Points, curves, and hypersurfaces: Reassessing the historical geometric object concept

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Abstract

In contemporary philosophy of physics, there has recently been a renewed interest in the theory of geometric objects—a programme developed originally by geometers such as Schouten, Veblen, and others in the 1920s and 30s. However, as yet, there has been little-to-no systematic investigation into the history of the geometric object concept. I discuss the early development of the geometric object concept, and show that geometers working on the programme in the 1920s and early 1930s had a more expansive conception of geometric objects than that which is found in later presentations—which, unlike the modern conception of geometric objects, included embedded submanifolds such as points, curves, and hypersurfaces. I reconstruct and critically evaluate their arguments for this more expansive geometric object concept, and also locate and assess the transition to the more restrictive modern geometric object concept.

1 Introduction

> Geometrisches Gebilde und geometrisches Objekt sind also dasselbe, von verschiedenen Gesichtspunkte aus betrachtet.[1] (Schouten and van Dantzig 1935, 46)

Philosophy of physics has recently seen a renewed interest in the theory of geometric objects—a programme developed originally by geometers such as Schouten, Veblen, and others in the 1920s and early 1930s, and brought to full maturity by Nijenhuis (1952). The geometric objects programme was, to my knowledge, first introduced into textbook general relativity in the 1960s by Trautman (1965) and Anderson (1967), and originally made its way into the philosophy of physics literature in connection with the Anderson-Friedman absolute objects programme (Anderson 1967; Friedman 1983), which aimed to

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1. “Geometric figure and geometric object are therefore the same thing, considered from different perspectives.” (All translations are my own unless otherwise stated.)

determine a sense in which general relativity was substantively generally covariant (and other theories were not). Whilst the absolute objects programme lost much of its interest with the Geroch-Giulini volume element and Geroch-Jones dust counterexamples (see e.g. Pitts (2006)), interest in the theory of geometric objects has continued, and the geometric vs. non-geometric objects distinction has recently been discussed by e.g. Pitts (2010, 2012, 2022), Read (2022), Jacobs and Read (2024), and Dürr (2021) in connection with issues relating to the definition of spinor fields and gravitational stress-energy in general relativity, whilst Dewar (2020) and March and Weatherall (2025) discuss the modern cousin of the geometric objects programme—natural bundles—as an explication of 'general covariance'.

Despite the clear foundational significance of geometric objects for philosophy of physics, the history of the geometric object concept remains regrettably under-explored. Here, one can isolate a number of philosophically-interesting questions: was the early geometric object concept the same as the later one; if not, how did this concept change; what heuristics and principles did geometers working on the programme employ in developing it; to what extent does the extensive body of work on geometric objects from the mid 1930s to the advent of the natural bundles programme (Nijenhuis 1972) carry over to that latter programme? And so forth.

This paper aims to take some first steps towards remedying this gap in the literature. In particular, it aims to partly address the first two questions raised above: whether the early geometric object concept was the same as the later one, and, if not, how this concept changed. My focus will be on the definitions of geometric objects suggested by geometers working on the programme during the period 1920 to 1936—a time where there was little precise consensus on how geometric objects should be defined, and the formal theory of geometric objects was still in its infancy. On the one hand, I will argue that geometers working on the programme during this period in fact had a more expansive conception of geometric objects than the modern one, which included embedded submanifolds such as points, curves, and hypersurfaces.[2] On the other hand, I will show that this conception changed abruptly around the year 1936, when a more formal theory of geometric objects began to be developed. The reason for this, or so I will suggest, was simple: with a need for more formality came a need for greater precision on the definition of components of an object, which previously had been disambiguated in a variety of different ways, and this led precisely to a scope restriction.

Why is this project philosophically interesting? Or, put somewhat differently, why does this episode in the historical development of the geometric object concept belong to the history of *philosophy* of physics (or mathematics), rather than just the history of mathematics? I can think of at least three (related) reasons, all of which point to slightly different ways one might situate this episode with respect to broader history of philosophy of science discussions.

2. More generally, it also included collections of embedded submanifolds, collections of collections, etc.

First, because geometers such as Schouten and Veblen working on the geometric objects programme saw the project of developing a geometric object concept as intimately connected with—perhaps even constitutive of—the project of conceptualising what 'geometry', or the 'problem of geometry' is. Indeed, this is precisely the framing of Schouten and van Dantzig's (1935) paper, "Was ist Geometrie?"[3] (see also Schouten (1926), Veblen (1929), and Veblen and Whitehead (1932)), which begins by tracing back the development of the geometric object concept through to Klein's *Erlanger Programm*, and whose final third, which I will engage with extensively in this paper, addresses precisely the question how and whether embedded submanifolds of a manifold such as points, curves, hypersurfaces, etc. (which they call "geometric figures") are to count as geometric objects. Schouten and van Dantzig's motivations for addressing this question are worth unpacking in some detail. In brief, the worry seems to have been that in identifying 'geometry' with the theory of geometric objects,"die mehr intuitive Seite der Geometrie"[4]—which they spell out as meaning the theory of geometric figures, or more informally, "etwas, was man zeichnen kann"[5]—has been cast aside "ist [...] in den Hintergrund gedrängt"[6]. Schouten and van Dantzig give a particularly forceful statement of this worry:

> Wenn es uns nicht gelänge vom geometrischen Objekte ausgehend den Weg zur gezeichneten Figur zurückzufinden, so müssten wir tatsächlich befürchten, dass die „kompetenten Leute“ aus dem Zitat Veblens uns eines Tages vorwerfen würden, wir hätten überhaupt keine Geometrie getrieben, sondern „Orgien des Formalismus“ gefeiert.[7] (Schouten and van Dantzig 1935, 39)

In other words, Schouten and van Dantzig see the recovery of geometric figures from geometric objects, here, as an adequacy condition on the project of conceptualising 'geometry', or the 'problem of geometry' in terms of geometric objects.[8] Just how strong an adequacy condition they had in mind is not entirely clear. It is possible that Schouten and van Dantzig saw the recovery of

3. "What is geometry?"

