An “Absolute” Type of Logic

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ABSTRACT
This paper proposes an alternative to standard first-order logic that seeks greater naturalness, generality, and semantic self-containment. The system removes the first-order restriction, avoids type hierarchies, and dispenses with external structures, making the meaning of expressions depend solely on their constituent symbols. Terms and formulas are unified into a single notion of expression, with set-builder notation integrated as a primitive construct. Connectives and quantifiers are treated as operators among others rather than as privileged primitives. The deductive framework is minimal and intuitive, with soundness and consistency established and completeness examined. While computability requirements may limit universality, the system offers a unified and potentially more faithful model of human mathematical deduction, providing an alternative foundation for formal reasoning.

KEYWORDS
mathematical logic, foundations, foundations of mathematics, nonclassical logic, absolute logic

1. Introduction

This paper outlines a system or approach to mathematical logic which is different from the standard one. By ‘the standard approach to logic’ I mean the one presented in chapter 2 of Enderton’s book [2] and there named ‘First-Order Logic’. The same approach is also outlined in chapter 2 of Mendelson’s book [5], where it is named ‘Quantification Theory’.

An online article by W. Ewald [3], in the Stanford Encyclopedia of Philosophy, describes the process that led to the establishment of first-order logic as the standard system of mathematical logic. However, the conclusion is that there are no clear reasons why this occurred.

> How did first-order logic come to be regarded as a privileged logical system—that is, as (in some sense) the “correct” logic for investigations in foundations of mathematics? That question, too, is highly complicated. Even after the Gödel results were widely understood, logicians continued to work in type theory, and it took years before first-order logic attained canonical status. The transition was gradual, and cannot be

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given a specific date.

First-order logic has been around for many decades, but to date no absolute evidence has been found that first-order logic is the best possible logic system. In this regard I may quote a stronger statement at the beginning of Josè Ferreirós' paper 'The road to modern logic – an interpretation' ([4]).

> It will be my contention that, contrary to a frequent assumption (at least among philosophers), First-Order Logic is *not* a 'natural unity', i.e. a system the scope and limits of which could be justified solely by rational argument.

Honestly, in my opinion, the approach to logic I am going to propose seems to be a 'natural unity' much more than first-order logic is. The basic idea behind this system is indeed to build a logical system that is as natural, general, and absolute as possible, and to have a faithful model of the human deductive process, as far as possible.

The proposed 'system' seems 'natural' enough to me in many respects, but I can't say for sure it's a truly general and absolute approach, or the only valid approach to logic. In fact, for instance, I believe that a logical system must satisfy some computability requirements. Although computability theory was born in the 1930s, therefore after mathematical logic and the formalization of first-order logic by Hilbert and Ackermann, when formalizing a logical system it is not possible to ignore basic concepts inherent in computability theory. I suspect that this very requirement could be an obstacle to the possibility of obtaining a general and absolute logical system, or a unique approach to logic.

Anyway, in our approach we do not want to use some features of first-order logic which we don't like, because they make it too 'limited' or 'relative' without any obvious necessity.

These features are primarily the constraint on the 'order' of the expressions (which have to be 'first order' as suggested by the name of the system) and the need of external 'structures' to associate a meaning to the expressions.

Let's first discuss these two features.

In first-order logic variables range over individuals, but in mathematics there are statements in which both quantifiers over individuals and quantifiers over sets of individuals occur. One simple example is the following condition:

for each subset X of $\mathbb{N}$ and for each x $\in \mathbb{N}$ we have x $\in$ X or x $\notin$ X .

We will explicitly show in section 14 that this is a valid expression of our language. Another example is the condition in which we state that every bounded, non empty set of real numbers has a supremum. Formalisms which are better suited than first-order logic to express such conditions are second-order logic and type theory, but these systems have a certain level of complexity and are based on different types of variable. In our system we can express the conditions we mentioned above, and we absolutely don't need different types of variables, the set to which the quantifier refers is explicitly written in the expression, this ultimately makes things easier and allows a more general approach. If we read the statement of a theorem in a mathematics book, usually in this statement some variables are introduced, and when introducing them often the set in which they are varying is explicitly specified, so from this point of view our approach is consistent with the actual processes of mathematics.

Our logic is not a first-order or second-order or n-order logic, it doesn't involve types, so from this point of view it is an 'absolute' type of logic.

Let's examine how our system behaves when giving a meaning and possibly a truth value to expressions. Standard logic doesn't plainly associate meanings and truth values to formulas. It introduces some related notion as the concepts of 'structure' (defined in section 2.2 of Enderton's book), truth in a structure, validity, satisfiability. Within first-order logic a structure is used, first of all, to define the collection of things to which a quantifier refers to. Moreover, some symbols such as connectives and quantifiers have a fixed meaning, while for other symbols the meaning is given by the structure. Notions such as validity and satisfiability reveal a question-based approach: 'what happens when we change the meaning of some symbols?' Although this may be an interesting perspective, this is not our approach, understanding what happens when we change the meaning of the symbols does not have a primary interest for us, although it's quite obvious that we'll also try to enunciate some results that are valid regardless of the meaning of the symbols. In this regard, if we had this perspective, in the first place it would have to be discussed if there are anyway symbols (e.g. connectives, quantifiers and others too) whose meaning cannot change.

Therefore, if a symbol is in our system, it has its own meaning, and we don't feature a notion of structure like the one of first-order logic. Also, the set of expressions in our language depends on the meaning of symbols. We'll simply speak of the meaning of an expression and when possible of the truth value of that meaning. As we've already said, the meaning of a sentence will depend solely on the meaning of the symbols it contains, it will not depend on external 'structures'. Therefore, from this view too, our logic is an 'absolute' type of logic.

We now list other features of our system, pointing out the differences and improvements with respect to standard logic.

In first-order logic there exist two different concepts of term and formula, in place of these two concepts in our approach we have just one notion of expression. Each expression is referred to a certain 'context'. A context can be seen as a (possibly empty) sequence of ordered pairs (x, φ), where x is a variable and φ is itself an expression. Given a context $k = (x_1, \varphi_1) \dots (x_m, \varphi_m)$ we call a 'state on k' a function which assigns 'allowable values' (we'll explain this later) to the variables $x_1, \dots, x_m$. If t is an expression with respect to context k and σ is a state on k, we'll be able to define the meaning of t with respect to k and σ, which we'll denote by $\#(k, t, \sigma)$.
Our approach requires to build all at the same time, contexts, expressions, states and meanings. We'll call sentences those expressions which are related to an empty context and whose meaning is true or false. The meaning of a sentence depends solely on the meaning of the symbols it contains, it doesn't depend on external 'structures'.

In first-order logic we have terms and formulas and we cannot apply a predicate to one or more formulas, and it seems this can be a limitation. With our system we can apply predicates to formulas.

When we specify a set in mathematics we often use the 'set-builder notation'. Examples of sets defined with this notation are $\{x \in \mathbb{N}| \; \exists y \in \mathbb{N} : x = 2y\}$, $\{x \in \mathbb{R}|x = x^2\}$, and so on. In our system the set-builder notation is included as an expression-building pattern, and this is an advantage over standard logic.

Of course in our approach we allow connectives and quantifiers, but unlike first-order logic these are at the same level of other operators, such as equality, membership and more. While the set-builder notation is necessarily present with its role, connectives and quantifiers as 'operators' are not strictly mandatory and are part of a broader category. For instance the universal quantifier simply applies an operation of logical conjunction to a set of conditions, and so it can be classified as an operator.

Our deductive system seeks to provide a good model of human mathematical deductive process. The concept of proof we'll feature is probably the most simple and intuitive that comes to mind, we try to anticipate some of it.

If we define S as the set of sentences then an axiom is a subset of S, an n-ary rule is a subset of S^{n+1}. If φ is a sentence then a proof of φ is a sequence $(\psi_1, \ldots, \psi_m)$ of sentences such that

- there exists an axiom A such that $\psi_1 \in A$;
- if $m > 1$ then for each $j = 2 \ldots m$ one of the following holds
  - there exists an axiom A such that $\psi_j \in A$,
  - there exists an n-ary rule R and $i_1, \ldots, i_n < j$ such that $(\psi_{i_1}, \ldots, \psi_{i_n}, \psi_j) \in R$;
- $\psi_m = \varphi$.

As regards the soundness of the system, it is proved at the beginning of section 7. Consistency, also proved in section 7, is a direct consequence of soundness. We discuss (in paragraph 8.2) on the completeness of our deductive systems.

We have examined the main features of the system. If the reader will ask what is the basic idea behind a system of this type, in agreement with what I said earlier I could say that the principle is to try to provide something like a general, absolute and unifying approach to logic and a faithful model of human mathematical deductive process.

This statement about our system of course is not a mathematical statement, so I cannot give a mathematical proof of it. I'm not even sure that I have truly and fully achieved the declared objectives and that they are fully achievable. A key aspect in this regard is the computability requirements that a logical system must satisfy, and in this version of the manuscript we pay due attention to these requirements.

On the other hand, logic exists with the specific primary purpose of being a model to human deduction. In general, suppose we want to provide a mathematical model of some process or reality. The fairness of the model can be judged much more through experience than through mathematics. In fact, mathematics always has to do with models and not directly with reality.

This paper's purpose is to present an approach to logic, but clearly we cannot provide here all possible explanations and comparisons in any way related to the approach itself. The author believes that this paper provides a fairly comprehensive presentation of the approach in question, this introduction includes significant elements of explanation, justification and comparison with the standard approach to logic. Other material in this regard is presented in the subsequent parts.

Further investigations on this approach will be conducted, in the future, if and when possible, by the author and/or other people. If any claim of this introduction would seem inappropriate, the author is ready to reconsider and possibly fix it. In any case he believes the most important part of this paper is not in the introduction, but in the subsequent sections.

The paper is quite long, but the time required to get an idea of the content is not very high. In fact, the author has chosen to include all the proofs, but quite often they aren't difficult proofs. In addition, the most complex part is perhaps definition 6.1 which has a certain complexity, but at a first reading it is not necessary to take care of all the details.

2. Changes from previous version

Here we describe the main changes of the paper with respect to the previuos version.

First of all, we have introduced computability constraints in the definition of the system. The process with which we generate expressions in our language is an inductive process. At each step we must ensure that the set of the new expressions is a recursive set. This ensures that the global set of expressions is a recursively enumerable set and so are the set of sentences etc.. We also introduced the constraint that axioms and rules must be r.e. sets., which seems reasonable.

Besides this we also added a new example of deduction.

3. The language of our logic system

In this section we want to define the language, which is the entity that underlies our logic system. The language is actually made up of various elements including some sets of symbols.

First we need a set of symbols $\mathcal{V}$. $\mathcal{V}$ members are also called 'variables' and just play the role of variables in the construction of our expressions (this implies that $\mathcal{V}$ members have no meaning associated). We assume $\mathcal{V}$ is a finite or countable set.

In addition we need another set of symbols $\mathcal{C}$. $\mathcal{C}$ members are also called 'constants' and have a meaning. For each $c \in \mathcal{C}$ we denote by $\#(c)$ the meaning of c. We assume $\mathcal{C}$ is a finite set.

Let f be a member of $\mathcal{C}$. Being f endowed with meaning, f is always an expression of our language. However, the meaning of f could also be a function. In this case f can also play the role of an 'operator' in the construction of expressions that are more complex than the simple constant f.

Not all the operators that we need, however, are identifiable as functions. Think to the logical connectives (logical negation, logical implication, quantifiers, etc..), but also to the membership predicate '$\in$' and to the equality predicate '$=$'. The meaning of these operators cannot be mapped to a precise mathematical object, therefore these operators won't have a precise meaning in our language, but we'll need to give meaning to the application of the operator to objects, where the operator is applicable.

In mathematics and in the real world objects can have properties, such as having a certain color, or being true, or being false. A property is therefore something that can be assigned to an object, no object, more than one object. For example, with reference to color, one or more objects are red or have the property 'to be of red color'. But more generally one or more objects have a color. Suppose to indicate, for objects x that have a color, the color of x with $C(x)$. So we can say that C is a property applicable to a class of objects. On the same object class we can indicate with $R(x)$ the condition 'x has the red color'. R is in turn a property applicable to a class of objects, with the characteristic that for all x $R(x)$ is true or false. A property with this additional feature can be called a 'predicate'.

The class of objects to which a property may be assigned may be called the domain of the property. The members of that domain may be individual objects or sequences of objects, for example, if x is an object and X is a set, the condition '$x \in X$' involves two objects, and then the domain of the membership property consists of the ordered pairs (x, X), where x is an object and X is a set.
Generally we are dealing with properties such that the objects of their domain are all individual objects, or all ordered pairs. Theoretically there may also be properties such that the objects of their domain are sequences of more than two items or even the number of items in sequence may be different in different elements of the domain.

As mentioned above the concept of 'property' is similar to the concept of function, but in mathematics there are properties that are not functions. For example, the condition '$x \in X$' just introduced can be applied to an arbitrary object and an arbitrary set, so the 'membership property' has not a well determined domain and cannot be considered a function in a strict sense.

So, in order to build our language, we need another set of symbols $\mathcal{F}$, where each f in $\mathcal{F}$ represents a property P_f. Symbols in $\mathcal{F}$ are also called operators or 'property symbols'. We assume $\mathcal{F}$ is a finite set. We will not assign a meaning to operators, because a property cannot be mapped to a consistent mathematical object (function or other). However, for each f

- we need to determine a condition $A_f(x_1, \ldots, x_n)$ that given a positive integer n and $x_1, \ldots, x_n$ arbitrary objects indicates if P_f is applicable to $x_1, \ldots, x_n$. The condition $A_f(x_1, \ldots, x_n)$ does not have to be decidable in an absolute sense, but it must be so when it is used in the process by which we construct our expressions;

- for each positive integer n and $x_1, \ldots, x_n$ arbitrary objects such that $A_f(x_1, \ldots, x_n)$ holds we must be able to calculate the value of $P_f(x_1, \ldots, x_n)$. This doesn't mean that P_f must be a computable function in a strict sense, but we must be able to know the value of $P_f(x_1, \ldots, x_n)$ when this calculation is required in the construction of our expressions.

We immediately explain these concepts by specifying what are the most important operators that we may include in our language, providing for each of them the conditions $A_f(x_1, \ldots, x_n)$ and $P_f(x_1, \ldots, x_n)$ (in general $P_f(x_1, \ldots, x_n)$ is a generic value, but in these cases it is a condition, i.e. its value can be true or false).

- Logical conjunction: it's the symbol $\wedge$ and we have
  for $n \neq 2$ $A_\wedge(x_1, \ldots, x_n)$ is false ,
  $A_\wedge(x_1, x_2) =$ (x_1 is true or x_1 is false) and (x_2 is true or x_2 is false),
  $P_\wedge(x_1, x_2) =$ both x_1 and x_2 are true ;
- Logical disjunction: it's the symbol $\vee$ and we have
  for $n \neq 2$ $A_\vee(x_1, \ldots, x_n)$ is false ,
  $A_\vee(x_1, x_2) =$ (x_1 is true or x_1 is false) and (x_2 is true or x_2 is false),
  $P_\vee(x_1, x_2) =$ at least one between x_1 and x_2 is true ;
- Logical implication: it's the symbol $\rightarrow$ and we have
  for $n \neq 2$ $A_\rightarrow(x_1, \ldots, x_n)$ is false ,
  $A_\rightarrow(x_1, x_2) =$ (x_1 is true or x_1 is false) and (x_2 is true or x_2 is false),
  $P_\rightarrow(x_1, x_2) = x_1$ is false or x_2 is true ;
- Double logical implication: it's the symbol $\leftrightarrow$ and we have
  for $n \neq 2$ $A_\leftrightarrow(x_1, \ldots, x_n)$ is false ,
  $A_\leftrightarrow(x_1, x_2) =$ (x_1 is true or x_1 is false) and (x_2 is true or x_2 is false),
  $P_\leftrightarrow(x_1, x_2) = P_\rightarrow(x_1, x_2)$ and $P_\rightarrow(x_2, x_1)$;
- Logical negation: it's the symbol $\neg$ and we have
  for $n > 1$ $A_\neg(x_1, \ldots, x_n)$ is false ,
  $A_\neg(x_1)$ is true,
  $P_\neg(x_1) = x_1$ is false ;
- Universal quantifier: it's the symbol $\forall$ and we have
  for $n > 1$ $A_\forall(x_1, \ldots, x_n)$ is false ,
  $A_\forall(x_1) = x_1$ is a set and for each x in x_1 (x is true or x is false),
  $P_\forall(x_1) =$ for each x in x_1 (x is true) .
- Existential quantifier: it's the symbol $\exists$ and we have
  for $n > 1$ $A_\exists(x_1, \ldots, x_n)$ is false ,
  $A_\exists(x_1) = x_1$ is a set and for each x in x_1 (x is true or x is false),
  $P_\exists(x_1) =$ there exists x in x_1 such that (x is true) .
- Membership predicate: it's the symbol $\in$ and we have
  for $n \neq 2$ $A_\in(x_1, \ldots, x_n)$ is false ,
  $A_\in(x_1, x_2) = x_2$ is a set,
  $P_\in(x_1, x_2) = x_1$ is a member of x_2 ;
- Equality predicate: it's the symbol $=$ and we have
  for $n \neq 2$ $A_=(x_1, \ldots, x_n)$ is false ,

$A_=(x_1, x_2)$ is true,
$P_=(x_1, x_2) = x_1$ is equal to x_2 .

In principle we can think and use also other operators, for instance operations between sets such as union or intersection can be represented through an operator, etc.. In any case, we must choose our operators in such a way as to guarantee computability in the construction of our expressions, and for this reason we must impose limits on the choice of operators. For example, set operators of the type just mentioned will not be used.

Our set $\mathcal{F}$ will typically be contained in the set $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in, =\}$, where each of the just mentioned symbols has been defined above. However, we want to have a more general approach than the one in which the operators are explicitly indicated, so we will also allow other types of operators, as long as they fall into one of the following categories.

The first admitted category of operators is the category of the symbols f such that

- for $n \neq 2$ $A_f(x_1, \ldots, x_n)$ is false,
- $A_f(x_1, x_2) =$ (x_1 is true or x_1 is false) and (x_2 is true or x_2 is false),
- $P_f(x_1, x_2)$ is true or false.

Since for $n \neq 2$ $A_f(x_1, \ldots, x_n)$ is false, we say the symbols in this category have a multiplicity of 2.

All of the symbols $\wedge, \vee, \rightarrow, \leftrightarrow$ fall within this category.

Another admitted category of operators is the category of the symbols f such that

- for $n > 1$ $A_f(x_1, \ldots, x_n)$ is false,
- $A_f(x_1)$ is true,
- $P_f(x_1)$ is true or false.

Since for $n > 1$ $A_f(x_1, \ldots, x_n)$ is false, we say the symbols in this category have a multiplicity of 1.

The symbol $\neg$ falls within this category.

Another admitted category of operators is the category of the symbols f such that

- for $n > 1$ $A_f(x_1, \ldots, x_n)$ is false,
- $A_f(x_1) = x_1$ is a set and for each x in x_1 (x is true or x is false),
- $P_f(x_1)$ is true or false.

Clearly the symbols in this category have a multiplicity of 1.

The symbols $\forall, \exists$ fall within this category.

Another admitted category of operators is the category of the symbols f such that

- for $n \neq 2$ $A_f(x_1, \ldots, x_n)$ is false,

- $A_f(x_1, x_2) = x_2$ is a set,
- $P_f(x_1, x_2)$ is true or false.

Clearly the symbols in this category have a multiplicity of 2.

The symbol $\in$ falls within this category.

Finally, another admitted category of operators is the category of the symbols f such that

- for $n \neq 2$ $A_f(x_1, \ldots, x_n)$ is false,
- $A_f(x_1, x_2)$ is true,
- $P_f(x_1, x_2)$ is true or false.

Clearly the symbols in this category have a multiplicity of 2.

The symbol $=$ falls within this category.

We require that all the symbols in $\mathcal{F}$ fall within one of the mentioned categories, and so they must have a multiplicity of 1 or 2.

In the standard approach to logic, quantifiers are not treated like the other logical connectives, but in this system we mean to separate the operation of applying a quantifier from the operation whereby we build the set to which the quantifier is applied, and therefore the quantifier is treated as the other logical operators (altogether, the universal quantifier is simply an extension of logical conjunction, the existential quantifier is simply an extension of logical disjunction).

With regard to the operation of building a set, we need a specific symbol to indicate that we are doing this, this symbol is the symbol '$\{\}$' which we will consider as a unique symbol.

Besides the set builder symbol, we need parentheses and commas to avoid ambiguity in the reading of our expressions; to this end we use the following symbols: left parenthesis '(', right parenthesis ')', comma ',' and colon ':'. We can indicate this further set of symbols with $\mathcal{Z}$.

To avoid ambiguity in reading our expressions we require that the sets $\mathcal{V}$, $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{Z}$ are disjoint. It's also requested that a symbol does not correspond to any chain of more symbols of the language. More generally, given $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ symbols of our language, and using the symbol '$\|$' to indicate the concatenation of characters and strings, we assume that equality of the two chains $\alpha_1 \| \ldots \| \alpha_n$ and $\beta_1 \| \ldots \| \beta_m$ is achieved when and only when $m = n$ and for each $i = 1 \ldots n$ $\alpha_i = \beta_i$. We also specify that by 'string' we mean a concatenation of symbols of our language.

While the set $\mathcal{Z}$ will be always the same, the sets $\mathcal{V}$, $\mathcal{C}$, $\mathcal{F}$ may change according to what is the language that we describe. If we think to our language as a language as defined in languages theory, once we have chosen $\mathcal{V}$, $\mathcal{C}$ and $\mathcal{F}$ the alphabet Σ of our language is given by $\Sigma = \mathcal{V} \cup \mathcal{C} \cup \mathcal{F} \cup \mathcal{Z}$.

Another variable element that we add to our language is made by a finite number of sets $D_1, \ldots, D_p$ such that:

- for each $i, j = 1 \ldots p$ such that $i \neq j$ $D_i \neq D_j$ and $D_i \cap D_j \neq \emptyset$;
- for each $i = 1 \ldots p$ and for each $x \in D_i$ x is not a set;
- for each $i = 1 \ldots p$ and for each $x \in D_i$ x is not true and x is not false.

Here we have to specify that we can also not need this additional sets and in this case we can say that $p = 0$.

A notion that we will soon use in the continuation is the notion of power set. Given a set A we'll indicate with $\mathcal{P}(A)$ the set of the subsets of A, but in our definition the empty set will not be a member of $\mathcal{P}(A)$, so $\mathcal{P}(A)$ for us is the set of the non empty subsets of A.

We also define $\mathcal{P}^q(A)$ for any positive integer q. Of course $\mathcal{P}^1(A) = \mathcal{P}(A)$ by definition, and given a positive integer q $\mathcal{P}^{q+1}(A) = \mathcal{P}(\mathcal{P}^q(A))$.

A specific language of our logic system is described by its variable elements which are the sets $\mathcal{V}$, $\mathcal{C}$, $\mathcal{F}$, the function $\#$ which associates a meaning to every element of $\mathcal{C}$ and in addition the (potentially empty) set of sets $\{D_1, \ldots, D_p\}$. Moreover somewhere we will be in the condition to define new expressions for our language with reference to the sets $\mathcal{P}^q(D_i)$ (or $(\mathcal{P}^q(D_i))^m$ for $m \geqslant 2$) where q is potentially unlimited. Since this could be a problem in the perspective of the recursivity of the set of expressions that we define, we also need a positive integer q_{max} which we want to use as an upper bound of q is this situation.

Therefore our language is identified by the 6-tuple $(\mathcal{V}, \mathcal{F}, \mathcal{C}, \#, \{D_1, \ldots, D_p\}, q_{max})$. Since the 'meaning' of an operator is not a mathematical object, operators must be seen as symbols which are tightly coupled with their meaning.

We also need to set some constraints on our constants, which must not refer to the empty set or to a set which has the empty set as a member and so on. In order to do that we want to define formally some predicates that we'll soon use in the continuation. We actually define the following predicates.

$Set_1(x) = x$ is a set.

$Event_1(x) = x$ is true or x is false.

Given a positive integer q

- $Set_{q+1}(x) = x$ is a set and for each $u \in x$ $Set_q(u)$;
- $Event_{q+1}(x) = x$ is a set and for each $u \in x$ $Event_q(u)$.

If $Set_1(x)$ holds we define

$NotEmpty_1(x) = (x \neq \emptyset)$.

Given a positive integer q, if $Set_{q+1}(x)$ holds we define

$NotEmpty_{q+1}(x) = NotEmpty_1(x)$ and for each $u \in x$ $NotEmpty_q(u)$.

The constraints we want to put on our constants can now be stated as follows: for each $c \in \mathcal{C}$

- if $Set_1(\#(c))$ then $NotEmpty_1(\#(c))$;
- for each $q > 1$ if $Set_q(\#(c))$ then $NotEmpty_q(\#(c))$.

3.1. Other definitions and results

Before we can describe the process of constructing expressions we still need to introduce some notation. In fact in that process we'll use the notion of 'context' and the notion of 'state'. Context and states have a similar form, here we define a notion of state-like pair and related results that well'apply to states, but similar definitions and results will be given for contexts.

We define $\mathcal{D} = \{\emptyset\} \cup \{\{1, \dots, m\} |\ m$ is a positive integer$\}$.

Suppose x is a function whose domain $dom(x)$ belongs to $\mathcal{D}$. Suppose $C \in \mathcal{D}$ is such that $C \subseteq dom(x)$. Then we define $x_{/C}$ as a function whose domain is C and such that for each $j \in C$ $x_{/C}(j) = x(j)$.

Suppose x and φ are two functions with the same domain D, and $D \in \mathcal{D}$. Then we say that (x, φ) is a '*state-like pair*'.

Given a state-like pair $k = (x, \varphi)$ the domain of x will be also called the *domain of* k. Therefore $dom(k) = dom(x) = dom(\varphi)$.

Furthermore $dom(k) \in \mathcal{D}$ and given $C \in \mathcal{D}$ such that $C \subseteq dom(k)$ we can define $k_{/C} = (x_{/C}, \varphi_{/C})$. Clearly $k_{/C}$ is a state-like pair.

We define $\mathcal{R}(k) = \{k_{/C} |\ C \in \mathcal{D}, C \subseteq dom(k)\}$.

Given another state-like pair h we write $h \sqsubseteq k$ if and only if $h \in \mathcal{R}(k)$.

Suppose $h \in \mathcal{R}(k)$, then there exists $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $h = k_{/C} = (x_{/C}, \varphi_{/C})$. Therefore $dom(h) = C$ and $k_{/dom(h)} = k_{/C} = h$.

Suppose $h \in \mathcal{R}(k)$ and $g \in \mathcal{R}(h)$. This means there exist $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $h = k_{/C}$, and there exist $D \in \mathcal{D}$ such that $D \subseteq dom(h)$, $g = h_{/D}$. So $D \subseteq dom(h) = C \subseteq dom(k)$, $g = (k_{/C})_{/D} = (x_{/C}, \varphi_{/C})_{/D} = (x_{/D}, \varphi_{/D}) = k_{/D}$. Therefore $g \in \mathcal{R}(k)$.

Suppose $k = (x, \varphi)$ is a state-like pair whose domain is D. Suppose (y, ψ) is an ordered pair. Then we can define the 'addition' of (y, ψ) to k.
Suppose $D = \{1, \dots, m\}$, then we define $D' = \{1, \dots, m+1\}$. We define x' as a function whose domain is D' such that for each $\alpha = 1 \dots m$ $x'(\alpha) = x(\alpha)$, and $x'(m+1) = y$. We define φ' as a function whose domain is D' such that for each $\alpha = 1 \dots m$ $\varphi'(\alpha) = \varphi(\alpha)$, $\varphi'(m+1) = \psi$. Then we define $k + (y, \psi) = (x', \varphi')$. Obviously $(k + (y, \psi))_{/\{1,\dots,m\}} = k$, so $k \in \mathcal{R}(k + (y, \psi))$.
If $D = \emptyset$ then clearly $D' = \{1\}$. We define x' as a function whose domain is D' such that $x'(1) = y$. We define φ' as a function whose domain is D' such that $\varphi'(1) = \psi$. Then we define $k + (y, \psi) = (x', \varphi')$. Obviously $(k + (y, \psi))_{/\emptyset} = \emptyset = k$, so $k \in \mathcal{R}(k + (y, \psi))$.
In both cases $k + (y, \psi)$ is a state-like pair, and $k \in \mathcal{R}(k + (y, \psi))$, which implies

$dom(k) \subseteq dom(k + (y, \psi))$.
We have also seen that $(k + (y, \psi))_{/dom(k)} = (k + (y, \psi))_{/D} = k$.

We also define $\epsilon = (\emptyset, \emptyset)$, so ϵ is a state-like pair.

Given a state-like pair $k = (x, \varphi)$ we define $var(k)$ as the image of the function x. In other words if $k = \epsilon$ then $x = \emptyset$, so $var(k) = \emptyset$, otherwise x has a domain $\{1, \ldots, m\}$ and $var(k) = \{x_i | i = 1 \ldots m\}$.

Clearly, if we assume that $k + (y, \psi) = (x', \varphi')$, we can easily see that

$$var(k + (y, \psi)) = \{x'_i | i \in dom(x'_i)\} = \{x_i | i \in dom(x_i)\} \cup \{y\} = var(k) \cup \{y\}.$$

In the next lemma we prove that, when a state-like pair is obtained as $k + (y, \psi)$, then k, y, and ψ are univocally determined.

Lemma 3.1. *Suppose $k_1 = (x_1, \varphi_1)$ is a state-like pair whose domain is D_1, and (y_1, ψ_1) is an ordered pair. Suppose $k_2 = (x_2, \varphi_2)$ is a state-like pair whose domain is D_2, and (y_2, ψ_2) is an ordered pair. Finally suppose $k_1 + (y_1, \psi_1) = k_2 + (y_2, \psi_2)$. Under these assumptions we can prove that $k_1 = k_2, y_1 = y_2, \psi_1 = \psi_2$.*

Proof. We define $h = k_1 + (y_1, \psi_1) = k_2 + (y_2, \psi_2)$. Since $h = k_1 + (y_1, \psi_1)$ we can have two possibilities:

- $D_1 = \emptyset, D'_1 = \{1\}$ and there exist two functions x'_1 and φ'_1 whose domain is D'_1 such that $h = (x'_1, \varphi'_1)$;
- there exists a positive integer m_1 such that $D_1 = \{1, \ldots, m_1\}, D'_1 = \{1, \ldots, m_1 + 1\}$ and there exist two functions x'_1 and φ'_1 whose domain is D'_1 such that $h = (x'_1, \varphi'_1)$.

Similarly, since $h = k_2 + (y_2, \psi_2)$ we can have two possibilities:

- $D_2 = \emptyset, D'_2 = \{1\}$ and there exist two functions x'_2 and φ'_2 whose domain is D'_2 such that $h = (x'_2, \varphi'_2)$;
- there exists a positive integer m_2 such that $D_2 = \{1, \ldots, m_2\}, D'_2 = \{1, \ldots, m_2 + 1\}$ and there exist two functions x'_2 and φ'_2 whose domain is D'_2 such that $h = (x'_2, \varphi'_2)$.

It follows that $(x'_1, \varphi'_1) = h = (x'_2, \varphi'_2)$, so $x'_1 = x'_2$ and $\varphi'_1 = \varphi'_2$, and $D'_1 = D'_2$.

Suppose $D_1 = \emptyset$. This implies that $D'_2 = D'_1 = \{1\}$, thus $D_2 = \emptyset$.
In this case $k_1 = \epsilon = k_2$, $y_1 = x'_1(1) = x'_2(1) = y_2$, $\psi_1 = \varphi'_1(1) = \varphi'_2(1) = \psi_2$.

Suppose there exists a positive integer m_1 such that $D_1 = \{1, \ldots, m_1\}$. This implies that $D'_2 = D'_1 = \{1, \ldots, m_1 + 1\}$, thus $D_2 = \{1, \ldots, m_1\}$.
In this case for each $\alpha = 1 \ldots m_1$ $x_1(\alpha) = x'_1(\alpha) = x'_2(\alpha) = x_2(\alpha)$, $\varphi_1(\alpha) = \varphi'_1(\alpha) = \varphi'_2(\alpha) = \varphi_2(\alpha)$. So $k_1 = (x_1, \varphi_1) = (x_2, \varphi_2) = k_2$; and moreover $y_1 = x'_1(m_1 + 1) = x'_2(m_1 + 1) = y_2$, $\psi_1 = \varphi'_1(m_1 + 1) = \varphi'_2(m_1 + 1) = \psi_2$. □

Other useful results are the following.

Lemma 3.2. *Suppose $h = (x, \varphi)$, $k = (z, \psi)$ are state-like pairs such that $h \in \mathcal{R}(k)$. Then, for each $j \in dom(h)$ $x_j = z_j$ and $\varphi_j = \psi_j$.*

Proof. There exists $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $h = k_{/C} = (z_{/C}, \psi_{/C})$. Therefore $x = z_{/C}$ and $\varphi = \psi_{/C}$. For each $j \in dom(h) = C$ $x_j = z_j$ and $\varphi_j = \psi_j$. □

Lemma 3.3. *Suppose $h = (x, \varphi)$, $k = (z, \psi)$ are state-like pairs such that $h \in \mathcal{R}(k)$ and for each $i, j \in dom(k)$ $i \neq j \to z_i \neq z_j$. Then, for each $i \in dom(k)$, $j \in dom(h)$ $z_i = x_j \to \psi_i = \varphi_j$.*

Proof. Let $i \in dom(k)$, $j \in dom(h)$ and $z_i = x_j$. Clearly $j \in dom(k)$, $x_j = z_j$, thus $z_i = z_j$, $i = j$, $\varphi_j = \psi_j = \psi_i$. □

Lemma 3.4. *Suppose $k = (x, \varphi)$ and $h = (y, \psi)$ are state-like pairs such that for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$. Suppose (u, θ) is an ordered pair and $u \notin var(k)$, $u \notin var(h)$. Let $k' = k + (u, \theta)$ and $h' = h + (u, \theta)$. Let also $k' = (x', \varphi')$ and $h' = (y', \psi')$, then for each $i \in dom(k')$, $j \in dom(h')$ $x'_i = y'_j \to \varphi'_i = \psi'_j$.*

Proof. Let $i \in dom(k')$, $j \in dom(h')$ such that $x'_i = y'_j$.

Suppose $i \in dom(k)$. If $j \notin dom(h)$ then $x'_i = x_i \in var(k)$, $y'_j = u \notin var(k)$ so $x'_i \neq y'_j$. So $j \in dom(h)$ and $\varphi'_i = \varphi_i = \psi_j = \psi'_j$.

Suppose $i \notin dom(k)$. If $j \in dom(h)$ then $x'_i = u \notin var(h)$ and $y'_j = y_j \in var(h)$, so $x'_i \neq y'_j$. Then obviously also $j \notin dom(h)$ and $\varphi'_i = \theta = \psi'_j$. □

Lemma 3.5. *Suppose $k = (x, \varphi)$ and $h = (y, \vartheta)$ are state-like pairs such that for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \vartheta_j$. Suppose $\kappa = (z, \phi) \sqsubseteq k$ and $g = (w, \theta) \sqsubseteq h$. Then for each $i \in dom(\kappa)$, $j \in dom(g)$ $z_i = w_j \to \phi_i = \theta_j$.*

Proof. There exists $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $\kappa = k_{/C} = (x_{/C}, \varphi_{/C})$. Therefore $dom(\kappa) = C \subseteq dom(k)$.

Similarly there exists $D \in \mathcal{D}$ such that $D \subseteq dom(h)$, $g = h_{/D} = (y_{/D}, \vartheta_{/D})$. Therefore $dom(g) = D \subseteq dom(h)$.

Let $i \in dom(\kappa)$, $j \in dom(g)$, $z_i = w_j$, then $i \in dom(k)$, $j \in dom(h)$,

$$x_i = (x_{/C})_i = z_i = w_j = (y_{/D})_j = y_j \ .$$

Then

$$\phi_i = (\varphi_{/C})_i = \varphi_i = \vartheta_j = (\vartheta_{/D})_j = \theta_j \ .$$

□

Lemma 3.6. *Suppose $h = (x, \varphi)$ is a state-like pair, (y, ϕ) is an ordered pair and define $k = h + (y, \phi)$. Suppose $g \in \mathcal{R}(k)$ is such that $g \neq k$. Then $g \in \mathcal{R}(h)$.*

Proof. Let $D = dom(h)$.

Suppose m is a positive integer and $D = \{1, \ldots, m\}$. Then $k = (x', \varphi')$ has a domain $\{1, \ldots, m{+}1\}$. Moreover there exists $C \in \mathcal{D}$ such that $C \subseteq \{1, \ldots, m{+}1\}$ and $g = k_{/C}$. Since $g \neq k$ we must have $C \subseteq \{1, \ldots, m\}$. We have

$$g = k_{/C} = (x'_{/C}, \varphi'_{/C}) = ((x'_{/D})_{/C}, (\varphi'_{/D})_{/C}) = (x_{/C}, \varphi_{/C}) = h_{/C} \ .$$

Now suppose $D = \emptyset$. Then $k = (x', \varphi')$ has a domain $\{1\}$. Moreover there exists $C \in \mathcal{D}$ such that $C \subseteq \{1\}$ and $g = k_{/C}$. Since $g \neq k$ we must have $C = \emptyset$ and $g = (\emptyset, \emptyset) = h$.

In both cases $g \in \mathcal{R}(h)$. □

Lemma 3.7. *Let x be a function such that $dom(x) \in \mathcal{D}$, let $C, D \in \mathcal{D}$ such that $C \subseteq D \subseteq dom(x)$. Then we can define $x_{/C}$ and $(x_{/D})_{/C}$, and we have $(x_{/D})_{/C} = x_{/C}$.*

Proof. Define $y = x_{/D}$, we have $dom(y) = D$ and for each $j \in D$ $y(j) = x(j)$. Moreover $dom(y_{/C}) = C = dom(x_{/C})$ and for each $j \in dom(C)$ $y_{/C}(j) = y(j) = x(j) = x_{/C}(j)$. □

Lemma 3.8. *Let $k = (x, \varphi)$ be a state-like pair, let $C, D \in \mathcal{D}$ such that $C \subseteq D \subseteq dom(k)$. Then we can define $k_{/C}$ and $(k_{/D})_{/C}$, and we have $(k_{/D})_{/C} = k_{/C}$.*

Proof.

$$(k_{/D})_{/C} = (x_{/D}, \varphi_{/D})_{/C} = ((x_{/D})_{/C}, (\varphi_{/D})_{/C}) = (x_{/C}, \varphi_{/C}) = k_{/C}.$$

□

Lemma 3.9. *Let g, h, k be state-like pairs, let $g \sqsubseteq h$, $h \sqsubseteq k$. Then $g \sqsubseteq k$.*

Proof. There exists $C \in \mathcal{D}$ such that $C \subseteq dom(h)$, $g = h_{/C}$.
There exists $D \in \mathcal{D}$ such that $D \subseteq dom(k)$, $h = k_{/D}$.

This implies that $C \subseteq dom(h) = D$, so $g = h_{/C} = (k_{/D})_{/C} = k_{/C}$.

Since $C \subseteq dom(k)$, $g \sqsubseteq k$. □

Lemma 3.10. *Let g, h and $k = (x, \varphi)$ be state-like pairs such that $g, h \in \mathcal{R}(k)$, $dom(g) \subseteq dom(h)$. Then $g \in \mathcal{R}(h)$.*

Proof. There exists $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $g = k_{/C}$. And there exists $D \in \mathcal{D}$ such that $D \subseteq dom(k)$, $h = k_{/D}$. It results $C = dom(g) \subseteq dom(h) = D$. Then, clearly

$$g = (x, \varphi)_{/C} = (x_{/C}, \varphi_{/C}) = ((x_{/D})_{/C}, (\varphi_{/D})_{/C}) = (x_{/D}, \varphi_{/D})_{/C} = h_{/C} \ .$$

□

Lemma 3.11. *Suppose $h = (x, \varphi)$ is a state-like pair, (y, ϕ) is an ordered pair and define $k = h + (y, \phi)$. Then $k_{/dom(h)} = h$.*

Proof. Let $D = dom(h)$ and $k = (x', \varphi')$. Then $k_{/dom(h)} = (x'_{/D}, \varphi'_{/D}) = (x, \varphi) = h$. □

We also need some notation referred to generic strings, this notation will be useful when applied to our expressions, which are non-empty strings. If t is a string we can indicate with $\ell(t)$ t's length, i.e. the number of characters in t. If $\ell(t) > 0$ then for each $\alpha \in \{1, \ldots, \ell(t)\}$ at position α within t there is a character, this symbol will be indicated with $t[\alpha]$. We call 'depth of α within t' (briefly $d(t, \alpha)$) the number which is obtained by subtracting the number of right round brackets ')' that occur in t before position α from the number of left round brackets '(' that occur in t before position α .
The following lemma will be useful later within proofs of unique readability. Its proof is so simple that we feel free to omit it.

Lemma 3.12. *Let ϑ, φ, η be strings with $\ell(\vartheta) > 0$, $\ell(\varphi) > 0$, and let $t = \vartheta\|\varphi\|\eta$; let also $\alpha \in \{1, \ldots, \ell(\varphi)\}$. The following result clearly holds:*

$$d(t, \ell(\vartheta) + \alpha) = d(t, \ell(\vartheta) + 1) + d(\varphi, \alpha).$$

□

Before we describe the process of constructing expressions for our language we must also prove some useful lemmas related to the predicates we have defined above.

