_j = \delta_{ij}$ and $e_i e_i^\dagger = P_i$ as in (33), and let $\tilde{e}_1, \dots, \tilde{e}_N$ be an orthonormal basis related to the new projectors $\tilde{P}_i$ in the analogous way, with $\tilde{e}_i^\dagger \tilde{e}_j = \delta_{ij}$ and $\tilde{e}_i \tilde{e}_i^\dagger = \tilde{P}_i$. Then the $N \times N$ matrix defined by $V \equiv \sum_i e_i \tilde{e}_i^\dagger$ is unitary and satisfies $V^\dagger P_i V = \tilde{P}_i$. Going the other way, if V is an $N \times N$ unitary matrix, then the $N \times N$ matrices defined for $i = 1, \dots, N$ by $\tilde{P}_i \equiv V^\dagger P_i V$ are guaranteed to constitute a PVM. QED

dynamics—invariant. The required condition is precisely that the right-hand side of the stochastic-quantum dictionary (39) should remain unchanged:

$$\mathrm{tr}(\Theta^{\dagger}(t \leftarrow 0)\tilde{P}_i\Theta(t \leftarrow 0)\tilde{P}_j) = \mathrm{tr}(\Theta^{\dagger}(t \leftarrow 0)P_i\Theta(t \leftarrow 0)P_j). \tag{83}$$

This condition is equivalent to the statement that

$$\mathrm{tr}(\tilde{\Theta}^{\dagger}(t \leftarrow 0)P_i\tilde{\Theta}(t \leftarrow 0)P_j) = \mathrm{tr}(\Theta^{\dagger}(t \leftarrow 0)P_i\Theta(t \leftarrow 0)P_j), \tag{84}$$

where

$$\tilde{\Theta}(t \leftarrow 0) \equiv V\Theta(t \leftarrow 0)V^{\dagger}. \tag{85}$$

Re-expressing both sides of the equivalent condition (84) in terms of modulus-squared values, as in (25), one sees that (85) is a dynamical symmetry precisely if

$$|\tilde{\Theta}_{ij}(t \leftarrow 0)|^2 = |\Theta_{ij}(t \leftarrow 0)|^2. \tag{86}$$

It follows immediately that $\tilde{\Theta}(t \leftarrow 0)$ can differ from $\Theta(t \leftarrow 0)$ by at most a Schur-Hadamard gauge transformation (29), so a necessary and sufficient condition for a unitary matrix V to give a dynamical symmetry is that

$$V\Theta(t \leftarrow 0)V^{\dagger} = \Theta(t \leftarrow 0) \odot \begin{pmatrix} e^{i\theta_{11}(t)} & e^{i\theta_{12}(t)} & \\ e^{i\theta_{21}(t)} & \ddots & \\ & & e^{i\theta_{NN}(t)} \end{pmatrix}. \tag{87}$$

As special cases, this condition includes *unitary* dynamical symmetries,

$$V\Theta(t \leftarrow 0)V^{\dagger} = \Theta(t \leftarrow 0), \tag{88}$$

as well as *anti-unitary* dynamical symmetries,

$$V\Theta(t \leftarrow 0)V^{\dagger} = \overline{\Theta(t \leftarrow 0)}, \tag{89}$$

where the overline notation denotes complex-conjugation—which is, ultimately, just a change of phases.

For the specific case of an anti-unitary dynamical symmetry, note that if one redefines $V \mapsto \overline{V}$, which is still unitary, then one can re-express (89) in the somewhat more conventional form

$$VK\Theta(t \leftarrow 0)KV^{\dagger} = \Theta(t \leftarrow 0). \tag{90}$$

Here K denotes the complex-conjugation operator, meaning that K is an involution,

$$K^2 = 1, \tag{91}$$

and, for any $N \times N$ matrix X, one has

$$KXK = \overline{X}. \tag{92}$$

The composite operator VK as a whole is then said to be an *anti-unitary operator*. Anti-unitary operators play an important role in describing time-reversal symmetries.[19]