4. "the more intuitive side of geometry"

5. "something which one can draw"

6. lit. "has been pushed into the background"

7. "If we do not succeed in finding our way back from geometric objects to drawn figures, then we must indeed fear that the "competent people" from Veblen's quote will one day charge us with having not done geometry at all, but having reveled in "orgies of formalism" instead."

8. There is a further exegetical issue, here, which is worth mentioning. In the opening remarks of their paper, Schouten and van Dantzig (1935, 15) caution—extensively!—against the idea that the meaning of 'geometry' is an a priori question, or even one which merits a priori considerations—"Geometrie ist *nicht* ein scharf gestecktes genau für alle Zeiten definierbares Feld mathematischer Untersuchung, die Frage „Was ist Geometrie“ ist daher überhaupt unvernünftig und muss durch die vernünftige Frage „Was versteht man heutzutage unter Geometrie“ ersetzt werden, eine Frage, deren Beantwortung Aufdeckung eines Tatsachenbestandes und nachträgliche Aufstellung einer Definition a posteriori, nicht aber Aufstellung einer Definition a priori verlangt." (emphasis in original) ("Geometry is *not* a sharply-demarcated, exactly-definable for all time field of mathematical inquiry, the question "what is geometry" is as such completely unreasonable and must be replaced by the reasonable question "what do we understand today by geometry", a question whose answer calls for the discovery of a contingent fact and subsequent construction of a definition a posteriori, not construction of a

geometric figures from geometric objects as playing an explanatory role ('given that geometry is the theory of geometric objects, why should we have thought that it was something to do with geometric figures?'), or a justificatory role ('that one can recover geometric figures from geometric objects gives us reason to think that this conception of geometry is on the right track'), or even a semantic-constitutive role ('necessarily, any conception of what 'geometry' is must encompass geometric figures')—though my own preferred reading is somewhere between the first and third. But in any case, the point is that whether the geometric object concept included embedded submanifolds was once seen as an important question for the project of conceptualising what 'geometry' is. From the perspective of history of philosophy discussions of the meaning of 'geometry', then, the change in the geometric object concept I will identify and assess in this paper is evidently of considerable interest.

A second reason comes from later discussions in the 1960s, when the geometric objects programme began to be introduced in earnest into the general relativity literature. Despite the fact that the definition of components of an object, as of around the year 1936 and subsequently, ruled out embedded submanifolds such as points, curves, and hypersurfaces as geometric objects, the idea that e.g. manifold points are geometric objects persisted into at least the work of Anderson (1967, 15), who claims that "[the] simplest example of a geometrical object is just a point of the manifold." Indeed, Anderson immediately goes on to claim—as far as I can tell, by his own lights incorrectly—that what he calls "local geometrical objects" (i.e. objects characterised by associating a set of components to each point of the manifold at which the object is defined) include manifold points as a special case. So in some sense, the question whether and how embedded submanifolds should count as geometric objects never went away, or perhaps went away only with the move towards coordinate-free differential-geometric methods (Friedman (1983), in his discussion of Anderson on geometric and absolute objects, does not mention it). On the other hand, Trautman (1965, 84–85), writing at approximately the same time as Anderson, is explicit in his discussion of geometric objects that these are "geometric object fields" (85)—i.e., local fields on a manifold, which do not include points, curves, etc. But understanding why the idea that embedded submanifolds are geometric objects persisted in this way—definitions notwithstanding!—requires, in my view, precisely an understanding of the definitional issues that geometers working on the geometric objects programme had grappled with some three decades earlier, and why they had grappled with them.

The third reason has a more contemporary outlook. As mentioned above, part of the recent philosophy of physics interest in the theory of geometric ob-

definition a priori.") But to my mind, at least, Schouten and van Dantzig's discussion in this last third of their paper—and in particular, the importance they attach to showing that the notion of a geometric object can recover that of a geometric figure—suggests that they had not liberated themselves from this *a priori* perspective on geometry to the extent they might have hoped. Whether the lack of discussion of geometric figures in subsequent work on the definition of geometric objects, e.g. Schouten and Haantjes (1936), marked the eventual point of liberation is an interesting question to speculate on, but I will not do so here.

jects has been as an explication of general covariance (Dewar 2020; March and Weatherall 2025; Pitts 2012). But the fact that embedded submanifolds of a manifold are not geometric objects, under the modern conception, has gone (at least to my knowledge) almost entirely unappreciated in these discussions. Moreover, this fact has non-trivial consequences for how we should understand the general covariance, or lack thereof, of theories set on different spacetime structures. This is nicely illustrated with the example of Earman's (1989) Aristotelian spacetime. Aristotelian spacetime is a structure $\langle M, t_a, h^{ab}, \nabla, \xi^a, \gamma \rangle$, where t_a and h^{ab} are orthogonal temporal and spatial metrics, ∇ is a flat compatible affine connection, ξ^a is a (twist-free, rigid, and geodesic) unit timelike vector field, and γ is an integral curve of ξ^a. Here, the point is that, in view of what I am going to say, the timelike curve γ which features in Aristotelian spacetime is not a geometric object. This has the consequence that Aristotelian spacetime, and theories set thereon, fail to be generally covariant (in even the 'trivial' sense) in a rather different way from theories set on weaker spacetime structures, such as Newtonian, Galilean, or Minkowski spacetime. In the terminology of Weatherall and March (2025), they are not natural, nor do they admit a naturalization. Now, I am not claiming that the episode in the historical development of the geometric object concept which I will discuss in this paper contains all the answers as to how we should understand precisely what this difference in failures of general covariance amounts to. Indeed it does not. But I do think that it can assist us in thinking about just what this difference is—both because it offers an alternative conception of geometric objects which does include embedded submanifolds, and because it forces us to think carefully about just what is changing with the move to the modern geometric object concept. I will return to this issue in §4.