Lemma 3.13. *Given $i = 1 \ldots p$ and a positive integer q, for each $x \in \mathcal{P}^q(D_i)$ we have*

- *$Set_q(x)$,*
- *for each $r > q$ $\neg Set_r(x)$*

Proof. We proceed by induction on q.

Let $q = 1$. We assume $x \in \mathcal{P}(D_i)$, then clearly $Set_1(x)$.

Given a positive integer r, we assume $Set_{r+1}(x)$ and try to derive a contradiction. Since $x \neq \emptyset$ we can take $z \in x$, we have $Set_r(z)$ and $z \in D_i$, this actually is a contradiction. So we have proved $\neg(Set_{r+1}(x))$.

Let now q be a positive integer and assume for each $x \in \mathcal{P}^q(D_i)$ we have

- $Set_q(x)$,
- for each $r > q$ $\neg Set_r(x)$.

We want to show that for each $x \in \mathcal{P}^{q+1}(D_i)$ we have

- $Set_{q+1}(x)$,
- for each $r > q + 1$ $\neg Set_r(x)$.

We have that $x \neq \emptyset$, for each $z \in x$ $z \in \mathcal{P}^q(D_i)$, so for each $z \in x$ $Set_q(z)$. Therefore $Set_{q+1}(x)$.

Given a positive integer $r > q+1$, we assume $Set_r(x)$ and try to derive a contradiction. Let $z \in x$, we have $z \in \mathcal{P}^q(D_i)$ and $Set_{r-1}(z)$. Since $r-1 > q$ $\neg Set_{r-1}(z)$ should hold, and we have derived a contradiction. □

Lemma 3.14. *Given $i, j = 1 \dots p$, q, r positive integers such that $q \neq r$ $\mathcal{P}^q(D_i) \cap \mathcal{P}^r(D_j) = \emptyset$.*

Proof. Let's suppose, absurdly, $x \in \mathcal{P}^q(D_i) \cap \mathcal{P}^r(D_j)$. Suppose $q < r$.

Using lemma 3.14 we have both $Set_r(x)$ and $\neg Set_r(x)$. Therefore we must have $\mathcal{P}^q(D_i) \cap \mathcal{P}^r(D_j) = \emptyset$.

In the case $q > r$ we can apply the same type of reasoning. □

Lemma 3.15. *Given $i, j = 1 \dots p$ such that $i \neq j$ and a positive integer q we have $\mathcal{P}^q(D_i) \cap \mathcal{P}^q(D_j) = \emptyset$.*

Proof. We proceed by induction on q.

Let $q = 1$. Assume $\mathcal{P}(D_i) \cap \mathcal{P}(D_j) \neq \emptyset$ and let $x \in \mathcal{P}(D_i) \cap \mathcal{P}(D_j)$.

We have $x \neq \emptyset$, $x \subseteq D_i$, $x \subseteq D_j$, so $x \subseteq D_i \cap D_j$, and $D_i \cap D_j \neq \emptyset$, against our assumptions.

In order to perform the inductive step, let q be a positive integer, we assume $\mathcal{P}^q(D_i) \cap \mathcal{P}^q(D_j) = \emptyset$ and we try to show $\mathcal{P}^{q+1}(D_i) \cap \mathcal{P}^{q+1}(D_j) = \emptyset$.

We assume, absurdly, $x \in \mathcal{P}^{q+1}(D_i) \cap \mathcal{P}^{q+1}(D_j)$. We have $x \neq \emptyset$, $x \subseteq \mathcal{P}^q(D_i)$, $x \subseteq \mathcal{P}^q(D_j)$. So $x \subseteq \mathcal{P}^q(D_i) \cap \mathcal{P}^q(D_j)$ and $\mathcal{P}^q(D_i) \cap \mathcal{P}^q(D_j) \neq \emptyset$ against our assumptions. □

Lemma 3.16. *Given $i = 1 \dots p$ and a positive integer q for each $x \in \mathcal{P}^q(D_i)$ and $r \leqslant q$ we have $Set_r(x)$ and $NotEmpty_r(x)$.*

Proof. We proceed by induction on q.

Let $q = 1$ and let $x \in \mathcal{P}(D_i)$. Clearly $Set_1(x)$ and $NotEmpty_1(x)$.

In order to perform the inductive step, let q be a positive integer, we assume for each $x \in \mathcal{P}^q(D_i)$ and $r \leqslant q$ we have $Set_r(x)$ and $NotEmpty_r(x)$.

Let now $x \in \mathcal{P}^{q+1}(D_i)$, we want to show that for each $r \leqslant q+1$ we have $Set_r(x)$ and $NotEmpty_r(x)$.

Cleary $Set_1(x)$ and $NotEmpty_1(x)$ both hold true.

Given $r \leqslant q+1$ such that $r > 1$ we want to prove $Set_r(x)$ and $NotEmpty_r(x)$.

In order to prove this we just need to prove that for each $u \in x$ $Set_{r-1}(u)$ and $NotEmpty_{r-1}(u)$.

Since $x \in \mathcal{P}^{q+1}(D_i)$ then $x \subseteq \mathcal{P}^q(D_i)$ and for each $u \in x$ $u \in \mathcal{P}^q(D_i)$. Since $r-1 \leqslant q$ we have indeed $Set_{r-1}(u)$ and $NotEmpty_{r-1}(u)$. □

Lemma 3.17. *Given $i = 1 \dots p$ and a positive integer q for each $x \in \mathcal{P}^q(D_i)$ and $r \leqslant q+1$ we have $\neg Event_r(x)$.*

Proof. We proceed by induction on q.

Let $q = 1$. Let $x \in \mathcal{P}(D_i)$, x is a set and I think we can assume $\neg Event_1(x)$. Moreover $x \subseteq D_i$ so for each $u \in x$ $u \in D_i$, for each $u \in x$ $\neg Event_1(x)$, so it is false that for each $u \in x$ $Event_1(x)$, and it follows that $\neg(Event_2(x))$.

For the inductive step, let q be a positive integer and we assume for each $x \in \mathcal{P}^q(D_i)$ and $r \leqslant q+1$ we have $\neg Event_r(x)$. Let $x \in \mathcal{P}^{q+1}(D_i)$, let $r \leqslant q+2$, we want to show that $\neg Event_r(x)$ holds.

If $r = 1$ since x is a set we can assume $\neg Event_1(x)$ holds.

If $r > 1$ we have $x \subseteq \mathcal{P}^q(D_i)$, so for each $u \in x$ $u \in \mathcal{P}^q(D_i)$, and then for each $u \in x$ $\neg Event_{r-1}(u)$. So it is false that for each $u \in x$ $Event_{r-1}(u)$, and $Event_r(x)$ is false. □

Lemma 3.18. *For each positive integer q and for each x if $Event_{q+1}(x)$ then $Set_q(x)$.*

Proof. For the initial step of the proof, if $Event_2(x)$ then $Set_1(x)$.

For the inductive step, let $q > 1$ and $Event_{q+1}(x)$, then x is a set and for each $u \in x$ $Event_q(u)$. It then follows that for each $u \in x$ $Set_{q-1}(u)$, and then $Set_q(x)$. □

Lemma 3.19. *Given $i = 1 \dots p$ and a positive integer q for each $x \in \mathcal{P}^q(D_i)$ and $r > q+1$ we have $\neg Event_r(x)$.*

Proof. Let $x \in \mathcal{P}^q(D_i)$ and let $r > q+1$. Since $r-1 > q$ by lemma 3.13 $\neg Set_{r-1}(x)$ and by lemma 3.18 $\neg Event_r(x)$. □

4. Computability theory

In our logic system we want to satisfy every requirement that must be desirable with respect to the Computabilty Theory. In order to be able to understand and satisfy such requirements, clearly we must have at least a basic knowledge of the Computability Theory.

We use Cutland's book [1] as the main reference for this. The book defines the concept of computable function: given a set A of natural numbers and a function $f : A \to \mathbb{N}$, we say that f is *computable* when it is URM-computable. We will not define here the concept of URM-computability, the reader can find the definition in the mentioned book.

As suggested by the book we use the symbol $\mathscr{C}$ to indicate the set of the computable functions from a subset of $\mathbb{N}$ to $\mathbb{N}$ (also called the 'partial functions' from $\mathbb{N}$ to $\mathbb{N}$).

The book also provides many alternative definitions of the notion of effective computability and affirms that 'the remarkable result of investigation by many researchers is the following: Each of the above proposals for a characterisation of the notion of effective computability gives rise to the same class of functions, the class that we have denoted with $\mathscr{C}$'.

Finally the book also states the famous 'Church's thesis' in the following terms: 'The intuitively and informally defined class of effectively computable partial functions coincides exactly with the class $\mathscr{C}$ of URM-computable functions'.

If A is a subset of $\mathbb{N}$ we can define the *characteristic function* of A as the function c_A given by: if $x \in A$ $c_A(x) = 1$; if $x \notin A$ $c_A(x) = 0$. Then A is said to be *recursive* if c_A is computable.

If A is a subset of $\mathbb{N}$ we can define the *semi-characteristic function* of A as the function s_A given by: if $x \in A$ $s_A(x) = 1$; if $x \notin A$ $s_A(x)$ is undefined. Then A is said to be *recursively enumerable (r.e.)* if s_A is computable.

A recursive set is obvioulsy also recursively enumerable.

Given a subset A of $\mathbb{N}$ the following statements are equivalent:

- A is r.e.;
- $A = \emptyset$ or A is the range of a total computable function;
- A is the range of a partial computable function.

Please refer to Cutland's book for the proof of the equivalence.

We now state and prove a theorem which is important for us, but is not present in Cutland's text.

Theorem 4.1. *Let A be a r.e. subset of $\mathbb{N}$, let f be a function defined on A such that for each $x \in A$ $f(x)$ is a r.e. subset of $\mathbb{N}$. Then $\bigcup_{x \in A} f(x)$ is r.e..*

Proof. There exists a partial computable function ξ such that $A = ran(\xi)$. For each $x \in A$ there also exists a partial function χ_x such that $f(x) = ran(\chi_x)$.

Let's consider the function $\pi : \mathbb{N}^2 \to \mathbb{N}$ (named the Cantor's pairing function) defined by

$$\pi(x,y) = (x+y)(x+y+1)/2 + y \ .$$

This function is a bijection and the inverse function $\zeta : \mathbb{N} \to \mathbb{N}^2$ is a computable function itself (cfr. Wikipedia 'https://en.wikipedia.org/wiki/Pairing_function').

Let's now consider a function ϕ defined over $\mathbb{N}$ such that $\phi(z)$ is calculated as follows: we first calculate $\zeta(z) = (z_1, z_2)$, then we calculate $\xi(z_1)$, if it terminates $\xi(z_1) \in A$ and we can set $\phi(z) = \chi_{\xi(z_1)}(z_2)$.

The function ϕ is a partial computable function and we can show that $\bigcup_{x\in A} f(x) = ran(\phi)$.

Given $y \in \bigcup_{x\in A} f(x)$ we will prove that $y \in ran(\phi)$. In fact there exists $x \in A$ such that $y \in f(x)$. There exists $z_1 \in \mathbb{N}$ such that $x = \xi(z_1)$, and there exists $z_2 \in \mathbb{N}$ such that $y = \chi_x(z_2) = \chi_{\xi(z_1)}(z_2)$. There exists $z \in \mathbb{N}$ such that $\zeta(z) = (z_1, z_2)$ and therefore $y = \phi(z)$.

Vice versa given $y \in ran(\phi)$ we want to show that $y \in \bigcup_{x\in A} f(x)$. There exists $z \in \mathbb{N}$ such that $y = \phi(z)$. If we set $(z_1, z_2) = \zeta(z)$ then $y = \phi(z) = \chi_{\xi(z_1)}(z_2)$. We have that $\xi(z_1) \in A$ and $y = \chi_{\xi(z_1)}(z_2) \in f(\xi(z_1))$.

□

Our reference book also explains how to apply the definition of computability and the related ones to a domain D which is different from $\mathbb{N}$. This requires the availability of a coding.

A *coding* of a domain D of objects is and explicit and effective injection $\alpha : D \to \mathbb{N}$.

We can actually assume that the range of α is $\mathbb{N}$ (and in this case α is a bijection) or at least that $ran(\alpha)$ is recursive.

A partial function $f : D \to D$ is coded by the function $f^* = \alpha \circ f \circ \alpha^{-1}$, so f^* is a partial function $\mathbb{N} \to \mathbb{N}$. We say that f is *computable* if and only if f^* is computable.

Given a set $A \subseteq D$ we can define $A^* = \{\alpha(d) | d \in A\}$. We say that A is *recursive* iff A^* is recursive, and that A is *recursively enumerable* iff A^* is recursively enumerable.

Given $A \subseteq D$ we can define the *characteristic function* of A as the function c_A whose domain is D given by: if $x \in A$ $c_A(x) = 1$; if $x \notin A$ $c_A(x) = 0$. We can also define the *semi-characteristic function* of A as the function s_A whose domain is A, such that for each $x \in A$ $s_A(x) = 1$.

In relation to the former definitions, we can prove the following lemma.

Lemma 4.2. *Let $A \subseteq D$, then*

- *A is recursive if and only if c_A is computable;*

- *A is r.e. if and only if s_A is computable.*

Proof. First of all we notice that given $n \in \mathbb{N}$

- if $n \in A^*$ then $\alpha^{-1}(n) \in A$;
- if $\alpha^{-1}(n) \in A$ then $n \in A^*$.

We also notice that given $x \in D$, if $x \notin A$ then $\alpha(x) \notin A^*$. In fact if $\alpha(x) \in A^*$ then there exists $y \in A$ such that $\alpha(x) = \alpha(y)$, but since $x \neq y$ and α is injective we cannot have $\alpha(x) = \alpha(y)$.

Let's assume A is recursive, we want to show that c_A is computable.

We know that c_{A^*} is computable. Given $x \in D$

- if $x \in A$ then $\alpha(x) \in A^*$ $c_A(x) = 1 = c_{A^*}(\alpha(x))$;
- if $x \notin A$ then $\alpha(x) \notin A^*$ $c_A(x) = 0 = c_{A^*}(\alpha(x))$.

Therefore in every case $c_A(x) = c_{A^*}(\alpha(x))$, and then c_A is computable.

Vice versa we now assume c_A is computable and we want to show that A is recursive.

Given $n \in \mathbb{N}$,

- if $n \notin ran(\alpha)$ we have $c_{ran(\alpha)}(n) = 0$, so $c_{A^*}(n) = 0 = c_{ran(\alpha}(n)$.
- if $n \in ran(\alpha)$ we have $c_{ran(\alpha)}(n) = 1$ and
  - if $n \in A^*$ then $\alpha^{-1}(n) \in A$, $c_{A^*}(n) = 1 = c_A(\alpha^{-1}(n))$;
  - if $n \notin A^*$ then $\alpha^{-1}(n) \notin A$, $c_{A^*}(n) = 0 = c_A(\alpha^{-1}(n))$.

Clearly we can compute $c_{A^*}(n)$ as follows:

If $c_{ran(\alpha)}(n) = 0$ then $c_{A^*}(n) = 0$;

if $c_{ran(\alpha)}(n) = 1$ then $c_{A^*}(n) = c_A(\alpha^{-1}(n))$.

Let's assume A is r.e., we wanto to show that s_A is computable.

Given $x \in D$,

- if $x \in A$ then $\alpha(x) \in A^*$, $x \in dom(s_A)$, $\alpha(x) \in dom(s_{A^*})$ $s_A(x) = 1 = s_{A^*}(\alpha(x))$;
- if if $x \notin A$ then $\alpha(x) \notin A^*$ $x \notin dom(s_A)$, $\alpha(x) \notin dom(s_{A^*})$, $s_A(x)$ and $s_{A^*}(\alpha(x))$ are both divergent.

Therefore $s_A(x)$ can be calculated by $s_{A^*}(\alpha(x))$, and s_A is computable.

Vice versa we now assume s_A is computable and we want to show that A is r.e..

Given $n \in \mathbb{N}$,

- if $n \notin ran(\alpha)$ we have $n \notin A^* = dom(s_{A^*})$, therefore $s_{A^*}(n)$ is divergent, and $s_A(\alpha^{-1}(n))$ is divergent too.
- if $n \in ran(\alpha)$ we have $c_{ran(\alpha)}(n) = 1$ and

- ◦ if $n \in A^*$ then $\alpha^{-1}(n) \in A$, $s_{A^*}(n) = 1 = s_A(\alpha^{-1}(n))$;
- ◦ if $n \notin A^*$ then $\alpha^{-1}(n) \notin A$, $s_{A^*}(n)$ and $s_A(\alpha^{-1}(n))$ are both divergent.

Therefore in all cases $s_{A^*}(n)$ can be calculated as $s_A(\alpha^{-1}(n))$, and so s_{A^*} is computable and A is r.e..

□

In the theorem 4.1 above we proved that a r.e. union of r.e. sets is still a r.e. set. This theorem was stated for subsets of ℕ, and we will generalize it to generic domains.

Theorem 4.3. *Let D_1 and D_2 be two 'domains' to which we can apply the notions of computability using two codings $\alpha_1 : D_1 \to \mathbb{N}$ and $\alpha_2 : D_2 \to \mathbb{N}$. Let A be a r.e. subset of D_1, let f be a function defined on A such that for each $x \in A$ $f(x)$ is a r.e. subset of D_2. Then $\bigcup_{x \in A} f(x)$ is a r.e. subset of D_2.*

Proof. We call W the set $\bigcup_{x\in A} f(x) \subseteq D_2$. We'll prove that W is r.e. by proving that $W^* = \{\alpha_2(y) | y \in W\}$ is r.e.. Let's assume that actually $W^* = \bigcup_{z \in A^*} f(\alpha_1^{-1}(z))^*$.

If the equality we have assumed holds, then we can consider that A^* is a r.e. subset of ℕ, and that for each $z \in A^*$ $\alpha_1^{-1}(z) \in A$, $f(\alpha_1^{-1}(z))$ is a r.e. subset of D_2, $f(\alpha_1^{-1}(z))^*$ is a r.e. subset of ℕ. Therefore, if the equality holds, we have proved that W^* is r.e. and our proof is finished.

Let's then show that $W^* = \bigcup_{z \in A^*} f(\alpha_1^{-1}(z))^*$ actually holds.

Let $w \in W^*$ then there exists $y \in W$: $w = \alpha_2(y)$, and there exists $x \in A$: $y \in f(x)$. Let $z = \alpha_1(x) \in A^*$, then $y \in f(\alpha_1^{-1}(z))$ and $w = \alpha_2(y) \in f(\alpha_1^{-1}(z))^*$. So we can confirm that $w \in \bigcup_{z \in A^*} f(\alpha_1^{-1}(z))^*$.

Conversely let $w \in \bigcup_{z\in A^*} f(\alpha_1^{-1}(z))^*$ and we want to prove that $w \in W^*$. There exists $z \in A^*$ such that $w \in f(\alpha_1^{-1}(z))^*$. Let $x = \alpha_1^{-1}(z) \in A$ then $w \in f(x)^*$. Let $y = \alpha_2^{-1}(w) \in f(x)$. We have $y \in W$ and so $w = \alpha_2(y) \in W^*$.

□

Typically we will be dealing with a finite or countable alphabet Σ, and the domain to which we will have to apply the concepts of computability will be the set Σ^* of all the empty or finite strings with characters in the mentioned alphabet. But we may also need to apply those concepts e.g. to $(\Sigma^*)^n$. So let us examine some sets to which we can actually apply computability notions, in order to be able to apply such concepts wherever we need them.

First of all we consider the set $\mathbb{N}^2$. There is a coding $\pi : \mathbb{N}^2 \to \mathbb{N}$ and this coding is the Cantor pairing function defined by

$$\pi(k_1, k_2) = (k_1 + k_2)(k_1 + k_2 + 1)/2 + k_2 \ .$$

So, obviously, we can apply computability notions to $\mathbb{N}^2$, and actually we are able to apply them also to $\mathbb{N}^n$ for an arbitrary integer $n > 2$. In fact if we assume π_n is a coding $\mathbb{N}^n \to \mathbb{N}$, with $n \geqslant 2$, then we can define a function $\pi_{n+1} : \mathbb{N}^{n+1} \to \mathbb{N}$ as follows:

$$\pi_{n+1}(x_1, \ldots, x_n, x_{n+1}) = \pi(\pi_n(x_1, \ldots, x_n), x_{n+1}) \ .$$

And this function is actually a coding $\mathbb{N}^{n+1} \to \mathbb{N}$.

At this point given n domains $D_1, \ldots, D_n$ such that for each $i = 1 \ldots n$ there exists a coding $\alpha_i : D_i \to \mathbb{N}$ we can build a coding $\alpha : D_1 \times \cdots \times D_n \to \mathbb{N}$. Our coding will be defined as follows:

$$\alpha(d_1, \ldots, d_n) = \pi_n(\alpha_1(d_1), \ldots, \alpha_n(d_n)) \ .$$

We said earlier that typically we will be dealing with a finite or countable alphabet Σ, and the domain to which we will have to apply the concepts of computability will be the set Σ^*. With respect to this, we notice that $\mathbb{N}$ can be itself considered as an alphabet, so we first try to find a coding $\mathbb{N}^* \to \mathbb{N}$.

Here we notice that $\mathbb{N}^* = \{\epsilon\} \cup \bigcup_{i \geqslant 1} \mathbb{N}^i$.

We have seen that for each $i \geqslant 2$ π_i is a coding $\mathbb{N}^i \to \mathbb{N}$, so we can create a coding $\gamma : \bigcup_{i \geqslant 1} \mathbb{N}^i \to \mathbb{N}^2$ as follows:

- for each $x \in \mathbb{N}$ $\gamma(x) = (0, x)$;
- for each $i > 1$, $(x_1, \ldots, x_i) \in \mathbb{N}^i$ $\gamma(x_1, \ldots, x_i) = (i - 1, \pi_i(x_1, \ldots, x_i))$.

We now want to create a coding $\alpha : \mathbb{N}^* \to \mathbb{N}$. We define our coding α as follows:

- $\alpha(\epsilon) = 0$;
- for each $x \in \bigcup_{i \geqslant 1} \mathbb{N}^i$ $\alpha(x) = \pi_2(\gamma(x)) + 1$.

Given a finite or countable alphabet Σ we now want to define a coding $\Sigma^* \to \mathbb{N}$. Of course there exists a coding $\sigma : \Sigma \to \mathbb{N}$. We first want to define a coding $\delta : \Sigma^* \to \mathbb{N}^*$, and, since $\Sigma^* = \{\epsilon\} \cup \bigcup_{i \geqslant 1} \Sigma^i$, we can define it as follows.

- $\delta(\epsilon) = \epsilon$;
- for each $i \geqslant 1$ $(x_1, \ldots, x_i) \in \Sigma^i$ $\delta(x_1, \ldots, x_i) = (\sigma(x_1), \ldots, \sigma(x_i)) \in \mathbb{N}^i$.

At this point if α is a coding $\mathbb{N}^* \to \mathbb{N}$ then $\gamma = \alpha \circ \delta$ is a coding $\Sigma^* \to \mathbb{N}$.

We can notice that if Σ is finite then σ is not surjective, and so also δ and γ are not surjective. Anyway $ran(\gamma)$ is still recursive since given $x \in \mathbb{N}$ we can decide whether $x \in ran(\gamma)$. In order to do this we can calculate $\alpha^{-1}(x) \in N^*$, and here we can determine if $\alpha^{-1}(x) \in ran(\delta)$, if this is true since $x = \alpha(\alpha^{-1}(x))$ then $x \in ran(\gamma)$. If on the contrary $\alpha^{-1}(x) \notin ran(\delta)$ then $x \notin ran(\gamma)$. In fact if $x \in ran(\gamma)$ then exists $y \in \Sigma^*$ such that $x = \gamma(y) = \alpha(\delta(y))$, so $\alpha(\alpha^{-1}(x)) = \alpha(\delta(y))$, and $\alpha^{-1}(x) = \delta(y) \in ran(\delta)$.

Once we have a coding for Σ^* we have it also for $(\Sigma^*)^n$, where n is a positive integer, and if Γ is another alphabet we have a coding for $(\Sigma^*)^n \times (\Gamma^*)^m$, where m is another positive integer.

Given n domains $D_1, \ldots, D_n$ and given $A_1 \subseteq D_1, \ldots, A_n \subseteq D_n$ if $A_1, \ldots, A_n$ are r.e. then $A_1 \times \cdots \times A_n$ is also r.e.. In fact given $(x_1, \ldots, x_n) \in D_1 \times \cdots \times D_n$ we can compute $s_{A_1 \times \cdots \times A_n}(x_1, \ldots, x_n)$ as follows: for each $i = 1 \ldots n$ we calculate $s_{A_i}(x_i)$ and if we obtain a result for each i then we emit the result 1.

Given a domain D and a coding $\alpha : D \to \mathbb{N}$ and given a function $f : \mathbb{N} \to D$ we can say that f is computable when $\alpha \circ f : \mathbb{N} \to \mathbb{N}$ is computable.

We can prove the following lemma:

Lemma 4.4. *Given a set $A \subseteq D$ A is r.e. if and only if $A = \emptyset$ or there exists a total computable function $f : N \to D$ such that $A = ran(f)$.*

Proof. Let $A^* = \{\alpha(d) | d \in A\}$.

If A is r.e. then A^* is r.e. and so $A^* = \emptyset$ or there exists a total computable function $g : \mathbb{N} \to \mathbb{N}$ such that $A^* = ran(g)$. If $A^* = \emptyset$ then clearly $A = \emptyset$, otherwise since $A^* \subseteq ran(\alpha)$ we can define $f = \alpha^{-1} \circ g$, f is a function $\mathbb{N} \to D$ and $ran(f) = A$.

In fact if $d \in ran(f)$ then there exists $x \in \mathbb{N}$: $d = \alpha^{-1}(g(x))$. We know that $g(x) \in A^*$ and so $d = \alpha^{-1}(g(x)) \in A$. Conversely if $d \in A$ then $\alpha(d) \in A^*$ and there exists $x \in \mathbb{N}$: $g(x) = \alpha(d)$, $d = \alpha^{-1}(g(x)) = f(x) \in ran(f)$.

Conversely if $A = \emptyset$ then $A^* = \emptyset$, A^* is r.e. and A is r.e.. If there exists a total computable function $f : \mathbb{N} \to D$ such that $A = ran(f)$ then $g = \alpha \circ f : \mathbb{N} \to \mathbb{N}$ is computable and $A^* = ran(g)$.

In fact if $y \in A^*$ then there exists $d \in A$: $y = \alpha(d)$ and there exists $x \in \mathbb{N}$: $d = f(x)$, so $y = \alpha(f(x)) \in ran(g)$. Conversely if $y \in ran(g)$ then there exists $x \in \mathbb{N}$: $y = \alpha(f(x))$, and since $f(x) \in A$ then $y \in A^*$.

So in the latest case too A^* is r.e. and A is r.e.. □

Let D_1 and D_2 be two 'domains' to which we can apply the notions of computability using two codings $\alpha_1 : D_1 \to \mathbb{N}$ and $\alpha_2 : D_2 \to \mathbb{N}$. Using codings, we can define the notion of 'computable function' also for a function $f : D_1 \to D_2$. We say that f is computable if and only if $\alpha_2 \circ f \circ \alpha_1^{-1} : \mathbb{N} \to \mathbb{N}$ is computable. We can notice that since the domain of α_1^{-1} could be a proper subset of $\mathbb{N}$ the mentioned function could actually be a partial function $\mathbb{N} \to \mathbb{N}$.

Using the just introduced notion, we can prove the following lemma:

Lemma 4.5. *Let A be a r.e. subset of D_1 and let $f : D_1 \to D_2$ be a computable function. If we define $B = \{f(a) | a \in A\}$, then $B \subseteq D_2$ is r.e..*

Proof. If $A = \emptyset$ then $B = \emptyset$ is r.e..

Otherwise there exists a computable function $g : \mathbb{N} \to D_1$ such that $A = ran(g)$. We have $f \circ g : \mathbb{N} \to D_2$ and $B = ran(f \circ g)$.

In fact if $d_2 \in B$ then there exists $a \in A$ such that $d_2 = f(a)$ and there exists $x \in \mathbb{N}$

such that $a = g(x)$, so $d_2 = f(g(x))$. Conversely if $d_2 \in ran(f \circ g)$ then there exists $x \in \mathbb{N}$ such that $d_2 = f(g(x))$, so $g(x) \in A$ and $d_2 = f(g(x)) \in B$. $\square$

5. Premise: description of contexts

We want to be able to show that a certain set of contexts is recursive or recursively enumerable. To this end we are going to define contexts as strings. Given a finite or countable alphabet Σ which contains a finite or countable set of variables $\mathcal{V}$, the symbols ':' and ',' and doesn't contain the symbols '<' and '>' we can define an alphabet $\Gamma = \Sigma \cup \{<,>\}$.

Let ϵ be the string '<>' and let $\Theta(\Sigma, \mathcal{V})$ (henceforth Θ) be the set

$$\{\epsilon\} \cup \{<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >> \mid x_1, \ldots, x_m \in \mathcal{V},\ \varphi_1 \ldots \varphi_m \in \Sigma^*\}\,.$$

Lemma 5.1. *Let $k \in \Theta - \{\epsilon\}$, let m positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $\varphi_1, \ldots, \varphi_m \in \Sigma^*$ such that $k =<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >>$. Let also p positive integer, $y_1, \ldots, y_p \in \mathcal{V}$, $\psi_1, \ldots, \psi_p \in \Sigma^*$ such that $k =<< y_1 : \psi_1 > \cdots < y_p : \psi_p >>$. Then $p = m$, for each $i = 1 \ldots m$ $y_i = x_i$ and $\psi_i = \varphi_i$.*

Proof. First of all, as in former parts of the paper, if t is a string we will indicate with $\ell(t)$ t's length, i.e. the number of characters in t.

Also, given $\alpha = 1 \ldots \ell(t)$ let $t[\alpha]$ indicate the character with position α inside t, and given $\alpha, \beta = 1 \ldots \ell(t)$ with $\alpha \leqslant \beta$ let $t[\alpha, \beta]$ be the substring of t which begins at character α and ends at character β.

For each $i = 1 \ldots m$ let $\varphi_i = k[\alpha_i, \mu_i]$. For each $j = 1 \ldots p$ let $\psi_j = k[\beta_j, \nu_j]$.

Clearly $y_1 = k[3] = x_1$.

If $\nu_1 > \mu_1$ then $\nu_1 \geqslant \mu_1 + 1 > \alpha_1 = 5 = \beta_1$ so '>' $= k[\mu_1 + 1]$ is a character in ψ_1: this cannot be true, so $\nu_1 > \mu_1$ is false and similarly $\mu_1 > \nu_1$ is false, so $\nu_1 = \mu_1$ and it follows that $\psi_1 = \varphi_1$.

If $m = 1$ then $k[\mu_1 + 2] =$ '>'. In this case if $p > 1$ then $k[\nu_1 + 2] =$ '<'. Therefore it must be $p = 1$ and our proof for $m = 1$ is finished.

Let's consider the case $m > 1$. In this case given $i = 1 \ldots m - 1$ we assume we have proved that for each $j = 1 \ldots i$ $p \geqslant j$ $y_j = x_j$, $\psi_j = \varphi_j$ and that $\nu_i = \mu_i$. We want to show that $p \geqslant i + 1$, $y_{i+1} = x_{i+1}$, $\psi_{i+1} = \varphi_{i+1}$, $\nu_{i+1} = \mu_{i+1}$.

Since $i < m$ $k[\mu_i + 2] =$ '<', therefore also $k[\nu_i + 2] =$ '<' and this implies that $p \geqslant i + 1$.

We also notice that $y_{i+1} = k[\nu_i + 3] = k[\mu_i + 3] = x_{i+1}$.

We also notice that $\beta_{i+1} = \nu_i + 5 = \mu_i + 5 = \alpha_{i+1}$.

If $\nu_{i+1} > \mu_{i+1}$ then $\nu_{i+1} \geqslant \mu_{i+1}+1 > \alpha_{i+1} = \beta_{i+1}$ so '>' $= k[\mu_{i+1}+1]$ is a character in ψ_{i+1}: this cannot be true, so $\nu_{i+1} > \mu_{i+1}$ is false and similarly $\mu_{i+1} > \nu_{i+1}$ is false, so $\nu_{i+1} = \mu_{i+1}$ and it follows that $\psi_{i+1} = \varphi_{i+1}$.

We have now proved that for each $i = 1 \ldots m$ $p \geqslant i$ $y_i = x_i$, $\psi_i = \varphi_i$ and $\nu_i = \mu_i$.

We have also $\ell(k) = \mu_m + 2$. If $p > m$ then $\ell(k) > \nu_m + 2 = \mu_m + 2$, and this is a contradiction, therefore $p = m$.

□

Given $k \in \Theta$ we define $dom(k)$ (i.e. the domain of k) as follows.

- if $k = \epsilon$ then $dom(k) = \emptyset$,
- if $k \neq \epsilon$ then $k =<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >>$ and we define $dom(k) = \{1, \ldots, m\}$.

We define $\mathcal{D} = \{\emptyset\} \cup \{\{1, \ldots, m\} |\ m$ is a positive integer$\}$.

Of course, given $k \in \Theta$, $dom(k) \in \mathcal{D}$.

Given $k \in \Theta$ and $C \in \mathcal{D}$ such that $C \subseteq dom(k)$ we can define $k_{/C}$, i.e. the 'restriction' of k to the domain C, as follows:

- if $k = \epsilon$ or $C = \emptyset$ then $k_{/C} = \epsilon \in \Theta$ (so $dom(k_{/C}) = \emptyset = C$),
- if $k \neq \epsilon$ and $C \neq \emptyset$ then $k =<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >>$ and $C = \{1, \ldots, p\}$ where $1 \leqslant p \leqslant m$, we define $k_{/C} =<< x_1 : \varphi_1 > \cdots < x_p : \varphi_p >> \in \Theta$ (so $dom(k_{/C}) = \{1, \ldots, p\} = C$).

We also define $\mathcal{R}(k) = \{k_{/C} |\ C \in \mathcal{D}, C \subseteq dom(k)\}$.

Given another $h \in \Theta$ we write $h \sqsubseteq k$ if and only if $h \in \mathcal{R}(k)$.

Suppose $h \in \mathcal{R}(k)$, then there exists $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $h = k_{/C}$. As we have seen in this case $dom(h) = C$ and $k_{/dom(h)} = k_{/C} = h$.

Given $k \in \Theta$ we define $var(k)$ as follows.

- if $k = \epsilon$ then $var(k) = \emptyset$,
- if $k \neq \epsilon$ then $k =<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >>$ and we define $var(k) = \{x_1, \ldots, x_m\}$.

We are now going to define the 'addition' of a new element to a context string.

Definition 5.2. Let $h \in \Theta$, $x \in \mathcal{V}$, $\varphi \in \Sigma^*$, we define $h+ < x, \varphi >$ as follows:

- if $h = \epsilon$ then $h+ < x, \varphi > =<< x, \varphi >> \in \Theta$;
- if $h \neq \epsilon$ then let m positive integer $x_1, \ldots, x_m \in \mathcal{V}$, $\varphi_1, \ldots, \varphi_m \in \Sigma^*$ such that $h =<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >>$, we define $h+ < x, \varphi > =<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >< x, \varphi >> \in \Theta$.

Lemma 5.3. *Let $h \in \Theta$, $x \in \mathcal{V}$, $\varphi \in \Sigma^*$, let $k = h+ < x, \varphi >$. Then the following hold true:*

- $dom(h) \subseteq dom(k)$,
- $h = k_{/dom(h)}$,
- $h \in \mathcal{R}(k)$,
- $var(k) = var(h) \cup \{x\}$.

Proof. It is obvious by the definition of k that $dom(h) \subseteq dom(k)$.

If $h = \epsilon$ then $dom(h) = \emptyset$ and $k_{/dom(h)} = \epsilon = h$.

if $h \neq \epsilon$ then let $h =<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >>$, this implies $k =<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >< x, \varphi >>$ and clearly $k_{/dom(h)} = h$.

It also follows that $h \in \mathcal{R}(k)$.

If $h = \epsilon$ then $var(k) = \{x\} = var(h) \cup \{x\}$.

If $h \neq \epsilon$ and $h =<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >>$ then

$$var(k) = \{x_1, \ldots, x_m\} \cup \{x\} = var(h) \cup \{x\} \ .$$

□

Lemma 5.4. *Given $k \in \Theta - \{\epsilon\}$ there exist $h \in \Theta$, $x \in \mathcal{V}$, $\varphi \in \Sigma^*$ such that $k = h+ < x, \varphi >$. Moreover h, x and φ are univocally determined.*

Proof. Let $k \in \Theta - \{\epsilon\}$, there exist a positive integer m, $x_1, \ldots, x_m \in \mathcal{V}$, $\varphi_1, \ldots, \varphi_m \in \Sigma^*$ such that $k =<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >>$.

If $m = 1$ then $k =<< x_1 : \varphi_1 >>= \epsilon+ < x_1, \varphi_1 >$.

If $m > 1$ then $k =<< x_1 : \varphi_1 > \cdots < x_{m-1} : \varphi_{m-1} >< x_m : \varphi_m >>$, and so $k =<< x_1 : \varphi_1 > \cdots < x_{m-1} : \varphi_{m-1} >> + < x_m : \varphi_m >$.

We have seen there exist $h \in \Theta$, $x \in \mathcal{V}$, $\varphi \in \Sigma^*$ such that $k = h+ < x, \varphi >$. Suppose there also exist $g \in \Theta$, $y \in \mathcal{V}$, $\psi \in \Sigma^*$ such that $k = g+ < y, \psi >$.

Suppose $h = \epsilon$ and $g \neq \epsilon$, then there exist a positive integer m and $y_1, \ldots, y_m \in \mathcal{V}$, $\psi_1, \ldots, \psi_m \in \Sigma^*$ such that $g =<< y_1 : \psi_1 > \cdots < y_m : \psi_m >>$. It follows that $k =<< x, \varphi >>$ and $k =<< y_1 : \psi_1 > \cdots < y_m : \psi_m >< y, \psi >>$, hence $1 = m + 1$, which is false. Therefore $h = \epsilon$ and $g \neq \epsilon$ is false and similarly $h \neq \epsilon$ and $g = \epsilon$ is false.

Let's consider the case where $h = g = \epsilon$. In this case $<< x, \varphi >>= k =<< y, \psi >>$, so $y = x$ and $\psi = \varphi$.

Finally we consider the case where both $h \neq \epsilon$ and $g \neq \epsilon$. There exist m positive integer $x_1, \ldots, x_m \in \mathcal{V}$, $\varphi_1, \ldots, \varphi_m \in \Sigma^*$ such that $h =<< x_1 : \varphi_1 > \cdots < x_m :$

$\varphi_m >>$. There also exist p positive integer, $y_1, \ldots, y_p \in \mathcal{V}$, $\psi_1, \ldots, \psi_p \in \Sigma^*$ such that $g =<< y_1 : \psi_1 > \cdots < y_p : \psi_p >>$. It follows that

- $k =<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >< x, \varphi >>$
- $k =<< y_1 : \psi_1 > \cdots < y_p : \psi_p >< y, \psi >>$

Hence $p + 1 = m + 1$, $p = m$, for each $i = 1 \ldots m$ $y_i = x_i$, $y = x$, $\psi = \varphi$. Finally $g = h$ also holds.

□

Lemma 5.5. *Let $h \in \Theta$, $x \in \mathcal{V}$, $\varphi \in \Sigma^*$, $k = h + < y, \phi >$. Suppose $g \in \mathcal{R}(k)$ is such that $g \neq k$. Then $g \in \mathcal{R}(h)$.*

Proof. Let $D = dom(h)$.

We first consider the case where $h \neq \epsilon$. In this case there exists a positive integer m such that $D = \{1, \ldots, m\}$, and clearly $dom(k) = \{1, \ldots, m+1\}$. Since $g \in \mathcal{R}(k)$ there exists $C \in \mathcal{D}$ such that $C \subseteq \{1, \ldots, m+1\}$ and $g = k_{/C}$. Since $g \neq k$ we must have $C \subseteq \{1, \ldots, m\}$. We have

$$g = k_{/C} = (k_{/D})_{/C} = h_{/C}\,.$$

Let's now consider the case where $h = \epsilon$. In this case $D = \emptyset$ and $dom(k) = \{1\}$. Moreover there exists $C \in \mathcal{D}$ such that $C \subseteq \{1\}$ and $g = k_{/C}$. Since $g \neq k$ we must have $C = \emptyset$ and $g = \epsilon = h$.

In both cases $g \in \mathcal{R}(h)$, of course. □

Lemma 5.6. *Let $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >> \in \Theta - \{\epsilon\}$, let $h \in \mathcal{R}(k)$. If $h \neq \epsilon$ then there exists $p = 1 \ldots m$ such that $h =<< x_1, \varphi_1 > \cdots < x_p, \varphi_p >>$.*

Proof. If $h \in \mathcal{R}(k)$ then there exists $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $h = k_{/C}$. If $C = \emptyset$ then $h = \epsilon$, so $C \neq \emptyset$, since $dom(k) = \{1, \ldots, m\}$ there exists $p = 1 \ldots m$ such that $C = \{1, \ldots, p\}$ and $h =<< x_1, \varphi_1 > \cdots < x_p, \varphi_p >>$. □

6. Building the expressions of our system

We can now describe the process of constructing expressions for our language $\mathcal{L}$. This is an inductive process in which not only we build expressions, but also we associate them with meaning, and in parallel also define the fundamental concept of 'context'. This process will be identified as 'Definition 6.1' although actually it is a process in which we give the definitions and prove properties which are needed in order to set up those definitions.