If $\Theta(t \leftarrow 0) = U(t \leftarrow 0)$ is unitary, then $V\Theta(t \leftarrow 0)V^\dagger$ will likewise be unitary. In that case, suppose either that V is continuously connected to the identity matrix $\mathbb{1}$ by some smooth parameter, with a corresponding self-adjoint generator $G = G^\dagger$, or, alternatively, that V is an involution, meaning that $V^2 = \mathbb{1}$, in which case $G \equiv V = V^\dagger$ is *itself* self-adjoint. Either way, G is self-adjoint, and therefore represents a candidate observable, so the expectation value $\langle G(t)\rangle$ is an empirically meaningful quantity at the level of measurement processes. If G commutes with $U(t \leftarrow 0)$, then Noether's theorem easily follows as the statement that this expectation value is constant in time, or conserved:

$$\langle G(t)\rangle = \mathrm{tr}(GU(t \leftarrow 0)\rho(0)U^\dagger(t \leftarrow 0)) = \langle G(0)\rangle. \tag{93}$$

As a potentially new result, the condition (87) may also open up the possibility of dynamical symmetries that are distinct from the unitary and anti-unitary cases. This possibility does not contradict Wigner's theorem (Wigner 1931), because the discussion at this point is pitched at the level of an *entire system*, which may explicitly include measuring devices as subsystems. As explained in other work (Barandes 2025), if one suitably models an entire system that includes a measuring device as an overall unistochastic process, then the measuring device will end up in one of its possible measurement-outcome configurations with the appropriate Born-rule probability for whatever observable is measured—whether a beable (represented by a self-adjoint matrix that is diagonal in the configuration basis) or an emergeable (represented by a non-diagonal self-adjoint matrix). Relative phases in this overall unistochastic process are immaterial gauge variables.

However, for Wigner's theorem, as with the traditional textbook formulation of the Dirac-von Neumann axioms, one assumes that an *implicit* measuring device is left out of the Hilbert-space formalism, and one can appeal to the Born rule and the collapse rule as bare posits. In particular, the implicit measuring device does not participate in the invertible transformation in question, and one needs to be mindful of relative phases in the quantum system that the measuring device is investigating. Because beables and emergeables in the quantum system appear on an essentially similar footing as far as the implicit measuring device is concerned, one therefore needs to specialize the dynamical symmetries (87) to a smaller class of transformations, which will be called *Wigner symmetries*.

After carrying out a Wigner symmetry, any emergeables exhibited by the quantum system should still be available to measuring devices. Each emergeable, being represented by a self-adjoint

[19]Intriguingly, because K anticommutes with i, meaning that $Ki = -iK$, the three mathematical objects i, K, and iK satisfy $-i^2 = K^2 = (iK)^2 = iK(iK) = 1$, and therefore generate a Clifford algebra isomorphic to the *pseudo-quaternions* (Stueckelberg 1960). In a sense, then, the Hilbert spaces of quantum systems are actually defined not over the complex numbers alone, but over the pseudo-quaternions, although K is not typically involved in defining observables.

operator, corresponds to a PVM of its own, so if P'_α is an element of this PVM before the symmetry transformation, then let $\tilde{P}'_\alpha$ denote the corresponding PVM element after carrying out the Wigner symmetry. By precisely the same reasoning that led to the result (82), it follows that these two PVMs are related by some unitary operator V':

$$\tilde{P}'_\alpha \mapsto \tilde{P}'_\alpha \equiv V'^\dagger P'_\alpha V'. \tag{94}$$

Ensuring that the Born rule is invariant in the emergeable's orthonormal basis leads to the condition

$$\mathrm{tr}(\Theta^\dagger(t \leftarrow 0)\tilde{P}'_\alpha \Theta(t \leftarrow 0)\tilde{P}'_\beta) = \mathrm{tr}(\Theta^\dagger(t \leftarrow 0)P'_\alpha \Theta(t \leftarrow 0)P'_\beta), \tag{95}$$

which ultimately leads to the statement that

$$|\tilde{\Theta}'_{\alpha\beta}(t \leftarrow 0)|^2 = |\Theta'_{\alpha\beta}(t \leftarrow 0)|^2, \tag{96}$$

a formula that must hold for all choices of orthonormal basis. This equation is equivalent to the usual starting point for Wigner's theorem,[20] whose conclusion is that Wigner symmetries must be implemented either by unitary or anti-unitary operators.