As such, the plan for this paper is as follows. In §2, I briefly recall some details of the modern geometric object concept (in §2.1) as presented by e.g. Nijenhuis (1952), and show that this excludes embedded submanifolds such as points, curves, and hypersurfaces. I then, in §2.2, introduce some definitions of geometric objects from the 1920s and early 1930s, and in §2.3, reconstruct and assess these geometers' arguments—in particular, an argument given by Schouten and van Dantzig (1935); to my knowledge, the only detailed such argument in this early literature—that embedded submanifolds of a manifold, collections thereof, etc. count as geometric objects. In doing so, I build a picture of this as a consensus view in the early geometric objects literature. I then, in §3, discuss the transition to the modern geometric object concept, which I locate in the work of Schouten and Haantjes (1936), and suggest an explanation for the reasons why this transition—in particular, its ruling out embedded submanifolds as geometric objects—went almost entirely unremarked. §4 concludes.

2 The case for a more expansive historical geometric object concept

2.1 The modern geometric object concept

I will begin with a self-contained outline of the modern geometric object concept—which, despite the terminology, was well established at least by the time of (Nijenhuis 1952), and (as we shall see) well underway by 1936. Here, and throughout, let M be a differential n-manifold (assumed connected, Hausdorff, and paracompact), and (U, ϕ) a coordinate system, where $U \subset M$ is an open region in M, $\phi : \mathbb{R}^n \to U$ is a diffeomorphism, and $\mathbb{R}^n$ is assumed equipped with a (fixed) choice of basis.

In the modern theory of geometric objects, an *object* on U is characterised by a set of N components $\Omega_{\phi,1}$, $\Omega_{\phi,2}$, ..., $\Omega_{\phi,N}$ relative to each coordinate system (U, ϕ) on U,[9] where a component $\Omega_{\phi,i}$ is a smooth map $\Omega_{\phi,i} : U \to \mathbb{R}$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth analytic coordinate transformation,[10] and let $\Omega_{\phi',1}$, $\Omega_{\phi',2}$, ..., $\Omega_{\phi',N}$ denote the components of the object in the coordinate system $(U, \phi' = \phi \circ f)$. The object is a *geometric object* iff, for each such f, for all i, and for each $p \in U$, $\Omega_{\phi',i}(p)$ is an analytic function(al) of $(f, \Omega_{\phi,1}(p), \Omega_{\phi,2}(p), ..., \Omega_{\phi,N}(p))$.

It is straightforward to see that embedded submanifolds of U, such as points, curves, and hypersurfaces, are not geometric objects on this definition. This is to do with the way we have defined objects. On the above definition, an object is specified (relative to a coordinate system (U, ϕ)) by a collection of assignments, to each $p \in U$, of real numbers (i.e. components of the object). But embedded submanifolds of U do not have components in this sense. For one, an embedded submanifold of U is defined via a (smooth) injection $\psi : S \to U$, where S is some set (smooth manifold). Second, in general, such objects (or rather, their images in U) may be closed subsets of U, and so one cannot always pick out such an image by a smooth assignment of collections of real numbers to all points in U (consider, e.g., the case $\dim(S) < \dim(M)$). And third, even in special cases where we can use a smooth assignment of collections of real numbers to all points in U to distinguish the image of S under ψ in U (e.g., when there exist smooth scalar fields on U with support exactly on the image of S under ψ), there is nothing to distinguish such components from the components associated not with embedded submanifolds of U but with, e.g., scalar fields, tensor fields, affine connections etc.

2.2 The early geometric object concept

Let me now begin to lay out my case that early geometers working on the theory of geometric objects had a more expansive conception of geometric objects than the modern one presented above, which *did* include embedded submanifolds

9. For simplicity, I am restricting attention to objects on U; the generalisation to objects on M is obvious.

10. Analytic, here, is in the sense that for any $\mathbf{v} = v^i \mathbf{e}_i \in \mathbb{R}^n$, $i = 1, 2, ..., n$ where $\mathbf{e}_i$ denote the (fixed) basis for $\mathbb{R}^n$, $f(\mathbf{v}) = v'^i \mathbf{e}_i$ for v'^i all analytic functions of $(v^1, v^2, ..., v^n)$

such as points, curves, and hypersurfaces. A clear statement of this idea is given by Veblen and Whitehead (1932):

> Anything which is unaltered by transformations of coordinates is called an invariant [...].[11] Thus a point is an invariant and so is a curve or a system of curves. Also, strictly speaking, anything, such as a plant or an animal, which is unrelated to the space which we are talking about, is an invariant. For an invariant, which is related to the space, i.e. a property of the space [...], we shall also use the term *geometric object*. (46)

As Schouten and van Dantzig (1935, 19) note, Veblen and Whitehead are using the word "property" here in a very general sense—for Veblen, "[ein] Punkt, ein System von Punkten oder ein System von Beziehungen ist [...] eine Eigenschaft des Raumes."[12] Indeed, immediately afterwards, Veblen and Whitehead cite manifold points as their first example of a geometric object:

> A point is an example of a geometric object which determines a set of numbers in each allowable coordinate system in which it is represented. (Veblen and Whitehead 1932, 46)

Of course, Veblen and Whitehead are being somewhat loose with the term "set" here—really, a point determines an ordered n-tuple of (real) numbers in each coordinate system in which it is represented (i.e. a set with some further structure). But setting this aside, Veblen and Whitehead are surely right that a manifold point is an object which is "unaltered by transformations of coordinates" (something which is easily seen in a modern differential-geometric setting, since the points, i.e. base set of a manifold, are specified prior to the introduction of coordinate charts on that manifold).