Within this definition we will define the expressions of our language. Such expressions are finite sequences of characters of the alphabet $\Sigma = \mathcal{V} \cup \mathcal{C} \cup \mathcal{F} \cup \mathcal{Z}$. In other words they are members of Σ^*.

Since this is a complex definition, we will first try to provide an informal idea of the entities we'll define in it. The definition is by induction on positive integers, we now introduce the sets and concepts we'll define for a generic positive integer n (this first listing is not the true definition, it's just to introduce the concepts, to enable the reader to understand their role).

$K(n)$ is the set of 'contexts' at step n. If we define $\Gamma = \Sigma \cup \{<, >\}$, contexts will be defined as members of Γ^*, and they will be strings of the form $<< x_1, \varphi_1 >, \dots, < x_m, \varphi_m >>$, where $x_1, \dots, x_m \in \mathcal{V}$ and $\varphi_1, \dots, \varphi_m$ are expressions. The string $<>$ which we'll also name ϵ is also a possibile context, and when we use the symbol ϵ with respect to a context we actually mean $<>$.

For each $k \in K(n)$ $\Xi(k)$ is the set of 'states' bound to context k. If $n > 1$ and $k \in K(n-1)$ then $\Xi(k)$ has already been defined at step $n-1$ or formerly, otherwise it will be defined at step n.

If $k = << x_1, \varphi_1 >, \dots, < x_m, \varphi_m >>$ is a context, a state on k is a state-like pair $\sigma = (x, s)$ where of course x is the function which associates x_i to each $i = 1 \dots m$ and (roughly speaking) for each $i = 1 \dots m$ s_i is a member of the meaning of the corresponding expression φ_i .

For each $k \in K(n)$ $E(n, k)$ is the set of expressions bound to step n and context k. And here it is important to underline that **we need to ensure that** $E(n, k)$ **(as a subset of** Σ^***) is a recursive set**.

$E(n)$ is the union of $E(n, k)$ for $k \in K(n)$ (this will not be explicitly recalled on each iteration in the definition).

For each $k \in K(n)$, $t \in E(n, k)$, $\sigma \in \Xi(k)$ we'll define $\#(k, t, \sigma)$ which stands for 'the meaning of t bound to k and σ'.

The following set $E_s(n, k)$ should be defined in the same way at each step, we put here its definition, to avoid to repeat that definition each time. For each $k \in K(n)$ we define

$$E_s(n, k) = \{t | t \in E(n, k), \forall \sigma \in \Xi(k)\, \#(k, t, \sigma) \text{ is a set}\}.$$

6.1. Definition process

This section contains only definition 6.1. This definition is an inductive definition process within which we have assumptions, lemmas etc.. Symbols like $\Box$ within this definition are not intended to terminate the definition, they just terminate an assumption or lemma etc. which is internal to the definition.

Definition 6.1. We are now ready to begin the actual definition process, so we perform the simple initial step of our inductive process.

We define $K(1) = \{\epsilon\}$, $\Xi(\epsilon) = \{\epsilon\}$, $E(1, \epsilon) = \mathcal{C}$.
Cleary when we define $K(1)$ with ϵ we mean the string $<>$, while when defining $\Xi(\epsilon) = \{\epsilon\}$ the ϵ on the left side is $<>$ and the ϵ on the right side is $(\emptyset, \emptyset)$.

For each $t \in E(1, \epsilon)$ we define $\#(\epsilon, t, \epsilon) = \#(t)$.

The inductive step is a bit more complex. Suppose all our definitions have been given at step n and let's proceed with step $n+1$. In this inductive step we'll need some assumptions which will be identified with a title like 'Assumption 5.1.x'. Each assumption is a statement that must be valid at step 1, we suppose is valid at step n and needs to be proved true at step $n+1$ at the end of our definition process.

The first assumptions we need are the following.

Assumption 6.1.1. $K(n) \subseteq \Theta$.

□

Assumption 6.1.2. $K(n)$ is recursive and $\epsilon \in K(n)$.

□

Assumption 6.1.3. For each $k \in K(n)$ $\Xi(k) \neq \emptyset$.

□

Assumption 6.1.4. If $n > 1$ then for each $m < n$ $K(m) \subseteq K(n)$.

□

Assumption 6.1.5. For each $k \in K(n)$ $E(n,k) \subseteq \Sigma^*$.

□

Assumption 6.1.6. For each $k \in K(n)$ $E(n,k)$ is recursive.

□

Assumption 6.1.7. For each $k \in K(n)$ $k \in \Theta$ and for each $\sigma \in \mathit{\Xi}(k)$ σ is a state-like pair and $dom(\sigma) = dom(k)$.

□

Assumption 6.1.8. For each $k \in K(n)$ $k = \epsilon$ and $\mathit{\Xi}(k) = \{\epsilon\}$ or
($n > 1$ and there exist $m < n$, $h \in K(m)$, $\phi \in E_s(m,h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h+ < y, \phi >$, $\Xi(k) = \{\sigma + (y,s) | \, \sigma \in \Xi(h), s \in \#(h,\phi,\sigma)\}$).

□

Assumption 6.1.9. If $n > 1$ then for each $k \in K(n) : k \neq \epsilon$, $\sigma \in \Xi(k)$, $h \in \mathcal{R}(k)$: $h \neq k$, there exists $m < n$ such that $h \in K(m)$ and it results $\sigma_{/dom(h)} \in \Xi(h)$.

□

Assumption 6.1.10. For each $k \in K(n)$ each of the following predicates over $E(n, k)$ (where q is a positive integer and $\varphi \in E(n, k)$) is decidable:

- for each $\sigma \in \Xi(k)$ $Set_q(\#(k, \varphi, \sigma))$;
- for each $\sigma \in \Xi(k)$ $Event_q(\#(k, \varphi, \sigma))$;
- for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma) \in D_i$;
- for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma) \in \mathcal{P}^q(D_i)$;
- if (for each $\sigma \in \Xi(k)$ $Set_q(\#(k, \varphi, \sigma))$) then (for each $\sigma \in \Xi(k)$ $NotEmpty_q(\#(k, \varphi, \sigma))$).

Moreover the last predicate holds true.

□

Clearly assumption 6.1.10 is valid with $n = 1$, in fact in this case $k = \epsilon$, $E(n, k) = E(1, \epsilon) = \mathcal{C}$, $\Xi(k) = \{\epsilon\}$, so $\varphi \in \mathcal{C}$ and $\#(k, \varphi, \sigma) = \#(\varphi)$, and the predicates are the following:

- $Set_q(\#(\varphi))$;
- $Event_q(\#(\varphi))$;
- $\#(\varphi) \in D_i$;
- $\#(\varphi) \in \mathcal{P}^q(D_i)$;
- if $(Set_q(\#(\varphi)))$ then $(NotEmpty_q(\#(\varphi)))$.

We can go on with the inductive step and define

$$K(n)^+ = \{h+ < y, \phi > \mid h \in K(n), \phi \in E_s(n, h), y \in (\mathcal{V} - var(h))\} - K(n)\,,$$

$$K(n + 1) = K(n) \cup K(n)^+\,.$$

Let $k \in K(n)^+$. Then there exist $h \in K(n), \phi \in E_s(n, h), y \in (\mathcal{V} - var(h))$ such that $k = h+ < y, \phi >$. By lemma 5.4 we know that h, ϕ, y are *univocally determined.*

We can assume that $\Xi(k)$ is defined for $k \in K(n)$, and we need to define this for $k \in K(n + 1) - K(n)$, i.e. for $k \in K(n)^+$. If $k \in K(n)^+$ there exist $h \in K(n), \phi \in E_s(n, h), y \in (\mathcal{V} - var(h))$ such that $k = h+ < y, \phi >$; and h, ϕ, y are univocally determined. So we can define

$$\Xi(k) = \{\sigma + (y, s) \mid \sigma \in \Xi(h), s \in \#(h, \phi, \sigma)\}\,.$$

A consequence of lemma 3.1 is the following: for each $k \in K(n)^+$ and $\sigma + (y, s)$ in $\Xi(k)$, σ, y and s are *univocally determined*.

To ensure the unique readability of our expressions we need the following assumption (which is clearly satisfied for $n = 1$).

Assumption 6.1.11. For each $t \in E(n)$

- $t[\ell(t)] \neq$ '(' ;
- if $t[\ell(t)] =$ ')' then $d(t, \ell(t)) = 1$, else $d(t, \ell(t)) = 0$;
- for each $\alpha \in \{1, \ldots, \ell(t)\}$ if $(t[\alpha] = $ ':'$) \vee (t[\alpha] = $ ','$) \vee (t[\alpha] = $ ')'$)$ then $d(t, \alpha) \geqslant 1$.

□

We immediately prove the following.

Proof of 6.1.1. Given that $K(n) \subseteq \Theta$ we have to show that $K(n+1) \subseteq \Theta$.

Let $k \in K(n+1)$, if $k \in K(n)$ then $k \in \Theta$, else there exist $h \in K(n) \subseteq \Theta$, $\phi \in E_s(n, h) \subseteq \Sigma^*$, $y \in (\mathcal{V} - var(h))$ such that $k = h + < y, \phi > \in \Theta$. □

Proof of 6.1.2. We have to show that $K(n+1)$ is recursive and $\epsilon \in K(n+1)$.

We have assumed by inductive hypothesis that $K(n)$ is recursive and that $\epsilon \in K(n)$.

First of all it is obvious that $\epsilon \in K(n+1)$ because $K(n) \subseteq K(n+1)$.

Let $k \in \Gamma^*$ and let's try to decide whether $k \in K(n+1)$. We can decide whether $k \in K(n)$, if this holds then $k \in K(n+1)$, otherwise we know $k \neq \epsilon$.

At this point we can verify whether k has the form $<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >>$ where $x_1, \ldots, x_m \in \mathcal{V}$, $\varphi_1, \ldots, \varphi_m \in \Sigma^*$.

In fact in order to verify that $x_i \in \mathcal{V}$ we just need to verify that x_i is not a character in our alphabet that is not a variable (there are finite such characters). In order to verify that $\varphi_i \in \Sigma^*$ we just need to verify that the characters in φ_i are not $<$ or $>$.

If k has not the form $<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >>$ clearly $k \notin \Theta$ and we can decide that $k \notin K(n+1)$.

If k has the form $<< x_1 : \varphi_1 > \cdots < x_m : \varphi_m >>$ then $k \in \Theta - \{\epsilon\}$, so there exist $h \in \Theta$, $y \in \mathcal{V}$, $\psi \in \Sigma^*$ such that $k = h + < y, \psi >$, we know how to calculate h, y, ψ, and they are univocally determined.

Now consider the following conditions

- $h \in K(n)$,
- $y \in \mathcal{V} - var(h)$,
- $\psi \in E_s(n, h)$.

If all these conditions hold, then $k \in K(n+1)$, else (knowing that $k \notin K(n)$) $k \notin K(n+1)$.

All of the mentioned conditions are decidable. In fact $K(n)$ is recursive and so we can decide whether $h \in K(n)$. Moreover $E(n,h)$ is recursive, and given $\psi \in E(n,h)$ the condition 'for each $\rho \in \Xi(h)$ $Set_1(\#(h,\psi,\rho))$' is decidable. Therefore $\psi \in E_s(n,h)$ is decidable.

As regards the condition $y \in \mathcal{V} - var(h)$, we know that y is a variable, so if it doesn't belong to $var(h)$ this means it belongs to $\mathcal{V} - var(h)$, and so we can also decide this condition.

Therefore we have proved that $K(n+1)$ is recursive. □

Proof of 6.1.3. Let $k \in K(n+1)$, we have to show $\Xi(k) \neq \emptyset$.

If $k \in K(n)$ then $\Xi(k) \neq \emptyset$, else there exist $h \in K(n)$, $\phi \in E_s(n,h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h + < y, \phi >$, and $\Xi(k) = \{\sigma + (y,s) | \, \sigma \in \Xi(h), s \in \#(h,\phi,\sigma)\}$.

By the inductive hypothesis $\Xi(h) \neq \emptyset$, let's then take $\sigma \in \Xi(h)$, then $Set_1(\#(h,\phi,\sigma))$ and $NotEmpty_1(\#(h,\phi,\sigma))$. If we take $s \in \#(h,\phi,\sigma)$ then $\sigma + (y,s) \in \Xi(k)$. □

It is time to define $E(n+1,k)$, for each k in $K(n+1)$. Then for each t in $E(n+1,k)$ and σ in $\Xi(k)$ we need to define $\#(k,t,\sigma)$. We begin to do this by defining some new sets of expressions bound to context k, and for the expressions in each new set we define the proposed value of $\#(k,t,\sigma)$.

For each $k = h + < y, \phi > \in K(n)^+$ we define

$$E_a(n+1,k) = \{y\}.$$

Clearly $E_a(n+1,k) \subseteq \Sigma^*$ and $E_a(n+1,k)$ recursive.

For each $t \in E_a(n+1,k)$, $\sigma = \rho + (y,s) \in \Xi(k)$ we define:

$$\#(k,t,\sigma)_{(n+1,k,a)} = s.$$

We notice that $\epsilon \in K(n)$ and define $E_b(n+1,\epsilon) = \emptyset$.

For each $k = h + < y, \phi > \in K(n) - \{\epsilon\}$ we define

$$E_b(n+1,k) = \{t | t \in E(n,h), t \notin E(n,k)\}.$$

Clearly $E_b(n+1,k) \subseteq E(n,h) \subseteq \Sigma^*$ and $E_b(n+1,k)$ recursive.

For each $t \in E_b(n+1,k)$, $\sigma = \rho + (y,s) \in \Xi(k)$ we define the proposed value of $\#(k,t,\sigma)$:

$$\#(k,t,\sigma)_{(n+1,k,b)} = \#(h,t,\rho).$$

Given $k \in K(n)$ and a constant $c \in \mathcal{C}$ we can define the following set

$$H_c(n+1,k) = \{(c)(\varphi_1,\ldots,\varphi_m) |\, \varphi_1,\ldots,\varphi_m \in E(n,k)\}.$$

and we can prove it is recursive using some auxiliary lemma.

Lemma 6.1.12. Let $\psi \in \Sigma^*$ and let $\varphi = (c)(\psi) \in \Sigma^*$. Suppose for each r positive integer such that $4 < r < \ell(\varphi)$ and $\varphi[r] =$ ‘,’ we have $d(\varphi,r) > 1$. Then $\varphi \in H_c(n+1,k)$ if and only if $\psi \in E(n,k)$.

Proof. It is obvious that if $\psi \in E(n,k)$ then $\varphi \in H_c(n+1,k)$.

Conversely, if $\varphi \in H_c(n+1,k)$ then there exist a positive integer m and $\psi_1,\ldots,\psi_m \in E(n,k)$ such that $\varphi = (c)(\psi_1,\ldots,\psi_m)$.

If $m > 1$ then let r be the first explicit occurrence of ‘,’ in $(c)(\psi_1,\ldots,\psi_m)$. Clearly we have $d(\varphi,r) > 1$, so it cannot be $m > 1$.

It follows that $m = 1$ then $(c)(\psi) = \varphi = (c)(\psi_1)$ and so $\psi = \psi_1 \in E(n,k)$. □

Lemma 6.1.13. Let $\psi \in \Sigma^*$ and let $\varphi = (c)(\psi) \in \Sigma^*$. Consider the set of the positive integers r such that $4 < r < \ell(\varphi)$, $\varphi[r] =$ ‘,’ and $d(\varphi,r) = 1$. Assume this set is not empty and let's name $r_1,\ldots,r_h$ its members (in increasing order).
Let's also define $\psi_1 = \varphi[5, r_1 - 1]$ (if $r_1 - 1 < 5$ then $\psi_1 = \epsilon$ where ϵ is the empty string over the alphabet Σ).
If $h > 1$ then for each $i = 1 \ldots h-1$ we define $\psi_{i+1} = \varphi[r_i + 1, r_{i+1} - 1]$ (if $r_{i+1} - 1 < r_i + 1$ then $\psi_{i+1} = \epsilon$).
Finally we define $\psi_{h+1} = \varphi[r_h + 1, \ell(\varphi) - 1]$ (if $\ell(\varphi) - 1 < r_h + 1$ then $\psi_{h+1} = \epsilon$).

With these definitions we have $\varphi = (c)(\psi_1,\ldots,\psi_{h+1})$ and $\varphi \in H_c(n+1,k)$ if and only if for each $i = 1 \ldots h+1$ $\psi_i \in E(n,k)$.

Proof. Clearly if for each $i = 1 \ldots h+1$ $\psi_i \in E(n,k)$ then $\varphi \in H_c(n+1,k)$.

Conversely, if $\varphi \in H_c(n+1,k)$ then there exist a positive integer m and $\chi_1,\ldots,\chi_m \in E(n,k)$ such that $\varphi = (c)(\chi_1,\ldots,\chi_m)$.

If $m = 1$ then $\varphi = (c)(\chi_1)$, we have $d(\varphi,r_1) = 1$. Since $4 < r_1 < \ell(\varphi_1)$ $\chi_1[r_1 - 4] = \varphi[r_1] =$ ‘,’, $d(\chi_1,r_1) = d(\varphi,r_1) - 1 = 0$. But this contradicts assumption 6.1.11 and therefore we cannot have $m = 1$.

Since $m > 1$ we can indicate with $q_1,\ldots,q_{m-1}$ the positions of the explicit occurrences of ‘,’ in the representation $(c)(\chi_1,\ldots,\chi_m)$ of φ.

For each $j = 1 \dots m-1$ $d(\varphi, q_j) = 1$, therefore $\{q_1, \dots, q_{m-1}\} \subseteq \{r_1, \dots, r_h\}$.

Suppose there exists $i = 1 \dots h$ such that $r_i \notin \{q_1, \dots, q_{m-1}\}$. In this case one of these conditions will occur:

- $r_i < q_1$,
- $r_i > q_{m-1}$,
- $m - 1 > 1$ and there exists $j = 1 \dots m-2$ such that $q_j < r_i < q_{j+1}$.

If $r_i < q_1$ then $4 < r_i$ also holds, $\chi_1 = \varphi[5, q_1 - 1]$, $\ell(\chi_1) = q_1 - 1 - 5 + 1 = q_1 - 5$, for each $\alpha = 1 \dots q_1 - 5$ $\chi_1[\alpha] = \varphi[4 + \alpha]$. So $r_i - 4 \geqslant 1$, $r_i - 4 < q_1 - 4$ and then $r_i - 4 \leqslant q_1 - 5 = \ell(\chi_1)$. Then also $\chi_1[r_i - 4] = \varphi[r_i] =$ ‘,’ and $d(\chi_1, r_i - 4) = d(\varphi, r_i) - 1 = 0$. This contradicts assumption 6.1.11 and therefore we cannot have $r_i < q_1$.

If $r_i > q_{m-1}$ then $r_i < \ell(\varphi)$ also holds, $\chi_m = \varphi[q_{m-1} + 1, \ell(\varphi) - 1]$, $\ell(\chi_m) = \ell(\varphi) - 1 - (q_{m-1} + 1) + 1 = \ell(\varphi) - q_{m-1} - 1$. For each $\alpha = 1 \dots \ell(\varphi) - q_{m-1} - 1$ $\chi_m[\alpha] = \varphi[q_{m-1} + \alpha]$. So $r_i - q_{m-1} \geqslant 1$, $r_i - q_{m-1} < \ell(\varphi) - q_{m-1}$ and then $r_i - q_{m-1} \leqslant \ell(\varphi) - q_{m-1} - 1 = \ell(\chi_m)$. Then also $\chi_m[r_i - q_{m-1}] = \varphi[r_i] =$ ‘,’ and $d(\chi_m, r_i - q_{m-1}) = d(\varphi, r_i) - 1 = 0$. This contradicts assumption 6.1.11 and therefore we cannot have $r_i > q_{m-1}$.

Finally assume $m - 1 > 1$ and there exists $j = 1 \dots m-2$ such that $q_j < r_i < q_{j+1}$. In this case $\chi_{j+1} = \varphi[q_j + 1, q_{j+1} - 1]$, $\ell(\chi_{j+1}) = q_{j+1} - 1 - (q_j + 1) + 1 = q_{j+1} - q_j - 1$. For each $\alpha = 1 \dots q_{j+1} - q_j - 1$ $\chi_{j+1}[\alpha] = \varphi[q_j + \alpha]$. So $r_i - q_j \geqslant 1$, $r_i - q_j < q_{j+1} - q_j$ and then $r_i - q_j \leqslant q_{j+1} - q_j - 1 = \ell(\chi_{j+1})$. Then also $\chi_{j+1}[r_i - q_j] = \varphi[r_i] =$ ‘,’ and $d(\chi_{j+1}, r_i - q_j) = d(\varphi, r_i) - 1 = 0$. This contradicts assumption 6.1.11 and therefore we cannot have that $m-1 > 1$ and there exists $j = 1 \dots m-2$ such that $q_j < r_i < q_{j+1}$.

So we have to conclude that $\{q_1, \dots, q_{m-1}\} = \{r_1, \dots, r_h\}$. This means that $h + 1 = m$ and for each $i = 1 \dots h+1$ $\psi_i = \chi_i \in E(n, k)$. □

Lemma 6.1.14. Given $k \in K(n)$ and $c \in \mathcal{C}$ $H_c(n+1, k)$ is recursive.

Proof. Let $\varphi \in \Sigma^*$. If φ doesn't begin with the four characters $(c)($ or doesn't end with the character $)$ then $\varphi \notin H_c(n+1, k)$.

Then assume we are in the case $\varphi = (c)(\psi)$ where $\psi \in \Sigma^*$. Consider the set of the positive integers r such that $4 < r < \ell(\varphi)$, $\varphi[r] =$ ‘,’ and $d(\varphi, r) = 1$.

If the mentioned set is empty then $\varphi \in H_c(n+1, k)$ if and only if $\psi \in E(n, k)$.

If the mentioned set is not empty then let's name $r_1, \dots, r_h$ its members (in increasing order).
Let's also define $\psi_1 = \varphi[5, r_1 - 1]$ (if $r_1 - 1 < 5$ then $\psi_1 = \epsilon$ where ϵ is the empty string over the alphabet Σ).
If $h > 1$ then for each $i = 1 \dots h-1$ we define $\psi_{i+1} = \varphi[r_i + 1, r_{i+1} - 1]$ (if $r_{i+1} - 1 < r_i + 1$ then $\psi_{i+1} = \epsilon$.
Finally we define $\psi_{h+1} = \varphi[r_h + 1, \ell(\varphi) - 1]$ (if $\ell(\varphi) - 1 < r_h + 1$ then $\psi_{h+1} = \epsilon$).

With these definitions we have $\varphi = (c)(\psi_1, \ldots, \psi_{h+1})$ and $\varphi \in H_c(n+1, k)$ if and only if for each $i = 1 \ldots h+1$ $\psi_i \in E(n, k)$. □

Lemma 6.1.15. Let $k \in K(n)$, $c \in \mathcal{C}$. There exists an algorithm that given $\varphi \in \Sigma^*$

- determines if $\varphi \in H_c(n+1, k)$,
- if $\varphi \in H_c(n+1, k)$ it also identifies a positive integer m and $\psi_1, \ldots, \psi_m \in E(n, k)$ such that $\varphi = (c)(\psi_1, \ldots, \psi_m)$.

Proof. See the proof of lemma 6.1.14. □

Lemma 6.1.16. Given $k \in K(n)$, $c \in \mathcal{C}$, $\varphi \in H_c(n+1, k)$ there exist m positive integer, $\psi_1, \ldots, \psi_m \in E(n, k)$ such that $\varphi = (c)(\psi_1, \ldots, \psi_m)$ and m and $\psi_1, \ldots, \psi_m$ are univocally determined.

Proof. It is obvious by the definition of $H_c(n+1, k)$ there exist m positive integer, $\psi_1, \ldots, \psi_m \in E(n, k)$ such that $\varphi = (c)(\psi_1, \ldots, \psi_m)$.

Suppose there are also p positive integer and $\varphi_1, \ldots, \varphi_p$ such that $\varphi = (c)(\varphi_1, \ldots, \varphi_p)$. Of course we want to show that $p = m$ and for each $i = 1 \ldots m$ $\varphi_i = \psi_i$.

To this end we consider there exists $\psi \in \Sigma^*$ such that $\varphi = (c)(\psi)$. Consider the set of the positive integers r such that $4 < r < \ell(\varphi)$, $\varphi[r] =$ ',' and $d(\varphi, r) = 1$.

Suppose the mentioned set is empty. In this case if $m > 1$ then let r be the first explicit occurrence of ',' in $(c)(\psi_1, \ldots, \psi_m)$. Clearly we would have $d(\varphi, r) = 1$, so it cannot be $m > 1$. Similarly it cannot be $p > 1$, so $m = 1 = p$ and $\varphi_1 = \psi = \psi_1$.

Now assume this set is not empty and let's name $r_1, \ldots, r_h$ its members (in increasing order).
Let's also define $\chi_1 = \varphi[5, r_1 - 1]$ (if $r_1 - 1 < 5$ then $\chi_1 = \epsilon$ where ϵ is the empty string over the alphabet Σ).
If $h > 1$ then for each $i = 1 \ldots h-1$ we define $\chi_{i+1} = \varphi[r_i + 1, r_{i+1} - 1]$ (if $r_{i+1} - 1 < r_i + 1$ then $\chi_{i+1} = \epsilon$.
Finally we define $\chi_{h+1} = \varphi[r_h + 1, \ell(\varphi) - 1]$ (if $\ell(\varphi) - 1 < r_h + 1$ then $\chi_{h+1} = \epsilon$).

We have seen in lemma 6.1.13 that we cannot have $m = 1$ and that since $m > 1$ we can indicate with $q_1, \ldots, q_{m-1}$ the positions of the explicit occurrences of ',' in the representation $(c)(\psi_1, \ldots, \psi_m)$ of φ.

For each $j = 1 \ldots m-1$ $d(\varphi, q_j) = 1$, therefore $\{q_1, \ldots, q_{m-1}\} \subseteq \{r_1, \ldots, r_h\}$. In the mentioned lemma we have seen that actually $\{q_1, \ldots, q_{m-1}\} = \{r_1, \ldots, r_h\}$. This means that $h + 1 = m$ and for each $i = 1 \ldots h+1$ $\chi_i = \psi_i \in E(n, k)$.

Similarly we obtain that $h + 1 = p$ and for each $i = 1 \ldots h+1$ $\chi_i = \varphi_i \in E(n, k)$.

Therefore finally $p = h + 1 = m$ and for each $i = 1 \ldots m$ $\varphi_i = \chi_i = \psi_i$. □

Given a constant $c \in \mathcal{C}$ if $\#(c)$ is a particular type of function then for each $k \in K(n)$ we can define a set of expressions related to c and k, and we'll call $E^c(n+1,k)$ this set of expressions.

Let's examine the categories of functions to which we refer.

If there exist $i = 1 \dots p$ and a positive integer m such that $\#(c)$ is a function whose domain is $(D_i)^m$ and whose range is D_i then we define $E^c(n+1,k)$ as the set of the strings $(c)(\varphi_1, \dots, \varphi_m) \in H_c(n+1,k)$ such that:

- $\varphi_1, \dots, \varphi_m \in E(n,k)$;
- for each $j = 1 \dots m$, $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in D_i$;
- $(c)(\varphi_1, \dots, \varphi_m) \notin E(n,k)$;
- $(c)(\varphi_1, \dots, \varphi_m) \notin E_b(n+1,k)$.

The set $E^c(n+1,k)$ is recursive since given $\psi \in \Sigma^*$ we can determine if $\psi \in H_c(n+1,k)$ and if so we can identify a positive integer u and $\varphi_1, \dots, \varphi_u \in E(n,k)$ such that $\psi = (c)(\varphi_1, \dots, \varphi_u)$. As we have seen u and $\varphi_1, \dots, \varphi_u$ are univocally determined, so if $u \neq m$ then $\psi \notin E^c(n+1,k)$. If $u = m$ then, for each $j = 1 \dots m$, we can decide if for each $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in D_i$, and we can also decide if the following conditions hold:

- $(c)(\varphi_1, \dots, \varphi_m) \notin E(n,k)$,
- $(c)(\varphi_1, \dots, \varphi_m) \notin E_b(n+1,k)$.

For each $t = (c)(\varphi_1, \dots, \varphi_m) \in E^c(n+1,k)$ we define

$$\#(k,t,\sigma)_{(n+1,k,<c>)} = \#(c)(\#(k,\varphi_1,\sigma), \dots, \#(k,\varphi_m,\sigma)).$$

If there exist $i = 1 \dots p$, a positive integer q and a positive integer m such that $\#(c)$ is a function whose domain is $(\mathcal{P}^q(D_i))^m$ and whose range is $\mathcal{P}^q(D_i)$ then we define $E^c(n+1,k)$ as the set of the strings $(c)(\varphi_1, \dots, \varphi_m) \in H_c(n+1,k)$ such that:

- $\varphi_1, \dots, \varphi_m \in E(n,k)$;
- for each $j = 1 \dots m$, $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in \mathcal{P}^q(D_i)$;
- $(c)(\varphi_1, \dots, \varphi_m) \notin E(n,k)$;
- $(c)(\varphi_1, \dots, \varphi_m) \notin E_b(n+1,k)$.

The set $E^c(n+1,k)$ is recursive in this case too since given $\psi \in \Sigma^*$ we can determine if $\psi \in H_c(n+1,k)$ and if so we can identify a positive integer u and $\varphi_1, \dots, \varphi_u \in E(n,k)$ such that $\psi = (c)(\varphi_1, \dots, \varphi_u)$. As we have seen u and $\varphi_1, \dots, \varphi_u$ are univocally determined, so if $u \neq m$ then $\psi \notin E^c(n+1,k)$. If $u = m$ then, for each $j = 1 \dots m$, we can decide if for each $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in \mathcal{P}^q(D_i)$, and we can also decide if the following conditions hold:

- $(c)(\varphi_1, \dots, \varphi_m) \notin E(n,k)$,
- $(c)(\varphi_1, \dots, \varphi_m) \notin E_b(n+1,k)$.

For each $t = (c)(\varphi_1, \dots, \varphi_m) \in E^c(n+1,k)$ we define

$$\#(k,t,\sigma)_{(n+1,k,<c>)} = \#(c)(\#(k,\varphi_1,\sigma), \dots, \#(k,\varphi_m,\sigma)).$$

If there exist $i = 1 \dots p$ and a positive integer m such that $\#(c)$ is a function whose domain is $(D_i)^m$ and such that for each $(d_1, \dots, d_m) \in (D_i)^m$ $\#(c)(d_1, \dots, d_m)$ is true or false, then we define $E^c(n+1, k)$ as the set of the strings $(c)(\varphi_1, \dots, \varphi_m) \in H_c(n+1, k)$ such that:

- $\varphi_1, \dots, \varphi_m \in E(n, k)$;
- for each $j = 1 \dots m$, $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in D_i$;
- $(c)(\varphi_1, \dots, \varphi_m) \notin E(n, k)$;
- $(c)(\varphi_1, \dots, \varphi_m) \notin E_b(n+1, k)$.

The set $E^c(n+1, k)$ is recursive since given $\psi \in \Sigma^*$ we can determine if $\psi \in H_c(n+1, k)$ and if so we can identify a positive integer u and $\varphi_1, \dots, \varphi_u \in E(n, k)$ such that $\psi = (c)(\varphi_1, \dots, \varphi_u)$. As we have seen u and $\varphi_1, \dots, \varphi_u$ are univocally determined, so if $u \neq m$ then $\psi \notin E^c(n+1, k)$. If $u = m$ then, for each $j = 1 \dots m$, we can decide if for each $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in D_i$, and we can also decide if the following conditions hold:

- $(c)(\varphi_1, \dots, \varphi_m) \notin E(n, k)$,
- $(c)(\varphi_1, \dots, \varphi_m) \notin E_b(n+1, k)$.

For each $t = (c)(\varphi_1, \dots, \varphi_m) \in E^c(n+1, k)$ we define

$$\#(k, t, \sigma)_{(n+1, k, <c>)} = \#(c)(\#(k, \varphi_1, \sigma), \dots, \#(k, \varphi_m, \sigma)).$$

If there exist $i = 1 \dots p$, a positive integer q and a positive integer m such that $\#(c)$ is a function whose domain is $(\mathcal{P}^q(D_i))^m$ and such that for each $(d_1, \dots, d_m) \in (\mathcal{P}^q(D_i))^m$ $\#(c)(d_1, \dots, d_m)$ is true or false, then we define $E^c(n+1, k)$ as the set of the strings $(c)(\varphi_1, \dots, \varphi_m) \in H_c(n+1, k)$ such that:

- $\varphi_1, \dots, \varphi_m \in E(n, k)$;
- for each $j = 1 \dots m$, $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in \mathcal{P}^q(D_i)$;
- $(c)(\varphi_1, \dots, \varphi_m) \notin E(n, k)$;
- $(c)(\varphi_1, \dots, \varphi_m) \notin E_b(n+1, k)$.

The set $E^c(n+1, k)$ is recursive in this case too since given $\psi \in \Sigma^*$ we can determine if $\psi \in H_c(n+1, k)$ and if so we can identify a positive integer u and $\varphi_1, \dots, \varphi_u \in E(n, k)$ such that $\psi = (c)(\varphi_1, \dots, \varphi_u)$. As we have seen u and $\varphi_1, \dots, \varphi_u$ are univocally determined, so if $u \neq m$ then $\psi \notin E^c(n+1, k)$. If $u = m$ then, for each $j = 1 \dots m$, we can decide if for each $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in \mathcal{P}^q(D_i)$, and we can also decide if the following conditions hold:

- $(c)(\varphi_1, \dots, \varphi_m) \notin E(n, k)$,
- $(c)(\varphi_1, \dots, \varphi_m) \notin E_b(n+1, k)$.

For each $t = (c)(\varphi_1, \dots, \varphi_m) \in E^c(n+1, k)$ we define

$$\#(k, t, \sigma)_{(n+1, k, <c>)} = \#(c)(\#(k, \varphi_1, \sigma), \dots, \#(k, \varphi_m, \sigma)).$$

Assume m is a positive integer and $\#(c)$ is a function whose domain is $\bigcup_{q \geqslant 1}(\bigcup_{i=1 \dots p}(\mathcal{P}^q(D_i))^m)$ such that for each $q \geqslant 1$, $i = 1 \dots p$, $(A_1, \dots, A_m) \in$

$(\mathcal{P}^q(D_i))^m$ $\#(c)(A_1,\ldots,A_m) \in \mathcal{P}^q(D_i)$. Then we define $E^c(n+1,k)$ as the set of the strings $(c)(\varphi_1,\ldots,\varphi_m) \in H_c(n+1,k)$ such that:

- $\varphi_1,\ldots,\varphi_m \in E(n,k)$;
- there exist $i = 1\ldots p$, $q = 1\ldots q_{max}$ such that for each $j = 1\ldots m$, $\sigma \in \Xi(k)$ $\#(k,\varphi_j,\sigma) \in \mathcal{P}^q(D_i)$;
- $(c)(\varphi_1,\ldots,\varphi_m) \notin E(n,k)$;
- $(c)(\varphi_1,\ldots,\varphi_m) \notin E_b(n+1,k)$.

The set $E^c(n+1,k)$ is recursive in this case too since given $\psi \in \Sigma^*$ we can determine if $\psi \in H_c(n+1,k)$ and if so we can identify a positive integer u and $\varphi_1,\ldots,\varphi_u \in E(n,k)$ such that $\psi = (c)(\varphi_1,\ldots,\varphi_u)$. As we have seen u and $\varphi_1,\ldots,\varphi_u$ are univocally determined, so if $u \neq m$ then $\psi \notin E^c(n+1,k)$. If $u = m$ then, for each $i = 1\ldots p$ and $q = 1\ldots q_{max}$ we can decide if for each $j = 1\ldots m$ and $\sigma \in \Xi(k)$ $\#(k,\varphi_j,\sigma) \in \mathcal{P}^q(D_i)$, and we can also decide if the following conditions hold:

- $(c)(\varphi_1,\ldots,\varphi_m) \notin E(n,k)$,
- $(c)(\varphi_1,\ldots,\varphi_m) \notin E_b(n+1,k)$.

For each $t = (c)(\varphi_1,\ldots,\varphi_m) \in E^c(n+1,k)$ we define

$$\#(k,t,\sigma)_{(n+1,k,<c>)} = \#(c)(\#(k,\varphi_1,\sigma),\ldots,\#(k,\varphi_m,\sigma)).$$

We may also include in our language a ‘special’ constant Π whose meaning $\#(\Pi)$ is a function over the domain $\bigcup_{q\geqslant 1}(\bigcup_{i=1\ldots p}\mathcal{P}^q(D_i))$ such that for each $q \geqslant 1$, $i = 1\ldots p$ $A \in \mathcal{P}^q(D_i)$ $\#(\Pi)(A) = \mathcal{P}(A)$. Then we define $E^{\Pi}(n+1,k)$ as the set of the strings $(\Pi)(\varphi_1) \in H_{\Pi}(n+1,k)$ such that:

- $\varphi_1 \in E(n,k)$;
- there exist $i = 1\ldots p$, $q = 1\ldots q_{max}$ such that for each $\sigma \in \Xi(k)$ $\#(k,\varphi_1,\sigma) \in \mathcal{P}^q(D_i)$;
- $(\Pi)(\varphi_1) \notin E(n,k)$;
- $(\Pi)(\varphi_1) \notin E_b(n+1,k)$.

The set $E^{\Pi}(n+1,k)$ is recursive since given $\psi \in \Sigma^*$ we can determine if $\psi \in H_{\Pi}(n+1,k)$ and if so we can identify a positive integer u and $\varphi_1,\ldots,\varphi_u \in E(n,k)$ such that $\psi = (\Pi)(\varphi_1,\ldots,\varphi_u)$. As we have seen u and $\varphi_1,\ldots,\varphi_u$ are univocally determined, so if $u \neq 1$ then $\psi \notin E^{\Pi}(n+1,k)$. If $u = 1$ then, for each $i = 1\ldots p$ and $q = 1\ldots q_{max}$ we can decide if for each $\sigma \in \Xi(k)$ $\#(k,\varphi_1,\sigma) \in \mathcal{P}^q(D_i)$, and we can also decide if the following conditions hold:

- $(\Pi)(\varphi_1) \notin E(n,k)$;
- $(\Pi)(\varphi_1) \notin E_b(n+1,k)$.

For each $t = (\Pi)(\varphi_1) \in E^{\Pi}(n+1,k)$ we define

$$\#(k,t,\sigma)_{(n+1,k,<\Pi>)} = \#(\Pi)(\#(k,\varphi_1,\sigma)).$$

Given $k \in K(n)$ and $f \in \mathcal{F}$ we can define the set $H_f(n+1,k)$ as follows. If f has multiplicity 1 then

$$H_f(n+1,k) = \{f(\varphi_1) |\, \varphi_1 \in E(n,k)\}\,.$$

If f has multiplicity 2 then

$$H_f(n+1,k) = \{f(\varphi_1,\varphi_2) |\, \varphi_1,\varphi_2 \in E(n,k)\}\,.$$

We can prove $H_f(n+1,k)$ is recursive using some auxiliary lemma.

Lemma 6.1.17. Let $f \in \mathcal{F}$ and assume f has multiplicity 1. Let $\psi \in \Sigma^*$ and let $\varphi = f(\psi) \in \Sigma^*$. Then $\varphi \in H_f(n+1,k)$ if and only if $\psi \in E(n,k)$.

Proof. It is obvious that if $\psi \in E(n,k)$ then $\varphi \in H_f(n+1,k)$.

Conversely, if $\varphi \in H_f(n+1,k)$ then there exists $\chi \in E(n,k)$ such that $\varphi = f(\chi)$.

Therefore $\psi = \chi \in E(n,k)$. □

Lemma 6.1.18. Let $f \in \mathcal{F}$ and assume f has multiplicity 2. Let $\psi \in \Sigma^*$ and let $\varphi = f(\psi) \in \Sigma^*$.

Consider the set of the positive integers r such that $2 < r < \ell(\varphi)$, $\varphi[r] =$ ‘,’ and $d(\varphi, r) = 1$. If this set has just one member r_1 then we can define $\psi_1 = \varphi[3, r_1 - 1]$ (if $r_1 - 1 < 3$ then $\psi_1 = \epsilon$ where ϵ is the empty string over the alphabet Σ). We also define $\psi_2 = \varphi[r_1+1, \ell(\varphi) - 1]$ (if $\ell(\varphi) - 1 < r_1 + 1$ then $\psi_2 = \epsilon$).

With these definitions we have that $\varphi \in H_f(n+1,k)$ if and only if

- the set of the positive integers r such that $2 < r < \ell(\varphi)$, $\varphi[r] =$ ‘,’ and $d(\varphi, r) = 1$ has just one member r_1,
- $\psi_1, \psi_2 \in E(n,k)$.

Proof. If the two conditions

- the set of the positive integers r such that $2 < r < \ell(\varphi)$, $\varphi[r] =$ ‘,’ and $d(\varphi, r) = 1$ has just one member r_1,
- $\psi_1, \psi_2 \in E(n,k)$.

both hold then clearly $\varphi = f(\psi_1, \psi_2) \in H_f(n+1,k)$.