4.2 Dilations

In most textbook treatments of quantum theory, a quantum system is axiomatically defined as a particular Hilbert space, together with a preferred set of self-adjoint operators designated as observables with predetermined physical meanings, along with a particular Hamiltonian to define the system's time evolution.[21] From that point of view, modifying a system's Hilbert-space formulation in any nontrivial way would necessarily mean fundamentally modifying the system itself.

From the alternative point of view developed in this paper, by contrast, a Hilbert-space formulation is merely a collection of mathematical tools for constructing the dynamics of a given indivisible stochastic process or carrying out calculations more efficiently, and is no more fundamental than a Lagrangian or Hamiltonian description of a Newtonian system. The indivisible stochastic process itself is ultimately defined by a configuration space and a dynamical law that stand apart from any arbitrary choice of Hilbert-space formulation. As a consequence, one is free to modify the Hilbert-space formulation for a given indivisible stochastic process as needed, much like changing from one gauge choice to another in a gauge theory, or like adding physically meaningless variables to the Lagrangian formulation of a Newtonian system.

With this motivation in place, recall again the basic stochastic-quantum dictionary (39):

$$\Gamma_{ij}(t \leftarrow 0) = \mathrm{tr}(\Theta^\dagger(t \leftarrow 0)P_i \Theta(t \leftarrow 0)P_j). \tag{97}$$

[20] See, for instance, eq. (2.A.1) in Weinberg's textbook on quantum field theory (Weinberg 1996, Chapter 2, Appendix A).

[21] In some circumstances, it may turn out to be more convenient to define a quantum system by a formal C*-algebra of observables alone, without picking a specific Hilbert-space representation (Haag 1993; Clifton, Halvorson 2001; Strocchi 2008; Feintzeig 2016).

The Hilbert-space formulation expressed by the right-hand side can be manipulated for convenience, provided that the left-hand side of the dictionary remains unchanged.

In particular, for any integer $D \geq 2$, one can freely enlarge, or *dilate*, the Hilbert-space formulation to a larger dimension ND by the following dilation transformation, as first explained in Subsection 3.5:

$$\left.\begin{aligned} \Theta(t \leftarrow 0) &\mapsto \Theta(t \leftarrow 0) \otimes \mathbb{1}^{\mathcal{I}}, \\ P_i &\mapsto P_i \otimes \mathbb{1}^{\mathcal{I}}, \\ P_j &\mapsto P_j \otimes P_\gamma^{\mathcal{I}}. \end{aligned}\right\} \tag{98}$$

Here $\mathbb{1}^{\mathcal{I}}$ is the $D \times D$ identity matrix on a new *internal* Hilbert space $\mathcal{H}_{\mathcal{I}}$, and $P_1^{\mathcal{I}}, \dots, P_D^{\mathcal{I}}$ collectively form any PVM on that internal Hilbert space satisfying the usual conditions of mutual exclusivity,

$$P_\gamma^{\mathcal{I}} P_{\gamma'}^{\mathcal{I}} = \delta_{\gamma\gamma'} P_\gamma^{\mathcal{I}}, \tag{99}$$

and completeness,

$$\sum_{\gamma=1}^{D} P_\gamma^{\mathcal{I}} = \mathbb{1}^{\mathcal{I}}. \tag{100}$$