However, one might wonder if the inclusion of embedded submanifolds of M under Veblen and Whitehead's definition of geometric objects was an artefact of their particular definition—in the 1920s and early 1930s, geometers had not yet united around a "standard" definition of geometric objects. As motivation for this, consider the following two approximately contemporaneous definitions of geometric objects, due respectively to Veblen and Thomas (1926) and Schouten and van Dantzig (1935):

> An invariant[13] [...] is an entity with definite determining components in any coordinate system, such that the transformations of the components from one coordinate system to another form a group isomor-

11. Note that by "unaltered by transformations of coordinates", Veblen and Whitehead do not mean that the coordinate components of the object (on some region) must be preserved, i.e. fixed identically, under coordinate transformations—though that is somewhat unclear from the language used in this passage.

12. "A point, a system of points, or a system of relations is [...] a property of the space."

13. Here, Veblen and Thomas are adopting the earlier terminology "invariant"; the term "geometric object" was proposed later as a substitute by Schouten and van Kampen (1930, 758), in part, to avoid the connotation that the coordinate components of such objects should be preserved, i.e. fixed identically, under coordinate transformations, as mentioned in fn. 11.

phic with the group of analytic transformations of the coordinates. (Veblen and Thomas 1926, 279)

> Die Punkte einer Mannigfaltigkeit sind festgelegt durch Koordinate, die „Urvariablen", die den Transformationen einer gewissen Gruppe, der „Basisgruppe" unterworfen sind. Ein geometrisches Object ist ein System von in irgendeinem Bereiche definierten Funktionen („Bestimmungszahlen" genannt) der Urvariablen, das sich bei Transformationen der Basisgruppe „in sich" mittransformiert, d.h. sich so transformiert, dass die neuen Bestimmungszahlen lediglich von den alten Bestimmungszahlen und den Transformationsfunktionen abhängen.[14] (Schouten and van Dantzig 1935, 19)

In both these definitions, one sees a focus on components of an object not dissimilar to the modern conception. Indeed, we saw above that part of the problem with assimilating embedded submanifolds of M under the modern geometric object concept was precisely this focus on components. As such, one might worry that the geometric object concept of these authors could not have been similarly as expansive as that of Veblen and Whitehead.

There are two reasons to be sceptical about this worry. The first is that Veblen and Whitehead were writing *after* Veblen and Thomas, which gives reason to suspect that at the very least *Veblen's* conception of geometric objects at the time included points, curves, hypersurfaces etc. (This is also suggested by the above-quoted remark of Schouten and van Dantzig about Veblen's views on geometric objects.) The second is that Schouten and van Dantzig, towards the end of their paper, supply a detailed argument—to my knowledge, the only such argument in this early literature—that what they call "geometric figures" (i.e. embedded submanifolds of M, collections thereof, etc.) count as geometric objects under their definition. It is to the assessment of this argument which I now turn.

2.3 Schouten and van Dantzig on geometric figures as geometric objects: an assessment

To understand Schouten and van Dantzig's argument that every geometric figure is a geometric object, we first need to understand what they mean by a geometric figure. Here, Schouten and van Dantzig begin with the idea that constructing a geometric figure consists of a process of "Auszeichnen gewisser Punkte des Raumes den andern gegenüber"[15] and a process of "Bezeichnen dieser Punkte"[16]

14. "The points of a manifold are specified via coordinates, the "urvariables", on which the transformations of a particular group, the "basis group", act. A geometric object is a system of functions of the urvariables (called "components" [of the object]) defined on some region, which transforms with the transformations of the basis group "into itself", i.e. transforms in such a way that the new components depend only on the old components and the functions of transformation."

15. "distinguishing particular points of the space with respect to others"

16. "labelling of these points"

(Schouten and van Dantzig 1935, 39). For our purposes, we can skip over their discussion of the first; on the second, they then elaborate:

> Was das Bezeichnen betrifft, bemerken wir, dass es sich dabei zunächst immer handelt um eine (bei unendlichen Mengen vorzugsweise stetige) Abbildung einer gegebenen Menge, der *Urbildmenge* auf irgendeine im Raum ausgezeichnete Menge, so dass jedem Punkt der zu bezeichnenden Menge (oder auch nur eines Teiles derselben) ein Punkt der Urbildmenge zugeordnet wird.[17] (40, emphasis in original)

Schouten and van Dantzig then give the following inductive definition of a geometric figure:

> Ein geometrisches Gebilde ist erstens ein Punkt in dem mit den gegebenen Urbildmengen erweiterten gegebenen Raume und zweitens jede Menge von schon definierten geometrishen Gebilden.[18] (42)

Before I discuss this definition further, let me remark on what Schouten and van Dantzig appear to have in mind here. Immediately prior to this definition, Schouten and van Dantzig claim that