Conversely if $\varphi \in H_f(n+1,k)$ then there exist $\chi_1, \chi_2 \in E(n,k)$ such that $\varphi = f(\chi_1, \chi_2)$. Let's call q_1 the position of the explicit occurrence of ‘,’ in the representation $f(\chi_1, \chi_2)$ of φ. Clearly $d(\varphi, q_1) = 1$ and q_1 is a member of the set of the positive integers r such that $2 < r < \ell(\varphi)$, $\varphi[r] =$ ‘,’ and $d(\varphi, r) = 1$.

Let's then call $r_1, \ldots, r_h$ the members of the set of the positive integers r such that $2 < r < \ell(\varphi)$, $\varphi[r] =$ ‘,’ and $d(\varphi, r) = 1$. We have already seen that $q_1 \in \{r_1, \ldots, r_h\}$. Suppose $h > 1$ and there exists $i = 1 \ldots h$ such that $r_i \neq q_1$. In this case one of the following conditions will occur:

- $r_i < q_1$,
- $r_i > q_1$.

If $r_i < q_1$ then $2 < r_i$ also holds, $\chi_1 = \varphi[3, q_1 - 1]$, $\ell(\chi_1) = q_1 - 1 - 2 = q_1 - 3$, for each $\alpha = 1 \ldots q_1 - 3$ $\chi_1[\alpha] = \varphi[2 + \alpha]$. So $r_i - 2 \geqslant 1$, $r_i - 2 < q_1 - 2$ and then $r_i - 2 \leqslant q_1 - 3 = \ell(\chi_1)$. Then also $\chi_1[r_i - 2] = \varphi[r_i] =$ ',' and $1 = d(\varphi, r_i) = d(\chi_1, r_i - 2) + 1$, so $d(\chi_1, r_i - 2) = 0$. This contradicts assumption 6.1.11 and therefore we cannot have $r_i < q_1$.

If $r_i > q_1$ then $r_i < \ell(\varphi)$ also holds, $\chi_2 = \varphi[q_1 + 1, \ell(\varphi) - 1]$, $\ell(\chi_2) = \ell(\varphi) - 1 - q_1$. For each $\alpha = 1 \ldots \ell(\varphi) - 1 - q_1$ $\chi_2[\alpha] = \varphi[q_1 + \alpha]$. So $r_i - q_1 \geqslant 1$, $r_i - q_1 < \ell(\varphi) - q_1$ and then $r_i - q_1 \leqslant \ell(\varphi) - 1 - q_1 = \ell(\chi_2)$. Then also $\chi_2[r_i - q_1] = \varphi[r_i] =$ ','.

Moreover if we define $\vartheta = \varphi[1, q_1]$ then φ is the concatenation of ϑ, χ_2 and). Then $d(\varphi, r_i) = d(\varphi, q_1 + (r_i - q_1)) = d(\varphi, \ell(\vartheta) + (r_i - q_1)) = d(\varphi, \ell(\vartheta) + 1) + d(\chi_2, (r_i - q_1))$. It follows that $d(\varphi, r_i) = d(\varphi, q_1) + d(\chi_2, (r_i - q_1))$, and so $d(\chi_2, (r_i - q_1)) = 0$. This contradicts assumption 6.1.11 and therefore we cannot have $r_i > q_1$.

So we have to conclude that $h = 1$, $r_1 = q_1$, $\psi_i = \chi_i \in E(n, k)$. □

Lemma 6.1.19. Let $f \in \mathcal{F}$ and assume f has multiplicity 2. Then $H_f(n + 1, k)$ is recursive.

Proof. Let $\varphi \in \Sigma^*$ and let's see how we decide if $\varphi \in H_f(n + 1, k)$.

If φ doesn't begin with the characters $f($ or doesn't end with the character) then $\varphi \notin H_f(n + 1, k)$.

Then assume we are in the case $\varphi = f(\psi)$ where $\psi \in \Sigma^*$. Consider the set of the positive integers r such that $2 < r < \ell(\varphi)$, $\varphi[r] =$ ',' and $d(\varphi, r) = 1$.

If the mentioned set is empty or has not exactly one member then $\varphi \notin H_f(n+1, k)$.

If this set has just one member r_1 then we can define $\psi_1 = \varphi[3, r_1 - 1]$ (if $r_1 - 1 < 3$ then $\psi_1 = \epsilon$ where ϵ is the empty string over the alphabet Σ). We also define $\psi_2 = \varphi[r_1 + 1, \ell(\varphi) - 1]$ (if $\ell(\varphi) - 1 < r_1 + 1$ then $\psi_2 = \epsilon$).

If $\psi_1, \psi_2 \in E(n, k)$ then we can decide $\varphi \in H_f(n + 1, k)$, otherwise $\varphi \notin H_f(n + 1, k)$. □

Lemma 6.1.20. Let $f \in \mathcal{F}$ and assume f has multiplicity 2. There exists an algorithm that given $\varphi \in \Sigma^*$

- determines if $\varphi \in H_f(n + 1, k)$,
- if $\varphi \in H_f(n + 1, k)$ it also identifies $\psi_1, \psi_2 \in E(n, k)$ such that $\varphi = f(\psi_1, \psi_2)$.

Proof. See the proof of lemma 6.1.19. □

Lemma 6.1.21. Let $f \in \mathcal{F}$ and assume f has multiplicity 2. Given $\varphi \in H_f(n+1,k)$ there exist $\chi_1, \chi_2 \in E(n,k)$ such that $\varphi = f(\chi_1, \chi_2)$ and χ_1, χ_2 are univocally determined.

Proof. It is obvious by the definition of $H_f(n+1,k)$ that there exist $\chi_1, \chi_2 \in E(n,k)$ such that $\varphi = f(\chi_1, \chi_2)$. We have also seen in lemma 6.1.18 that the set of the positive integers r such that $2 < r < \ell(\varphi)$, $\varphi[r] =$ ',' and $d(\varphi, r) = 1$ has just one member r_1.

By the same lemma if we define $\psi_1 = \varphi[3, r_1 - 1]$ (if $r_1 - 1 < 3$ then $\psi_1 = \epsilon$ where ϵ is the empty string over the alphabet Σ) and $\psi_2 = \varphi[r_1 + 1, \ell(\varphi) - 1]$ (if $\ell(\varphi) - 1 < r_1 + 1$ then $\psi_2 = \epsilon$), then $\psi_1 = \chi_1, \psi_2 = \chi_2$.

We can assume there also exist $\phi_1, \phi_2 \in E(n,k)$ such that $\varphi = f(\phi_1, \phi_2)$. Clearly we can apply lemma 6.1.18 also in this case and obtain $\psi_1 = \phi_1$, $\psi_2 = \phi_2$.

It obviously follow that $\phi_1 = \chi_1$, $\phi_2 = \chi_2$.

□

Lemma 6.1.22. Let $f \in \mathcal{F}$ and assume f has multiplicity 1. Then $H_f(n+1,k)$ is recursive.

Proof. Let $\varphi \in \Sigma^*$ and let's see how we decide if $\varphi \in H_f(n+1,k)$.

If φ doesn't begin with the characters $f($ or doesn't end with the character $)$ then $\varphi \notin H_f(n+1,k)$.

Then assume we are in the case $\varphi = f(\psi)$ where $\psi \in \Sigma^*$.

In this case using lemma 6.1.17 if $\psi \in E(n,k)$ we'll decide that $\varphi \in H_f(n+1,k)$, otherwise we'll decide that $\varphi \notin H_f(n+1,k)$. □

Lemma 6.1.23. Let $f \in \mathcal{F}$ and assume f has multiplicity 1. There exists an algorithm that given $\varphi \in \Sigma^*$

- determines if $\varphi \in H_f(n+1,k)$,
- if $\varphi \in H_f(n+1,k)$ it also identifies $\psi \in E(n,k)$ such that $\varphi = f(\psi)$.

Proof. See the proof of lemma 6.1.22. □

Lemma 6.1.24. Let $f \in \mathcal{F}$ and assume f has multiplicity 1. Given $\varphi \in H_f(n+1,k)$ there exists $\chi \in E(n,k)$ such that $\varphi = f(\chi)$ and χ is univocally determined.

Proof. It is obvious by the definition of $H_f(n+1,k)$ that there exists $\chi \in E(n,k)$ such that $\varphi = f(\chi)$.

We can also assume there exists $\phi \in E(n,k)$ such that $\varphi = f(\phi)$, then obviously $\phi = \chi$. □

For each $k \in K(n)$ and $f \in \mathcal{F}$ if f has multiplicity 2 we define $E^f(n+1,k)$ as the set of the strings $f(\varphi_1, \varphi_2) \in H_f(n+1,k)$ such that:

- $\varphi_1, \varphi_2 \in E(n,k)$;
- for each $\sigma \in \Xi(k)$ $A_f(\#(k,\varphi_1,\sigma), \#(k,\varphi_2,\sigma))$ is true;
- $f(\varphi_1,\varphi_2) \notin E(n,k)$;
- $f(\varphi_1,\varphi_2) \notin E_b(n+1,k)$.

For instance, this means that if f is the 'logical conjunction' symbol '$\wedge$' and it belongs to $\mathcal{F}$, φ_1, φ_2 belong to $E(n,k)$, for each $\sigma \in \Xi(k)$ both $\#(k,\varphi_1,\sigma)$ and $\#(k,\varphi_2,\sigma)$ are true or false, $\wedge(\varphi_1,\varphi_2) \notin E(n,k)$, $\wedge(\varphi_1,\varphi_2) \notin E_b(n+1,k)$ then $\wedge(\varphi_1,\varphi_2)$ belongs to $E^f(n+1,k)$.

We now show that $E^f(n+1,k)$ is recursive. Given $\varphi \in \Sigma^*$ we can determine if $\varphi \in H_f(n+1,k)$. Clearly if $\varphi \notin H_f(n+1,k)$ then $\varphi \notin E^f(n+1,k)$. If $\varphi \in H_f(n+1,k)$ then we can identify $\psi_1, \psi_2 \in E(n,k)$ such that $\varphi = f(\psi_1,\psi_2)$. We have seen that ψ_1, ψ_2 are univocally determined.

For f with multiplicity 2 $A_f(\#(k,\varphi_1,\sigma), \#(k,\varphi_2,\sigma))$ can be one of the following

- $Event_1(\#(k,\varphi_1,\sigma))$ and $Event_1(\#(k,\varphi_2,\sigma))$,
- $Set_1(\#(k,\varphi_2,\sigma))$,
- 'something which is true' (e.g. $1=1$)

In every mentioned case the predicate 'for each $\sigma \in \Xi(k)$ $A_f(\#(k,\varphi_1,\sigma), \#(k,\varphi_2,\sigma))$' is decidable, and we can also decide if the following conditions hold

- $f(\varphi_1,\varphi_2) \notin E(n,k)$,
- $f(\varphi_1,\varphi_2) \notin E_b(n+1,k)$.

For each f with multiplicity 2, $t = f(\varphi_1,\varphi_2) \in E^f(n+1,k)$ we define

$$\#(k,t,\sigma)_{(n+1,k,<f>)} = P_f(\#(k,\varphi_1,\sigma), \#(k,\varphi_2,\sigma)).$$

If f has multiplicity 1 we define $E^f(n+1,k)$ as the set of the strings $f(\varphi_1) \in H_f(n+1,k)$ such that:

- $\varphi_1 \in E(n,k)$;
- for each $\sigma \in \Xi(k)$ $A_f(\#(k,\varphi_1,\sigma))$ is true;
- $f(\varphi_1) \notin E(n,k)$.
- $f(\varphi_1) \notin E_b(n+1,k)$.

We now show that $E^f(n+1,k)$ is recursive. Given $\varphi \in \Sigma^*$ we can determine if $\varphi \in H_f(n+1,k)$. Clearly if $\varphi \notin H_f(n+1,k)$ then $\varphi \notin E^f(n+1,k)$. If $\varphi \in H_f(n+1,k)$ then we can identify $\psi \in E(n,k)$ such that $\varphi = f(\psi)$. We have seen that ψ is univocally determined.

For f with multiplicity 1 $A_f(\#(k,\varphi_1,\sigma))$ can be one of the following

- 'something which is true' (e.g. $1=1$),
- $Event_2(\#(k,\varphi_1,\sigma))$.

In every mentioned case the predicate 'for each $\sigma \in \Xi(k)$ $A_f(\#(k, \varphi_1, \sigma)$' is decidable, and we can also decide if the following conditions hold

- $f(\varphi_1) \notin E(n, k)$,
- $f(\varphi_1) \notin E_b(n+1, k)$.

For each f with multiplicity 1, $t = f(\varphi_1) \in E^f(n+1, k)$ we define

$$\#(k, t, \sigma)_{(n+1,k,<f>)} = P_f(\#(k, \varphi_1, \sigma)).$$

Let $k \in K(n)$, m a positive integer, x a function whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \ldots m$ $i \neq j \rightarrow x_i \neq x_j$, φ a function whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $\varphi_i \in E(n)$, and finally let $\phi \in E(n)$. We write

$$\mathcal{E}(n, k, m, x, \varphi, \phi)$$

to indicate the following condition (where $k'_1 = k+ < x_1, \varphi_1 >$, and if $m > 1$ for each $i = 1 \ldots m-1$ $k'_{i+1} = k'_i + < x_{i+1}, \varphi_{i+1} >$):

- $\varphi_1 \in E_s(n, k)$;
- if $m > 1$ then for each $i = 1 \ldots m-1$ $k'_i \in K(n) \wedge \varphi_{i+1} \in E_s(n, k'_i)$;
- $k'_m \in K(n) \wedge \phi \in E(n, k'_m)$.

For each $k \in K(n)$ we define $H_e(n+1, k)$ as the set of the strings

$$\{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi)$$

such that:

- m is a positive integer;
- x is a function whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \ldots m$ $i \neq j \rightarrow x_i \neq x_j$;
- φ is a function whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $\varphi_i \in E(n)$;
- $\phi \in E(n)$;
- $\mathcal{E}(n, k, m, x, \varphi, \phi)$;

Let $t \in \Sigma^*$. If t doesn't begin with the characters '{}(' or it doesn't end with ')' then $t \notin H_e(n+1, k)$. Let $\psi \in \Sigma^*$ and let $t = \{\}(\psi)$. Consider the set of the positive integers r such that $2 < r < \ell(t)$, $t[r] =$ ',' and $d(t, r) = 1$. If $t \in H_e(n+1, k)$ then this set is not empty and by contraposition if this set is empty then $t \notin H_e(n+1, k)$. The following lemma will helps us to state the recursivity of $H_e(n+1, k)$.

Lemma 6.1.25. Let $\psi \in \Sigma^*$ and let $t = \{\}(\psi) \in \Sigma^*$. Consider the set of the positive integers r such that $2 < r < \ell(t)$, $t[r] =$ ',' and $d(t, r) = 1$. Assume this set is not empty and let's name $r_1, \ldots, r_h$ its members (in increasing order).

Let's also define $\psi_1 = t[3, r_1 - 1]$ (if $r_1 - 1 < 3$ then $\psi_1 = \epsilon$ where ϵ is the empty string over the alphabet Σ).
If $h > 1$ then for each $i = 1 \dots h-1$ we define $\psi_{i+1} = t[r_i + 1, r_{i+1} - 1]$ (if $r_{i+1} - 1 < r_i + 1$ then $\psi_{i+1} = \epsilon$.
Finally we define $\psi_{h+1} = t[r_h + 1, \ell(t) - 1]$ (if $\ell(t) - 1 < r_h + 1$ then $\psi_{h+1} = \epsilon$).

With these definitions $t \in H_e(n+1, k)$ if and only if

- for each $i = 1 \dots h$ $\ell(\psi_i) \geqslant 3$, $\psi_i[2] =$ ':'; $\ell(\psi_{h+1}) \geqslant 1$;
- let's define a function x over the domain $\{1, \dots, h\}$ by setting $x(i) = \psi_i[1]$; let's define a function φ over the domain $\{1, \dots, h\}$ by setting $\varphi(i) = \psi_i[3, \ell(\psi_i)]$; let's define $\phi = \psi_{h+1}$ then
  - for each $i = 1 \dots h$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \dots h$ $i \neq j \rightarrow x_i \neq x_j$,
  - for each $i = 1 \dots h$ $\varphi_i \in E(n)$,
  - $\phi \in E(n)$;
  - $\mathcal{E}(n, k, h, x, \varphi, \phi)$.

Proof. Suppose $t \in H_e(n+1, k)$, then there exist

- a positive integer m;
- a function y whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $y_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \dots m$ $i \neq j \rightarrow y_i \neq y_j$;
- a function χ whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $\chi_i \in E(n)$;
- $\theta \in E(n)$;

such that $\mathcal{E}(n, k, m, y, \chi, \theta)$ and $t = \{\}(y_1 : \chi_1, \dots, y_m : \chi_m, \theta)$.

Let's indicate with $q_1, \dots, q_m$ the positions of the explicit occurrences of ',' in the representation $\{\}(y_1 : \chi_1, \dots, y_m : \chi_m, \theta)$ of t. For each $i = 1 \dots m$ $d(t, q_i) = 1$ therefore $\{q_1, \dots, q_m\} \subseteq \{r_1, \dots, r_h\}$.

Suppose there exists $i = 1 \dots h$ such that $r_i \notin \{q_1, \dots, q_m\}$. In this case one of the following conditions will occur:

- $r_i < q_1$,
- $r_i > q_m$,
- $m > 1$ and there exists $j = 1 \dots m-1$ such that $q_j < r_i < q_{j+1}$.

If $r_i < q_1$ then $4 < r_i$ also holds, $\chi_1 = t[5, q_1 - 1]$, $\ell(\chi_1) = q_1 - 1 - 4 = q_1 - 5$, for each $\alpha = 1 \dots q_1 - 5$ $\chi_1[\alpha] = t[4 + \alpha]$. So $r_i - 4 \geqslant 1$, $r_i - 4 < q_1 - 4$ and then $r_i - 4 \leqslant q_1 - 5 = \ell(\chi_1)$. Then also $\chi_1[r_i - 4] = t[r_i] =$ ',' and $1 = d(t, r_i) = 1 + d(\chi_1, r_i - 4)$, therefore $d(\chi_1, r_i - 4) = 0$. This contradicts assumption 6.1.11 and therefore we cannot have $r_i < q_1$.

If $r_i > q_m$ then $r_i < \ell(t)$ also holds, $\theta = t[q_m + 1, \ell(t) - 1]$, $\ell(\theta) = \ell(t) - 1 - q_m = \ell(t) - q_m - 1$. For each $\alpha = 1 \dots \ell(t) - q_m - 1$ $\theta[\alpha] = t[q_m + \alpha]$. So $r_i - q_m \geqslant 1$, $r_i - q_m < \ell(t) - q_m$ and then $r_i - q_m \leqslant \ell(t) - q_m - 1 = \ell(\theta)$. Then also $\theta[r_i - q_m] = t[r_i] =$ ',' and $1 = d(t, r_i) = d(t, q_m + 1) + d(\theta, r_i - q_m) = 1 + d(\theta, r_i - q_m)$, therefore $d(\theta, r_i - q_m) = 0$. This contradicts assumption 6.1.11 and therefore we cannot have

$r_i > q_m$.

Finally assume $m > 1$ and there exists $j = 1 \ldots m-1$ such that $q_j < r_i < q_{j+1}$. In this case we have also $q_j + 2 < r_i$, $\chi_{j+1} = t[q_j+3, q_{j+1}-1]$, $\ell(\chi_{j+1}) = q_{j+1} - 1 - (q_j + 2) = q_{j+1} - q_j - 3$. For each $\alpha = 1 \ldots q_{j+1} - q_j - 3$ $\chi_{j+1}[\alpha] = t[q_j + 2 + \alpha]$. So $r_i - q_j - 2 \geqslant 1$, $r_i - q_j - 2 < q_{j+1} - q_j - 2$ and then $r_i - q_j - 2 \leqslant q_{j+1} - q_j - 3 = \ell(\chi_{j+1})$. Then also $\chi_{j+1}[r_i - q_j - 2] = t[r_i] =$ ',' and $1 = d(t, r_i) = d(t, q_j + 3) + d(\chi_{j+1}, r_i - q_j - 2) = 1 + d(\chi_{j+1}, r_i - q_j - 2)$. Therefore $d(\chi_{j+1}, r_i - q_j - 2) = 0$ and this contradicts assumption 6.1.11 and therefore we cannot have that $m > 1$ and there exists $j = 1 \ldots m-1$ such that $q_j < r_i < q_{j+1}$.

So we have to conclude that $\{q_1, \ldots, q_m\} = \{r_1, \ldots, r_h\}$. This means that $h = m$.

It also follows that for each $i = 1 \ldots h$ $\psi_i =$ '$y_i : \chi_i$', $\psi_{h+1} = \theta$. And it also follows that for each $i = 1 \ldots h$ $\ell(\psi_i) \geqslant 3$, $\psi_i[2] =$ ':'; $\ell(\psi_{h+1}) \geqslant 1$.

Let's now define a function x over the domain $\{1, \ldots, h\}$ by setting $x(i) = \psi_i[1] = y(i)$; let's define a function φ over the domain $\{1, \ldots, h\}$ by setting $\varphi(i) = \psi_i[3, \ell(\psi_i)] = \chi(i)$; let's define $\phi = \psi_{h+1} = \theta$ then

- for each $i = 1 \ldots h$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \ldots h$ $i \neq j \to x_i \neq x_j$,
- for each $i = 1 \ldots h$ $\varphi_i \in E(n)$,
- $\phi \in E(n)$;
- $\mathcal{E}(n, k, h, x, \varphi, \phi)$.

Conversely assume the following:

- for each $i = 1 \ldots h$ $\ell(\psi_i) \geqslant 3$, $\psi_i[2] =$ ':'; $\ell(\psi_{h+1}) \geqslant 1$;
- let's define a function x over the domain $\{1, \ldots, h\}$ by setting $x(i) = \psi_i[1]$; let's define a function φ over the domain $\{1, \ldots, h\}$ by setting $\varphi(i) = \psi_i[3, \ell(\psi_i)]$; let's define $\phi = \psi_{h+1}$ then
  - for each $i = 1 \ldots h$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \ldots h$ $i \neq j \to x_i \neq x_j$,
  - for each $i = 1 \ldots h$ $\varphi_i \in E(n)$,
  - $\phi \in E(n)$;
  - $\mathcal{E}(n, k, h, x, \varphi, \phi)$.

Notice that $t = \{\}(\psi_1, \ldots, \psi_h, \psi_{h+1}) = \{\}(x_1 : \varphi_1, \ldots, x_h : \varphi_h, \phi)$, so clearly $t \in H_e(n+1, k)$. □

Lemma 6.1.26. Given $k \in K(n)$ $H_e(n+1, k)$ is recursive.

Proof. Let $t \in \Sigma^*$. If t doesn't begin with the characters '{}(' or it doesn't end with ')' then $t \notin H_e(n+1, k)$. Let $\psi \in \Sigma^*$ and let $t = \{\}(\psi)$. Consider the set of the positive integers r such that $2 < r < \ell(t)$, $t[r] =$ ',' and $d(t, r) = 1$. If $t \in H_e(n+1, k)$ then this set is not empty and by contraposition if this set is empty then $t \notin H_e(n+1, k)$.

So we can consider the case of the former lemma where the just mentioned set is not empty. And we define $\psi_1 = t[3, r_1 - 1]$ (if $r_1 - 1 < 3$ then $\psi_1 = \epsilon$ where ϵ is the empty string over the alphabet Σ).

If $h > 1$ then for each $i = 1 \dots h-1$ we define $\psi_{i+1} = t[r_i + 1, r_{i+1} - 1]$ (if $r_{i+1} - 1 < r_i + 1$ then $\psi_{i+1} = \epsilon$.
Finally we define $\psi_{h+1} = t[r_h + 1, \ell(t) - 1]$ (if $\ell(t) - 1 < r_h + 1$ then $\psi_{h+1} = \epsilon$).

At this point we can verify the following: for each $i = 1 \dots h$ $\ell(\psi_i) \geqslant 3$, $\psi_i[2] =$ ':'; $\ell(\psi_{h+1}) \geqslant 1$: if this is false then $t \notin H_e(n+1, k)$.

If the condition is true we can go on with our verifications and first of all we define a function x over the domain $\{1, \dots, h\}$ by setting $x(i) = \psi_i[1]$; let's define a function φ over the domain $\{1, \dots, h\}$ by setting $\varphi(i) = \psi_i[3, \ell(\psi_i)]$; let's define $\phi = \psi_{h+1}$.

At this point we just need to check this condition

- for each $i = 1 \dots h$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \dots h$ $i \neq j \rightarrow x_i \neq x_j$,
- $\varphi_1 \in E_s(n, k)$;
- if $m > 1$ then for each $i = 1 \dots m-1$ $k'_i \in K(n) \wedge \varphi_{i+1} \in E_s(n, k'_i)$;
- $k'_m \in K(n) \wedge \phi \in E(n, k'_m)$.

The condition is decidable. In fact $E(n, k)$ is recursive and so are $K(n)$, $E(n, k'_i)$ and $E(n, k'_m)$. If the condition holds then clearly $t \in H_e(n+1, k)$, else $t \notin H_e(n+1, k)$. □

Lemma 6.1.27. Let $k \in K(n)$, given $t \in H_e(n+1, k)$ there exist

- a positive integer m;
- a function y whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $y_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \dots m$ $i \neq j \rightarrow y_i \neq y_j$;
- a function χ whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $\chi_i \in E(n)$;
- $\theta \in E(n)$;

such that $\mathcal{E}(n, k, m, y, \chi, \theta)$ and $t = \{\}(y_1 : \chi_1, \dots, y_m : \chi_m, \theta)$.

Moreover m, y, χ, θ are univocally determined.

Proof. Since $t \in H_e(n+1, k)$ we are in the case of lemma 6.1.25 and we can define $r_1, \dots, r_h$ and $\psi_1, \dots, \psi_{h+1}$ as in that lemma. As shown in the lemma the following holds

- for each $i = 1 \dots h$ $\ell(\psi_i) \geqslant 3$, $\psi_i[2] =$ ':'; $\ell(\psi_{h+1}) \geqslant 1$;
- let's define a function x over the domain $\{1, \dots, h\}$ by setting $x(i) = \psi_i[1]$; let's define a function φ over the domain $\{1, \dots, h\}$ by setting $\varphi(i) = \psi_i[3, \ell(\psi_i)]$; let's define $\phi = \psi_{h+1}$ then
  - for each $i = 1 \dots h$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \dots h$ $i \neq j \rightarrow x_i \neq x_j$,
  - for each $i = 1 \dots h$ $\varphi_i \in E(n)$,
  - $\phi \in E(n)$;
  - $\mathcal{E}(n, k, h, x, \varphi, \phi)$.

As seen in the lemma it also happens that $m = h$ and for each $i = 1 \dots m$ $y(i) = x(i)$, $\chi(i) = \varphi(i)$, and finally $\theta = \phi$.

If we had identified some potentially different (p, z, η, ϑ) such that

- p is a positive integer;

- z is a function whose domain is $\{1, \ldots, p\}$ such that for each $i = 1 \ldots p$ $z_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \ldots p$ $i \neq j \rightarrow z_i \neq z_j$;
- η is a function whose domain is $\{1, \ldots, p\}$ such that for each $i = 1 \ldots p$ $\eta_i \in E(n)$;
- $\vartheta \in E(n)$;
- $\mathcal{E}(n, k, p, z, \eta, \vartheta)$;
- $t = \{\}(z_1 : \theta_1, \ldots, z_p : \chi_p, \vartheta)$;

we can still conclude that $p = h = m$ and for each $i = 1 \ldots p$ $z(i) = x(i) = y(i)$, $\eta(i) = \varphi(i) = \chi(i)$, and finally $\vartheta = \phi = \theta$. □

For each $k \in K(n)$ we define $E_e(n+1, k)$ as the set of the strings

$$\{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi) \in H_e(n+1, k)$$

such that:

- m is a positive integer;
- x is a function whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \ldots m$ $i \neq j \rightarrow x_i \neq x_j$;
- φ is a function whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $\varphi_i \in E(n)$;
- $\phi \in E(n)$;
- $\{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi) \notin E(n, k)$;
- $\{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi) \notin E_b(n+1, k)$.

Lemma 6.1.28. $E_e(n+1, k)$ is recursive.

Proof. As we have seen in lemma 6.1.26, given $t \in \Sigma^*$ we can decide if $t \in H_e(n+1, k)$ and if we decide it is true then we also identify in the process what follows:

- a positive integer h;
- a function x over the domain $\{1, \ldots, h\}$ such that for each $i = 1 \ldots h$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \ldots h$ $i \neq j \rightarrow x_i \neq x_j$;
- a function φ over the domain $\{1, \ldots, h\}$ such that for each $i = 1 \ldots h$ $\varphi_i \in E(n)$;
- $\phi \in E(n)$

such that $\mathcal{E}(n, k, h, x, \varphi, \phi)$ and $t = \{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi)$.

Clearly we can also decide if the following conditions hold:

- $\{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi) \notin E(n, k)$;
- $\{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi) \notin E_b(n+1, k)$.

If the conditions both hold then $t \in E_e(n+1, k)$, otherwise $t \notin E_e(n+1, k)$. □

For every $t = \{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi) \in E_e(n+1, k)$ we define

$$\#(k, t, \sigma)_{(n+1,k,e)} = \{\#(k'_m, \phi, \sigma'_m) |\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\},$$

where $k'_1 = k + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \ldots m-1$ $k'_{i+1} = k'_i + (x_{i+1}, \varphi_{i+1})$.

Notice that the set $\{\#(k'_m, \phi, \sigma'_m) |\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\}$ is specified using a standard mathematical notation. We could specify it using a notation closer to the one of our expressions, in this case we should define a set $\mathcal{Q}$ as the set of $\sigma'_m \in \Xi(k'_m)$ such that $\sigma \sqsubseteq \sigma'_m$, then we could write the above mentioned set as $\{\}(\sigma'_m : \mathcal{Q}, \#(k'_m, \phi, \sigma'_m))$.

Actually, it might still be a bit unclear what is the intended meaning of the expression

$$\{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi).$$

This is the same meaning that the expression

$$\{\phi |\ x_1 \in \varphi_1, \ldots, x_m \in \varphi_m\}$$

is intended to have when used in most mathematics books.

We have terminated the definition of the 'new sets' (of expressions bound to context k) and the related work, we are now ready to define $E(n+1, k)$ for $k \in K(n+1)$.

First of all let $\mathcal{C}'$ be the set of the constants $c \in \mathcal{C}$ for which, given $k \in K(n)$, we can define $E^c(n+1, k)$.

To be precise we list here all the type of constant that belong to $\mathcal{C}'$. The following constants, and only those which are listed here, belong to $\mathcal{C}'$.

- the constants $c \in \mathcal{C}$ such that $\#(c)$ is a function whose domain is $(D_i)^m$ and whose range is D_i;
- the constants $c \in \mathcal{C}$ such that $\#(c)$ is a function whose domain is $(\mathcal{P}^q(D_i))^m$ and whose range is $\mathcal{P}^q(D_i)$;
- the constants $c \in \mathcal{C}$ such that there exist $i = 1 \ldots p$ and a positive integer m such that $\#(c)$ is a function whose domain is $(D_i)^m$ and such that for each $(d_1, \ldots, d_m) \in (D_i)^m$ $\#(c)(d_1, \ldots, d_m)$ is true or false;
- the constants $c \in \mathcal{C}$ such that there exist $i = 1 \ldots p$, a positive integer q and a positive integer m such that $\#(c)$ is a function whose domain is $(\mathcal{P}^q(D_i))^m$ and such that for each $(d_1, \ldots, d_m) \in (\mathcal{P}^q(D_i))^m$ $\#(c)(d_1, \ldots, d_m)$ is true or false;
- the constants $c \in \mathcal{C}$ such that $\#(c)$ is a function whose domain is $\bigcup_{q \geqslant 1}(\bigcup_{i=1 \ldots p}(\mathcal{P}^q(D_i))^m)$ such that for each $q \geqslant 1$, $i = 1 \ldots p$, $(A_1, \ldots, A_m) \in (\mathcal{P}^q(D_i))^m$ $\#(c)(A_1, \ldots, A_m) \in \mathcal{P}^q(D_i)$;
- if $\mathcal{C}$ includes a constant Π whose meaning $\#(\Pi)$ is a function over the domain $\bigcup_{q \geqslant 1}(\bigcup_{i=1 \ldots p} \mathcal{P}^q(D_i))$ such that for each $q \geqslant 1$, $i = 1 \ldots p$ $A \in \mathcal{P}^q(D_i)$ $\#(\Pi)(A) = \mathcal{P}(A)$, then Π also belongs to $\mathcal{C}'$.

If $k \in K(n)^+$ we have defined $E_a(n+1, k)$, we also define

- $E(n+1, k) = E_a(n+1, k)$.

If $k \in K(n)$ we have defined $E_b(n+1, k)$, $E^c(n+1, k)$ (for each $c \in \mathcal{C}'$), $E^f(n+1, k)$ (for each $f \in \mathcal{F}$), $E_e(n+1, k)$, we also define

$$\mathcal{H}(n+1, k) = \{E(n, k), E_b(n+1, k), E_e(n+1, k)\} \cup \{E^c(n+1, k) | c \in \mathcal{C}'\} \cup \{E^f(n+1, k) | f \in \mathcal{F}\},$$

$$E(n+1,k) = \bigcup_{A \in \mathcal{H}(n+1,k)} A$$

Lemma 6.1.29. Given $A, B \in \mathcal{H}(n+1,k)$ $A \cap B = \emptyset$

Proof. It is obvious by definition that $E_b(n+1,k) \cap E(n,k) = \emptyset$.

It is also obvious that $E_e(n+1,k) \cap E(n,k) = \emptyset$ and $E_e(n+1,k) \cap E_b(n+1,k) = \emptyset$.

Given $c \in \mathcal{C}'$

- $E^c(n+1,k) \cap E(n,k) = \emptyset$,
- $E^c(n+1,k) \cap E_b(n+1,k) = \emptyset$,
- $E^c(n+1,k) \cap E_e(n+1,k) = \emptyset$.

Given $c_1, c_2 \in \mathcal{C}'$ $E^{c_1}(n+1,k) \cap E^{c_2}(n+1,k) = \emptyset$.

Given $f \in \mathcal{F}$

- $E^f(n+1,k) \cap E(n,k) = \emptyset$,
- $E^f(n+1,k) \cap E_b(n+1,k) = \emptyset$,
- $E^f(n+1,k) \cap E_e(n+1,k) = \emptyset$.

Given $f \in \mathcal{F}$, $c \in \mathcal{C}'$ $E^f(n+1,k) \cap E^c(n+1,k) = \emptyset$.

Given $f_1, f_2 \in \mathcal{F}$ $E^{f_1}(n+1,k) \cap E^{f_2}(n+1,k) = \emptyset$.

□

For every $k \in K(n+1)$, $t \in E(n+1,k)$ and $\sigma \in \Xi(k)$ we need that $\#(k,t,\sigma)$ is defined.

If $k \in K(n)^+$ we just need to define $\#(k,t,\sigma)$ for each $t \in E_a(n+1,k)$. Obviously we define $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,a)}$.

If $k \in K(n)$, how do we define $\#(k,t,\sigma)$ for each $t \in E(n+1,k)$? We have seen that $E(n+1,k) = \bigcup_{A \in \mathcal{H}(n+1,k)} A$ and that given $A, B \in \mathcal{H}(n+1,k)$ $A \cap B = \emptyset$.

Given $t \in E(n,k)$ $\#(k,t,\sigma)$ is already defined and we don't need to redefine it.

Given $t \in E_b(n+1,k)$ we define $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,b)}$.

Given $t \in E_e(n+1,k)$ we define $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,e)}$.

Given $c \in \mathcal{C}'$, $t \in E^c(n+1,k)$ we define $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,<c>)}$.

Given $f \in \mathcal{F}$, $t \in E^f(n+1,k)$ we define $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,<f>)}$.

Notice that if $k \in K(n)^+$ we have not defined $E_b(n+1,k)$, $E_e(n+1,k)$ given $c \in \mathcal{C}'$ we have not defined $E^c(n+1,k)$ and given $f \in \mathcal{F}$ we have not defined $E^f(n+1,k)$. We conventionally define all of these sets as the empty set.

Also notice that if $k \in K(n)$ we have not defined $E_a(n+1,k)$ and we conventionally define it as the empty set.

In the last part of our definition we need to prove that all the assumptions we have made at step n are true at step $n+1$.

Proof of 6.1.4. Let $m < n+1$. If $m = n$ then clearly $K(m) = K(n) \subseteq K(n+1)$. Else $m < n$ so $K(m) \subseteq K(n) \subseteq K(n+1)$.

□

Proof of 6.1.5 and 6.1.6 . Let $k \in K(n+1)$, if $k \in K(n)^+$ then $E(n+1,k) = E_a(n+1,k) \subseteq \Sigma^*$.

If $k \in K(n)$ then $E(n+1,k) = \bigcup_{A \in \mathcal{H}(n+1,k)} A$. Since $\mathcal{H}(n+1,k)$ is finite, in order to prove that $E(n+1,k) \subseteq \Sigma^*$ we just need to prove that for each $A \in \mathcal{H}(n+1,k)$ $A \subseteq \Sigma^*$.

We actually have the following:

- $E(n,k) \subseteq \Sigma^*$,
- $E_b(n+1,k) \subseteq \Sigma^*$,
- $E_e(n+1,k) \subseteq \Sigma^*$,
- for each $c \in \mathcal{C}'$ $E^c(n+1,k) \subseteq \Sigma^*$,
- for each $f \in \mathcal{F}$ $E^f(n+1,k) \subseteq \Sigma^*$.

Let's now see how we prove that $E(n+1,k)$ is recursive.

Let $t \in \Sigma^*$, we have to decide if $t \in E(n+1,k)$. First we can decide if $k \in K(n)$, if this is false then we just need fo decide if $t \in E_a(n+1,k)$.

If instead $k \in K(n)$ holds true, we check the following conditions

- $t \in E(n,k)$,
- $t \in E_b(n+1,k)$,
- $t \in E_e(n+1,k)$,
- the condition $t \in E^c(n+1,k)$ (for each $c \in \mathcal{C}'$),
- the condition $t \in E^f(n+1,k)$ (for each $f \in \mathcal{F}$).

If at least one of the conditions is true then we can decide $t \in E(n+1,k)$, otherwise $t \notin E(n+1,k)$.

□

Proof of 6.1.7. We have to show that for each $k \in K(n+1)$ $k \in \Theta$ and for each $\sigma \in \Xi(k)$ σ is a state-like pair and $dom(\sigma) = dom(k)$.

If $k \in K(n)$ this is clearly true because it is precisely our assumption.

If $k \in K(n)^+$ then there exist $h \in K(n), \phi \in E_s(n,h), z \in (\mathcal{V} - var(h))$ such that $k = h+ < z, \phi >$ and $\Xi(k) = \{\rho + (z,s) | \, \rho \in \Xi(h), s \in \#(h, \phi, \rho)\}$.

For each $\sigma \in \Xi(k)$ $\sigma = \rho+(z,s)$ with $\rho \in \Xi(h), s \in \#(h,\phi,\rho)$, so σ is a state-like pair.

Moreover, we can assume $dom(h) = dom(\rho) = \emptyset$ or $dom(h) = dom(\rho) = \{1, \ldots, m\}$ for a positive integer m. In the first case $dom(\sigma) = \{1\} = dom(k)$, else

$$dom(\sigma) = dom(\rho) \cup \{m+1\} = dom(h) \cup \{m+1\} = dom(k) \ .$$

□

Proof of 6.1.8. We have to show that for each $k \in K(n+1)$ $k = \epsilon$ and $\varXi(k) = \{\epsilon\}$ or
(there exist $m < n+1$, $h \in K(m)$, $\phi \in E_s(m,h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h+ < y, \phi >$, $\Xi(k) = \{\sigma + (y,s) | \, \sigma \in \Xi(h), s \in \#(h,\phi,\sigma)\}$).

If $k \in K(n)$ by the inductive hypothesis $k = \epsilon$ and $\varXi(k) = \{\epsilon\}$ or
($n > 1$ and there exist $m < n < n+1$, $h \in K(m)$, $\phi \in E_s(m,h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h+ < y, \phi >$, $\Xi(k) = \{\sigma + (y,s) | \, \sigma \in \Xi(h), s \in \#(h,\phi,\sigma)\}$).

Otherwise $k \in K(n)^+$ so there exist $h \in K(n), \phi \in E_s(n,h), y \in (\mathcal{V} - var(h))$ such that $k = h+ < y, \phi >$, $\Xi(k) = \{\sigma + (y,s) | \, \sigma \in \Xi(h), s \in \#(h,\phi,\sigma)\}$. □

Proof of 6.1.9. We have to show that for each $k \in K(n+1) : k \neq \epsilon$, $\sigma \in \Xi(k)$, $h \in \mathcal{R}(k) : h \neq k$, there exists $m < n+1$ such that $h \in K(m)$ and it results $\sigma_{/dom(h)} \in \Xi(h)$.

We first consider the case where $n+1 = 2$. Here we have to show that for each $k \in K(2) : k \neq \epsilon$, $\sigma \in \Xi(k)$, $h \in \mathcal{R}(k) : h \neq k$, $h \in K(1)$ and it results $\sigma_{/dom(h)} \in \Xi(h)$.