It is then a mathematical identity that one can rewrite the stochastic-quantum dictionary (39) as

$$\left.\begin{aligned} \Gamma_{ij}(t \leftarrow 0) = \operatorname{tr}\Big(\operatorname{tr}_{\mathcal{I}}\Big(\Big[\Theta^\dagger(t \leftarrow 0) \otimes \mathbb{1}^{\mathcal{I}}\Big]\big[P_i \otimes \mathbb{1}^{\mathcal{I}}\big] \\ \times \big[\Theta(t \leftarrow 0) \otimes \mathbb{1}^{\mathcal{I}}\big]\big[P_j \otimes P_\gamma^{\mathcal{I}}\big]\Big)\Big), \end{aligned}\right\} \tag{101}$$

with a second trace, or partial trace, over the internal Hilbert space $\mathcal{H}_{\mathcal{I}}$. The choice of value for the label γ here is immaterial, with different choices of γ related by gauge transformations.

One can equivalently write the dilated form (101) of the dictionary in *block-matrix form* as

$$\Gamma_{ij}(t \leftarrow 0) = \operatorname{tr}_{\mathcal{I}}\Big([\Theta_{ij}(t \leftarrow 0)]^{\mathcal{I}\dagger}[\Theta_{ij}(t \leftarrow 0)]^{\mathcal{I}} P_\gamma^{\mathcal{I}}\Big). \tag{102}$$

Here $[\Theta_{ij}(t \leftarrow 0)]^{\mathcal{I}}$ is a diagonal $D \times D$ matrix consisting of repeated copies of the specific entry $\Theta_{ij}(t \leftarrow 0)$ (for fixed i, j) along the diagonal:

$$[\Theta_{ij}(t \leftarrow 0)]^{\mathcal{I}} \equiv \Theta_{ij}(t \leftarrow 0)\, \mathbb{1}^{\mathcal{I}}. \tag{103}$$

Meanwhile, the adjoint operation $\dagger$ in (102) acts on this $D \times D$ block matrix $[\Theta_{ij}(t \leftarrow 0)]^{\mathcal{I}}$, so it does not transpose the indices i and j on the $N \times N$ matrix $\Theta_{ij}(t \leftarrow 0)$ itself:

$$[\Theta_{ij}(t \leftarrow 0)]^{\mathcal{I}\dagger} \equiv \overline{[\Theta_{ij}(t \leftarrow 0)]}^{\mathcal{I}}. \tag{104}$$

It follows that

$$[\Theta_{ij}(t \leftarrow 0)]^{\mathcal{I}\dagger}[\Theta_{ij}(t \leftarrow 0)]^{\mathcal{I}} P_\gamma^{\mathcal{I}} = |\Theta_{ij}(t \leftarrow 0)|^2 P_\gamma^{\mathcal{I}}, \tag{105}$$

so the trace over $\mathcal{H}_{\mathcal{I}}$ indeed yields $|\Theta_{ij}(t \leftarrow 0)|^2 = \Gamma_{ij}(t \leftarrow 0)$, as required by (25).

In this dilated version of the Hilbert-space formulation, Schur-Hadamard gauge transformations (29) are enhanced to the following local-in-time gauge transformations, which have not yet been described in the research literature and therefore constitute another new result:

$$[\Theta_{ij}(t \leftarrow 0)]^{\mathcal{I}} \mapsto V^{\mathcal{I}}_{(ij)}(t)[\Theta_{ij}(t \leftarrow 0)]^{\mathcal{I}}. \tag{106}$$

Here $V^{\mathcal{I}}_{(ij)}(t)$ are a set of N^2 unitary, $D \times D$ matrices, where each such unitary matrix as a whole is labeled by a specific pair (ij) of configuration labels:

$$V^{\mathcal{I}\dagger}_{(ij)}(t) = (V^{\mathcal{I}}_{(ij)}(t))^{-1}. \tag{107}$$