> Der Prozess der Parametrisierung kann nun, falls erwünscht, auf den Prozess der Auszeichnung zurückgeführt werden, indem man den Raum erweitert durch Hinzunahme aller verwendeten Urbildmengen. Jede feste Parametrisierung einer Menge wird dann ersetzt durch Auszeichnen der Menge derjenigen Elementenpaare, die bestehen aus dem zu parametrisierenden Element der Menge und dem ihm entsprechenden Element der verwendeten Urbildmenge.[19] (41–42)

But then it immediately becomes clear that Schouten and van Dantzig's "definition" of a geometric figure given above cannot quite be what they have in mind. The following example will illustrate this nicely. Let M be an n-dimensional smooth manifold with $n > 0$ and S a singleton set, and consider the product space $S \times M$. A point in $S \times M$ corresponds to a map from S to M; thus, a set of distinct such points corresponds to a set of maps from S to M. But this is not a parameterisation of points in M, nor a parameterisation of parameterisations etc., because the first element in each such ordered pair (i.e., point in $S \times M$) is the same. I therefore suggest that we replace Schouten and van Dantzig's inductive definition of "geometric figure" with the following inductive definition:

17. "As for what the labelling [of points] is concerned with, we remark that it always is to do with a (in the case of infinite sets preferably continuous) map from a given set, the *preimage*, into any distinguished set in the space, so that to every point of the distinguished set (or of only a part of it) there is assigned a point in the preimage."

18. "A geometric figure is in the first instance a point in the given space extended by the given preimages, and in the second instance any set of already defined geometric figures."

19. "The process of parameterisation can now, if desired, be assimilated back into the process of distinguishing [points in a manifold], in which one extends the space through the addition of all the preimages used. Any fixed parameterisation of a set is then replaced through distinguishing the set of respective pairs of elements, which consist of the parameterised element of the set and the element of the relevant preimage corresponding to it."

- Given any smooth manifold M, any open $U \subset M$, and any set (smooth manifold) S, any (smooth) map $\psi : S \to U$ (i.e. a parameterisation) is a geometric figure on U.
- Given any set Ψ of geometric figures on U, any (smooth) parameterisation of (a subset of) Ψ with respect to some set (smooth manifold) T is a geometric figure on U.
- Nothing else is a geometric figure on U.

Note that this definition captures all the standard examples: points (where S is a singleton set), curves (where S is diffeomorphic to (a segment of) $\mathbb{R}$), hypersurfaces etc. It also captures examples such as congruences of curves, collections of points, etc.

There is one final component of Schouten and van Dantzig's definition of a geometric figure which I have not yet discussed: the notion of a *parameterisation group*. Let $\psi : S \to U$ be a geometric figure (in the above sense). Then given any (smooth, if S is a smooth manifold) map $f : S \to S$, $f \in \mathrm{Aut}(S)$ where $\mathrm{Aut}(S)$ is the automorphism group of S, the map $\psi \circ f : S \to U$ is also a geometric figure with the same image in U. A parameterisation group is a subgroup of $\mathrm{Aut}(S)$. A geometric figure in the above sense, along with a choice of parameterisation group, is Schouten and van Dantzig's final definition of a geometric figure.

With this in hand, let us now turn to Schouten and van Dantzig's argument that every geometric figure is a geometric object. I will quote the relevant passages in full:

> Die angegebenen Beispiele haben zur Genüge gezeigt, wie man von den geometrischen Objekten zu den geometrischen Gebilden gelangen kann. Wie ist es nun aber umgekehrt, ist auch jedes geometrische Gebilde ein geometrisches Objekt? Wir dürfen annehmen, dass die als „Raum“ eingeführte stetiges Abbild einer Urbildmenge ist, denn wir haben es ja in Hand die Urbildmenge diesem Zwecke entsprechend zu wählen. Diese Abbildung ist aber, wie wir oben sahen, ein Koordinatensystem (im weitesten Sinne) und jedes geometrische Gebilde, das ja letzten Endes aus Punkten aufgebaut ist, lässt sich in Bezug auf ein solches System irgendwie zahlenmässig festlegen. Die Parametrisierungen, die eventuell in dem geometrischen Objekt enthalten sind, bilden dabei von der Wahl des Koordinatensystems unabhängige Zahlenmengen. Natürlich dürfen wir nicht verlangen, dass die Menge der erforderlichen Bestimmungszahlen immer endlich sei, ja sogar nicht einmal abzählbar unendlich oder endlich dimensional. Jedenfalls haben wir aber Bestimmungszahlen erhalten, die sich bei Koordinatentransformationen in bestimmter Weise transformieren werden, also ein geometrisches Objekt.[20] (Schouten and van Dantzig 1935, 45–46)

20. “The examples given have shown sufficiently how one can get from geometric objects to geometric figures. But now how about the converse, is every geometric figure also a geometric object? We may assume that it [the geometric figure] acts as an induced “space” of a

Summarising, Schouten and van Dantzig continue:

> Kurz gefasst ist der Sachverhalt folgender. Das geometrische Gebilde entsteht aus dem Raume durch den Prozess der Teilmengenbildung und den Prozess der Parametrisierung mit Hilfe der Urbildmengen und der Parametrisierungsgruppen. Der Prozess der Parametrisierung erzeugt aber anderseits, auf den Raum selbst (eventuell stückweise) angewandt, die Koordinatensysteme und ihre Transformationsgruppen und sobald nun das geometrische Gebilde zahlenmässig in Bezug auf jedes dieser Koordinatensysteme festgelegt wird (was stets möglich ist), so erhält es Bestimmungszahlen und eine Transformationsweise und wird somit zum geometrischen Objekt.[21] (Schouten and van Dantzig 1935, 46)

In fact, there are two arguments which Schouten and van Dantzig could be read as making in this passage, and it is important to be clear about the difference between them. I will begin by reconstructing them both; here is the first:

1. Let $\psi : S \to U \subset M$ be a geometric figure, with parameterisation group $G \subset \mathrm{Aut}(S)$. Let the set S be the components of the geometric figure, (S, ψ) be the coordinate system, and G the basis group.