Let $k \in K(2) : k \neq \epsilon$, $\sigma \in \Xi(k)$, $h \in \mathcal{R}(k) : h \neq k$. Clearly $k \in K(1)^+$, so there exist $g \in K(1)$, $\phi \in E_s(1,g)$, $y \in \mathcal{V} - var(g)$ such that $k = g+ < y, \phi >$. By lemma 5.5 we obtain that $h \in \mathcal{R}(g)$. Since $g = \epsilon$ then also $h = \epsilon \in K(1)$, so $\sigma_{/dom(h)} = \sigma_{/\emptyset} = \epsilon \in \Xi(\epsilon) = \Xi(h)$.

Let's now examine the case where $n+1 > 2$. Let $k \in K(n+1) : k \neq \epsilon$, let $\sigma \in \Xi(k)$, $h \in \mathcal{R}(k) : h \neq k$, we have to show there exists $m < n+1$ such that $h \in K(m)$ and it results $\sigma_{/dom(h)} \in \Xi(h)$.

As we have just proved in relation to assumption 6.1.8, there exist $m < n+1$, $g \in K(m)$, $\phi \in E_s(m,g)$, $y \in (\mathcal{V} - var(h))$ such that $k = g+ < y, \phi >$, $\Xi(k) = \{\rho + (y,s) | \, \rho \in \Xi(g), s \in \#(g,\phi,\sigma)\}$).

This implies there exist $\rho \in \Xi(g), s \in \#(g,\phi,\rho)$ such that $\sigma = \rho + (y,s)$. By assumption 6.1.7 and lemma 3.11 we have that $\sigma_{/dom(g)} = \sigma_{/dom(\rho)} = \rho$.

If $h = g$ then $\sigma_{/dom(h)} = \sigma_{/dom(g)} = \rho \in \Xi(h)$.

Otherwise we have $h \neq g$. Since $k = g+ < y, \phi >$, $h \in \mathcal{R}(k)$, $h \neq k$ by lemma 5.5 we have that $h \in \mathcal{R}(g)$. If $g = \epsilon$ we would have $h = \epsilon = g$, so $g \neq \epsilon$. This implies that $m \geqslant 2$. By our inductive hypothesis we obtain there exists $q < m \leqslant n$ such that $h \in K(q)$ and $\rho_{/dom(h)} \in \Xi(h)$. Now by lemma 3.8

$$\sigma_{/dom(h)} = (\sigma_{/dom(g)})_{/dom(h)} = \rho_{/dom(h)} \in \Xi(h).$$

□

Proof of 6.1.10. Given $k \in K(n+1)$ we have to show that a certain set of predicates over $E(n+1,k)$ are decidable.

We recall that the predicates are the following

- for each $\sigma \in \Xi(k)$ $Set_q(\#(k,\varphi,\sigma))$;
- for each $\sigma \in \Xi(k)$ $Event_q(\#(k,\varphi,\sigma))$;
- for each $\sigma \in \Xi(k)$ $\#(k,\varphi,\sigma) \in D_i$;
- for each $\sigma \in \Xi(k)$ $\#(k,\varphi,\sigma) \in \mathcal{P}^q(D_i)$;
- if (for each $\sigma \in \Xi(k)$ $Set_q(\#(k,\varphi,\sigma))$) then
  (for each $\sigma \in \Xi(k)$ $NotEmpty_q(\#(k,\varphi,\sigma))$).

And we also need to verify that the last predicate holds true.

We have seen that if $k \in K(n)^+$ $E(n+1,k) = E_a(n+1,k)$, and if $k \in K(n)$

$$E(n+1,k) = \bigcup_{A \in \mathcal{H}(n+1,k)} A\,.$$

So let P be one of the predicates which we want to examine.

If $k \in K(n)^+$ P is a predicate over $E_a(n+1,k)$, suppose we can show in this case P is decidable over $E_a(n+1,k)$.

If instead $k \in K(n)$ P is a predicate over $\bigcup_{A \in \mathcal{H}(n+1,k)} A$. Suppose in this case for each $A \in \mathcal{H}(n+1,k)$ we can show P is decidable over A.

If we can show the above properties for P, then P is decidable over $E(n+1,k)$.

In fact given $t \in E(n+1,k)$ to decide $P(t)$ we first decide if $k \in K(n)$, if this is false then $t \in E_a(n+1,k)$ and we can decide $P(t)$.

If $k \in K(n)$ is true then $t \in \bigcup_{A \in \mathcal{H}(n+1,k)} A$. For each $A \in \mathcal{H}(n+1,k)$ we can decide if $t \in A$, this will be true for just one set A and since $t \in A$ we can decide $P(t)$.

So in order to prove the decidability of P we must prove the following:

- if $k \notin K(n)$ then P is decidable over $E_a(n+1,k)$,

- if $k \in K(n)$ then for each $A \in \mathcal{H}(n+1,k)$ P is decidable over A.

There is also a predicate Q over $E(n+1,k)$ which we want to prove true. In order to prove the truthness of this predicate we must prove the following:

- if $k \notin K(n)$ then Q is true over $E_a(n+1,k)$,
- if $k \in K(n)$ then for each $A \in \mathcal{H}(n+1,k)$ Q is true over A.

Finally, in order to prove the decidability of all the predicates we want to declare decidable and the truthness of the predicate Q we will proceed as follows.

- we prove that if $k \notin K(n)$ then for each of our predicates P P is decidable over $E_a(n+1,k)$, Q is true over $E_a(n+1,k)$,
- we prove that if $k \in K(n)$ then for each $A \in \mathcal{H}(n+1,k)$ and for each of our predicates P P is decidable over A, Q is true over A.

For the first step, let $\mathbf{k} \notin \mathbf{K}(\mathbf{n})$ and let's try to prove that for each of our predicates P P is decidable over $\mathbf{E_a}(\mathbf{n}+\mathbf{1},\mathbf{k})$.

Since $k \in K(n)^+$ there exist $h \in K(n), \phi \in E_s(n,h), y \in (\mathcal{V} - var(h))$ such that $k = h+ < y, \phi >$, $\Xi(k) = \{\rho + (y,s) | \, \rho \in \Xi(h), s \in \#(h,\phi,\rho)\}$. Moreover h, y and ϕ are clearly identifiable within k and $E_a(n+1,k) = \{y\}$.

Given $t \in E_a(n+1,k)$, $\sigma = \rho + (y,s) \in \Xi(k)$ $\#(k,t,\sigma) = s \in \#(h,\phi,\rho)$.

We first consider the predicate 'for each $\sigma \in \Xi(k)$ $Set_q(\#(k,\varphi,\sigma))$' (where q is a positive integer).

We consider that $\phi \in E(n,h)$ and by inductive hypothesis we can decide whether 'for each $\rho \in \Xi(h)$ $Set_{q+1}(\#(h,\phi,\rho))$'.

If we decide this is true then for each $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h,\phi,\rho)$ such that $\sigma = \rho + (y,s)$, $\#(k,\varphi,\sigma) = s \in \#(h,\phi,\rho)$, and since $Set_{q+1}(\#(h,\phi,\rho))$ we have $Set_q(\#(k,\varphi,\sigma))$.

So if we decide 'for each $\rho \in \Xi(h)$ $Set_{q+1}(\#(h,\phi,\rho))$' is true we can use this to decide 'for each $\sigma \in \Xi(k)$ $Set_q(\#(k,\varphi,\sigma))$' is true.

If instead we decide 'for each $\rho \in \Xi(h)$ $Set_{q+1}(\#(h,\phi,\rho))$' is false this means there exists $\rho \in \Xi(h)$ such that $\neg(Set_{q+1}(\#(h,\phi,\rho)))$. Since $\phi \in E_s(n,h)$ we have that $\#(h,\phi,\rho)$ is a set and so there exists $s \in \#(h,\phi,\rho)$ such that $\neg(Set_q(s))$. If we set $\sigma = \rho + (y,s)$ then $\sigma \in \Xi(k)$ and $\#(k,\varphi,\sigma) = s$ so $\neg(Set_q(\#(k,\varphi,\sigma))$.

So if we decide 'for each $\rho \in \Xi(h)$ $Set_{q+1}(\#(h,\phi,\rho))$' is false we can use this to decide 'for each $\sigma \in \Xi(k)$ $Set_q(\#(k,\varphi,\sigma))$' is false too.

We now want to prove the following:

if (for each $\sigma \in \Xi(k)$ $Set_q(\#(k,\varphi,\sigma))$) then
(for each $\sigma \in \Xi(k)$ $NotEmpty_q(\#(k,\varphi,\sigma))$).

We assume (for each $\sigma \in \Xi(k)$ $Set_q(\#(k,\varphi,\sigma))$), clearly this implies (for each $\rho \in \Xi(h)$ $Set_{q+1}(\#(h,\phi,\rho))$), which (by inductive hypothesis) implies (for each $\rho \in \Xi(h)$ $NotEmpty_{q+1}(\#(h,\phi,\rho))$).

We can then consider that for each $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h,\phi,\rho)$ such that $\sigma = \rho + (y,s)$, $\#(k,\varphi,\sigma) = s \in \#(h,\phi,\rho)$. Since $NotEmpty_{q+1}(\#(h,\phi,\rho))$ holds then $NotEmpty_q(\#(k,\varphi,\sigma))$ holds too.

Given $i = 1 \dots p$ we now want to consider the predicate 'for each $\sigma \in \Xi(k)$ $\#(k,\varphi,\sigma) \in D_i$'.

By the inductive hypothesis we are able to decide the predicate 'for each $\rho \in \Xi(h)$ $\#(h,\phi,\rho) \in \mathcal{P}(D_i)$'.

If we decide the last condition is true then as seen above for each $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h,\phi,\rho)$ such that $\sigma = \rho + (y,s)$, $\#(k,\varphi,\sigma) = s \in \#(h,\phi,\rho) \in \mathcal{P}(D_i)$, therefore $\#(k,\varphi,\sigma) \in D_i$.

If instead we decide the mentioned condition is false, then there exists $\rho \in \Xi(h)$: $\#(h,\phi,\rho) \notin \mathcal{P}(D_i)$. Since $\#(h,\phi,\rho)$ is a set and it is not empty, this means there exists $s \in \#(h,\phi,\rho)$: $s \notin D_i$. If we set $\sigma = \rho + (y,s)$ then $\sigma \in \Xi(k)$ and $\#(k,\varphi,\sigma) = s \notin D_i$, so there exists $\sigma \in \Xi(k)$: $\#(k,\varphi,\sigma) \notin D_i$.

Given $i = 1 \dots p$ and a positive integer q we now want to consider the predicate 'for each $\sigma \in \Xi(k)$ $\#(k,\varphi,\sigma) \in \mathcal{P}^q(D_i)$'.

By the inductive hypothesis we are able to decide the predicate 'for each $\rho \in \Xi(h)$ $\#(h,\phi,\rho) \in \mathcal{P}^{q+1}(D_i)$'.

If we decide the last condition is true then as seen above for each $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h,\phi,\rho)$ such that $\sigma = \rho + (y,s)$, $\#(k,\varphi,\sigma) = s \in \#(h,\phi,\rho) \in \mathcal{P}^{q+1}(D_i)$, therefore $\#(k,\varphi,\sigma) \in \#(h,\phi,\rho) \subseteq \mathcal{P}^q(D_i)$.

If instead we decide the mentioned condition is false, then there exists $\rho \in \Xi(h)$: $\#(h,\phi,\rho) \notin \mathcal{P}^{q+1}(D_i)$. Since $\#(h,\phi,\rho)$ is a set and it is not empty, this means there exists $s \in \#(h,\phi,\rho)$: $s \notin \mathcal{P}^q(D_i)$. If we set $\sigma = \rho + (y,s)$ then $\sigma \in \Xi(k)$ and $\#(k,\varphi,\sigma) = s \notin \mathcal{P}^q(D_i)$, so there exists $\sigma \in \Xi(k)$: $\#(k,\varphi,\sigma) \notin \mathcal{P}^q(D_i)$.

Given a positive integer q we now want to consider the predicate 'for each $\sigma \in \Xi(k)$ $Event_q(\#(k,\varphi,\sigma))$'.

By the inductive hypothesis we are able to decide the predicate 'for each $\rho \in \Xi(h)$ $Event_{q+1}(\#(h,\phi,\rho))$'.

If we decide the last condition is true then as seen above for each $\sigma \in \Xi(k)$ there exists $\rho \in \Xi(h)$ such that $\#(k,\varphi,\sigma) \in \#(h,\phi,\rho)$. Since $Event_{q+1}(\#(h,\phi,\rho))$ we have $Event_q(\#(k,\varphi,\sigma))$.

If instead we decide the mentioned condition is false, then there exists $\rho \in \Xi(h)$: $\neg(Event_{q+1}(\#(h,\phi,\rho)))$. This implies there exists $s \in \#(h,\phi,\rho)$: $\neg(Event_q(s))$. If we set $\sigma = \rho + (y, s)$ then $\sigma \in \Xi(k)$ and $\#(k, \varphi, \sigma) = s$, so $\neg(Event_q(\#(k, \varphi, \sigma)))$. This means there exists $\sigma \in \Xi(k)$: $\neg(Event_q(\#(k, \varphi, \sigma)))$.

Let's now move to the second step of our proof, where we expect to prove that if $\mathbf{k} \in \mathbf{K}(\mathbf{n})$ then for each $A \in \mathcal{H}(n+1, k)$ and for each of our predicates P P is decidable over A.

By the inductive hypothesis (i.e. what we assumed true at level n) we can assume that each of our predicates P is decidable over $\mathbf{E}(\mathbf{n}, \mathbf{k})$.

Let's now try to prove that for each of our predicates P P is decidable over $\mathbf{E_b}(\mathbf{n+1}, \mathbf{k})$.

If $k = \epsilon$ we have $E_b(n+1, k) = \emptyset$, so our predicates are trivially decidabile over such empty domain.

We'll then consider the case $k \neq \epsilon$. Here by our assumption 6.1.8 $n > 1$ and there exist $m < n$, $h \in K(m)$, $\phi \in E_s(m, h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h + < y, \phi >$, $\Xi(k) = \{\rho + (y, s) | \, \rho \in \Xi(h), s \in \#(h, \phi, \rho)\})$.

For each $\varphi \in E_b(n+1, k)$, $\sigma = \rho + (y, s) \in \Xi(k)$ $\#(k, \varphi, \sigma) = \#(h, \varphi, \rho)$.

We first consider the predicate 'for each $\sigma \in \Xi(k)$ $Set_q(\#(k, \varphi, \sigma))$' (where q is a positive integer).

By the inductive hypothesis we can decide if the following condition holds: 'for each $\rho \in \Xi(h)$ $Set_q(\#(h, \varphi, \rho))$'.

If the just mentioned condition holds we can consider that for each $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$ and $\#(k, \varphi, \sigma) = \#(h, \varphi, \rho)$. Since $Set_q(\#(h, \varphi, \rho))$ then $Set_q(\#(k, \varphi, \sigma))$ holds too.

If the mentioned condition is decided as false then there exists $\rho \in \Xi(h)$: $\neg(Set_q(\#(h, \varphi, \rho)))$. We have that for each $\delta \in \Xi(h)$ $Set_1(\#(h, \phi, \delta))$, so for each $\delta \in \Xi(h)$ $NotEmpty_1(\#(h, \phi, \delta))$. So let $s \in \#(h, \phi, \rho)$ and let $\sigma = \rho + (y, s)$, then $\sigma \in \Xi(k)$ and $\#(k, \varphi, \sigma) = \#(h, \varphi, \rho)$ and so $\neg(Set_q(\#(k, \varphi, \sigma)))$.

We now want to prove the following:

if (for each $\sigma \in \Xi(k)$ $Set_q(\#(k, \varphi, \sigma))$) then
(for each $\sigma \in \Xi(k)$ $NotEmpty_q(\#(k, \varphi, \sigma))$).

We assume (for each $\sigma \in \Xi(k)$ $Set_q(\#(k, \varphi, \sigma))$), clearly this implies
(for each $\rho \in \Xi(h)$ $Set_q(\#(h, \phi, \rho))$), which (by inductive hypothesis) implies
(for each $\rho \in \Xi(h)$ $NotEmpty_q(\#(h, \phi, \rho))$).

We can then consider that for each $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$, $\#(k, \varphi, \sigma) = \#(h, \phi, \rho)$. Since $NotEmpty_q(\#(h, \phi, \rho))$ holds then $NotEmpty_q(\#(k, \varphi, \sigma))$ holds too.

Given $i = 1 \dots p$ we now want to consider the predicate 'for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma) \in D_i$'.

By the inductive hypothesis we can decide the condition 'for each $\rho \in \Xi(h)$ $\#(h, \varphi, \rho) \in D_i$'.

If the just mentioned condition holds then we consider that for each $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$ and $\#(k, \varphi, \sigma) = \#(h, \varphi, \rho)$. Therefore clearly $\#(k, \varphi, \sigma) \in D_i$.

If on the contrary the mentioned condition is decided as false then there exists $\rho \in \Xi(h)$: $\#(h, \varphi, \rho) \notin D_i$. We have that for each $\delta \in \Xi(h)$ $Set_1(\#(h, \phi, \delta))$, so for each $\delta \in \Xi(h)$ $NotEmpty_1(\#(h, \phi, \delta))$. So let $s \in \#(h, \phi, \rho)$ and let $\sigma = \rho + (y, s)$, then $\sigma \in \Xi(k)$ and $\#(k, \varphi, \sigma) = \#(h, \varphi, \rho) \notin D_i$.

Given $i = 1 \dots p$ and a positive integer q we now want to consider the predicate 'for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma) \in \mathcal{P}^q(D_i)$'.

By the inductive hypothesis we are able to decide the predicate 'for each $\rho \in \Xi(h)$ $\#(h, \varphi, \rho) \in \mathcal{P}^q(D_i)$'.

If the just mentioned condition holds then we consider that for each $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$ and $\#(k, \varphi, \sigma) = \#(h, \varphi, \rho)$. Therefore clearly $\#(k, \varphi, \sigma) \in \mathcal{P}^q(D_i)$.

If on the contrary the mentioned condition is decided as false then there exists $\rho \in \Xi(h)$: $\#(h, \varphi, \rho) \notin \mathcal{P}^q(D_i)$. We have that for each $\delta \in \Xi(h)$ $Set_1(\#(h, \phi, \delta))$, so for each $\delta \in \Xi(h)$ $NotEmpty_1(\#(h, \phi, \delta))$. So let $s \in \#(h, \phi, \rho)$ and let $\sigma = \rho + (y, s)$, then $\sigma \in \Xi(k)$ and $\#(k, \varphi, \sigma) = \#(h, \varphi, \rho) \notin \mathcal{P}^q(D_i)$.

Given a positive integer q we now want to consider the predicate 'for each $\sigma \in \Xi(k)$ $Event_q(\#(k, \varphi, \sigma))$'.

By the inductive hypothesis we are able to decide the predicate 'for each $\rho \in \Xi(h)$ $Event_q(\#(h, \varphi, \rho))$'.

If the just mentioned condition holds then we consider that for each $\sigma \in \Xi(k)$ there exist $\rho \in \Xi(h)$, $s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$ and $\#(k, \varphi, \sigma) = \#(h, \varphi, \rho)$. Therefore clearly $Event_q(\#(k, \varphi, \sigma))$.

If on the contrary the mentioned condition is decided as false then there exists $\rho \in \Xi(h)$: $\neg(Event_q(\#(h, \varphi, \rho)))$. We have that for each $\delta \in \Xi(h)$ $Set_1(\#(h, \phi, \delta))$, so for each $\delta \in \Xi(h)$ $NotEmpty_1(\#(h, \phi, \delta))$. So let $s \in \#(h, \phi, \rho)$ and let $\sigma = \rho + (y, s)$, then $\sigma \in \Xi(k)$ and $\#(k, \varphi, \sigma) = \#(h, \varphi, \rho)$, so $\neg(Event_q(\#(k, \varphi, \sigma)))$.

Let's now try to prove that for each of our predicates P P is decidable over $\mathbf{E_e(n+1, k)}$.

We recall that for every $t = \{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi) \in E_e(n+1, k)$ we have defined

$$\#(k,t,\sigma) = \{\#(k'_m, \phi, \sigma'_m) |\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\},$$

where $k'_1 = k + < x_1, \varphi_1 >$, and if $m > 1$ for each $i = 1 \dots m-1$ $k'_{i+1} = k'_i + < x_{i+1}, \varphi_{i+1} >$.

We first consider the predicate 'for each $\sigma \in \Xi(k)$ $Set_q(\#(k,t,\sigma))$' (where q is a positive integer).

It is clear that for each $\sigma \in \Xi(k)$ $Set_1(\#(k,t,\sigma))$ holds true.

Let's then examine the condition 'for each $\sigma \in \Xi(k)$ $Set_{q+1}(\#(k,t,\sigma))$' (where q is a positive integer).

We have $\phi \in E(n, k'_m)$ so we can decide the condition 'for each $\sigma'_m \in \Xi(k'_m)$ $Set_q(\#(k'_m, \phi, \sigma'_m))$'.

If the just mentioned condition holds true then we can observe that for each $\sigma \in \Xi(k)$ and for each $u \in \#(k,t,\sigma)$ there exists $\sigma'_m \in \Xi(k'_m)$ such that $u = \#(k'_m, \phi, \sigma'_m)$, and so $Set_q(u)$. It follows that $Set_{q+1}(\#(k,t,\sigma))$ holds true.

If on the contrary the mentioned condition is decided as false then there exists $\sigma'_m \in \Xi(k'_m)$ such that $\neg(Set_q(\#(k'_m, \phi, \sigma'_m)))$.

Let $\sigma = (\sigma'_m)_{/dom(k)}$, we can apply assumption 6.1.9 to show that $\sigma \in \Xi(k)$. In fact $k'_m \neq \epsilon$ so $n > 1$, $\sigma'_m \in \Xi(k'_m)$, $k \in \mathcal{R}(k'_m)$, $k \neq k'_m$. It is also obvious that $\sigma \sqsubseteq \sigma'_m$. So $\#(k'_m, \phi, \sigma'_m) \in \#(k,t,\sigma)$, and so there exists $u \in \#(k,t,\sigma)$ such that $\neg(Set_q(u))$. Finally, there exists $\sigma \in \Xi(k)$ such that $\neg(Set_{q+1}(\#(k,t,\sigma)))$.

We have seen that for each $\sigma \in \Xi(k)$ $Set_1(\#(k,t,\sigma))$. So we also need to show that for each $\sigma \in \Xi(k)$ $NotEmpty_1(\#(k,t,\sigma))$.

Given $\sigma \in \Xi(k)$, in order to show $NotEmpty_1(\#(k,t,\sigma))$ we have to prove there exists $\sigma'_m \in \Xi(k'_m)$ such that $\sigma \sqsubseteq \sigma'_m$, in this case in fact $\#(k'_m, \phi, \sigma'_m) \in \#(k,t,\sigma)$.

First we will show there exists $\sigma'_1 \in \Xi(k'_1)$ such that $\sigma \sqsubseteq \sigma'_1$.

Indeed $\varphi_1 \in E_s(n,k)$ so for each $\delta \in \Xi(k)$ $Set_1(\#(k, \varphi_1, \delta))$. So for each $\delta \in \Xi(k)$ $NotEmpty_1(\#(k, \varphi_1, \delta))$ and this implies $\#(k, \varphi_1, \sigma) \neq \emptyset$. So given $s_1 \in \#(k, \varphi_1, \sigma)$ we can define $\sigma'_1 = \sigma + (x_1, s_1)$ and clearly $\sigma'_1 \in \Xi(k'_1)$. Obviously $\sigma \sqsubseteq \sigma'_1$.

If $m > 1$ given $j = 1 \dots m-1$ we can assume the existence of $\sigma'_j \in \Xi(k'_j)$ such that $\sigma \sqsubseteq \sigma'_j$ and prove the existence of $\sigma'_{j+1} \in \Xi(k'_{j+1})$ such that $\sigma \sqsubseteq \sigma'_{j+1}$. Indeed $\varphi_{j+1} \in E_s(n, k'_j)$ so for each $\delta \in \Xi(k'_j)$ $Set_1(\#(k'_j, \varphi_{j+1}, \delta))$. So for each $\delta \in \Xi(k'_j)$ $NotEmpty_1(\#(k'_j, \varphi_{j+1}, \delta))$ and this implies $\#(k'_j, \varphi_{j+1}, \sigma'_j) \neq \emptyset$. So given $s_{j+1} \in \#(k'_j, \varphi_{j+1}, \sigma'_j)$ we can define $\sigma'_{j+1} = \sigma'_j + (x_{j+1}, s_{j+1})$ and clearly $\sigma'_{j+1} \in \Xi(k'_{j+1})$. Obviously $\sigma \sqsubseteq \sigma'_j \sqsubseteq \sigma'_{j+1}$.

So it is proved that there exists $\sigma'_m \in \Xi(k'_m)$ such that $\sigma \sqsubseteq \sigma'_m$, and thus $\#(k'_m, \phi, \sigma'_m) \in \#(k, t, \sigma)$.

Let's now consider the case where for each $\sigma \in \Xi(k)$ $Set_{q+1}(\#(k, t, \sigma))$. We want to show that for each $\sigma \in \Xi(k)$ $NotEmpty_{q+1}(\#(k, t, \sigma))$.

The following condition holds true:
'for each $\sigma'_m \in \Xi(k'_m)$ $Set_q(\#(k'_m, \phi, \sigma'_m))$'.

Consequently the following also holds:
'for each $\sigma'_m \in \Xi(k'_m)$ $NotEmpty_q(\#(k'_m, \phi, \sigma'_m))$'.

Given $\sigma \in \Xi(k)$ we have to show $NotEmpty_1(\#(k, t, \sigma))$ and for each $u \in \#(k, t, \sigma)$ $NotEmpty_q(u)$.

Since for each $\delta \in \Xi(k)$ $Set_{q+1}(\#(k, t, \delta))$ then for each $\delta \in \Xi(k)$ $Set_1(\#(k, t, \delta))$ also holds. This implies $NotEmpty_1(\#(k, t, \sigma))$.

Moreover given $u \in \#(k, t, \sigma)$ there exists $\sigma'_m \in \Xi(k'_m)$ such that $u = \#(k'_m, \phi, \sigma'_m)$, so $NotEmpty_q(u)$ holds true.

Given $i = 1 \ldots p$ we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in D_i$'.

Clearly for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma)$ is a set and so $\#(k, t, \sigma) \notin D_i$. Since $\Xi(k) \neq \emptyset$ the condition 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in D_i$' is false.

Given $i = 1 \ldots p$ we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in \mathcal{P}(D_i)$'.

We have $\phi \in E(n, k'_m)$ so we can decide the condition 'for each $\sigma'_m \in \Xi(k'_m)$ $\#(k'_m, \phi, \sigma'_m) \in D_i$'.

If this condition is true then we can prove that 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in \mathcal{P}(D_i)$' is also true.

In fact given $u \in \#(k, t, \sigma)$ we have there exists $\sigma'_m \in \Xi(k'_m)$ such that $u = \#(k'_m, \phi, \sigma'_m) \in D_i$ and so $\#(k, t, \sigma) \subseteq D_i$, and since $\#(k, t, \sigma) \neq \emptyset$, $\#(k, t, \sigma) \in \mathcal{P}(D_i)$.

If on the contrary the mentioned condition is decided as false then there exists $\sigma'_m \in \Xi(k'_m)$ such that $\#(k'_m, \phi, \sigma'_m) \notin D_i$.

Let $\sigma = (\sigma'_m)_{/dom(k)}$, we can apply assumption 6.1.9 to show that $\sigma \in \Xi(k)$. In fact $k'_m \neq \epsilon$ so $n > 1$, $\sigma'_m \in \Xi(k'_m)$, $k \in \mathcal{R}(k'_m)$, $k \neq k'_m$. It is also obvious that $\sigma \sqsubseteq \sigma'_m$. So $\#(k'_m, \phi, \sigma'_m) \in \#(k, t, \sigma)$, and so there exists $u \in \#(k, t, \sigma)$ such that $u \notin D_i$. So $\#(k, t, \sigma)$ is not a subset of D_i, and so $\#(k, t, \sigma) \notin \mathcal{P}(D_i)$.
Finally, there exists $\sigma \in \Xi(k)$ such that $\#(k, t, \sigma) \notin \mathcal{P}(D_i)$.

Given $i = 1 \ldots p$ and a positive integer q we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^{q+1}(D_i)$'.

We have $\phi \in E(n,k'_m)$ so we can decide the condition 'for each $\sigma'_m \in \Xi(k'_m)$ $\#(k'_m,\phi,\sigma'_m) \in \mathcal{P}^q(D_i)$'.

If this condition is true then we can prove that 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^{q+1}(D_i)$' is also true.

In fact given $u \in \#(k,t,\sigma)$ we have there exists $\sigma'_m \in \Xi(k'_m)$ such that $u = \#(k'_m,\phi,\sigma'_m) \in \mathcal{P}^q(D_i)$ and so $\#(k,t,\sigma) \subseteq \mathcal{P}^q(D_i)$, and since $\#(k,t,\sigma) \neq \emptyset$, $\#(k,t,\sigma) \in \mathcal{P}^{q+1}(D_i)$.

If on the contrary the mentioned condition is decided as false then there exists $\sigma'_m \in \Xi(k'_m)$ such that $\#(k'_m,\phi,\sigma'_m) \notin \mathcal{P}^q(D_i)$.

Let $\sigma = (\sigma'_m)_{/dom(k)}$, we can apply assumption 6.1.9 to show that $\sigma \in \Xi(k)$. In fact $k'_m \neq \epsilon$ so $n > 1$, $\sigma'_m \in \Xi(k'_m)$, $k \in \mathcal{R}(k'_m)$, $k \neq k'_m$. It is also obvious that $\sigma \sqsubseteq \sigma'_m$. So $\#(k'_m,\phi,\sigma'_m) \in \#(k,t,\sigma)$, and so there exists $u \in \#(k,t,\sigma)$ such that $u \notin \mathcal{P}^q(D_i)$. So $\#(k,t,\sigma)$ is not a subset of $\mathcal{P}^q(D_i)$, and so $\#(k,t,\sigma) \notin \mathcal{P}^{q+1}(D_i)$.
Finally, there exists $\sigma \in \Xi(k)$ such that $\#(k,t,\sigma) \notin \mathcal{P}^{q+1}(D_i)$.

Given a positive integer q we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Event_q(\#(k,t,\sigma))$'.

Actually for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma)$ is a set, so $\neg(Event_1(\#(k,t,\sigma))$. Therefore the condition 'for each $\sigma \in \Xi(k)$ $Event_1(\#(k,t,\sigma))$' is false.

Given a positive integer q we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Event_{q+1}(\#(k,t,\sigma))$'.

We have $\phi \in E(n,k'_m)$ so we can decide the condition 'for each $\sigma'_m \in \Xi(k'_m)$ $Event_q(\#(k'_m,\phi,\sigma'_m))$'.

If this condition is true then for each $\sigma \in \Xi(k)$ for each $u \in \#(k,t,\sigma)$ there exists $\sigma'_m \in \Xi(k'_m)$ such that $u = \#(k'_m,\phi,\sigma'_m)$, and so $Event_q(u)$. Therefore $Event_{q+1}(\#(k,t,\sigma))$.

If on the contrary the mentioned condition is decided as false then there exists $\sigma'_m \in \Xi(k'_m)$ such that $\neg(Event_q(\#(k'_m,\phi,\sigma'_m)))$.

Let $\sigma = (\sigma'_m)_{/dom(k)}$, we can apply assumption 6.1.9 to show that $\sigma \in \Xi(k)$. In fact $k'_m \neq \epsilon$ so $n > 1$, $\sigma'_m \in \Xi(k'_m)$, $k \in \mathcal{R}(k'_m)$, $k \neq k'_m$. It is also obvious that $\sigma \sqsubseteq \sigma'_m$. So $\#(k'_m,\phi,\sigma'_m) \in \#(k,t,\sigma)$, and so there exists $u \in \#(k,t,\sigma)$ such that $\neg(Event_q(u))$. So $\neg(Event_{q+1}(\#(k,t,\sigma)))$.
Finally, there exists $\sigma \in \Xi(k)$ such that $\neg(Event_{q+1}(\#(k,t,\sigma)))$.

Let's now try to prove that for each of our predicates P and for each $c \in C'$ P is decidable over $\mathbf{E^c(n+1,k)}$.

§. We first consider the case where $\#(c)$ is a function whose domain is $(D_i)^m$ and whose range is D_i, and $E^c(n+1,k)$ is defined as the set of the strings $(c)(\varphi_1,\ldots,\varphi_m) \in H_c(n+1,k)$ such that:

- $\varphi_1,\ldots,\varphi_m \in E(n,k)$;
- for each $j = 1\ldots m$, $\sigma \in \Xi(k)$ $\#(k,\varphi_j,\sigma) \in D_i$;
- $(c)(\varphi_1,\ldots,\varphi_m) \notin E(n,k)$;
- $(c)(\varphi_1,\ldots,\varphi_m) \notin E_b(n+1,k)$.

We recall that for each $t = (c)(\varphi_1,\ldots,\varphi_m) \in E^c(n+1,k)$ we have defined

$$\#(k,t,\sigma) = \#(c)(\#(k,\varphi_1,\sigma),\ldots,\#(k,\varphi_m,\sigma)).$$

It is immediately clear that the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_i$' is true and that the corresponding predicate over $E^c(n+1,k)$ is decidable.

Given $j = 1\ldots p$ such that $j \neq i$ we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_j$'.

Since for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_i$ and $D_i \cap D_j = \emptyset$ then for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \notin D_j$ and we can decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_j$' is false.

Given $j = 1\ldots p$ and a positive integer q we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^q(D_j)$'.

For each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_i$, so $\#(k,t,\sigma)$ is not a set and $\#(k,t,\sigma) \notin \mathcal{P}^q(D_j)$, therefore the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^q(D_j)$' is false.

We now want to decide the condition 'for each $\sigma \in \Xi(k)$ $Set_q(\#(k,t,\sigma))$'.

For each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_i$, so $\#(k,t,\sigma)$ is not a set and $\neg(Set_q(\#(k,t,\sigma)))$, therefore the mentioned condition must be false.

We now want to decide the condition 'for each $\sigma \in \Xi(k)$ $Event_1(\#(k,t,\sigma))$'.

For each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_i$, so $\neg(Event_1(\#(k,t,\sigma)))$. Therefore the mentioned condition is false.
Given a positive integer q, we now want to decide the condition 'for each $\sigma \in \Xi(k)$ $Event_{q+1}(\#(k,t,\sigma))$'.

For each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_i$, so $\neg(Set_1(\#(k,t,\sigma)))$. Therefore the mentioned condition is false.

§. We now consider the case where there exist $i = 1\ldots p$, a positive integer q and a positive integer m such that $\#(c)$ is a function whose domain is $(\mathcal{P}^q(D_i))^m$ and whose range is $\mathcal{P}^q(D_i)$. In this case we defined $E^c(n+1,k)$ as the set of the strings $(c)(\varphi_1,\ldots,\varphi_m) \in H_c(n+1,k)$ such that:

- $\varphi_1,\ldots,\varphi_m \in E(n,k)$;
- for each $j = 1\ldots m$, $\sigma \in \Xi(k)$ $\#(k,\varphi_j,\sigma) \in \mathcal{P}^q(D_i)$;

- $(c)(\varphi_1, \ldots, \varphi_m) \notin E(n,k)$;
- $(c)(\varphi_1, \ldots, \varphi_m) \notin E_b(n+1,k)$.

By our definitions, for each $t = (c)(\varphi_1, \ldots, \varphi_m) \in E^c(n+1,k)$

$$\#(k,t,\sigma) = \#(c)(\#(k,\varphi_1,\sigma), \ldots, \#(k,\varphi_m,\sigma)).$$

It is immediately clear that the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^q(D_i)$' is true and that the corresponding predicate over $E^c(n+1,k)$ is decidable.

Given $j = 1 \ldots p$ we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_j$'.

For each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^q(D_i)$, so $\#(k,t,\sigma)$ is a set and $\#(k,t,\sigma) \notin D_j$, therefore the mentioned condition is false.

Given $j = 1 \ldots p$ and a positive integer r such that $r \neq q$, we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^r(D_j)$'. By lemma 3.14 $\mathcal{P}^r(D_j) \cap \mathcal{P}^q(D_i) = \emptyset$, so for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \notin \mathcal{P}^r(D_j)$, and the mentioned condition is false.

Finally, given $j = 1 \ldots p$ such that $j \neq i$ we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^q(D_j)$'. By lemma 3.15 $\mathcal{P}^q(D_j) \cap \mathcal{P}^q(D_i) = \emptyset$, so for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \notin \mathcal{P}^q(D_j)$, and the mentioned condition is false.

Given a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Set_r(\#(k,t,\sigma))$', and when this condition is decided as true we must also be able to decide that for each $\sigma \in \Xi(k)$ $NotEmpty_r(\#(k,t,\sigma))$.

We first consider the case $r \leqslant q$. We know that for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^q(D_i)$, so by lemma 3.16 for each $\sigma \in \Xi(k)$ $Set_r(\#(k,t,\sigma))$ and $NotEmpty_r(\#(k,t,\sigma))$.

Let's now consider the case $r > q$. Here by lemma 3.13 for each $\sigma \in \Xi(k)$ $\neg Set_r(\#(k,t,\sigma))$, so the condition 'for each $\sigma \in \Xi(k)$ $Set_r(\#(k,t,\sigma))$' is false.

Given a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Event_r(\#(k,t,\sigma))$'.

We know that for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^q(D_i)$, so by lemmas 3.17 and 3.19 $\neg Event_r(\#(k,t,\sigma))$. This obvioulsy implies that our condition 'for each $\sigma \in \Xi(k)$ $Event_r(\#(k,t,\sigma))$' is false.

§. We now consider the case where there exist $i = 1 \ldots p$ and a positive integer m such that $\#(c)$ is a function whose domain is $(D_i)^m$ and such that for each $(d_1, \ldots, d_m) \in (D_i)^m$ $\#(c)(d_1, \ldots, d_m)$ is true or false. In this case we defined $E^c(n+1,k)$ as the set of the strings $(c)(\varphi_1, \ldots, \varphi_m) \in H_c(n+1,k)$ such that:

- $\varphi_1, \ldots, \varphi_m \in E(n,k)$;
- for each $j = 1 \ldots m$, $\sigma \in \Xi(k)$ $\#(k,\varphi_j,\sigma) \in D_i$;

- $(c)(\varphi_1, \ldots, \varphi_m) \notin E(n, k)$;
- $(c)(\varphi_1, \ldots, \varphi_m) \notin E_b(n + 1, k)$.

By our definitions, for each $t = (c)(\varphi_1, \ldots, \varphi_m) \in E^c(n + 1, k)$

$$\#(k, t, \sigma) = \#(c)(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma)).$$

Clearly for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma)$ is true or false.

Given $\alpha = 1 \ldots p$ we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in D_\alpha$'.

Since for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma)$ is true or false, then for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \notin D_\alpha$, and so then condition 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in D_\alpha$' is false.

Given $\alpha = 1 \ldots p$ and a positive integer q we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in \mathcal{P}^q(D_\alpha)$'.

Given $\sigma \in \Xi(k)$ $Event_1(\#(k, t, \sigma))$ and by lemma 3.17 this implies $\#(k, t, \sigma) \notin \mathcal{P}^q(D_\alpha)$. Therefore the condition 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in \mathcal{P}^q(D_\alpha)$' is false.

Given a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Set_r(\#(k, t, \sigma))$', and when this condition is decided as true we must also be able to decide that for each $\sigma \in \Xi(k)$ $NotEmpty_r(\#(k, t, \sigma))$.

For each $\sigma \in \Xi(k)$ $Event_1(\#(k, t, \sigma))$ so $\neg Set_1(\#(k, t, \sigma))$ and then also $\neg Set_r(\#(k, t, \sigma))$. Therefore the condition 'for each $\sigma \in \Xi(k)$ $Set_r(\#(k, t, \sigma))$' is false.

Given a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Event_r(\#(k, t, \sigma))$'.

Clearly the condition is true for $r = 1$, while for $r > 1$ given $\sigma \in \Xi(k)$ $\neg Set_1(\#(k, t, \sigma))$ and so $\neg Event_r(\#(k, t, \sigma))$, so the condition is false for $r > 1$.

§. We now consider the case where there exist $i = 1 \ldots p$, a positive integer q and a positive integer m such that $\#(c)$ is a function whose domain is $(\mathcal{P}^q(D_i))^m$ and such that for each $(d_1, \ldots, d_m) \in (\mathcal{P}^q(D_i))^m$ $\#(c)(d_1, \ldots, d_m)$ is true or false. In this case we defined $E^c(n + 1, k)$ as the set of the strings $(c)(\varphi_1, \ldots, \varphi_m) \in H_c(n + 1, k)$ such that:

- $\varphi_1, \ldots, \varphi_m \in E(n, k)$;
- for each $j = 1 \ldots m$, $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in \mathcal{P}^q(D_i)$;
- $(c)(\varphi_1, \ldots, \varphi_m) \notin E(n, k)$;
- $(c)(\varphi_1, \ldots, \varphi_m) \notin E_b(n + 1, k)$.

By our definitions, for each $t = (c)(\varphi_1, \ldots, \varphi_m) \in E^c(n + 1, k)$

$$\#(k, t, \sigma) = \#(c)(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma)).$$

Clearly for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma)$ is true or false.

Given $\alpha = 1 \dots p$ we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_\alpha$'.

Since for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma)$ is true or false, then for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \notin D_\alpha$, and so then condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_\alpha$' is false.

Given $\alpha = 1 \dots p$ and a positive integer q we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^q(D_\alpha)$'.