The gauge transformations (106) will not generally preserve the factorization $\Theta(t \leftarrow 0) \otimes \mathbb{1}^{\mathcal{I}}$ appearing in (101), so they motivate considering more general $ND \times ND$ time-evolution operators $\tilde{\Theta}(t \leftarrow 0)$, in terms of which the dilated dictionary (101) takes the form

$$\Gamma_{ij}(t \leftarrow 0) = \mathrm{tr}\Big(\mathrm{tr}_{\mathcal{I}}\Big(\tilde{\Theta}^{\dagger}(t \leftarrow 0)\big[P_i \otimes \mathbb{1}^{\mathcal{I}}\big]\tilde{\Theta}(t \leftarrow 0)\big[P_j \otimes P^{\mathcal{I}}_{\gamma}\big]\Big)\Big). \tag{108}$$

Any $ND \times ND$ matrix $\tilde{\Theta}(t \leftarrow 0)$ appearing on the right-hand side of this dictionary and satisfying the natural generalization of the summation condition (26) is guaranteed to lead to a valid transition matrix $\Gamma_{ij}(t \leftarrow 0)$ on the left-hand side, so working with a dilated Hilbert-space formulation essentially provides a larger 'canvas' for designing transition matrices.

As a simple example of a dilation for the case $D = 2$, one can formally eliminate the complex numbers from a quantum system's Hilbert space (Myrheim 1999). Specifically, by increasing the system's Hilbert-space dimension from N to $2N$, one can replace the imaginary unit $i \equiv \sqrt{-1}$ with the real-valued 2×2 matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, with the enhanced version (106) of Schur-Hadamard gauge transformations now consisting of two-dimensional rotations of the internal Hilbert space $\mathcal{H}_{\mathcal{I}}$.[22] One can then represent the complex-conjugation operator K appearing in (90) as the real-valued 2×2 matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The result is that all unitary and anti-unitary operators become $2N \times 2N$ real orthogonal matrices. One cost of using this 'real' representation, however, is that the Hilbert spaces of composite systems will not factorize as neatly into Hilbert spaces for their constituent subsystems.

As a much more significant application of dilations, recall that any transition matrix $\Gamma_{ij}(t \leftarrow 0)$ has a Kraus decomposition (60), which one can equivalently write as

$$\Gamma_{ij}(t \leftarrow 0) = \sum_{\beta=1}^{N} \mathrm{tr}(K^{\dagger}_{\beta}(t \leftarrow 0)P_i K_{\beta}(t \leftarrow 0)P_j). \tag{109}$$

As explained in Subsection 3.5 and proved in other work (Barandes 2025, 2023), the Stinespring dilation theorem (Stinespring 1955, Keyl 2002) then guarantees that by an appropriate dilation to

[22] Importantly, one can prove the uncertainty principle just as well with the imaginary unit i represented by a 2×2 matrix in this way.

a larger Hilbert space if necessary, one can express $\Gamma_{ij}(t \leftarrow 0)$ in terms of a unitary time-evolution operator $\tilde{U}(t \leftarrow 0)$:

$$\Gamma_{ij}(t \leftarrow 0) = \mathrm{tr}\Big(\mathrm{tr}_{\mathcal{I}}\Big(\tilde{U}^{\dagger}(t \leftarrow 0)\big[P_i \otimes \mathbb{1}^{\mathcal{I}}\big]\tilde{U}(t \leftarrow 0)\big[P_j \otimes P_{\gamma}^{\mathcal{I}}\big]\Big)\Big). \tag{110}$$

As yet another key application of dilations, a dilated Hilbert-space formulation can make it possible to describe new kinds of emergeables. Some of these *dilation-emergeables* may be observables that can yield definite results in measurement processes, along the lines described in other work (Barandes 2025), despite not having a direct meaning solely at the level of the system's underlying configuration space.

In this way, an indivisible stochastic process based on a configuration space can easily accommodate emergent observables that describe empirically meaningful patterns in the dynamics and that model all kinds of quantum phenomena. Indeed, obtaining a unitary time-evolution operator for a given system may require dilating the Hilbert space in just this way, as in (110).