2. Under any coordinate transformation $g \in G$, the geometric figure gets mapped to the geometric figure $\psi \circ g : S \to U$.

3. Under any coordinate transformation thus defined, i.e. any $g \in G$, S is mapped to itself (by definition).

4. Thus the components of the geometric figure under the coordinate transformation are the same as the old components, and therefore depend only on the old components.

C. Therefore, $\psi : S \to U$ is a geometric object.

continuous mapping of some preimage, because we have the freedom to choose the preimage in accordance with this purpose. But, as we saw above, this representation is a coordinate system (in the broadest sense), and every geometric figure, which ultimately is build out of points, can be specified numerically with reference to such a system. The parameterisations, which eventually are retained in the geometric object, form sets of numbers which are independent of the choice of coordinate system. Of course, we may not demand that the sets of the necessary components are always finite, nor even that they are denumerably infinite or finite-dimensional. But in any case, we have obtained components, which will transform in a determinate way with coordinate transformations, and thus a geometric object."

21. "In brief, the situation is as follows. The geometric figure is constructed out of the space through the process of the subset construction and the process of parameterisation with the help of the preimages and the parameterisation groups. But on the other hand, the process of parameterisation, which itself made use of the space (eventually piecewise), induces the coordinate systems and their transformation groups, and as soon as the geometric figure is specified numerically with reference to each of these coordinate systems (which is always possible), it possesses components and ways of transforming, and as such becomes a geometric object."

The second reconstruction is as follows:

1′. Let $\psi : S \to U \subset M$ be a geometric figure, and (U, ϕ) a coordinate system.

2′. Let the components of $\psi : S \to U$ in the coordinate system (U, ϕ) be the set $\mathrm{Im}(\phi^{-1} \circ \psi)$, and let the basis group be the group of smooth analytic maps from $\mathbb{R}^n$ to itself.

3′. By definition, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is an element of the basis group, then the components of $\psi : S \to U$ under f are the set $\mathrm{Im}(f \circ \phi^{-1} \circ \psi)$.

4′. By construction, the components of the geometric figure under the coordinate transformation depend only on the old components and functions of transformation (and in fact, the group of such transformations of the components of $\psi : S \to U$ is isomorphic to the basis group).

C′. Therefore, $\psi : S \to U$ is a geometric object.

Under both reconstructions, Schouten and van Dantzig leave implicit the inductive step necessary to show that every geometric figure is a geometric object; however, this is not difficult to fill in. In the first case, the inductive step goes through unchanged from the original argument; in the second, one only needs to note that the new "components" of the geometric figure will be a set of subsets (of subsets, to some finite order) of $\mathbb{R}^n$.

What can be said in favour of either of these two reconstructions? On the one hand, the first reconstruction has going for it that it is arguably more faithful to the letter of Schouten and van Dantzig's text. Passages such as "[diese] Abbildung ist aber [...] ein Koordinatensystem (im weitesten Sinne)" and "[der] Prozess der Parametrisierung erzeugt [...] die Koordinatensysteme und ihre Transformationsgruppen" are difficult to make sense of if the coordinate system and parameterisation are two different maps (from different spaces into U). That said, however, there are also some passages which are more difficult to make sense of on the first reconstruction. For example, "jedes geometrische Gebilde [...] lässt sich in Bezug auf ein solches System irgendwie zahlenmässig festlegen" makes little sense if $\psi : S \to U$ is the coordinate system in question, since there is no *a priori* reason to take the elements of S to be numbers (and thus it is unclear in what sense the geometric object is specified *numerically*, unless "zahlenmässig" is intended here in a very loose sense).

On the other hand, the second reconstruction has going for it that it avoids a number of important disanalogies between the components of "standard" examples of geometric objects, such as scalar fields, tensor fields, affine connections etc. and the "components" which Schouten and van Dantzig associate with geometric figures. One of these has already been alluded to: if the components of a geometric figure are the points of S (and there will be multiple isomorphic such S available), there is no guarantee that these "components" are anything to do with real numbers. (Of course, the disanalogy would remain that such components are sets (of sets, etc. to some finite order) of n-tuples of real numbers, rather than real numbers.) A second, perhaps more pressing disanalogy has to do with the parameterisation group. On the first reading of Schouten and van Dantzig, the parameterisation group is a subgroup of $\mathrm{Aut}(S)$, and therefore need not have anything to do with the group of smooth analytic coordinate transformations on $\mathbb{R}^n$ (though it will, of course, still have its representations on $\mathbb{R}^n$ as a subgroup of this group), which is the group of coordinate transformations which are relevant for objects such as scalar fields, tensor fields, affine connections etc. But on the second reading, this worry is avoided: the relevant automorphism groups for *any* geometric object are the (subgroups) of the group of smooth analytic coordinate transformations on $\mathbb{R}^n$. I will return to these remaining disanalogies, especially in the definition of "components defined on some region", in §3.