Given $\sigma \in \Xi(k)$ $Event_1(\#(k,t,\sigma))$ and by lemma 3.17 this implies $\#(k,t,\sigma) \notin \mathcal{P}^q(D_\alpha)$. Therefore the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^q(D_\alpha)$' is false.

Given a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Set_r(\#(k,t,\sigma))$', and when this condition is decided as true we must also be able to decide that for each $\sigma \in \Xi(k)$ $NotEmpty_r(\#(k,t,\sigma))$.

For each $\sigma \in \Xi(k)$ $Event_1(\#(k,t,\sigma))$ so $\neg Set_1(\#(k,t,\sigma))$ and then also $\neg Set_r(\#(k,t,\sigma))$. Therefore the condition 'for each $\sigma \in \Xi(k)$ $Set_r(\#(k,t,\sigma))$' is false.

Given a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Event_r(\#(k,t,\sigma))$'.

Clearly the condition is true for $r = 1$, while for $r > 1$ given $\sigma \in \Xi(k)$ $\neg Set_1(\#(k,t,\sigma))$ and so $\neg Event_r(\#(k,t,\sigma))$, so the condition is false for $r > 1$.

§. We now consider the case where $\#(c)$ is a function whose domain is $\bigcup_{q \geqslant 1}(\bigcup_{i=1\dots p}(\mathcal{P}^q(D_i))^m)$ such that for each $q \geqslant 1$, $i = 1 \dots p$, $(A_1, \dots, A_m) \in (\mathcal{P}^q(D_i))^m$ $\#(c)(A_1, \dots, A_m) \in \mathcal{P}^q(D_i)$. In this case we defined $E^c(n+1,k)$ as the set of the strings $(c)(\varphi_1, \dots, \varphi_m) \in H_c(n+1,k)$ such that:

- $\varphi_1, \dots, \varphi_m \in E(n,k)$;
- there exist $i = 1 \dots p$, $q = 1 \dots q_{max}$ such that for each $j = 1 \dots m$, $\sigma \in \Xi(k)$ $\#(k,\varphi_j,\sigma) \in \mathcal{P}^q(D_i)$;
- $(c)(\varphi_1, \dots, \varphi_m) \notin E(n,k)$;
- $(c)(\varphi_1, \dots, \varphi_m) \notin E_b(n+1,k)$.

By our definitions, for each $t = (c)(\varphi_1, \dots, \varphi_m) \in E^c(n+1,k)$

$$\#(k,t,\sigma) = \#(c)(\#(k,\varphi_1,\sigma), \dots, \#(k,\varphi_m,\sigma)).$$

Clearly, by the inductive hypothesis, given $t = (c)(\varphi_1, \dots, \varphi_m) \in E^c(n+1,k)$ for each $i = 1 \dots p$ and $q = 1 \dots q_{max}$ we can decide if for each $j = 1 \dots m$ and $\sigma \in \Xi(k)$ $\#(k,\varphi_j,\sigma) \in \mathcal{P}^q(D_i)$. There must exist $i = 1 \dots p$ and $q = 1 \dots q_{max}$ such that for each $j = 1 \dots m$ and $\sigma \in \Xi(k)$ $\#(k,\varphi_j,\sigma) \in \mathcal{P}^q(D_i)$, and so we can determine i and q with such requirements.

Given $\alpha = 1 \dots p$ we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_\alpha$'.

As we have just seen, there must exist $i = 1 \ldots p$ and $q = 1 \ldots q_{max}$ such that for each $j = 1 \ldots m$ and $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in \mathcal{P}^q(D_i)$, so $\#(k, t, \sigma) \in \mathcal{P}^q(D_i)$, $\#(k, t, \sigma)$ is a set and $\#(k, t, \sigma) \notin D_\alpha$, therefore the mentioned condition is false.

Given $\alpha = 1 \ldots p$ and a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in \mathcal{P}^r(D_\alpha)$'.

We first consider the case where $r \leqslant q_{max}$. In this case we have two subcases: the first subcase is when for each $j = 1 \ldots m$ and $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in \mathcal{P}^r(D_\alpha)$, in this case for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in \mathcal{P}^r(D_\alpha)$.
Otherwise there must exist $i = 1 \ldots p$ and $q = 1 \ldots q_{max}$ such that for each $j = 1 \ldots m$ and $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in \mathcal{P}^q(D_i)$ and so $\#(k, t, \sigma) \in \mathcal{P}^q(D_i)$. Of course in this case $i \neq \alpha$ or $r \neq q$.
In the case $r \neq q$ by lemma 3.14 $\mathcal{P}^r(D_\alpha) \cap \mathcal{P}^q(D_i) = \emptyset$, so for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \notin \mathcal{P}^r(D_\alpha)$, and the condition we are discussing 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in \mathcal{P}^r(D_\alpha)$' is false.
In the case $r = q$ and $i \neq \alpha$ by lemma 3.15 $\mathcal{P}^r(D_\alpha) \cap \mathcal{P}^q(D_i) = \emptyset$, so for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \notin \mathcal{P}^r(D_\alpha)$, and the condition we are discussing 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in \mathcal{P}^r(D_\alpha)$' is false.

Let's now consider the case where $r > q_{max}$. Here there must exist $i = 1 \ldots p$ and $q = 1 \ldots q_{max}$ such that for each $j = 1 \ldots m$ and $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in \mathcal{P}^q(D_i)$. Clearly $r \neq q$, so by lemma 3.14 $\mathcal{P}^r(D_\alpha) \cap \mathcal{P}^q(D_i) = \emptyset$, so for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \notin \mathcal{P}^r(D_\alpha)$, and the condition we are discussing 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in \mathcal{P}^r(D_\alpha)$' is false.

Given a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Set_r(\#(k, t, \sigma))$', and when this condition is decided as true we must also be able to decide that for each $\sigma \in \Xi(k)$ $NotEmpty_r(\#(k, t, \sigma))$.

We have seen that, given $t = (c)(\varphi_1, \ldots, \varphi_m) \in E^c(n + 1, k)$, there must exist $i = 1 \ldots p$ and $q = 1 \ldots q_{max}$ such that for each $j = 1 \ldots m$ and $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in \mathcal{P}^q(D_i)$, and that we can determine such i and q.

Then, given $t = (c)(\varphi_1, \ldots, \varphi_m) \in E^c(n + 1, k)$, let $i = 1 \ldots p$ and $q = 1 \ldots q_{max}$ be such that for each $j = 1 \ldots m$ and $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in \mathcal{P}^q(D_i)$. We have that for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in \mathcal{P}^q(D_i)$.
If $r \leqslant q$ then by lemma 3.16 for each $\sigma \in \Xi(k)$ $Set_r(\#(k, t, \sigma))$ and $NotEmpty_r(\#(k, t, \sigma))$.
If instead $r > q$ then by lemma 3.13 for each $\sigma \in \Xi(k)$ $\neg Set_r(\#(k, t, \sigma))$, and then the condition 'for each $\sigma \in \Xi(k)$ $Set_r(\#(k, t, \sigma))$' is false.

Given a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Event_r(\#(k, t, \sigma))$'.

Given $t = (c)(\varphi_1, \ldots, \varphi_m) \in E^c(n + 1, k)$, let $i = 1 \ldots p$ and $q = 1 \ldots q_{max}$ be such that for each $j = 1 \ldots m$ and $\sigma \in \Xi(k)$ $\#(k, \varphi_j, \sigma) \in \mathcal{P}^q(D_i)$. We have that for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in \mathcal{P}^q(D_i)$.
By lemmas 3.17 and 3.19 we can conclude that for each $\sigma \in \Xi(k)$ $\neg Event_r(\#(k, t, \sigma))$.

Therefore the condition 'for each $\sigma \in \Xi(k)$ $Event_r(\#(k, t, \sigma))$' is false.

§. We now consider the case where c is the special constant Π whose meaning $\#(\Pi)$ is a function over the domain $\bigcup_{q \geqslant 1}(\bigcup_{i=1...p} \mathcal{P}^q(D_i))$ such that for each $q \geqslant 1$, $i = 1 \dots p$ $A \in \mathcal{P}^q(D_i)$ $\#(\Pi)(A) = \mathcal{P}(A)$. In this case we defined $E^{\Pi}(n+1, k)$ as the set of the strings $(\Pi)(\varphi_1) \in H_{\Pi}(n+1, k)$ such that:

- $\varphi_1 \in E(n, k)$;
- there exist $i = 1 \dots p$, $q = 1 \dots q_{max}$ such that for each $\sigma \in \Xi(k)$ $\#(k, \varphi_1, \sigma) \in \mathcal{P}^q(D_i)$;
- $(\Pi)(\varphi_1) \notin E(n, k)$;
- $(\Pi)(\varphi_1) \notin E_b(n+1, k)$.

By our definitions, for each $t = (\Pi)(\varphi_1) \in E^{\Pi}(n+1, k)$

$$\#(k, t, \sigma) = \#(\Pi)(\#(k, \varphi_1, \sigma)).$$

Clearly, by the inductive hypothesis, given $t = (\Pi)(\varphi_1) \in E^{\Pi}(n+1, k)$ for each $i = 1 \dots p$ and $q = 1 \dots q_{max}$ we can decide if for each $\sigma \in \Xi(k)$ $\#(k, \varphi_1, \sigma) \in \mathcal{P}^q(D_i)$. There must exist $i = 1 \dots p$ and $q = 1 \dots q_{max}$ such that for each $\sigma \in \Xi(k)$ $\#(k, \varphi_1, \sigma) \in \mathcal{P}^q(D_i)$, and so we can determine i and q with such requirements.

We can also notice that, given a set B, if $A \in \mathcal{P}(B)$ then $A \subseteq B$, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, $\mathcal{P}(A) \in \mathcal{P}(\mathcal{P}(B))$.
So if $A \in \mathcal{P}^q(D_i)$ then we have two cases: if $q = 1$ then $A \in \mathcal{P}(D_i)$ and so $\mathcal{P}(A) \in \mathcal{P}^2(D_i)$.
If $q > 1$ then $A \in \mathcal{P}(\mathcal{P}^{q-1}(D_i))$, so $\mathcal{P}(A) \in \mathcal{P}^2(\mathcal{P}^{q-1}(D_i))$, that is $\mathcal{P}(A) \in \mathcal{P}^{q+1}(D_i)$.
Actually in both cases $\mathcal{P}(A) \in \mathcal{P}^{q+1}(D_i)$.

Clearly given $t = (\Pi)(\varphi_1) \in E^{\Pi}(n+1, k)$, $i = 1 \dots p$ and $q = 1 \dots q_{max}$ such that for each $\sigma \in \Xi(k)$ $\#(k, \varphi_1, \sigma) \in \mathcal{P}^q(D_i)$, we have that for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) = \mathcal{P}(\#(k, \varphi_1, \sigma)) \in \mathcal{P}^{q+1}(D_i)$.

Given $\alpha = 1 \dots p$ we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in D_\alpha$'.

As we have just seen, there must exist $i = 1 \dots p$ and $q = 1 \dots q_{max}$ such that for each $\sigma \in \Xi(k)$ $\#(k, \varphi_1, \sigma) \in \mathcal{P}^q(D_i)$, so $\#(k, t, \sigma) \in \mathcal{P}^{q+1}(D_i)$, $\#(k, t, \sigma)$ is a set and $\#(k, t, \sigma) \notin D_\alpha$, therefore the mentioned condition is false.

Given $\alpha = 1 \dots p$ and a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in \mathcal{P}^r(D_\alpha)$'.

Let $t = (\Pi)(\varphi_1)$. We first consider the case where $2 \leqslant r \leqslant q_{max} + 1$. In this case we have two subcases: the first subcase is when for each $\sigma \in \Xi(k)$ $\#(k, \varphi_1, \sigma) \in \mathcal{P}^{r-1}(D_\alpha)$, in this case for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma) \in \mathcal{P}^r(D_\alpha)$.
Otherwise there must exist $i = 1 \dots p$ and $q = 1 \dots q_{max}$ such that for each $\sigma \in \Xi(k)$ $\#(k, \varphi_1, \sigma) \in \mathcal{P}^q(D_i)$ and so $\#(k, t, \sigma) \in \mathcal{P}^{q+1}(D_i)$. Of course in this case $i \neq \alpha$ or $r - 1 \neq q$.
In the case $r - 1 \neq q$ by lemma 3.14 $\mathcal{P}^r(D_\alpha) \cap \mathcal{P}^{q+1}(D_i) = \emptyset$, so for each $\sigma \in \Xi(k)$

$\#(k,t,\sigma) \notin \mathcal{P}^r(D_\alpha)$, and the condition we are discussing 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^r(D_\alpha)$' is false.
In the case $r-1=q$ and $i \neq \alpha$ by lemma 3.15 $\mathcal{P}^r(D_\alpha) \cap \mathcal{P}^{q+1}(D_i) = \emptyset$, so for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \notin \mathcal{P}^r(D_\alpha)$, and the condition we are discussing 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^r(D_\alpha)$' is false.

Let's now consider the case where $r=1$ or $r > q_{max}+1$. Here there must exist $i = 1 \dots p$ and $q = 1 \dots q_{max}$ such that for each $\sigma \in \Xi(k)$ $\#(k,\varphi_1,\sigma) \in \mathcal{P}^q(D_i)$ and so $\#(k,t,\sigma) \in \mathcal{P}^{q+1}(D_i)$. Clearly $r \neq q+1$, so by lemma 3.14 $\mathcal{P}^r(D_\alpha) \cap \mathcal{P}^{q+1}(D_i) = \emptyset$, so for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \notin \mathcal{P}^r(D_\alpha)$, and the condition we are discussing 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^r(D_\alpha)$' is false.

Given a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Set_r(\#(k,t,\sigma))$', and when this condition is decided as true we must also be able to decide that for each $\sigma \in \Xi(k)$ $NotEmpty_r(\#(k,t,\sigma))$.

As we have seen given $t = (\Pi)(\varphi_1) \in E^\Pi(n+1,k)$, $i = 1 \dots p$ and $q = 1 \dots q_{max}$ such that for each $\sigma \in \Xi(k)$ $\#(k,\varphi_1,\sigma) \in \mathcal{P}^q(D_i)$, we have that for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) = \mathcal{P}(\#(k,\varphi_1,\sigma)) \in \mathcal{P}^{q+1}(D_i)$.

Let's first consider the case $r \leqslant q+1$, here by lemma 3.16 we have that for each $\sigma \in \Xi(k)$ $Set_r(\#(k,t,\sigma))$ and $NotEmpty_r(\#(k,t,\sigma))$.

Let's then consider the case $r > q+1$, here by lemma 3.13 we have that for each $\sigma \in \Xi(k)$ $\neg Set_r(\#(k,t,\sigma))$, hence the condition 'for each $\sigma \in \Xi(k)$ $Set_r(\#(k,t,\sigma))$' is false.

Given a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Event_r(\#(k,t,\sigma))$'.

As we have seen given $t = (\Pi)(\varphi_1) \in E^\Pi(n+1,k)$, $i = 1 \dots p$ and $q = 1 \dots q_{max}$ such that for each $\sigma \in \Xi(k)$ $\#(k,\varphi_1,\sigma) \in \mathcal{P}^q(D_i)$, we have that for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) = \mathcal{P}(\#(k,\varphi_1,\sigma)) \in \mathcal{P}^{q+1}(D_i)$.

By lemmas 3.17 and 3.19 we can conclude that in both cases $r \leqslant q+2$ and $r > q+2$ for each $\sigma \in \Xi(k)$ $\neg Event_r(\#(k,t,\sigma))$. Therefore the condition 'for each $\sigma \in \Xi(k)$ $Event_r(\#(k,t,\sigma))$' is false.

In order to finish our proof we have to prove that for each of our predicates P and for each $f \in \mathcal{F}$ P is decidable over $\mathbf{E^f(n+1,k)}$.

§. We first consider the case where f has multiplicity 2. In this case we defined $E^f(n+1,k)$ as the set of the strings $f(\varphi_1,\varphi_2) \in H_f(n+1,k)$ such that:

- $\varphi_1, \varphi_2 \in E(n,k)$;
- for each $\sigma \in \Xi(k)$ $A_f(\#(k,\varphi_1,\sigma), \#(k,\varphi_2,\sigma))$ is true;
- $f(\varphi_1,\varphi_2) \notin E(n,k)$;
- $f(\varphi_1,\varphi_2) \notin E_b(n+1,k)$.

By our definitions, for each $t = f(\varphi_1, \varphi_2) \in E^f(n+1,k)$,

$$\#(k,t,\sigma) = P_f(\#(k,\varphi_1,\sigma), \#(k,\varphi_2,\sigma)).$$

Clearly for each $t \in E^f(n+1,k)$ and $\sigma \in \Xi(k)$ $\#(k,t,\sigma)$ is true or false.

Let's also consider the case where f has multiplicity 1. In this case we defined $E^f(n+1,k)$ as the set of the strings $f(\varphi_1) \in H_f(n+1,k)$ such that:

- $\varphi_1 \in E(n,k)$;
- for each $\sigma \in \Xi(k)$ $A_f(\#(k,\varphi_1,\sigma))$ is true;
- $f(\varphi_1) \notin E(n,k)$.
- $f(\varphi_1) \notin E_b(n+1,k)$.

By our definitions, for each $t = f(\varphi_1) \in E^f(n+1,k)$,

$$\#(k,t,\sigma) = P_f(\#(k,\varphi_1,\sigma)).$$

It is also true in this case that for each $t \in E^f(n+1,k)$ and $\sigma \in \Xi(k)$ $\#(k,t,\sigma)$ is true or false, and we can show that each of our predicate is decidable (in both cases of multiplicity 1 and 2) using this property.

Given $\alpha = 1 \dots p$ we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_\alpha$'.

Since for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma)$ is true or false, then for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \notin D_\alpha$, and so then condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in D_\alpha$' is false.

Given $\alpha = 1 \dots p$ and a positive integer q we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^q(D_\alpha)$'.

Given $\sigma \in \Xi(k)$ $Event_1(\#(k,t,\sigma))$ and by lemma 3.17 this implies $\#(k,t,\sigma) \notin \mathcal{P}^q(D_\alpha)$. Therefore the condition 'for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) \in \mathcal{P}^q(D_\alpha)$' is false.

Given a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Set_r(\#(k,t,\sigma))$', and when this condition is decided as true we must also be able to decide that for each $\sigma \in \Xi(k)$ $NotEmpty_r(\#(k,t,\sigma))$.

For each $\sigma \in \Xi(k)$ $Event_1(\#(k,t,\sigma))$ so $\neg Set_1(\#(k,t,\sigma))$ and then also $\neg Set_r(\#(k,t,\sigma))$. Therefore the condition 'for each $\sigma \in \Xi(k)$ $Set_r(\#(k,t,\sigma))$' is false.

Given a positive integer r we must be able to decide the condition 'for each $\sigma \in \Xi(k)$ $Event_r(\#(k,t,\sigma))$'.

Clearly the condition is true for $r = 1$, while for $r > 1$ given $\sigma \in \Xi(k)$ $\neg Set_1(\#(k,t,\sigma))$ and so $\neg Event_r(\#(k,t,\sigma))$, so the condition is false for $r > 1$.

□

Proof of 6.1.11. We need to prove that for each $k \in K(n+1),\ t \in E(n+1,k)$

- $t[\ell(t)] \neq$ '(' ;
- if $t[\ell(t)] =$ ')' then $d(t,\ell(t)) = 1$, else $d(t,\ell(t)) = 0$;
- for each $\alpha \in \{1,\ldots,\ell(t)\}$ if $(t[\alpha] = \text{`:'}) \vee (t[\alpha] = \text{`,'}) \vee (t[\alpha] = \text{`)'})$ then $d(t,\alpha) \geqslant 1$.

We have seen that if $k \in K(n)^+$ $E(n+1,k) = E_a(n+1,k)$, and if $k \in K(n)$

$$E(n+1,k) = \bigcup_{A \in \mathcal{H}(n+1,k)} A\,,$$

with the following definition of $\mathcal{H}(n+1,k)$:

$$\mathcal{H}(n{+}1,k) = \{E(n,k), E_b(n{+}1,k), E_e(n{+}1,k)\} \cup \{E^c(n{+}1,k) | c \in \mathcal{C}'\} \cup \{E^f(n{+}1,k) | f \in \mathcal{F}\}.$$

Let $k \in K(n)^+$ and $t \in \mathbf{E_a(n+1,k)}$. There exist $h \in K(n)$, $\phi \in E_s(n,h)$, $y \in \mathcal{V} - var(h)$ such that $k = h + < y, \phi >$. We also have $t = y$, so t has just one character, $t[1]$ differs from '(', ':', ',', ')' and $d(t,\ell(t)) = 0$.

Let $k \in K(n)$ and $\mathbf{t} \in \mathbf{E(n,k)}$, this means that $t \in E(n)$. In this case we just need to apply assumption 6.1.11.

Let $k = h + < y, \phi > \in K(n) - \{\epsilon\}$ and $t \in \mathbf{E_b(n+1,k)}$. We have $h \in K(n)$, $t \in E(n,h)$, so we can apply assumption 6.1.11 to finish.

Let $k \in K(n)$, $c \in \mathcal{C}'$ and $t \in \mathbf{E^c(n+1,k)}$. Then $t \in H_c(n+1,k)$, so there exist $\varphi_1,\ldots,\varphi_m$ in $E(n,k)$ such that $t = (c)(\varphi_1,\ldots,\varphi_m)$.

In this representation of t we see 'explicit occurrences' of the symbols '(' , ')' and ','. There are explicit occurrences of ',' only when $m > 1$. The first explicit occurrence of ')' is in position 3, and the second explicit occurrence of ')' is clearly in position $\ell(t)$. If $m > 1$ we indicate with $q_1,\ldots,q_{m-1}$ the positions of the explicit occurrences of ','.

We have $d(t,2) = 1$ and also $d(t,3) = 1$, moreover $d(t,5) = d(t,3) - 1 + 1 = 1$.

If $m > 1$ we can prove that for each $i = 1 \ldots m-1$ $d(t,q_i) = 1$.

We first consider that

$$d(t,q_1 - 1) = d(t, 4 + \ell(\varphi_1)) = d(t,4+1) + d(\varphi_1,\ell(\varphi_1)) = 1 + d(\varphi_1,\ell(\varphi_1)).$$

If $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t,q_1) = d(t,q_1-1) - 1 = d(\varphi_1,\ell(\varphi_1)) = 1$. Else $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{\text{`('}, \text{`)'}\}$ so $d(t,q_1) = d(t,q_1 - 1) = 1 + d(\varphi_1,\ell(\varphi_1)) = 1$.

If $m = 2$ we have finished this step. Now suppose $m > 2$. Let $i = 1 \ldots m-2$ and suppose $d(t,q_i) = 1$. We'll show that $d(t,q_{i+1}) = 1$ also holds.

In fact

$$\begin{aligned} d(t,q_{i+1} - 1) = d(t, q_i + \ell(\varphi_{i+1})) = d(t,q_i + 1) + d(\varphi_{i+1},\ell(\varphi_{i+1})) = \\ = 1 + d(\varphi_{i+1},\ell(\varphi_{i+1})). \end{aligned}$$

If $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then
$d(t,q_{i+1}) = d(t,q_{i+1} - 1) - 1 = d(\varphi_{i+1},\ell(\varphi_{i+1})) = 1$.

Else $t[q_{i+1}-1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin \{\text{‘(’}, \text{‘)’}\}$ so
$d(t, q_{i+1}) = d(t, q_{i+1}-1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

So it is shown that for each $i = 1 \dots m-1$ $d(t, q_i) = 1$.

We now want to show that $d(t, \ell(t)) = 1$.

If $m = 1$ then

$$d(t, \ell(t)-1) = d(t, 4+\ell(\varphi_1)) = d(t, 4+1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1)).$$

If $m > 1$ then

$$d(t, \ell(t)-1) = d(t, q_{m-1}+\ell(\varphi_m)) = d(t, q_{m-1}+1) + d(\varphi_m, \ell(\varphi_m)) = 1 + d(\varphi_m, \ell(\varphi_m)).$$

If $t[\ell(t)-1] = \varphi_m[\ell(\varphi_m)] = \text{‘)’}$ then $d(t, \ell(t)) = d(t, \ell(t)-1) - 1 = d(\varphi_m, \ell(\varphi_m)) = 1$. Else $t[\ell(t)-1] = \varphi_m[\ell(\varphi_m)] \notin \{\text{‘(’}, \text{‘)’}\}$ so $d(t, \ell(t)) = d(t, \ell(t)-1) = 1 + d(\varphi_m, \ell(\varphi_m)) = 1$.

Let's now examine the facts we have to prove. It is true that $t[\ell(t)] \neq \text{‘(’}$. It's also true that $t[\ell(t)] = \text{‘)’}$ and $d(t, \ell(t)) = 1$.

Now let $\alpha \in \{1, \dots, \ell(t)\}$ and ($t[\alpha] = \text{‘:’}$ or $t[\alpha] = \text{‘,’}$ or $t[\alpha] = \text{‘)’}$). This implies $\alpha \notin \{1, 2, 4\}$.

If $\alpha \in \{3, q_1, \dots, q_{m-1}, \ell(t)\}$ we have already shown that $d(t, \alpha) = 1$. Otherwise there are these alternative possibilities:

a. $(m = 1) \wedge (\alpha > 4) \wedge (\alpha < \ell(t))$,
b. $(m > 1) \wedge (\alpha > 4) \wedge (\alpha < q_1)$,
c. $(m > 2) \wedge (\exists i = 1 \dots m-2 : (\alpha > q_i) \wedge (\alpha < q_{i+1}))$,
d. $(m > 1) \wedge (\alpha > q_{m-1}) \wedge (\alpha < \ell(t))$.

In the situation a. we have

$$4 < \alpha < \ell(t),$$

$$0 < \alpha - 4 < \ell(t) - 4,$$

$$1 \leqslant \alpha - 4 \leqslant \ell(t) - 5 = \ell(\varphi_1),$$

$$\varphi_1[\alpha - 4] = t[\alpha],$$

$$\begin{aligned} d(t, \alpha) &= d(t, 4 + (\alpha - 4)) = d(t, 4+1) + d(\varphi_1, \alpha - 4) = \\ &= 1 + d(\varphi_1, \alpha - 4) \geqslant 2. \end{aligned}$$

In the situation b. we have

$$4 < \alpha < q_1,$$

$$0 < \alpha - 4 < q_1 - 4,$$

$$1 \leqslant \alpha - 4 \leqslant q_1 - 5 = \ell(\varphi_1),$$

$$\varphi_1[\alpha - 4] = t[\alpha],$$

$$\begin{aligned} d(t,\alpha) = d(t, 4 + (\alpha - 4)) = d(t, 4+1) + d(\varphi_1, \alpha - 4) = \\ = 1 + d(\varphi_1, \alpha - 4) \geqslant 2. \end{aligned}$$

In the situation c. we have

$$q_i < \alpha < q_{i+1},$$

$$0 < \alpha - q_i < q_{i+1} - q_i,$$

$$1 \leqslant \alpha - q_i \leqslant q_{i+1} - q_i - 1 = \ell(\varphi_{i+1}),$$

$$\varphi_{i+1}[\alpha - q_i] = t[\alpha],$$

$$\begin{aligned} d(t,\alpha) = d(t, q_i + (\alpha - q_i)) = d(t, q_i + 1) + d(\varphi_{i+1}, \alpha - q_i) = \\ = 1 + d(\varphi_{i+1}, \alpha - q_i) \geqslant 2. \end{aligned}$$

In the situation d. we have

$$q_{m-1} < \alpha < \ell(t),$$

$$0 < \alpha - q_{m-1} < \ell(t) - q_{m-1},$$

$$1 \leqslant \alpha - q_{m-1} \leqslant \ell(t) - q_{m-1} - 1 = \ell(\varphi_m),$$

$$\varphi_m[\alpha - q_{m-1}] = t[\alpha],$$

$$\begin{aligned} d(t,\alpha) = d(t, q_{m-1} + (\alpha - q_{m-1})) = d(t, q_{m-1} + 1) + d(\varphi_m, \alpha - q_{m-1}) = \\ = 1 + d(\varphi_m, \alpha - q_{m-1}) \geqslant 2. \end{aligned}$$

Let $k \in K(n)$, $f \in \mathcal{F}$ and $t \in \mathbf{E^f(n+1,k)}$. Then $t \in H_f(n+1,k)$, so if f has multiplicity 2 there exist $\varphi_1, \varphi_2 \in E(n,k)$ such that $t = f(\varphi_1, \varphi_2)$, if f has multiplicity

1 there exists $\varphi_1 \in E(n,k)$ such that $t = f(\varphi_1)$.

We first consider the case where f has multiplicity 1. Here we first want to show that $d(t,\ell(t)) = 1$.

We have

$$d(t,\ell(t)-1) = d(t,2+\ell(\varphi_1)) = d(t,2+1) + d(\varphi_1,\ell(\varphi_1)) = 1 + d(\varphi_1,\ell(\varphi_1)).$$

If $t[\ell(t)-1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t,\ell(t)) = d(t,\ell(t)-1) - 1 = d(\varphi_1,\ell(\varphi_1)) = 1$. Else $t[\ell(t)-1] = \varphi_1[\ell(\varphi_1)] \notin \{$'(', ')'$\}$ so $d(t,\ell(t)) = d(t,\ell(t)-1) = 1+d(\varphi_1,\ell(\varphi_1)) = 1$.

Let's now examine the facts we have to prove. It is true that $t[\ell(t)] \neq$ '('. It's also true that $t[\ell(t)] =$ ')' and $d(t,\ell(t)) = 1$.

Now let $\alpha \in \{1,\ldots,\ell(t)\}$ and ($t[\alpha] =$ ':' or $t[\alpha] =$ ',' or $t[\alpha] =$ ')'). This implies $\alpha \notin \{1,2\}$.

If $\alpha = \ell(t)$ we have already shown that $d(t,\alpha) = 1$. Otherwise clearly $2 < \alpha < \ell(t)$ and

$$0 < \alpha - 2 < \ell(t) - 2,$$

$$1 \leqslant \alpha - 2 \leqslant \ell(t) - 3 = \ell(\varphi_1),$$

$$\varphi_1[\alpha-2] = t[\alpha],$$

$$\begin{aligned} d(t,\alpha) = d(t,2+(\alpha-2)) = d(t,2+1) + d(\varphi_1,\alpha-2) = \\ = 1 + d(\varphi_1,\alpha-2) \geqslant 2. \end{aligned}$$

Let's then consider the case where f has multiplicity 2. Here we indicate with q_1 the position of the explicit occurrence of ',' within t. First of all we want to prove that $d(t,q_1) = 1$. To this end we consider that

$$d(t,q_1-1) = d(t,2+\ell(\varphi_1)) = d(t,2+1) + d(\varphi_1,\ell(\varphi_1)) = 1 + d(\varphi_1,\ell(\varphi_1)).$$

If $t[q_1-1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t,q_1) = d(t,q_1-1) - 1 = d(\varphi_1,\ell(\varphi_1)) = 1$. Else $t[q_1-1] = \varphi_1[\ell(\varphi_1)] \notin \{$'(', ')'$\}$ so $d(t,q_1) = d(t,q_1-1) = 1 + d(\varphi_1,\ell(\varphi_1)) = 1$.

We then want to show that $d(t,\ell(t)) = 1$. We have

$$d(t,\ell(t)-1) = d(t,q_1+\ell(\varphi_2)) = d(t,q_1+1) + d(\varphi_2,\ell(\varphi_2)) = 1 + d(\varphi_2,\ell(\varphi_2)).$$

If $t[\ell(t)-1] = \varphi_2[\ell(\varphi_2)] =$ ')' then $d(t,\ell(t)) = d(t,\ell(t)-1) - 1 = d(\varphi_2,\ell(\varphi_2)) = 1$.

Else $t[\ell(t)-1] = \varphi_2[\ell(\varphi_2)] \notin \{\text{`('}, \text{`)'}\}$ so $d(t,\ell(t)) = d(t,\ell(t)-1) = 1+d(\varphi_2,\ell(\varphi_2)) = 1$.

Let's now examine the facts we have to prove. It is true that $t[\ell(t)] \neq$ '('. It's also true that $t[\ell(t)] =$ ')' and $d(t,\ell(t)) = 1$.

Now let $\alpha \in \{1,\ldots,\ell(t)\}$ and ($t[\alpha] =$ ':' or $t[\alpha] =$ ',' or $t[\alpha] =$ ')'). This implies $\alpha \notin \{1,2\}$.

If $\alpha \in \{q_1,\ell(t)\}$ we have already shown that $d(t,\alpha) = 1$. Otherwise there are these alternative possibilities:

a. $(\alpha > 2) \wedge (\alpha < q_1)$,
b. $(\alpha > q_1) \wedge (\alpha < \ell(t))$.

In the situation a. we have

$$2 < \alpha < \ell(t),$$

$$0 < \alpha - 2 < \ell(t) - 2,$$

$$1 \leqslant \alpha - 2 \leqslant \ell(t) - 3 = \ell(\varphi_1),$$

$$\varphi_1[\alpha - 2] = t[\alpha],$$

$$\begin{aligned} d(t,\alpha) = d(t,2+(\alpha-2)) = d(t,2+1) + d(\varphi_1,\alpha-2) = \\ = 1 + d(\varphi_1,\alpha-2) \geqslant 2. \end{aligned}$$

In the situation b. we have

$$q_1 < \alpha < \ell(t),$$

$$0 < \alpha - q_1 < \ell(t) - q_1,$$

$$1 \leqslant \alpha - q_1 \leqslant \ell(t) - q_1 - 1 = \ell(\varphi_2),$$

$$\varphi_2[\alpha - q_1] = t[\alpha],$$

$$\begin{aligned} d(t,\alpha) = d(t,q_1+(\alpha-q_1)) = d(t,q_1+1) + d(\varphi_2,\alpha-q_1) = \\ = 1 + d(\varphi_2,\alpha-q_1) \geqslant 2. \end{aligned}$$

Let $k \in K(n)$ and $t \in \mathbf{E_e}(\mathbf{n+1},\mathbf{k})$. As a consequence to $t \in E_e(n+1,k)$ there exist

- a positive integer m,
- a function x whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \ldots m$ $i \neq j \rightarrow x_i \neq x_j$,
- a function φ whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $\varphi_i \in E(n)$,
- $\phi \in E(n)$

such that $t = \{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi)$.

In this representation we see 'explicit occurrences' of the symbols ',' and ':'. We indicate with $q_1, \ldots, q_m$ the positions of the explicit occurrences of ':' and with $r_1 \ldots r_m$ the positions of the explicit occurrences of ','. The only explicit occurrence of ')' has the position $\ell(t)$.

We want to show that for each $i = 1 \ldots m$ $d(t, q_i) = 1, d(t, r_i) = 1$ and that $d(t, \ell(t)) = 1$.

It is obvious that $d(t, q_1) = 1$. Moreover

$$\begin{aligned} d(t, r_1 - 1) &= d(t, q_1 + \ell(\varphi_1)) = d(t, q_1 + 1) + d(\varphi_1, \ell(\varphi_1)) = \\ &= 1 + d(\varphi_1, \ell(\varphi_1)). \end{aligned}$$

If $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t, r_1) = d(t, r_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$. Else $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{$'(', ')'$\}$ so $d(t, r_1) = d(t, r_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

If $m = 1$ we have shown that for each $i = 1 \ldots m$ $d(t, q_i) = 1, d(t, r_i) = 1$. Now suppose $m > 1$, let $i = 1 \ldots m - 1$ and suppose $d(t, q_i) = 1, d(t, r_i) = 1$. We show that $d(t, q_{i+1}) = 1, d(t, r_{i+1}) = 1$.

We have $q_{i+1} = r_i + 2$ and it is immediate that $d(t, q_{i+1}) = 1$. Moreover

$$\begin{aligned} d(t, r_{i+1} - 1) &= d(t, q_{i+1} + \ell(\varphi_{i+1})) = \\ &= d(t, q_{i+1} + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})). \end{aligned}$$

If $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then
$d(t, r_{i+1}) = d(t, r_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.
Else $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin \{$'(', ')'$\}$ so
$d(t, r_{i+1}) = d(t, r_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Furthermore

$$\begin{aligned} d(t, \ell(t) - 1) &= d(t, r_m + \ell(\phi)) = \\ &= d(t, r_m + 1) + d(\phi, \ell(\phi)) = 1 + d(\phi, \ell(\phi)). \end{aligned}$$

If $t[\ell(t) - 1] = \phi[\ell(\phi)] =$ ')' then $d(t, \ell(t)) = d(t, \ell(t) - 1) - 1 = d(\phi, \ell(\phi)) = 1$. Else $t[\ell(t) - 1] = \phi[\ell(\phi)] \notin \{$'(', ')'$\}$ so $d(t, \ell(t)) = d(t, \ell(t) - 1) = 1 + d(\phi, \ell(\phi)) = 1$.

Let's now examine the facts we have to prove. It is true that $t[\ell(t)] \neq$ '('. It's also true that $t[\ell(t)] =$ ')' and $d(t, \ell(t)) = 1$.

Now let $\alpha \in \{1, \ldots, \ell(t)\}$ and ($t[\alpha] =$ ':' or $t[\alpha] =$ ',' or $t[\alpha] =$ ')').

If $\alpha \in \{q_1, \ldots, q_m, r_1, \ldots, r_m, \ell(t)\}$ we have already shown that $d(t, \alpha) = 1$. Otherwise there are these alternative possibilities:

a. $\exists i = 1 \dots m$ such that $q_i < \alpha < r_i$,
b. $r_m < \alpha < \ell(t)$.

In the situation a. we have

$$q_i < \alpha < r_i,$$

$$0 < \alpha - q_i < r_i - q_i,$$

$$1 \leqslant \alpha - q_i \leqslant r_i - q_i - 1 = \ell(\varphi_i),$$

$$\varphi_i[\alpha - q_i] = t[\alpha],$$

$$\begin{aligned} d(t, \alpha) &= d(t, q_i + (\alpha - q_i)) = d(t, q_i + 1) + d(\varphi_i, \alpha - q_i) = \\ &= 1 + d(\varphi_i, \alpha - q_i) \geqslant 2. \end{aligned}$$

In the situation b. we have

$$r_m < \alpha < \ell(t),$$

$$0 < \alpha - r_m < \ell(t) - r_m,$$

$$1 \leqslant \alpha - r_m \leqslant \ell(t) - r_m - 1 = \ell(\phi),$$

$$\phi[\alpha - r_m] = t[\alpha],$$

$$\begin{aligned} d(t, \alpha) &= d(t, r_m + (\alpha - r_m)) = d(t, r_m + 1) + d(\phi, \alpha - r_m) = \\ &= 1 + d(\phi, \alpha - r_m) \geqslant 2. \end{aligned}$$

□

7. Deductive systems and proofs

In this section we will define deductive systems and proofs and we will introduce other concepts and results related to our deductive methodology. Given a language $\mathcal{L} = (\mathcal{V}, \mathcal{F}, \mathcal{C}, \#, \{D_1, \dots, D_n\}, q_{max})$, we begin with some preliminary definitions.

Let $K = \bigcup_{n \geqslant 1} K(n)$.

For each $k \in K$ let

$$E(k) = \bigcup_{n \geqslant 1 : k \in K(n)} E(n, k) \ ,$$

$$E_s(k) = \{t | t \in E(k), \forall \sigma \in \Xi(k) \ \#(k, t, \sigma) \text{ is a set } \} \ .$$

Let $E = \bigcup_{k \in K} E(k)$; E is the set of all expressions in our language.

One expression $t \in E(k)$ is a 'sentence with respect to k' when for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma)$ is true or $\#(k, t, \sigma)$ is false.

We define $S(k) = \{t | t \in E(k), t \text{ is a sentence with respect to } k\}$.

For each $t \in E(\epsilon)$ we define $\#(t) = \#(\epsilon, t, \epsilon)$.

A sentence with respect to ϵ will simply be called a 'sentence'.

At this point we can define what is a proof in our language. To define this we need to define the notions of axiom and rule. We first notice that the symbols of our language belong to the four disjoint sets $\mathcal{V}$, $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{Z}$. Let's call $\Sigma = \mathcal{V} \cup \mathcal{C} \cup \mathcal{F} \cup \mathcal{Z}$ the set of all the symbols (or alphabet) of our language and Σ^* the set of all the empty or finite strings built with the symbols in Σ. Clearly given $k \in K$ $S(k) \subseteq E(k) \subseteq \Sigma^*$.

An *axiom* is a set A such that

- $A \subseteq S(\epsilon) \subseteq \Sigma^*$,
- A is r.e.,
- for each $\varphi \in A$ $\#(\varphi)$ holds.

The property 'for each $\varphi \in A$ $\#(\varphi)$ holds' states that axiom A is 'sound'.

Given a positive integer n we indicate with $S(\epsilon)^n$ the set of all n-tuples $(\varphi_1, \ldots, \varphi_n)$ for $\varphi_1, \ldots, \varphi_n \in S(\epsilon)$. An n-ary *rule* is a set R such that

- $R \subseteq S(\epsilon)^{n+1} \subseteq (\Sigma^*)^{n+1}$,
- R is r.e.,
- for each $(\varphi_1, \ldots, \varphi_n, \varphi) \in R$ if $\#(\varphi_1), \ldots, \#(\varphi_n)$ hold then $\#(\varphi)$ holds.

The property 'for each $(\varphi_1, \ldots, \varphi_n, \varphi) \in R$ if $\#(\varphi_1), \ldots, \#(\varphi_n)$ hold then $\#(\varphi)$ holds' states that rule R is 'sound'.