It is important to keep in mind that whether or not one actually carries out this formal dilation of the Hilbert-space formulation, the stochastic dynamics of the underlying indivisible stochastic process will still be the same. Any emergent patterns in the system's stochastic dynamics that are made manifest or explicit by the dilation, as represented by any new dilation-emergeables that arise, were always there all along, albeit in a non-manifest or implicit way.

An important potential example of this last application is intrinsic spin. If one wished to to introduce spin as a dilation-emergeable, one could merely dilate the Hilbert space to ND dimensions, introduce a D-dimensional representation of $SO(3)$ for the internal Hilbert space, and then require that the dilated time-evolution operator had the appropriate form of rotation symmetry. This approach to representing spin would ensure that despite picking an arbitrary three-dimensional coordinate axis in the process of formally carrying out the dilation of the Hilbert space—such as by choosing the spin-z operator to be diagonal on the dilated Hilbert space—the underlying indivisible stochastic process would not fundamentally involve any preferred direction or entail any basic violation of rotation invariance.

5 Discussion and Future Work

The present paper discussed the basic theory of stochastic processes, with an emphasis on generalizations to accommodate non-Markovianity, before introducing the notion of an indivisible stochastic process. The paper then reviewed the stochastic-quantum correspondence between indivisible stochastic processes and quantum systems, leading to the indivisible interpretation of quantum theory, or indivisible quantum theory (Barandes 2025). This interpretation has a thoroughly realist orientation, and does not entail parallel universes, nor does it involve perspectival or relational notions of ontology.

The axioms of indivisible quantum theory are simpler to state and more physically transparent than the Dirac-von Neumann axioms of textbook quantum theory.

- Kinematical axiom: For each model under consideration, one picks an appropriate configuration space, whose members are the possible configurations of the system being modeled, treated as elementary according to the model. The configuration space is a fixed feature of the model, meaning that it does not vary between real-world runs or instantiations of the model.

- Dynamical axiom: For arbitrary target times, and for conditioning times corresponding to division events, the model's dynamical laws consist of transition probabilities that take the form of conditional probabilities for the system to be in a particular configuration at each target time, given that the system is in a particular configuration at each conditioning time. Division events may occur naturally within the model's own dynamics, and can also be generated spontaneously through interactions with other systems. For example, division events are generated during a measurement process, which can be modeled as just another stochastic process. At the level of the given model, the dynamical laws are fixed features.

- Epistemic axiom: The system has some time-dependent standalone probability distribution to be in a particular configuration at any given target time. This standalone probability distribution is connected between different times by the model's transition probabilities, and is contingent, meaning that it can vary between runs of the model.

It is worth noting that the kinematical and epistemic axioms here are essentially classical, in the sense that they involve classical notions of ontology and probability. The distinctly non-classical ingredient is the dynamical axiom, which replaces the effectively Markovian differential equations of classical theories with dynamical laws that consist of a sparse set of indivisible transition probabilities.

Notice that the configurations of an indivisible stochastic process play the role of what have historically been called 'hidden variables.' However, they are not hidden in a literal sense, because they are what one actually sees in experiments, and they do not supplement or augment the traditional wave function, but instead replace wave functions as the ontological ingredients of quantum theory.

Any mention of hidden variables may immediately bring to mind a number of no-go theorems about non-locality, including the various versions of Bell's theorem (Bell 1964, 1976, 1990). These theorems will be addressed in detail in future work. In addition to exploring implications for how to think about causation at a microphysical level, now that the basic dynamical laws are no longer Markovian differential equations, that other work will also argue that locality in space is preserved at the cost of non-Markovianity. One can view non-Markovianity roughly as a form of non-locality in time that is consistent with the light-cone structure of special relativity, and which is arguably unavoidable in quantum theory anyway, as pointed out by others (Glick, Adami 2020).

Future work will extend the analysis of symmetries and dilations to more examples. At a broader level, it would be interesting to examine what else this new formulation of quantum theory can teach us about laws, probability, causation, as well as explore new applications for quantum computing, and new ways of thinking about generalizing quantum theory to accommodate gravity.

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