The other point in favour of this second reading of Schouten and van Dantzig is that it also establishes that geometric figures are geometric objects according to the definition given by Veblen and Thomas (1926) in §2.2, and makes Veblen and Whitehead's (1932) argument that a manifold point is a geometric object a special case of Schouten and van Dantzig's more general argument. This is related to the above point about the disanalogies between the components of "standard" geometric objects and Schouten and van Dantzig's "components" for geometric figures. For the first, the group of analytic coordinate transformations can only be defined if the space of coordinates is $\mathbb{R}^n$ (or any space uniquely isomorphic to it, as exist for e.g. the case $n = 1$). For the second, Veblen and Whitehead identify the "components" of a point as a "set of [real] numbers", which is true on the second reading of Schouten and van Dantzig's argument, but not necessarily the case on the first reading (on which the set S of components could be any singleton set, which *might* have as its element an ordered n-tuple of real numbers, but could also have as its element a single real number or indeed any other object). This is a desirable consequence, since Schouten and van Dantzig were aware of Veblen, Thomas, and Whitehead's definitions (and quote these extensively in the first section of their paper).

However, regardless of which reading of Schouten and van Dantzig one adopts, it remains the case that this argument suffices to establish that geometric figures, with their components thus-defined, are geometric objects in the sense of Schouten and van Dantzig's definition in the previous section, and, on the second reading of their argument, also geometric objects in Veblen and Thomas's sense. This completes my argument that early geometers working on the theory of geometric objects had a more expansive conception of geometric objects than the modern one.

3 Locating the pointwise turn

Given that early work on the theory of geometric objects adopted this more expansive conception, one might ask, when and why did the situation change? The answer is to be found in a paper by Schouten and Haantjes, published only a year later in 1936. Here, Schouten and Haantjes begin by noting that there "is a certain lack of rigour" in previous definitions of the geometric object

(Schouten and Haantjes 1936, 360), and, building on a suggestion by Wundheiler (1934), propose an alternative formal definition of the geometric object. For our purposes here, we will only need to focus on their definition of an "object" (where I have changed and simplified their notation for ease of exposition):

> Corresponding to every coordinate system [...] and every [...] point p in U, let a finite set of numbers Ω_1, Ω_2, ... be given. We symbolise these numbers by Ω [...]. The numbers Ω are called components of an object.[22] (Schouten and Haantjes 1936, 363)

Here, again, the region U is a "geometric region", i.e. an open subregion of M diffeomorphic to $\mathbb{R}^n$ (360).

Most important is the general shape of the construction here: an object is specified by a finite collection of components (relative to some coordinate system), which are real numbers defined at each $p \in U$. This is striking, because it rules out precisely the components which Schouten and van Dantzig (1935) associated with geometric figures (embedded submanifolds, collections thereof, etc.) and so rules these out as "objects".

To see that this does indeed restrict the space of objects to what might be described as "local fields", and thus rules out Schouten and van Dantzig's "geometric figures", we just need to note that an object, on this definition, is an assignment of a (finite) collection of components *to each point* in a region, i.e. a component, relative to some coordinate system (U, ϕ), is a smooth map from U into $\mathbb{R}$. As we saw in the previous section, embedded submanifolds of U can be associated with "components", but not components in this sense. (Indeed, as we saw in the above passages from Schouten and van Dantzig (1935), the fact that the set of components must be finite already rules out extended objects such as curves, even before we look at the definition of the components.) As such, embedded submanifolds of U do not count as objects under Schouten and Haantjes's (1936) definition, by precisely the same arguments given in §2.1.

Why, then, did this transition go largely (if entirely) unremarked? I suggest that the reason for this is that the notion of "components defined on some region" which featured in previous definitions of geometric objects is ambiguous between two ways of associating components, i.e. collections of real numbers, to coordinatised regions. One way of associating components to regions is to understand components on some $U \subset M$ as a collection of smooth maps Ω_ϕ from U into $\mathbb{R}$. This is the notion of "components defined on some region" which is adopted by Schouten and Haantjes (1936) and then Nijenhuis (1952). It is also the notion of "components defined on some region" needed to capture standard examples such as the components of scalar fields, tensor fields, affine connections etc.

However, there is another way of understanding "components defined on some region", which is to notice that, if $\psi : S \to U$ is a (smooth, injective) map, then we have the resources to consider "components defined on some

22. Note that this is almost exactly the same as the modern definition of an "object" presented in §2.1.

region" in a quite different way—which is to take the "components defined on U" to be the set $\mathrm{Im}(\phi^{-1} \circ \psi) \subset \mathbb{R}^n$. Indeed, we have already seen that this is precisely kind of notion of "components defined on some region" adopted by Schouten and van Dantzig (1935) in their argument that every geometric figure is a geometric object (though recall that Schouten and van Dantzig could also be read as countenancing the idea that the set S itself could be taken as the "components"). It is also the notion implicitly adopted by Veblen and Whitehead (1932) in their claim that every point is a geometric object, for the special case where S is a singleton set. However, mathematically speaking, these two notions of "components defined on some region" are very different—the first is a collection of smooth assignments, to each $p \in U$, of a real number, the second is an assignment to *the region U as a whole* of a collection of n-tuples of real numbers.[23] As soon as one begins to be mathematically precise about the definition of "components", one is naturally led to jettison this second option in favour of the first, if one wants to capture standard examples such as the components of scalar fields, tensor fields, affine connections etc., which are all defined *pointwise*, as mentioned above. But since the first option is in fact a plausible precisification of "components defined on some region", it is natural to think of it as just that—i.e. an explication of what was previously meant by "components defined on some region", rather than narrowing of what had gone before.