Both in the definition of axiom and rule we have included a requirement of soundness.

A deductive system is built on top of our language $\mathcal{L}$, and is identified by a pair $(\mathcal{A}, \mathcal{R})$ where $\mathcal{A}$ is a finite set of axioms in $\mathcal{L}$ and $\mathcal{R}$ is a finite set of rules in $\mathcal{L}$.

We require that the set of the axioms and the set of the rules are finite since we need to be able to list each of them on a piece of paper.

Given a language $\mathcal{L}$, $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ deductive system in $\mathcal{L}$, φ, $\psi_1, \ldots, \psi_m$ sentences in $\mathcal{L}$, we say that $(\psi_1, \ldots, \psi_m)$ is a *proof* of φ in $\mathcal{D}$ if and only if

- there exists $A \in \mathcal{A}$ such that $\psi_1 \in A$;
- if $m > 1$ then for each $j = 2 \ldots m$ one of the following holds
  - there exists $A \in \mathcal{A}$ such that $\psi_j \in A$,
  - there exist an n-ary rule $R \in \mathcal{R}$ and $i_1, \ldots, i_n < j$ such that $(\psi_{i_1}, \ldots, \psi_{i_n}, \psi_j) \in R$;
- $\psi_m = \varphi$.

Given $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ deductive system in $\mathcal{L}$ and φ sentence in $\mathcal{L}$ we say that φ *is derivable in* $\mathcal{D}$ and write $\vdash_{\mathcal{D}} \varphi$ if and only if there exist $\psi_1, \ldots, \psi_m$ sentences in $\mathcal{L}$ such that $(\psi_1, \ldots, \psi_m)$ is a proof of φ in $\mathcal{D}$.

A deductive system $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ is said to be *sound* if and only if for each φ sentence in $\mathcal{L}$ if $\vdash_{\mathcal{D}} \varphi$ then $\#(\varphi)$ holds. In the next lemma we easily prove that each of our systems is sound.

Lemma 7.1. *Let $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ be a deductive system in $\mathcal{L}$. Then $\mathcal{D}$ is sound.*

Proof. Let φ be a sentence in $\mathcal{L}$. Suppose $\vdash_{\mathcal{D}} \varphi$. There exist $\psi_1, \ldots, \psi_m$ sentences in $\mathcal{L}$ such that $(\psi_1, \ldots, \psi_m)$ is a proof of φ in $\mathcal{D}$. We can show that for each $j = 1 \ldots m$ $\#(\psi_j)$ holds.

There exists $A \in \mathcal{A}$ such that $\psi_1 \in A$, so $\#(\psi_1)$ holds.

If $m > 1$ suppose $j = 2 \ldots m$.

If there exists $A \in \mathcal{A}$ such that $\psi_j \in A$ then $\#(\psi_j)$ holds.

Otherwise there exist an n-ary rule $R \in \mathcal{R}$ and $i_1, \ldots, i_n < j$ such that

$$(\psi_{i_1}, \ldots, \psi_{i_n}, \psi_j) \in R \ .$$

Since $\#(\psi_{i_1}), \ldots, \#(\psi_{i_n})$ all hold then $\#(\psi_j)$ also holds. □

We now want to point out some recursivity requirements with respect to the sets that we defined above: $E(k)$, $S(k)$, $E_s(k)$. We will prove these sets are recursively enumerable.

For each $k \in K$ we defined $E(k) = \bigcup_{n \geqslant 1 : k \in K(n)} E(n, k)$.

The set $\{n | n \in \mathbb{N}, n \geqslant 1, k \in K(n)\}$ is r.e.. In fact if we call n_0 the least $n \in \mathbb{N}$ such that $k \in K(n)$ we have that the just mentioned set is actually $\{n | n \in \mathbb{N}, n \geqslant n_0\}$, that is a recursive and r.e. set. Since for each n in the mentioned r.e. set $E(n, k)$ is r.e. then $E(k)$ is also r.e..

Given a positive integer n and $k \in K(n)$, let's define the following sets:

$$S(n,k) = \{\varphi |\ \varphi \in E(n,k), \text{ for each } \sigma \in \Xi(k)\ \#(k,\varphi,\sigma) \text{ is true or } \#(k,\varphi,\sigma) \text{ is false }\};$$
$$E_s(n,k) = \{\varphi |\ \varphi \in E(n,k), \text{ for each } \sigma \in \Xi(k)\ \#(k,\varphi,\sigma) \text{ is a set }\};$$
$$E_{D_i}(n,k) = \{\varphi |\ \varphi \in E(n,k), \text{ for each } \sigma \in \Xi(k)\ \#(k,\varphi,\sigma) \in D_i\}.$$

For each $\varphi \in E(n,k)$ we can decide the following conditions:

- for each $\sigma \in \Xi(k)$ $\#(k,\varphi,\sigma)$ is true or false;
- for each $\sigma \in \Xi(k)$ $\#(k,\varphi,\sigma)$ is a set;
- for each $\sigma \in \Xi(k)$ $\#(k,\varphi,\sigma) \in D_i$.

Therefore, since $E(n,k)$ is recursive, $S(n,k)$, $E_s(n,k)$ and $E_{D_i}(n,k)$ are recursive too.

It is easy to verify that $S(k) = \bigcup_{n\geqslant 1:k\in K(n)} S(n,k)$, therefore $S(k)$ is r.e..

Similarly, it is easy to verify that $E_s(k) = \bigcup_{n\geqslant 1:k\in K(n)} E_s(n,k)$, therefore $E_s(k)$ is r.e..

Moreover we can define $E_{D_i}(k) = \{\varphi |\ \varphi \in E(k), \text{ for each } \sigma \in \Xi(k)\ \#(k,\varphi,\sigma) \in D_i\}$

Then it is easy to verify that $E_{D_i}(k) = \bigcup_{n\geqslant 1:k\in K(n)} E_{D_i}(n,k)$, therefore $E_{D_i}(k)$ is r.e..

We now want to define some possible axiom and rule and prove they are recursively enumerable, in order to convince ourselves and the reader that we have correctly built our system. We first need to provide some definition.

Definition 7.2. Let $x \in \mathcal{V},\ \varphi \in E$. We define

$$H[x:\varphi] = \varphi \in E_s(\epsilon)\ .$$

If the condition $H[x:\varphi]$ holds then we define $k[x:\varphi] = \epsilon + < x, \varphi >$. Clearly $k[x:\varphi] \in K$. In fact there exists n positive integer such that $\epsilon \in K(n) \wedge \varphi \in E_s(n,\epsilon)$, $x \in \mathcal{V} - var(\epsilon)$, so $k[x:\varphi] = \epsilon + < x, \varphi > \in K(n) \cup K(n)^+ = K(n+1) \subseteq K$.

Moreover $k[x:\varphi] =<< x, \varphi >>$ and $var(k[x:\varphi]) = \{x\}$.

Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$. We can assume to have defined $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$ and if this holds to have defined also $k[x_1:\varphi_1,\ldots,x_m:\varphi_m] \in K$, such that

$$k[x_1:\varphi_1,\ldots,x_m:\varphi_m] =<< x_1, \varphi_1 >, \ldots, < x_m : \varphi_m >>$$
$$var(k[x_1:\varphi_1,\ldots,x_m:\varphi_m]) = \{x_1,\ldots,x_m\}$$

We define

$$H[x_1:\varphi_1,\ldots,x_{m+1}:\varphi_{m+1}] = H[x_1:\varphi_1,\ldots,x_m:\varphi_m] \wedge \varphi_{m+1} \in E_s(k[x_1:\varphi_1,\ldots,x_m:\varphi_m])\ .$$

If $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$ then we define

$$k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}] = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] + < x_{m+1}, \varphi_{m+1} > \ .$$

Clearly $k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}] \in K$. In fact there exists a positive integer n such that $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \in K(n)$ and $\varphi_{m+1} \in E_s(n, k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$, $x_{m+1} \in \mathcal{V} - var(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$, so $k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}] \in K(n) \cup K(n)^+ = K(n+1)$.

Moreover

$$k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}] =<< x_1, \varphi_1 >, \ldots, < x_{m+1} : \varphi_{m+1} >>$$
$$var(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]) = \{x_1, \ldots, x_{m+1}\} \ .$$

Lemma 7.3. *Let m positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$. Then $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ is defined and if $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ holds then $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ is also defined and belongs to K. Moreover*

$$var(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) = \{x_1, \ldots, x_m\} \ .$$

Proof. This is an obvious consequence of the previous definition and has been verified, by induction on m, in the definition itself. □

Remark 7.4. Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. In these assumptions we can easily see that for each $i = 1 \ldots m$ $H[x_1 : \varphi_1, \ldots, x_i : \varphi_i]$ holds and so $k[x_1 : \varphi_1, \ldots, x_i : \varphi_i]$ is defined, $k[x_1 : \varphi_1, \ldots, x_i : \varphi_i] \in K$, $var(k[x_1 : \varphi_1, \ldots, x_i : \varphi_i]) = \{x_1, \ldots, x_i\}$.

In fact this is clearly true for $i = m$. Given $i = 2 \ldots m$, if we suppose this is true for i, then we have $H[x_1 : \varphi_1, \ldots, x_{i-1} : \varphi_{i-1}]$, and so the remaining facts also hold.

In these assumptions we can define $k_0 = \epsilon$ and for each $i = 1 \ldots m$ $k_i = k[x_1 : \varphi_1, \ldots, x_i : \varphi_i]$. We have $k_0 \in K$, $var(k_0) = \emptyset$, for each $i = 1 \ldots m$ $k_i \in K$, $var(k_i) = \{x_1, \ldots, x_i\}$. Hereafter we'll often use this kind of simplified notation.

We can also easily see that for each $i = 1 \ldots m$ $\varphi_i \in E_s(k_{i-1})$ and $k_i = k_{i-1} + < x_i, \varphi_i >$, and $dom(k_i) = \{1, \ldots, i\}$.

For the following definition we need to assume that the symbol $\forall$ belongs to the set $\mathcal{F}$ of our language. This assumption applies to the remainder of this section.

Definition 7.5. Let m be a positive integer. Let $\varphi_1, \ldots, \varphi_m \in \Sigma^*$. Let $\psi_1, \ldots, \psi_m \in \Sigma^*$. Let $\varphi \in \Sigma^*$. Define

$$\gamma[\psi_m : \varphi_m, \varphi] = \forall(\{\}(\psi_m : \varphi_m, \varphi)) \ .$$

If $m > 1$ for each $i = 2 \dots m$ suppose we have defined $\gamma[\psi_i : \varphi_i, \dots, \psi_m : \varphi_m, \varphi]$ and define

$$\gamma[\psi_{i-1} : \varphi_{i-1}, \dots, \psi_m : \varphi_m, \varphi] = \forall(\{\}(\psi_{i-1} : \varphi_{i-1}, \gamma[\psi_i : \varphi_i, \dots, \psi_m : \varphi_m, \varphi])) \ .$$

With this we have also defined $\gamma[\psi_1 : \varphi_1, \dots, \psi_m : \varphi_m, \varphi]$.

□

We can define a function χ on the domain $(\Sigma^*)^{2m} \times \Sigma^*$ such that given $(\psi_1, \varphi_1, \dots, \psi_m, \varphi_m) \in (\Sigma^*)^{2m}$ and $\varphi \in \Sigma^*$, $\chi((\psi_1, \varphi_1, \dots, \psi_m, \varphi_m), \varphi) = \gamma[\psi_1 : \varphi_1, \dots, \psi_m : \varphi_m, \varphi]$. Clearly this function is computable since the result can be obtained by simply concatenating the elements of the input with other symbols of our language.

We can also observe the following.

Lemma 7.6. *Let m be a positive integer, $m > 1$. Let $\varphi_1, \dots, \varphi_m \in \Sigma^*$. Let $\psi_1, \dots, \psi_m \in \Sigma^*$. Let $\varphi \in \Sigma^*$. Then*

$$\gamma[\psi_1 : \varphi_1, \dots, \psi_m : \varphi_m, \varphi] = \gamma[\psi_1 : \varphi_1, \dots, \psi_{m-1} : \varphi_{m-1}, \forall(\{\}(\psi_m : \varphi_m, \varphi))] \ .$$

Proof. We want to prove that for each $i = 1 \dots m-1$

$$\gamma[\psi_i : \varphi_i, \dots, \psi_m : \varphi_m, \varphi] = \gamma[\psi_i : \varphi_i, \dots, \psi_{m-1} : \varphi_{m-1}, \forall(\{\}(\psi_m : \varphi_m, \varphi))] \ .$$

We start the proof at $m-1$ and we are then going backwards by induction to 1. So

$$\begin{aligned}\gamma[\psi_{m-1} : \varphi_{m-1}, \psi_m : \varphi_m, \varphi] &= \forall(\{\}(\psi_{m-1} : \varphi_{m-1}, \gamma[\psi_m : \varphi_m, \varphi])) \\ &= \forall(\{\}(\psi_{m-1} : \varphi_{m-1}, \forall(\{\}(\psi_m : \varphi_m, \varphi)))) \\ &= \gamma[\psi_{m-1} : \varphi_{m-1}, \forall(\{\}(\psi_m : \varphi_m, \varphi))]\end{aligned}$$

If $m = 2$ our proof is finished, whilst if $m > 2$ given $i = 2 \dots m-1$ we can assume

$$\gamma[\psi_i : \varphi_i, \dots, \psi_m : \varphi_m, \varphi] = \gamma[\psi_i : \varphi_i, \dots, \psi_{m-1} : \varphi_{m-1}, \forall(\{\}(\psi_m : \varphi_m, \varphi))] \ .$$

And in this case we have

$$\begin{aligned}\gamma[\psi_{i-1} : \varphi_{i-1}, \dots, \psi_m : \varphi_m, \varphi] &= \forall(\{\}(\psi_{i-1} : \varphi_{i-1}, \gamma[\psi_i : \varphi_i, \dots, \psi_m : \varphi_m, \varphi])) \\ &= \forall(\{\}(\psi_{i-1} : \varphi_{i-1}, \gamma[\psi_i : \varphi_i, \dots, \psi_{m-1} : \varphi_{m-1}, \forall(\{\}(\psi_m : \varphi_m, \varphi))])) \\ &= \gamma[\psi_{i-1} : \varphi_{i-1}, \dots, \psi_{m-1} : \varphi_{m-1}, \forall(\{\}(\psi_m : \varphi_m, \varphi))] \ .\end{aligned}$$

□

Given a positive integer m let's call R_m the set

$$\{(x_1, \varphi_1, \dots, x_m, \varphi_m) |\ x_1, \dots, x_m \in \mathcal{V} \text{ with } x_i \neq x_j \text{ for } i \neq j,\ \varphi_1, \dots, \varphi_m \in E, H[x_1 : \varphi_1, \dots, x_m : \varphi_m]\} \ .$$

Clearly given $(x_1, \varphi_1, \dots, x_m, \varphi_m) \in R_m$ $k[x_1 : \varphi_1, \dots, x_m : \varphi_m] \in K$.

Let's also define

$$Q_m = \bigcup_{(x_1,\varphi_1,\ldots,x_m,\varphi_m)\in R_m} \{(x_1,\varphi_1,\ldots,x_m,\varphi_m)\} \times S(k[x_1:\varphi_1,\ldots,x_m:\varphi_m]) \ .$$

Actually $Q_m \subseteq (\Sigma^*)^{2m} \times \Sigma^*$. Our goal is now to show that Q_m is r.e. in order to be able to show that the set $\{\chi((x_1,\varphi_1,\ldots,x_m,\varphi_m),\varphi) | ((x_1,\varphi_1,\ldots,x_m,\varphi_m),\varphi) \in Q_m\}$ is r.e. itself.

As a first remark in this proof we can notice that our set of variables $\mathcal{V}$ is recursive. In fact given a string $\varphi \in \Sigma^*$ if φ has not exactly one character then it doesn't belong to $\mathcal{V}$. If it has just one character then, since apart from the variables our alphabet has a finite number of symbols, we can decide if $\varphi \in \mathcal{V}$.

The first step in this proof is to show that R_m is r.e., i.e. the following lemma:

Lemma 7.7. *For each m positive integer R_m is r.e..*

Proof. In the initial step of the proof we have to show that R_1 is r.e.. We have

$$\begin{aligned} R_1 &= \{(x_1,\varphi_1) | \, x_1 \in \mathcal{V}, \varphi_1 \in E, H[x_1:\varphi_1]\} \\ &= \{(x_1,\varphi_1) | \, x_1 \in \mathcal{V}, \varphi_1 \in E_s(\epsilon)\} \\ &= \mathcal{V} \times E_s(\epsilon) \end{aligned}$$

and since both $\mathcal{V}$ and $E_s(\epsilon)$ are r.e. then R_1 is r.e..

Given a positive integer m we assume R_m is r.e. and want to show that R_{m+1} is r.e..

Actually

$$\begin{aligned} R_{m+1} &= \{(x_1,\varphi_1,\ldots,x_{m+1},\varphi_{m+1}) | \; x_1,\ldots,x_{m+1} \in \mathcal{V} \text{ with } x_i \neq x_j \text{ for } i \neq j, \; \varphi_1,\ldots,\varphi_{m+1} \in E, H[x_1:\varphi_1,\ldots,x_{m+1}:\varphi_{m+1}]\} \\ &= \{(x_1,\varphi_1,\ldots,x_{m+1},\varphi_{m+1}) | \; x_1,\ldots,x_{m+1} \in \mathcal{V} \text{ with } x_i \neq x_j \text{ for } i \neq j, \; \varphi_1,\ldots,\varphi_{m+1} \in E, \\ &\quad H[x_1:\varphi_1,\ldots,x_m:\varphi_m] \wedge \varphi_{m+1} \in E_s(k[x_1:\varphi_1,\ldots,x_m:\varphi_m])\} \\ &= \{(x_1,\varphi_1,\ldots,x_{m+1},\varphi_{m+1}) | \; (x_1,\varphi_1,\ldots,x_m,\varphi_m) \in R_m, x_{m+1} \in \mathcal{V} - \{x_1,\ldots,x_m\}, \varphi_{m+1} \in E_s(k[x_1:\varphi_1,\ldots,x_m:\varphi_m])\} \,. \end{aligned}$$

Let's now consider the set

$$U_{m+1} = \bigcup_{(x_1,\varphi_1,\ldots,x_m,\varphi_m)\in R_m} \{(x_1,\varphi_1,\ldots,x_m,\varphi_m)\} \times (\mathcal{V} - \{x_1,\ldots,x_m\}) \times E_s(k[x_1:\varphi_1,\ldots,x_m:\varphi_m]) \ .$$

Given $(x_1,\varphi_1,\ldots,x_m,\varphi_m) \in R_m$ the sets $\{(x_1,\varphi_1,\ldots,x_m,\varphi_m)\} \subseteq (\Sigma^*)^{2m}$, $(\mathcal{V} - \{x_1,\ldots,x_m\}) \subseteq \Sigma^*$ and $E_s(k[x_1:\varphi_1,\ldots,x_m:\varphi_m]) \subseteq \Sigma^*$ are r.e., so the cartesian product

$$\{(x_1,\varphi_1,\ldots,x_m,\varphi_m)\} \times (\mathcal{V} - \{x_1,\ldots,x_m\}) \times E_s(k[x_1:\varphi_1,\ldots,x_m:\varphi_m])$$

is also r.e., and $U_{m+1} \subseteq (\Sigma^*)^{2m} \times \Sigma^* \times \Sigma^*$ is r.e..

The set R_{m+1} is a subset of $(\Sigma^*)^{2m+2}$ which is not (necessarily) the same of $(\Sigma^*)^{2m} \times \Sigma^* \times \Sigma^*$. In fact a member of $(\Sigma^*)^{2m+2}$ can be expressed as $(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1})$ and a member of $(\Sigma^*)^{2m} \times \Sigma^* \times \Sigma^*$ can be expressed as $((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), \psi_{m+1}, \varphi_{m+1})$. Anyway we can easily map members of the first set to the ones of the second set and vice-versa. In fact we can define a function κ over $(\Sigma^*)^{2m+2}$ such that $\kappa(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}) = ((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), \psi_{m+1}, \varphi_{m+1})$, and the function κ is computable.

Given $(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}) \in (\Sigma^*)^{2m+2}$ if $(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}) \in R_{m+1}$ then $\kappa(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}) \in U_{m+1}$ and vice-versa if $\kappa(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}) \in U_{m+1}$ then $(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}) \in R_{m+1}$.

As we have seen U_{m+1} is r.e. so its semi-characteristic function s_U is computable. Let's now consider the function $s_U \circ \kappa$ which is defined over $(\Sigma^*)^{2m+2}$. Given $(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}) \in (\Sigma^*)^{2m+2}$ if $(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}) \in R_{m+1}$ then $\kappa(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}) \in U_{m+1}$ and $s_U(\kappa(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}) = 1$. If $(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}) \notin R_{m+1}$ then $\kappa(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}) \notin U_{m+1}$ and $s_U(\kappa(\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1})$ diverges. So $s_U \circ \kappa$ is actually the semicharacteristic function of R_{m+1} and it is clearly a computable function. This proves that R_{m+1} is r.e.. □

Now given $(x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m$ both $\{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\}$ and $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ are r.e., so $\{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ is r.e. too, and so Q_m is a r.e. subset of $(\Sigma^*)^{2m} \times \Sigma^*$.

We can now recall that we have defined a computable function $\chi : (\Sigma^*)^{2m} \times \Sigma^* \to \Sigma^*$. Because of lemma 4.5 we have that the set $\{\chi((x_1, \varphi_1, \ldots, x_m, \varphi_m), \varphi) | \, ((x_1, \varphi_1, \ldots, x_m, \varphi_m), \varphi) \in Q_m\}$ is a r.e. subset of Σ^*.

And finally the set

$$\bigcup_{m \geqslant 1} \{\chi((x_1, \varphi_1, \ldots, x_m, \varphi_m), \varphi) | \, ((x_1, \varphi_1, \ldots, x_m, \varphi_m), \varphi) \in Q_m\}$$

is itself a r.e. set. It seems this is not particularly significant to us because this set is not an axiom, but we'll see sets that are very similar to this one and that we can use as an axiom in our deductive system.

8. Deductive methodology

We now need to introduce some other fundamental notions and results relevant to our deductive methodology.

At the beginning of section 3 we have introduced the logical connectives. In our deductions, expressions will make an extensive use of the logical connectives, so we assume that all of these symbols: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists$ are in our set $\mathcal{F}$. For each of these operators f $A_f(x_1, \ldots, x_n)$ and $P_f(x_1, \ldots, x_n)$ are defined as specified at the beginning of section 3.

Lemma 8.1. *For each n positive integer such that $n \geqslant 2$, $k \in K(n)$: $k \neq \epsilon$ there exists $m < n$ such that $k \in K(m)^+$.*

Proof. We prove this by induction on n. Clearly if $k \in K(2)$ and $k \neq \epsilon$ then $k \in K(1)^+$.

Let $n \geqslant 2$, $k \in K(n+1)$: $k \neq \epsilon$. Clearly if $k \in K(n)^+$ our proof is finished. Otherwise $k \in K(n)$ and in this case we can apply the inductive hypothesis. □

Lemma 8.2. *For each n positive integer such that $n \geqslant 2$, $k \in K(n)$: $k \neq \epsilon$*

- *there exist $m < n$, $h \in K(m), \phi \in E_s(m,h), y \in (\mathcal{V} - var(h))$ such that $k = h + < y, \phi >$, $\Xi(k) = \{\sigma + (y,s) | \, \sigma \in \Xi(h), s \in \#(h,\phi,\sigma)\}$;*
- *for each $g \in K(n), \theta \in E_s(n,g), z \in (\mathcal{V} - var(g))$ such that $k = g + < z, \theta >$ $\Xi(k) = \{\sigma + (z,s) | \, \sigma \in \Xi(g), s \in \#(g,\theta,\sigma)\}$.*

Proof. The first part clearly follows from lemma 8.1. The second part holds because we have $g = h, z = y, \theta = \phi$. □

Lemma 8.3. *For each n positive integer such that $n \geqslant 2$, $k \in K(n) : k \neq \epsilon$, $h \in \mathcal{R}(k) : h \neq k$ there exists $m < n$ such that $h \in K(m)$.*

Proof. We prove this by induction on n. Let $k \in K(2)$: $k \neq \epsilon$, $h \in \mathcal{R}(k)$: $h \neq k$. There exist $m < n$, $g \in K(m), \phi \in E_s(m,g), y \in (\mathcal{V} - var(g))$ such that $k = g + (y,\phi)$. In this case $m = 1$, so $g = \epsilon$. By lemma 5.5 we have $h \in \mathcal{R}(\epsilon)$ and so $h = \epsilon \in K(1)$.

In order to perform the inductive step, let $k \in K(n+1) : k \neq \epsilon$, $h \in \mathcal{R}(k)$ such that $h \neq k$. There exist $m < n+1$, $g \in K(m), \phi \in E_s(m,g), y \in (\mathcal{V} - var(g))$ such that $k = g + (y,\phi)$. By lemma 5.5 we have $h \in \mathcal{R}(g)$. If $h = g \in K(m)$ our proof is finished. Otherwise $h \neq g$ and $g \neq \epsilon$, we can apply our inductive hypothesis and obtain that there exists $q < m < n+1$ such that $h \in K(q)$. □

Lemma 8.4. *For each n positive integer such that $n \geqslant 2$, $k \in K(n) : k \neq \epsilon$, $\sigma \in \Xi(k)$, $h \in \mathcal{R}(k) : h \neq k$, there exists $m < n$ such that $h \in K(m)$ and it results $\sigma_{/dom(h)} \in \Xi(h)$.*

Proof. We prove this by induction on n. Let $k \in K(2)$: $k \neq \epsilon$, $\sigma \in \Xi(k)$, $h \in \mathcal{R}(k)$ such that $h \neq k$. Clearly $k \in K(1)^+$, so there exist $g \in K(1)$, $\phi \in E_s(1,g)$, $y \in \mathcal{V} - var(g)$ such that $k = g + < y, \phi >$. By lemma 5.5 we obtain that $h \in \mathcal{R}(g)$. Since $g = \epsilon$ then also $h = \epsilon \in K(1)$, so $\sigma_{/dom(h)} = \sigma_{/\emptyset} = \epsilon \in \Xi(\epsilon) = \Xi(h)$.

In order to perform the inductive step, let $k \in K(n+1) : k \neq \epsilon$, $\sigma \in \Xi(k)$, $h \in \mathcal{R}(k)$ such that $h \neq k$. By lemma 8.1 there exists $m \leqslant n$ such that $k \in K(m)^+$. Then there exist $g \in K(m), \phi \in E_s(m,g), y \in (\mathcal{V} - var(g))$ such that $k = g + < y, \phi >$. Moreover

$$\Xi(k) = \{\rho + (y,s) | \, \rho \in \Xi(g), s \in \#(g,\phi,\rho)\} \, .$$

Therefore there exist $\rho \in \Xi(g), s \in \#(g, \phi, \rho)$ such that $\sigma = \rho + (y, s)$. By assumption 6.1.7 and lemma 3.11 we have that $\sigma_{/dom(g)} = \sigma_{/dom(\rho)} = \rho$.

If $h = g$ then $\sigma_{/dom(h)} = \sigma_{/dom(g)} = \rho \in \Xi(h)$.

Otherwise we have $h \neq g$. Since $k = g+ < y, \phi >$, $h \in \mathcal{R}(k)$, $h \neq k$ by lemma 5.5 we have that $h \in \mathcal{R}(g)$. If $g = \epsilon$ we would have $h = \epsilon = g$, so $g \neq \epsilon$. This implies that $m \geqslant 2$. By our inductive hypothesis we obtain there exists $q < m \leqslant n$ such that $h \in K(q)$ and $\rho_{/dom(h)} \in \Xi(h)$. Now

$$\sigma_{/dom(h)} = (\sigma_{/dom(g)})_{/dom(h)} = \rho_{/dom(h)} \in \Xi(h).$$

□

Lemma 8.5. *For each n positive integer* $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>\in K(n) - \{\epsilon\}$, *for each* $i, j = 1 \ldots m$ $i \neq j \to x_i \neq x_j$.

Proof. Since $K(1) - \{\epsilon\} = \emptyset$ the initial step is trivially verified.

Let n be a positive integer, let $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>\in K(n+1) - \{\epsilon\}$, we want to verify that for each $i, j = 1 \ldots m$ $i \neq j \to x_i \neq x_j$.

If $k \in K(n)$ this is obvioulsy verified.

Otherwise $k \in K(n)^+$, so there exist $h \in K(n), \phi \in E_s(n, h), y \in (\mathcal{V} - var(h))$ such that $k = h+ < y, \phi >$.

If $h = \epsilon$ then $k =<< y, \phi >>$, this implies $m = 1$ and we have finished.

If $h \neq \epsilon$ then $h =<< y_1, \psi_1 > \cdots < y_p, \psi_p >>$ and
$k =<< y_1, \psi_1 > \cdots < y_p, \psi_p >< y, \phi >>$.

Clearly this implies $m = p + 1$. Given $i, j = 1 \ldots m$ with $i \neq j$ if $i, j \leqslant p$ then $x_i = y_i \neq y_j = x_j$. If $i \leqslant p$ and $j = m$ then $x_i = y_i \neq y = x_m = x_j$. □

Lemma 8.6. *For each n positive integer,* $k \in K(n)$, $\sigma = (z, \xi) \in \Xi(k)$*:*

- *if* $k = \epsilon$ *then* $z = \emptyset$, $var(\sigma) = \emptyset = var(k)$*;*
- *if* $k \neq \epsilon$, $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>$ *then* $dom(z) = \{1, \ldots, m\}$, *for each* $i = 1 \ldots m$ $z_i = x_i$, $var(\sigma) = var(k)$.

Proof. The initial step is trivially verified.

Let n be a positive integer, let $k \in K(n+1)$, let $\sigma = (z, \xi) \in \Xi(k)$. If $k \in K(n)$ then we can assume the result is valid.

Otherwise $k \in K(n)^+$, so there exist $h \in K(n), \phi \in E_s(n, h), y \in (\mathcal{V} - var(h))$ such that $k = h+ < y, \phi >$ and

$$\Xi(k) = \{\rho + (y, s) |\, \rho \in \Xi(h), s \in \#(h, \phi, \rho)\}.$$

There exists $\rho = (u, \nu) \in \Xi(h), s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$.

If $h = \epsilon$ then $k =<< y, \phi >>$, so $m = 1$, $\rho = \epsilon$, $dom(z) = \{1\}$, $z(1) = y$. Moreover $x_1 = y = z(1)$, $var(k) = \{y\} = var(\sigma)$.

Otherwise let $h =<< y_1, \psi_1 > \cdots < y_p, \psi_p >>$, so
$k =<< y_1, \psi_1 > \cdots < y_p, \psi_p >< y, \phi >>$.

Using our inductive hypothesis we can state that $dom(u) = \{1, \ldots, p\}$ for each $i = 1 \ldots p$ $y_i = u_i$, $var(\rho) = var(h)$.

It follows that $dom(z) = \{1, \ldots, p+1\} = \{1, \ldots, m\}$.

For each $i = 1 \ldots p$ $x_i = y_i = u_i = z_i$, moreover $x_{p+1} = y = z_{p+1}$.

It also follows that $var(\sigma) = var(k)$. □

Lemma 8.7. *For each n positive integer, $k \in K(n)$, $\sigma = (z, \xi) \in \Xi(k)$, for each $i, j \in dom(\sigma)$ $i \neq j \rightarrow z_i \neq z_j$.*

Proof. Clearly in the case $k = \epsilon$ we have $\sigma = \epsilon$ and the result is trivially verified.

Now suppose $k \neq \epsilon$, $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>$.

From lemma 8.5 it follows that for each $i, j = 1 \ldots m$ $i \neq j \rightarrow x_i \neq x_j$. From lemma 8.6 $dom(z) = \{1, \ldots, m\}$, for each $i = 1 \ldots m$ $z_i = x_i$.

It follows that for each $i, j \in dom(\sigma)$ if $i \neq j$ then $z_i = x_i \neq x_j = z_j$.

□

Lemma 8.8. *For each n positive integer, $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>, h =<< y_1, \psi_1 > \cdots < y_q, \psi_q >> \in K(n) - \{\epsilon\}$ if $h \sqsubseteq k$ then for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \rightarrow \varphi_i = \psi_j$.*

Proof. From lemma 8.5 it follows that for each $i, j \in dom(k)$ $i \neq j \rightarrow x_i \neq x_j$. With this we can apply lemma 5.6 and obtain that there exists $p = 1 \ldots m$ such that $h =<< x_1, \varphi_1 > \cdots < x_p, \varphi_p >>$.

At this point for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j$ implies $x_i = x_j$ so $i = j$ and $\varphi_i = \varphi_j = \psi_j$. □

Lemma 8.9. *For each n positive integer, $h, k \in K(n)$, $\sigma = (x, \eta) \in \Xi(k)$, $\rho = (y, \theta) \in \Xi(h)$, if $\rho \sqsubseteq \sigma$ then for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \rightarrow \eta_i = \theta_j$.*

Proof. From lemma 8.7 it follows that for each $i, j \in dom(\sigma)$ $i \neq j \rightarrow x_i \neq x_j$. With this we can apply lemma 3.3 and obtain that
for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \rightarrow \eta_i = \theta_j$. □

Lemma 8.10. *For each n positive integer such that $n \geqslant 2$, $k \in K(n)$, $t \in E(n,k)$ such that $t \notin \mathcal{C}$ one of the following two alternatives holds:*

- $t \in E_a(n,k) \cup E_e(n,k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n,k) \cup \bigcup_{f \in \mathcal{F}} E^f(n,k)$*;*
- $n > 2$ *and there exist m positive integer such that $2 \leqslant m < n$, $h \in K(m)$ such that $h \sqsubseteq k$, $t \in E_a(m,h) \cup E_e(m,h) \cup \bigcup_{c \in \mathcal{C}'} E^c(m,h) \cup \bigcup_{f \in \mathcal{F}} E^f(m,h)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k,t,\sigma) = \#(h,t,\sigma_{/dom(h)})$.*

Proof. Of course we begin with the case $n = 2$. Let $k \in K(2)$, $t \in E(2,k)$ such that $t \notin \mathcal{C}$. We have $K(2) = K(1) \cup K(1)^+$.

If $k \in K(1)^+$ we have $E(2,k) = E_a(2,k)$, so $t \in E_a(2,k)$.

If $k \in K(1)$ we have

$$E(2,k) = E(1,k) \cup E_b(2,k) \cup E_e(2,k) \cup \bigcup_{c \in \mathcal{C}'} E^c(2,k) \cup \bigcup_{f \in \mathcal{F}} E^f(2,k) \ .$$

Since $k = \epsilon$ we have $E(1,k) = \mathcal{C}$, $E_b(2,k) = \emptyset$, so

$$E(2,k) = \mathcal{C} \cup E_e(2,k) \cup \bigcup_{c \in \mathcal{C}'} E^c(2,k) \cup \bigcup_{f \in \mathcal{F}} E^f(2,k) \ .$$

Therefore in this case we have

$$t \in E_e(2,k) \cup \bigcup_{c \in \mathcal{C}'} E^c(2,k) \cup \bigcup_{f \in \mathcal{F}} E^f(2,k) \ .$$

Let now $n \geqslant 2$ and we try to prove the result for $n+1$. So let $k \in K(n+1)$, $t \in E(n+1,k)$ such that $t \notin \mathcal{C}$. We have $K(n+1) = K(n) \cup K(n)^+$.

If $k \in K(n)^+$ we have $E(n+1,k) = E_a(n+1,k)$ so $t \in E_a(n+1,k)$.

We now need to examine the case $k \in K(n)$. Here we have

$$E(n+1,k) = E(n,k) \cup E_b(n+1,k) \cup E_e(n+1,k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1,k) \cup \bigcup_{f \in \mathcal{F}} E^f(n+1,k) \ .$$

If $t \in E_e(n+1,k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1,k) \cup \bigcup_{f \in \mathcal{F}} E^f(n+1,k)$ then our result is verified.

If $t \in E(n,k)$ and we can apply our inductive hypothesis, which leads to two alternatives:

- $t \in E_a(n,k) \cup E_e(n,k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n,k) \cup \bigcup_{f \in \mathcal{F}} E^f(n,k)$;
- $n > 2$ and there exist m positive integer such that $2 \leqslant m < n$, $h \in K(m)$ such that $h \sqsubseteq k$, $t \in E_a(m,h) \cup E_e(m,h) \cup \bigcup_{c \in \mathcal{C}'} E^c(m,h) \cup \bigcup_{f \in \mathcal{F}} E^f(m,h)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k,t,\sigma) = \#(h,t,\sigma_{/dom(h)})$.

In the first case we observe that $2 \leqslant n < n+1$, $k \in K(n)$, $k \sqsubseteq k$. Moreover for each $\sigma \in \Xi(k)$ $\sigma_{/dom(k)} = \sigma \in \Xi(k)$ and $\#(k,t,\sigma) = \#(k,t,\sigma_{/dom(k)})$.

So in the first case our result is verified.

Let's examine the second case. Here $2 \leqslant m < n < n+1$, $h \in K(m)$, $h \sqsubseteq k$, for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k,t,\sigma) = \#(h,t,\sigma_{/dom(h)})$. So everything is as expected and our result is verified in this case too.

We have still one case to examine, which is the case of $t \in E_b(n+1,k)$. Here we have $k \neq \epsilon$ so by assumption 6.1.8 there exist $m < n$, $h \in K(m)$, $\phi \in E_s(m,h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h+ < y, \phi >$. Moreover by the definition of $E_b(n+1,k)$ we know that $t \in E(n,h)$. So we can apply our inductive hypothesis, which again leads to two alternatives:

- $t \in E_a(n,h) \cup E_e(n,h) \cup \bigcup_{c \in \mathcal{C}'} E^c(n,h) \cup \bigcup_{f \in \mathcal{F}} E^f(n,h)$;
- $n > 2$ and there exist p positive integer such that $2 \leqslant p < n$, $g \in K(p)$ such that $g \sqsubseteq h$, $t \in E_a(p,g) \cup E_e(p,g) \cup \bigcup_{c \in \mathcal{C}'} E^c(p,g) \cup \bigcup_{f \in \mathcal{F}} E^f(p,g)$ and for each $\rho \in \Xi(h)$ $\rho_{/dom(g)} \in \Xi(g)$ and $\#(h,t,\rho) = \#(g,t,\rho_{/dom(g)})$.

In the first case we observe that $2 \leqslant n < n+1$, $h \in K(n)$, $h \sqsubseteq k$, $t \in E_a(n,h) \cup E_e(n,h) \cup \bigcup_{c \in \mathcal{C}'} E^c(n,h) \cup \bigcup_{f \in \mathcal{F}} E^f(n,h)$, moreover for each $\sigma = \rho + (y,s) \in \Xi(k)$ we have

- $\#(k,t,\sigma) = \#(h,t,\rho)$,
- $\sigma_{/dom(h)} = \sigma_{/dom(\rho)} = \rho$,
- therefore $\#(k,t,\sigma) = \#(h,t,\sigma_{/dom(h)})$.

Let's examine the second case. Here $2 \leqslant p < n < n+1$, $g \in K(p)$, $g \sqsubseteq h \sqsubseteq k$, $t \in E_a(p,g) \cup E_e(p,g) \cup \bigcup_{c \in \mathcal{C}'} E^c(p,g) \cup \bigcup_{f \in \mathcal{F}} E^f(p,g)$. Moreover for each $\sigma = \rho + (y,s) \in \Xi(k)$ we have

- $\#(k,t,\sigma) = \#(h,t,\rho)$,
- $\sigma_{/dom(h)} = \sigma_{/dom(\rho)} = \rho$,
- $\sigma_{/dom(g)} = (\sigma_{/dom(h)})_{/dom(g)} = \rho_{/dom(g)}$,
- $\#(k,t,\sigma) = \#(h,t,\rho) = \#(g,t,\rho_{/dom(g)}) = \#(g,t,\sigma_{/dom(g)})$.

□

Lemma 8.11. *For each n positive integer, $k \in K(n)$, $t \in E(n,k)$ if $t \in \mathcal{C}$ then for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) = \#(t)$.*

Proof. Let's verify the result for $n = 1$. Here $k = \epsilon$, for each $\sigma \in \Xi(\epsilon)$ $\sigma = \epsilon$ so $\#(k,t,\sigma) = \#(\epsilon,t,\epsilon) = \#(t)$.

Now let's examine the inductive step. Given $k \in K(n+1)$, $t \in E(n+1,k)$ such that $t \in \mathcal{C}$ and $\sigma \in \Xi(k)$ we want to show that $\#(k,t,\sigma) = \#(t)$.

If $k \in K(n)^+$ then $t \in E_a(n+1,k)$, but since $t \in \mathcal{C}$ this cannot happen, so $k \in K(n)^+$ cannot happen.

Therefore $k \in K(n)$ and
$t \in E(n,k) \cup E_b(n+1,k) \cup E_e(n+1,k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1,k) \cup \bigcup_{f \in \mathcal{F}} E^f(n+1,k)$.

Since $t \in \mathcal{C}$ it follows that $t \in E(n,k) \cup E_b(n+1,k)$.

If $t \in E(n,k)$ clearly $\#(k,t,\sigma) = \#(t)$ holds by the inductive hypothesis.

If $t \in E_b(n+1,k)$ then we have $k \neq \epsilon$ so by assumption 6.1.8 there exist $m < n$, $h \in K(m)$, $\phi \in E_s(m,h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h+ < y, \phi >$, $\Xi(k) =$

$\{\rho + (y,s) | \, \rho \in \Xi(h), s \in \#(h,\phi,\rho)\})$. Moreover by the definition of $E_b(n+1,k)$ we know that $t \in E(n,h)$.