However, there is something interesting about all this—which is that the problem with assimilating embedded submanifolds of U under Schouten and Haantjes's (1936) definition of a geometric object comes at the level of their definition of an *object*, and thus is, in some sense, prior to questions about how such objects behave under coordinate transformations. Indeed, this is to be expected, since embedded submanifolds of a differential manifold (and (smoothly) parameterised families thereof, etc.) have well-defined lifts under diffeomorphisms. So were early geometers working on the theory of geometric objects simply mistaken to think that these were geometric objects? I think that the answer to this question is "yes and no". "Yes", in that the sense in which embedded submanifolds such as points, curves, and hypersurfaces can be associated with "components" with respect to some coordinate system is very different from the sense in which scalar fields, tensor fields, affine connections etc. are, and discussions such as that of Schouten and van Dantzig (1935) gloss over this difference. But—and perhaps more substantively—"no", in that embedded submanifolds can be represented relative to a coordinate system, and these coordinate representations have well-defined transformations under transformations of that coordinate system.

23. Note that this worry about the assignment of components to each point in U vs. U as a whole remains if one understands Schouten and van Dantzig (1935) as arguing that the set S can be taken as the components of a geometric object.

4 Close

In this paper, I have argued that geometers working on the theory of geometric objects in the 1920s and early 1930s held a more expansive view of geometric objects than the modern definition. I have reconstructed and assessed their arguments for this, and also shown how the need for greater precision in the formal definition of geometric objects naturally leads from this early viewpoint to the modern one.

The abrupt change in the geometric object concept around the year 1936 I have identified in this paper raises several interesting questions. For example, one might see the transition from the early geometric object concept to the modern one as a potential Kuhn loss example, insofar as the former encompassed *any* coordinate-independent structure with well-defined lifts under (spacetime) diffeomorphisms, but the latter only those which are "local fields".[24]

This also sheds new and interesting light on the evolution of the concept of general covariance—in particular, the changing relationships between general covariance, in the guise of the geometric object concept,[25] and (i) "local fields", (ii) coordinate-independence, and (iii) the property of having well-defined lifts under (spacetime) diffeomorphisms. As has recently been noted by March and Weatherall (2025), these can come apart—for example, objects such as Yang-Mills fields can be coordinate-independent but fail to be geometric objects because they lack well-defined lifts under (spacetime) diffeomorphisms. My discussion here also points to a second way in which these can come apart: objects such as embedded submanifolds can be coordinate-independent and have well-defined lifts under (spacetime) diffeomorphisms, but fail to be geometric objects (under the modern conception) because they are not "local fields".

One of the morals of this paper has been that this relationship between general covariance and "local fields" did not always exist. From the current perspective, then, one might be tempted to ask: just what, if anything, is the significance of this relationship? To end this paper, I want to briefly say something to address this question. But I would be inclined to begin with a somewhat different question, namely: just what kind of structures was the criterion of "general covariance" supposed to apply to?

On one version of the criterion of general covariance, it is a property of certain kinds of objects defined on a spacetime manifold. Objects of that kind are said to be generally covariant just in case they can be characterised in a coordinate-independent way, and have well-defined lifts under spacetime diffeomorphisms. And, as discussed above, on this version of general covariance, there does not seem to be a particularly deep difference between objects which are "local fields" (tensor fields, metrics, affine connections, etc.), and objects such as embedded submanifolds which are not.

But when we talk about general covariance, we generally have in mind a property not of certain kinds of objects defined on a spacetime manifold, but

24. I am grateful to Brian Pitts for suggesting this connection.

25. Misner, Thorne, and Wheeler (1973, 48) attribute the first clear statement of this connection between general covariance and geometric objects to Veblen and Whitehead (1932).

a property of spacetime *theories*—which are of course not spacetime manifolds with certain kinds of objects defined thereon, but if anything, collections thereof (i.e., the collection of models of the theory), or probably better, systems of partial differential equations which pick out that collection of models as their solution space (cf. Weatherall and March (2025)). For our purposes, it will suffice simply that a spacetime theory asserts a certain relationship between certain kinds of objects defined on a spacetime manifold. The theory is said to be generally covariant just in case this relationship is preserved under the lift of those objects under spacetime diffeomorphisms.[26]

It is here that I think the idea that embedded submanifolds of a manifold are not geometric objects does have bite. This is perhaps clearest with the example of points. A generally covariant spacetime theory cannot assert a relationship between *a particular point* of a manifold and other objects—tensor fields, affine connections, or whatever—defined at that point, except insofar as that point is uniquely distinguished by its differential-topological properties, and except insofar as that same relationship is asserted to hold equally of every other point (with the same differential-topological properties) of the manifold. Likewise, a generally covariant spacetime theory cannot assert a relationship between *a particular curve* on a manifold and other objects defined along that curve, except insofar as that curve is uniquely distinguished by its differential-topological properties, and except insofar as that relationship is asserted to hold equally of all other curves (with the same differential-topological properties)—in which case we are really talking about congruences of curves, which are just complete vector fields, and so we are back to geometric objects. More generally: it is only insofar as embedded submanifolds can be thought of as corresponding to certain kinds of local fields (complete vector fields for congruences of curves, integrable one-forms for foliations of codimension one hypersurfaces, etc.) that these are candidate objects to enter into the relationships asserted by a generally covariant spacetime theory.

This is not to say that individual models of a generally covariant theory may not pick out particular collections of embedded submanifolds—points, curves, etc.—as distinguished in some way, e.g. by local field values there. Of course they may. But we should be careful to separate this from the property of a theory requiring, or specifying, distinguished embedded submanifolds as part of its formulation. A distinguished space of embedded submanifolds of a manifold cannot be invariant under (all) spacetime diffeomorphisms unless that space is just the manifold itself, whereas a distinguished space of geometric object field values on that manifold can. And that connection between "local fields" and general covariance is precisely what the modern geometric object concept allows us to express.

26. Note that a necessary condition for this is that these objects be "generally covariant" in the sense described above, cf. March and Weatherall (2025).

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