Clearly there exist $\rho \in \Xi(h)$, $s \in \#(h,\phi,\rho)$ such that $\sigma = \rho + (y,s)$ and $\#(k,t,\sigma) = \#(h,t,\rho)$. By the inductive hypothesis $\#(h,t,\rho) = \#(t)$, so $\#(k,t,\sigma) = \#(t)$. □

Lemma 8.12. *Let $k, h \in K(n)$ such that $h = \epsilon$ or $k = \epsilon$ or ($h, k \neq \epsilon$ and $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>$, $h =<< y_1, \psi_1 > \cdots < y_q, \psi_q >>$ and for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$). Let $u \in \mathcal{V} - var(k)$: $u \in \mathcal{V} - var(h)$ and $\vartheta \in E(n)$, let $k' = k + < u, \vartheta >$ and $h' = h + < u, \vartheta >$. Since $k, h \neq \epsilon$ there exist $x'_1, \ldots, x'_p \in \mathcal{V}$, $\varphi'_1, \ldots, \varphi'_p \in \Sigma^*$ such that $k' =<< x'_1, \varphi'_1 > \cdots < x'_p, \varphi'_p >>$, $y'_1, \ldots, y'_r \in \mathcal{V}$, $\psi'_1, \ldots, \psi'_r \in \Sigma^*$ such that $h' =<< y'_1, \psi'_1 > \cdots < y'_r, \psi'_r >>$ and for each $i \in dom(k')$, $j \in dom(h')$ $x'_i = y'_j \to \varphi'_i = \psi'_j$.*

Proof. If both $k, h = \epsilon$ then $k' =<< u, \vartheta >>= h'$ and our result is verified.

If $k \neq \epsilon$ and $h = \epsilon$ then let $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>$, clearly $k' =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >< u, \vartheta >>$ and $h =<< u, \vartheta >>$. Here we see that for each $i \in dom(k')$, $j \in dom(h')$ $x'_i = y'_j$ implies $j = 1$, $y'_j = u$, $x'_i = u$, so $\varphi'_i = \vartheta = \psi'_j$.

Finally if both $h, k \neq \epsilon$, $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>$, $h =<< y_1, \psi_1 > \cdots < y_q, \psi_q >>$ then $k' =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >< u, \vartheta >>$ and $h' =<< y_1, \psi_1 > \cdots < y_q, \psi_q >< u, \vartheta >>$. Given $i \in dom(k')$, $j \in dom(h')$ such that $x'_i = y'_j$ we have $i = 1 \ldots m+1$, $j = 1 \ldots q+1$.

If $i \leqslant m$ and $j \leqslant q$ then clearly $x_i = x'_i = y'_j = y_j$ and $\varphi'_i = \varphi_i = \psi_j = \psi'_j$.

If $i = m+1$ then $x_i = u$, so $y_j = u$ and $j = q+1$, so $\varphi'_i = \vartheta = \psi'_j$. □

Lemma 8.13. *Let $k, h \in K(n)$ such that $h = \epsilon$ or $k = \epsilon$ or ($h, k \neq \epsilon$ and $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>$, $h =<< y_1, \psi_1 > \cdots < y_q, \psi_q >>$ and for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$). Let $\kappa \sqsubseteq k$ and $g \sqsubseteq h$ then $\kappa = \epsilon$ or $g = \epsilon$ or*

- *$\kappa, g \neq \epsilon$ and so $h, k \neq \epsilon$,*
- *there exist p, r positive integers such that $p \leqslant m$, $r \leqslant q$, $\kappa =<< x_1, \varphi_1 > \cdots < x_p, \varphi_p >>$, $g =<< y_1, \psi_1 > \cdots < y_r, \psi_r >>$ and for each $i \in dom(\kappa)$, $j \in dom(g)$ $x_i = y_j \to \varphi_i = \psi_j$.*

Proof. Clearly we can have $\kappa = \epsilon$ or $g = \epsilon$, otherwise we have $\kappa, g \neq \epsilon$, so also ($h, k \neq \epsilon$ and $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>$, $h =<< y_1, \psi_1 > \cdots < y_q, \psi_q >>$ and for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$).

By lemma 5.6 there exist p, r positive integers such that $p \leqslant m$, $r \leqslant q$, $\kappa =<< x_1, \varphi_1 > \cdots < x_p, \varphi_p >>$, $g =<< y_1, \psi_1 > \cdots < y_r, \psi_r >>$.

Moreover $dom(\kappa) \subseteq dom(k)$ and $dom(g) \subseteq dom(h)$ so for each $i \in dom(\kappa)$, $j \in dom(g)$ $x_i = y_j \to \varphi_i = \psi_j$. □

Lemma 8.14. *Let $k, h \in K(n)$ such that $h = \epsilon$ or $k = \epsilon$ or ($h, k \neq \epsilon$ and $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>$, $h =<< y_1, \psi_1 > \cdots < y_q, \psi_q >>$ and for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$). Let $t \in E(n, k) \cap E(n, h)$. Let $\sigma = (x, z) \in \Xi(k)$, $\rho = (y, r) \in \Xi(h)$ such that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \to z_i = r_j$. Then $\#(k, t, \sigma) = \#(h, t, \rho)$.*

Proof. We prove this by induction on a positive integer n.

Let's verify the initial step. Here we have $k, h \in K(1)$. This implies $h = \epsilon = k$. We have $t \in E(1, \epsilon) = \mathcal{C}$. We have $\sigma = (x, s) \in \Xi(\epsilon)$, $\rho = (y, r) \in \Xi(\epsilon)$. Of course this implies $\sigma = \epsilon = \rho$. Then $\#(k, t, \sigma) = \#(\epsilon, t, \epsilon) = \#(h, t, \rho)$.

Let us see the inductive step, that is given a positive integer n we assume the result is true for each $m \leqslant n$ and we try to prove it for $n + 1$. In other words what we are trying to prove is that for each $k, h \in K(n + 1)$ such that one of the following conditions holds

- $h = \epsilon$
- $k = \epsilon$
- $h, k \neq \epsilon$ and $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>$, $h =<< y_1, \psi_1 > \cdots < y_q, \psi_q >>$ and for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$

and for each $t \in E(n+1, k) \cap E(n+1, h)$, $\sigma = (x, z) \in \Xi(k)$, $\rho = (y, r) \in \Xi(h)$ such that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \to z_i = r_j$ we have $\#(k, t, \sigma) = \#(h, t, \rho)$.

If $t \in \mathcal{C}$ then by lemma 8.11 $\#(k, t, \sigma) = \#(t) = \#(h, t, \rho)$.

Otherwise since $k \in K(n + 1)$ and $t \in E(n + 1, k)$ we can apply lemma 8.10 and obtain these two following alternative possibilities:

- $t \in E_a(n+1, k) \cup E_e(n+1, k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1, k) \cup \bigcup_{f \in \mathcal{F}} E^f(n+1, k)$;
- $n + 1 > 2$ and there exist μ positive integer such that $2 \leqslant \mu < n + 1$, $\kappa \in K(\mu)$ such that $\kappa \sqsubseteq k$, $t \in E_a(\mu, \kappa) \cup E_e(\mu, \kappa) \cup \bigcup_{c \in \mathcal{C}'} E^c(\mu, \kappa) \cup \bigcup_{f \in \mathcal{F}} E^f(\mu, \kappa)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(\kappa)} \in \Xi(\kappa)$ and $\#(k, t, \sigma) = \#(\kappa, t, \sigma_{/dom(\kappa)})$.

Since $h \in K(n+1)$ and $t \in E(n+1, h)$ we can also use lemma 8.10 to obtain these two other following alternative possibilities:

- $t \in E_a(n+1, h) \cup E_e(n+1, h) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1, h) \cup \bigcup_{f \in \mathcal{F}} E^f(n+1, h)$;
- $n + 1 > 2$ and there exist ν positive integer such that $2 \leqslant \nu < n + 1$, $g \in K(\nu)$ such that $g \sqsubseteq h$, $t \in E_a(\nu, g) \cup E_e(\nu, g) \cup \bigcup_{c \in \mathcal{C}'} E^c(\nu, g) \cup \bigcup_{f \in \mathcal{F}} E^f(\nu, g)$ and for each $\rho \in \Xi(h)$ $\rho_{/dom(g)} \in \Xi(g)$ and $\#(h, t, \rho) = \#(g, t, \rho_{/dom(g)})$.

So we have three possible cases to examine. The first is

- $t \in E_a(n+1, k) \cup E_e(n+1, k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1, k) \cup \bigcup_{f \in \mathcal{F}} E^f(n+1, k)$ and
- $t \in E_a(n+1, h) \cup E_e(n+1, h) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1, h) \cup \bigcup_{f \in \mathcal{F}} E^f(n+1, h)$.

The second case is

- $t \in E_a(n+1, k) \cup E_e(n+1, k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1, k) \cup \bigcup_{f \in \mathcal{F}} E^f(n+1, k)$ and
- $n + 1 > 2$ and there exist ν positive integer such that $2 \leqslant \nu < n + 1$, $g \in K(\nu)$ such that $g \sqsubseteq h$, $t \in E_a(\nu, g) \cup E_e(\nu, g) \cup \bigcup_{c \in \mathcal{C}'} E^c(\nu, g) \cup \bigcup_{f \in \mathcal{F}} E^f(\nu, g)$ and for each $\rho \in \Xi(h)$ $\rho_{/dom(g)} \in \Xi(g)$ and $\#(h, t, \rho) = \#(g, t, \rho_{/dom(g)})$.

Another case to examine would be the following

- $n+1>2$ and there exist μ positive integer such that $2 \leqslant \mu < n+1$, $\kappa \in K(\mu)$ such that $\kappa \sqsubseteq k$, $t \in E_a(\mu,\kappa) \cup E_e(\mu,\kappa) \cup \bigcup_{c \in \mathcal{C}'} E^c(\mu,\kappa) \cup \bigcup_{f \in \mathcal{F}} E^f(\mu,\kappa)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(\kappa)} \in \Xi(\kappa)$ and $\#(k,t,\sigma) = \#(\kappa,t,\sigma_{/dom(\kappa)})$ and
- $t \in E_a(n+1,h) \cup E_e(n+1,h) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1,h) \cup \bigcup_{f \in \mathcal{F}} E^f(n+1,h)$.

Anyway this case is practically equal to the second one, so we don't need to consider it. Finally the third case is the following.

- $n+1>2$ and there exist μ positive integer such that $2 \leqslant \mu < n+1$, $\kappa \in K(\mu)$ such that $\kappa \sqsubseteq k$, $t \in E_a(\mu,\kappa) \cup E_e(\mu,\kappa) \cup \bigcup_{c \in \mathcal{C}'} E^c(\mu,\kappa) \cup \bigcup_{f \in \mathcal{F}} E^f(\mu,\kappa)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(\kappa)} \in \Xi(\kappa)$ and $\#(k,t,\sigma) = \#(\kappa,t,\sigma_{/dom(\kappa)})$ and
- $n+1>2$ and there exist ν positive integer such that $2 \leqslant \nu < n+1$, $g \in K(\nu)$ such that $g \sqsubseteq h$, $t \in E_a(\nu,g) \cup E_e(\nu,g) \cup \bigcup_{c \in \mathcal{C}'} E^c(\nu,g) \cup \bigcup_{f \in \mathcal{F}} E^f(\nu,g)$ and for each $\rho \in \Xi(h)$ $\rho_{/dom(g)} \in \Xi(g)$ and $\#(h,t,\rho) = \#(g,t,\rho_{/dom(g)})$.

We now examine the three different cases we have distinguished. We start with the first one, where we have four different subcases:
$t \in E_a(n+1,k) \cup E_e(n+1,k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1,k) \cup \bigcup_{f \in \mathcal{F}} E^f(n+1,k)$.

We start with the subcase $t \in E_a(n+1,k)$. We must have $t \in E_a(n+1,h)$.

If $k \in K(n)$ then $E_a(n+1,k) = \emptyset$ so $k \in K(n)^+$ and there exist $\kappa \in K(n)$, $\theta \in E_s(n,\kappa)$, $u \in (\mathcal{V} - var(\kappa))$ such that $k = \kappa + < u,\theta >$, $E_a(n+1,k) = \{u\}$. Since $\sigma \in \Xi(k)$ there exist $\xi \in \Xi(\kappa)$, $s \in \#(\kappa,\theta,\xi)$ such that $\sigma = \xi + (u,s)$,
$\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,a)} = s$.

If $h \in K(n)$ then $E_a(n+1,h) = \emptyset$ so $h \in K(n)^+$ and there exist $\vartheta \in K(n)$, $\mu \in E_s(n,\vartheta)$, $v \in (\mathcal{V} - var(\vartheta))$ such that $h = \vartheta + < v,\mu >$, $E_a(n+1,k) = \{v\}$. Since $\rho \in \Xi(h)$ there exist $\zeta \in \Xi(\vartheta)$, $q \in \#(\vartheta,\mu,\zeta)$ such that $\rho = \zeta + (v,q)$,
$\#(h,t,\rho) = \#(h,t,\rho)_{(n+1,h,a)} = q$.

Since $t \in E_a(n+1,k)$ we have $t = u$, since $t \in E_a(n+1,h)$ we have $t = v$, therefore $u = v$.

There exists $i \in dom(\sigma)$ such that $u = x_i$, $s = z_i$, there exists $j \in dom(\rho)$ such that $v = y_j$, $q = r_j$.

Therefore $x_i = u = v = y_j$ and $\#(k,t,\sigma) = s = z_i = r_j = q = \#(h,t,\rho)$.

$\diamond$

We now consider the subcase $t \in \bigcup_{c \in \mathcal{C}'} E^c(n+1,k)$. This implies there exists $c_1 \in \mathcal{C}'$ such that $t \in E^{c_1}(n+1,k)$. This also implies $k \in K(n)$ and we have $t \in H_{c_1}(n+1,k)$.

Clearly $t \notin E_a(n+1,h) \cup E_e(n+1,h) \cup \bigcup_{f \in \mathcal{F}} E^f(n+1,h)$, so $t \in \bigcup_{c \in \mathcal{C}'} E^c(n+1,h)$. This implies there exists $c_2 \in \mathcal{C}'$ such that $t \in E^{c_2}(n+1,h)$. Clearly we must have $h \in K(n)$ and we have also $t \in H_{c_2}(n+1,h)$.

As we have seen in lemmas 6.1.12 and 6.1.13 since $t = (c_1)(\psi) = (c_2)(\psi)$ then $c_2 = c_1$ and t can be written as $(c_1)(\psi_1,\ldots,\psi_u)$. Since $t \in H_{c_1}(n+1,k) \cap H_{c_2}(n+1,h)$ then for each $i = 1 \ldots u$ $\psi_i \in E(n,k) \cap E(n,h)$. By the inductive hypothesis it follows immediately that for each $i = 1 \ldots u$ $\#(k,\psi_i,\sigma) = \#(h,\psi_i,\rho)$.

Finally it follows that

$$\begin{aligned}\#(k,t,\sigma) &= \#(c_1)(\#(k,\psi_1,\sigma),\ldots,\#(k,\psi_u,\sigma)) \\ &= \#(c_2)(\#(h,\psi_1,\rho),\ldots,\#(h,\psi_u,\rho)) = \#(h,t,\rho)\end{aligned}$$

◇

We now consider the subcase $t \in \bigcup_{f\in\mathcal{F}} E^f(n+1,k)$. This implies there exists $f_1 \in \mathcal{F}$ such that $t \in E^{f_1}(n+1,k)$. This also implies $k \in K(n)$ and we have $t \in H_{f_1}(n+1,k)$.

Clearly $t \notin E_a(n+1,h) \cup E_e(n+1,h) \cup \bigcup_{c\in\mathcal{C}'} E^c(n+1,k)$, so $t \in \bigcup_{f\in\mathcal{F}} E^f(n+1,h)$. This implies there exists $f_2 \in \mathcal{F}$ such that $t \in E^{f_2}(n+1,h)$. This also implies $h \in K(n)$ and we have $t \in H_{f_2}(n+1,h)$.

Since $t \in H_{f_1}(n+1,k)$ then $t = f_1(\psi)$ where $\psi \in \Sigma^*$. Since $t \in H_{f_2}(n+1,k)$ then $t = f_2(\varphi)$ where $\varphi \in \Sigma^*$. It follows that $f_2 = f_1$.

If f_1 has multiplicity 1 then there exists $\psi \in E(n,k)$ such that $t = f_1(\psi)$ and there exists $\varphi \in E(n,h)$ such that $t = f_2(\varphi)$. It follows that $\varphi = \psi$ and by the inductive hypothesis $\#(k,\psi,\sigma) = \#(h,\psi,\rho) = \#(h,\varphi,\rho)$. It also follows that

$$\#(k,t,\sigma) = P_{f_1}(\#(k,\psi,\sigma)) = P_{f_2}(\#(h,\varphi,\rho)) = \#(h,t,\rho).$$

If f_1 has multiplicity 2 we can consider that $t = f_1(\psi)$ where $\psi \in \Sigma^*$ and that $t \in H_{f_1}(n+1,k)$, so by lemma 6.1.18 we can determine $\psi_1,\psi_2 \in E(n,k)$ such that $t = f_1(\psi_1,\psi_2)$.

We have also $t = f_2(\psi)$ where $\psi \in \Sigma^*$ and $t \in H_{f_2}(n+1,h)$, so using the same lemma we can determine that $\psi_1,\psi_2 \in E(n,h)$ and $t = f_2(\psi_1,\psi_2)$.

By the inductive hypotesis $\#(k,\psi_1,\sigma) = \#(h,\psi_1,\rho)$ and $\#(k,\psi_2,\sigma) = \#(h,\psi_2,\rho)$, so

$$\#(k,t,\sigma) = P_{f_1}(\#(k,\psi_1,\sigma),\#(k,\psi_2,\sigma)) = P_{f_2}(\#(h,\psi_1,\rho),\#(h,\psi_2,\rho)) = \#(h,t,\rho).$$

◇

We now consider the subcase $t \in E_e(n+1,k)$ (which implies $k \in K(n)$). Clearly $t \notin E_a(n+1,h) \cup \bigcup_{c\in\mathcal{C}'} E^c(n+1,h) \cup \bigcup_{f\in\mathcal{F}} E^f(n+1,h)$, so $t \in E_e(n+1,h)$, which implies $h \in K(n)$.

We have $t \in H_e(n+1,k)$ and $t \in H_e(n+1,h)$.

Since $t = \{\}(\psi)$ for some $\psi \in \Sigma^*$ we can apply lemma 6.1.25.

Consider the set of the positive integers r such that $2 < r < \ell(t)$, $t[r] =$ ‘,’ and $d(t,r) = 1$. Since $t \in H_e(n+1,k)$ this set is not empty and let's name $r_1, \ldots, r_p$ its members (in increasing order).

Let's also define $\psi_1 = t[3, r_1 - 1]$ (if $r_1 - 1 < 3$ then $\psi_1 = \epsilon$ where ϵ is the empty string over the alphabet Σ).
If $p > 1$ then for each $i = 1 \ldots p-1$ we define $\psi_{i+1} = t[r_i + 1, r_{i+1} - 1]$ (if $r_{i+1} - 1 < r_i + 1$ then $\psi_{i+1} = \epsilon$.
Finally we define $\psi_{p+1} = t[r_p + 1, \ell(t) - 1]$ (if $\ell(t) - 1 < r_p + 1$ then $\psi_{p+1} = \epsilon$).

Since $t \in H_e(n+1,k)$ we have that for each $i = 1 \ldots p$ $\ell(\psi_i) \geqslant 3$, $\psi_i[2] =$ ‘:’; $\ell(\psi_{p+1}) \geqslant 1$ and we can define a function u over the domain $\{1, \ldots, p\}$ by setting $u(i) = \psi_i[1]$; we can define a function ϑ over the domain $\{1, \ldots, p\}$ by setting $\vartheta(i) = \psi_i[3, \ell(\psi_i)]$; let's also define $\theta = \psi_{p+1}$. With those definitions we have the following

- for each $i = 1 \ldots p$ $u_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \ldots p$ $i \neq j \to u_i \neq u_j$,
- for each $i = 1 \ldots p$ $\vartheta_i \in E(n)$,
- $\theta \in E(n)$,
- $\mathcal{E}(n, k, p, u, \vartheta, \theta)$.

Since $t \in H_e(n+1,h)$ we have also the following:

- for each $i = 1 \ldots p$ $u_i \in \mathcal{V} - var(h)$,
- $\mathcal{E}(n, h, p, u, \vartheta, \theta)$.

We have also $t = \{\}(\psi_1, \ldots, \psi_p, \psi_{p+1}) = \{\}(u_1 : \vartheta_1, \ldots, u_p : \vartheta_p, \theta)$, so as suggested by lemma 6.1.27 we have identified the elements p, u, ϑ, θ such that $t = \{\}(u_1 : \vartheta_1, \ldots, u_p : \vartheta_p, \theta)$.

We have

$$\#(k, t, \sigma) = \{\#(k'_p, \theta, \sigma'_p) |\ \sigma'_p \in \Xi(k'_p), \sigma \sqsubseteq \sigma'_p\} \ ,$$

where $k'_1 = k + < u_1, \vartheta_1 >$, and if $p > 1$ for each $i = 1 \ldots p-1$
$k'_{i+1} = k'_i + < u_{i+1}, \vartheta_{i+1} >$.

We have also

$$\#(h, t, \rho) = \{\#(h'_p, \theta, \rho'_p) |\ \rho'_p \in \Xi(h'_p), \rho \sqsubseteq \rho'_p\} \ ,$$

where $h'_1 = h + < u_1, \vartheta_1 >$, and if $p > 1$ for each $i = 1 \ldots p-1$
$h'_{i+1} = h'_i + < u_{i+1}, \vartheta_{i+1} >$.

We want to show that $\#(k, t, \sigma) = \#(h, t, \rho)$, thus we have to show

$$\{\#(k'_p, \theta, \sigma'_p) |\ \sigma'_p \in \Xi(k'_p), \sigma \sqsubseteq \sigma'_p\} = \{\#(h'_p, \theta, \rho'_p) |\ \rho'_p \in \Xi(h'_p), \rho \sqsubseteq \rho'_p\} \ .$$

To prove this we just need to prove the following two assertions:

- for each $\sigma'_p \in \Xi(k'_p)$ such that $\sigma \sqsubseteq \sigma'_p$ there exists $\rho'_p \in \Xi(h'_p)$ such that $\rho \sqsubseteq \rho'_p$ and $\#(h'_p, \theta, \rho'_p) = \#(k'_p, \theta, \sigma'_p)$;

- for each $\rho'_p \in \Xi(h'_p)$ such that $\rho \sqsubseteq \rho'_p$ there exists $\sigma'_p \in \Xi(k'_p)$ such that $\sigma \sqsubseteq \sigma'_p$ and $\#(k'_p, \theta, \sigma'_p) = \#(h'_p, \theta, \rho'_p)$.

It is clearly enough to prove the first one, since the second would be proved by simply substituting variables in the proof of the first.

Let $\sigma'_p \in \Xi(k'_p)$ such that $\sigma \sqsubseteq \sigma'_p$, we want to find $\rho'_p \in \Xi(h'_p)$ such that $\rho \sqsubseteq \rho'_p$ and $\#(h'_p, \theta, \rho'_p) = \#(k'_p, \theta, \sigma'_p)$.

If $p > 1$ we define $\sigma'_1 = (\sigma'_p)_{/dom(k'_1)}$, otherwise $\sigma'_1 = (\sigma'_p)_{/dom(k'_1)}$ holds all the same. We should be able to prove that:

- $\sigma'_1 \in \Xi(k'_1)$
- there exists $s_1 \in \#(k, \vartheta_1, \sigma)$ such that $\sigma'_1 = \sigma + (u_1, s_1)$.

If $p = 1$ then $\sigma'_1 \in \Xi(k'_1)$ clearly holds, else we have $k'_p \neq \epsilon$, $\sigma'_p \in \Xi(k'_p)$, $k'_1 \in \mathcal{R}(k'_p)$, $k'_1 \neq k'_p$, so by lemma 8.4 $\sigma'_1 = (\sigma'_p)_{/dom(k'_1)} \in \Xi(k'_1)$.

We have $k'_1 = k+ < u_1, \vartheta_1 >$ and $k'_1 \in K(n)$, clearly $k'_1 \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $k \in K(n)$, $\vartheta_1 \in E_s(n, k)$, $u_1 \in \mathcal{V} - var(k)$, so by lemma 8.2

$$\Xi(k'_1) = \{\xi + (u_1, s) |\, \xi \in \Xi(k), s \in \#(k, \vartheta_1, \xi)\}.$$

Then there exist $\xi \in \Xi(k), s \in \#(k, \vartheta_1, \xi)$ such that $\sigma'_1 = \xi + (u_1, s)$. Here we can see that

$$(\sigma'_1)_{/dom(k)} = (\sigma'_1)_{/dom(\xi)} = \xi$$

and at the same time, since $dom(k) \subseteq dom(k'_1) \subseteq dom(k'_p) = dom(\sigma'_p)$,

$$(\sigma'_1)_{/dom(k)} = ((\sigma'_p)_{/dom(k'_1)})_{/dom(k)} = (\sigma'_p)_{/dom(k)} = (\sigma'_p)_{/dom(\sigma)} = \sigma.$$

Therefore $\xi = \sigma$ and there exists $s \in \#(k, \vartheta_1, \sigma)$ such that $\sigma'_1 = \sigma + (u_1, s)$.

If $p > 1$ then for each $i = 1 \ldots p-1$ if $i+1 < p$ we can define $\sigma'_{i+1} = (\sigma'_p)_{/dom(k'_{i+1})}$, otherwise $\sigma'_{i+1} = \sigma'_p = (\sigma'_p)_{/dom(k'_p)} = (\sigma'_p)_{/dom(k'_{i+1})}$ is equally true. We can observe that with the definitions we have provided for each $i = 1 \ldots p$ $\sigma'_i = (\sigma'_p)_{/dom(k'_i)}$.

We should also be able to prove that for each $i = 1 \ldots p-1$

- $\sigma'_{i+1} \in \Xi(k'_{i+1})$,
- there exists $s_{i+1} \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$ such that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s_{i+1})$.

If $i+1 = p$ then $\sigma'_{i+1} \in \Xi(k'_{i+1})$ clearly holds, else $i+1 < p$ and $k'_p \neq \epsilon$, $\sigma'_p \in \Xi(k'_p)$, $k'_{i+1} \in \mathcal{R}(k'_p)$, $k'_{i+1} \neq k'_p$, so by lemma 8.4 $\sigma'_{i+1} = (\sigma'_p)_{/dom(k'_{i+1})} \in \Xi(k'_{i+1})$.

We have $k'_{i+1} = k'_i + < u_{i+1}, \vartheta_{i+1} >$ and $k'_{i+1} \in K(n)$, clearly $k'_{i+1} \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $k'_i \in K(n)$, $\vartheta_{i+1} \in E_s(n, k'_i)$, $var(k'_i) = var(k) \cup \{u_1, \ldots, u_i\}$, $u_{i+1} \in \mathcal{V} - var(k'_i)$, so by lemma 8.2

$$\Xi(k'_{i+1}) = \{\xi + (u_{i+1}, s) |\, \xi \in \Xi(k'_i), s \in \#(k'_i, \vartheta_{i+1}, \xi)\}.$$

Then there exist $\xi \in \Xi(k'_i), s \in \#(k'_i, \vartheta_{i+1}, \xi)$ such that $\sigma'_{i+1} = \xi + (u_{i+1}, s)$. Here we can see that

$$(\sigma'_{i+1})_{/dom(k'_i)} = (\sigma'_{i+1})_{/dom(\xi)} = \xi$$

and at the same time, since $dom(k'_i) \subseteq dom(k'_{i+1}) \subseteq dom(k'_p) = dom(\sigma'_p)$,

$$(\sigma'_{i+1})_{/dom(k'_i)} = ((\sigma'_p)_{/dom(k'_{i+1})})_{/dom(k'_i)} = (\sigma'_p)_{/dom(k'_i)} = \sigma'_i.$$

Therefore $\xi = \sigma'_i$ and there exists $s \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$ such that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s)$.

Then we define $\rho'_1 = \rho + (u_1, s_1)$, and we should be able to prove that $\rho'_1 \in \Xi(h'_1)$.

We have $\mathcal{E}(n, h, p, u, \vartheta, \theta)$. This implies $\vartheta_1 \in E_s(n, h)$. We have $h'_1 = h + < u_1, \vartheta_1 >$ and $h'_1 \in K(n)$, $h'_1 \neq \epsilon$, $n \geqslant 2$, moreover $h \in K(n)$, $u_1 \in \mathcal{V} - var(h)$ and therefore

$$\Xi(h'_1) = \{\xi + (u_1, s) |\, \xi \in \Xi(h), s \in \#(h, \vartheta_1, \xi)\}.$$

Since $\rho \in \Xi(h)$, to prove that $\rho'_1 \in \Xi(h'_1)$ we just need to prove that $s_1 \in \#(h, \vartheta_1, \rho)$. We know that $s_1 \in \#(k, \vartheta_1, \sigma)$. We have $\vartheta_1 \in E(n, k)$, $\vartheta_1 \in E(n, h)$.

With that we can apply the inductive hypothesis and obtain that $\#(k, \vartheta_1, \sigma) = \#(h, \vartheta_1, \rho)$, therefore $s_1 \in \#(h, \vartheta_1, \rho)$ and $\rho'_1 \in \Xi(h'_1)$.

We also notice that $k'_1 = k + < u_1, \vartheta_1 >$, $h'_1 = h + < u_1, \vartheta_1 >$, so if we set $k'_1 =<< x'_1, \varphi'_1 > \cdots < x'_{m'}, \varphi'_{m'} >>$ and $h'_1 =<< y'_1, \psi'_1 > \cdots < y'_{q'}, \psi'_{q'} >>$ then by lemma 8.12 for each $\alpha \in dom(k'_1)$, $\beta \in dom(h'_1)$ $x'_\alpha = y'_\beta \to \varphi'_\alpha = \psi'_\beta$.
Moreover we notice that $\sigma'_1 = \sigma + (u_1, s_1)$, $\rho'_1 = \rho + (u_1, s_1)$, and if we set $\sigma'_1 = (x', z')$, $\rho'_1 = (y', r')$ then by lemma 3.4 for each $\alpha \in dom(\sigma'_1)$, $\beta \in dom(\rho'_1)$ $x'_\alpha = y'_\beta \to z'_\alpha = r'_\beta$.

If $p > 1$ then for each $i = 1 \ldots p - 1$ we can define $\rho'_{i+1} = \rho'_i + (u_{i+1}, s_{i+1})$ and we expect to be able to prove that $\rho'_{i+1} \in \Xi(h'_{i+1})$.

We have $\mathcal{E}(n, h, p, u, \vartheta, \theta)$ and $h'_{i+1} = h'_i + < u_{i+1}, \vartheta_{i+1} >$. This implies $h'_{i+1} \in K(n)$, $h'_{i+1} \neq \epsilon$ and $n \geqslant 2$ holds too. Moreover $h'_i \in K(n)$, $\vartheta_{i+1} \in E_s(n, h'_i)$, and, since $var(h'_i) = var(h) \cup \{u_1, \ldots, u_i\}$, $u_{i+1} \in \mathcal{V} - var(h'_i)$. Therefore

$$\Xi(h'_{i+1}) = \{\xi + (u_{i+1}, s) |\, \xi \in \Xi(h'_i), s \in \#(h'_i, \vartheta_{i+1}, \xi)\}.$$

By inductive hypothesis we can assume that $\rho'_i \in \Xi(h'_i)$, therefore to prove $\rho'_{i+1} \in \Xi(h'_{i+1})$ we just need to prove $s_{i+1} \in \#(h'_i, \vartheta_{i+1}, \rho'_i)$. We know that $s_{i+1} \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$.

As an inductive hypothesis we can also assume that

- if we set $k'_i =<< x'_1, \varphi'_1 > \cdots < x'_{m'}, \varphi'_{m'} >>$ and $h'_i =<< y'_1, \psi'_1 > \cdots < y'_{q'}, \psi'_{q'} >>$ then for each $\alpha \in dom(k'_i)$, $\beta \in dom(h'_i)$ $x'_\alpha = y'_\beta \to \varphi'_\alpha = \psi'_\beta$.
- if we set $\sigma'_i = (x', z')$, $\rho'_i = (y', r')$ then for each $\alpha \in dom(\sigma'_i)$, $\beta \in dom(\rho'_i)$ $x'_\alpha = y'_\beta \to z'_\alpha = r'_\beta$.

We have $k'_i \in K(n)$, $h'_i \in K(n)$, $\vartheta_{i+1} \in E_s(n, k'_i)$, $\vartheta_{i+1} \in E_s(n, h'_i)$, $\sigma'_i \in \Xi(k'_i)$, $\rho'_i \in \Xi(h'_i)$, so we can apply the inductive hypothesis and obtain that

$\#(k'_i, \vartheta_{i+1}, \sigma'_i) = \#(h'_i, \vartheta_{i+1}, \rho'_i)$. Therefore $s_{i+1} \in \#(h'_i, \vartheta_{i+1}, \rho'_i)$ and we have proved $\rho'_{i+1} \in \Xi(h'_{i+1})$.

In this proof that $\rho'_{i+1} \in \Xi(h'_{i+1})$ we have used an inductive hypothesis which we still haven't proved, so we need to prove it now. What we need to prove is the following:

- if we set $k'_{i+1} =<< x'_1, \varphi'_1 > \cdots < x'_{m'}, \varphi'_{m'} >>$ and $h'_{i+1} =<< y'_1, \psi'_1 > \cdots < y'_{q'}, \psi'_{q'} >>$ then for each $\alpha \in dom(k'_{i+1})$, $\beta \in dom(h'_{i+1})$ $x'_\alpha = y'_\beta \rightarrow \varphi'_\alpha = \psi'_\beta$.
- if we set $\sigma'_{i+1} = (x', z')$, $\rho'_{i+1} = (y', r')$ then for each $\alpha \in dom(\sigma'_{i+1})$, $\beta \in dom(\rho'_{i+1})$ $x'_\alpha = y'_\beta \rightarrow z'_\alpha = r'_\beta$.

To prove the first item we consider that $k'_{i+1} = k'_i + < u_{i+1}, \vartheta_{i+1} >$, $h'_{i+1} = h'_i + < u_{i+1}, \vartheta_{i+1} >$, $u_{i+1} \in \mathcal{V} - var(k'_i)$, $u_{i+1} \in \mathcal{V} - var(h'_i)$, $\vartheta_{i+1} \in E(n)$. So we can apply lemma 8.12 and the first condition is proved.

To prove the second item we consider that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s_{i+1})$, $\rho'_{i+1} = \rho'_i + (u_{i+1}, s_{i+1})$, $u_{i+1} \in \mathcal{V} - var(\sigma'_i)$, $u_{i+1} \in \mathcal{V} - var(\rho'_i)$. So we can apply lemma 3.4 and the second condition is proved.

At this point we have defined ρ'_p such that $\rho \sqsubseteq \rho'_p$ and proved that $\rho'_p \in \Xi(h'_p)$. We have also that $k'_p \in K(n)$, $\theta \in E(n, k'_p)$, $h'_p \in K(n)$, $\theta \in E(n, h'_p)$, $\sigma'_p \in \Xi(k'_p)$. Moreover

- if we set $k'_p =<< x'_1, \varphi'_1 > \cdots < x'_{m'}, \varphi'_{m'} >>$ and $h'_p =<< y'_1, \psi'_1 > \cdots < y'_{q'}, \psi'_{q'} >>$ then for each $\alpha \in dom(k'_p)$, $\beta \in dom(h'_p)$ $x'_\alpha = y'_\beta \rightarrow \varphi'_\alpha = \psi'_\beta$.
- if we set $\sigma'_p = (x', z')$, $\rho'_p = (y', r')$ then for each $\alpha \in dom(\sigma'_p)$, $\beta \in dom(\rho'_p)$ $x'_\alpha = y'_\beta \rightarrow z'_\alpha = r'_\beta$.

With that, $\#(h'_p, \theta, \rho'_p) = \#(k'_p, \theta, \sigma'_p)$ follows by inductive hypothesis.

$\diamond$

Let's consider the second case, which as we recall is the following:

- $t \in E_a(n+1, k) \cup E_e(n+1, k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1, k) \cup \bigcup_{f \in \mathcal{F}} E^f(n+1, k)$ and
- $n+1 > 2$ and there exist ν positive integer such that $2 \leqslant \nu < n+1$, $g \in K(\nu)$ such that $g \sqsubseteq h$, $t \in E_a(\nu, g) \cup E_e(\nu, g) \cup \bigcup_{c \in \mathcal{C}'} E^c(\nu, g) \cup \bigcup_{f \in \mathcal{F}} E^f(\nu, g)$ and for each $\rho \in \Xi(h)$ $\rho_{/dom(g)} \in \Xi(g)$ and $\#(h, t, \rho) = \#(g, t, \rho_{/dom(g)})$.

Initially we consider the same four different subcases of the first case: $t \in E_a(n+1, k) \cup E_e(n+1, k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1, k) \cup \bigcup_{f \in \mathcal{F}} E^f(n+1, k)$.

We start with the subcase $t \in E_a(n+1, k)$. We must have $t \in E_a(\nu, g)$.

If $k \in K(n)$ then $E_a(n+1, k) = \emptyset$ so $k \in K(n)^+$ and there exist $\kappa \in K(n)$, $\theta \in E_s(n, \kappa)$, $u \in (\mathcal{V} - var(\kappa))$ such that $k = \kappa + < u, \theta >$, $E_a(n+1, k) = \{u\}$. Since $\sigma \in \Xi(k)$ there exist $\xi \in \Xi(\kappa)$, $s \in \#(\kappa, \theta, \xi)$ such that $\sigma = \xi + (u, s)$, $\#(k, t, \sigma) = \#(k, t, \sigma)_{(n+1, k, a)} = s$.

If $g \in K(\nu - 1)$ then $E_a(\nu, g) = \emptyset$ so $g \in K(\nu - 1)^+$ and there exist $\vartheta \in K(\nu - 1)$, $\mu \in E_s(\nu - 1, \vartheta)$, $v \in (\mathcal{V} - var(\vartheta))$ such that $g = \vartheta + < v, \mu >$, $E_a(\nu, g) = \{v\}$. We have $\rho \in \Xi(h)$ and $\rho_{/dom(g)} \in \Xi(g)$. Let $\eta = \rho_{/dom(g)}$, then there exist $\zeta \in \Xi(\vartheta)$, $q \in \#(\vartheta, \mu, \zeta)$ such that $\eta = \zeta + (v, q)$, $\#(g, t, \eta) = \#(g, t, \eta)_{(\nu, g, a)} = q$.

We have to prove that $\#(k,t,\sigma) = \#(h,t,\rho)$, and since $\#(h,t,\rho) = \#(g,t,\eta)$ it is enough to prove that $\#(k,t,\sigma) = \#(g,t,\eta)$.

Since $t \in E_a(n+1,k)$ we have $t = u$, since $t \in E_a(\nu,g)$ we have $t = v$, therefore $u = v$.

Since $\eta \sqsubseteq \rho$ we can apply lemma 3.5 to show that if $\eta = (w,\delta)$ then for each $i \in dom(\sigma),\ j \in dom(\eta)\ x_i = w_j \rightarrow z_i = \delta_j$.

There exists $i \in dom(\sigma)$ such that $u = x_i,\ s = z_i$, there exists $j \in dom(\eta)$ such that $v = w_j,\ q = \delta_j$.

Therefore $x_i = u = v = w_j$ and $\#(k,t,\sigma) = s = z_i = \delta_j = q = \#(g,t,\eta)$.

◇

We now consider the subcase $t \in \bigcup_{c\in\mathcal{C}'} E^c(n+1,k)$. This implies there exists $c_1 \in \mathcal{C}'$ such that $t \in E^{c_1}(n+1,k)$. This also implies $k \in K(n)$ and we have $t \in H_{c_1}(n+1,k)$.

Clearly $t \notin E_a(\nu,g) \cup E_e(\nu,g) \cup \bigcup_{f\in\mathcal{F}} E^f(\nu,g)$, so $t \in \bigcup_{c\in\mathcal{C}'} E^c(\nu,g)$. This implies there exists $c_2 \in \mathcal{C}'$ such that $t \in E^{c_2}(\nu,g)$. Clearly we must have $g \in K(\nu-1)$ and we have also $t \in H_{c_2}(\nu,g)$.

As we have seen in lemmas 6.1.12 and 6.1.13 since $t = (c_1)(\psi) = (c_2)(\psi)$ then $c_2 = c_1$ and t can be written as $(c_1)(\psi_1,\ldots,\psi_u)$. Since $t \in H_{c_1}(n+1,k) \cap H_{c_2}(\nu,g)$ then for each $i = 1\ldots u\ \psi_i \in E(n,k) \cap E(\nu-1,g)$. Clearly $g \in K(n)$ and $\psi_i \in E(n,g)$.

Let $\eta = \rho_{/dom(g)} \in \Xi(g)$. By lemma 8.13 we have that $k = \epsilon$ or $g = \epsilon$ or

- $k,g \neq \epsilon$ and so $k,h \neq \epsilon$,
- there exist p positive integer such that $p \leqslant q$, $g = << y_1,\psi_1 > \cdots < y_r,\psi_p >>$ and for each $i \in dom(k),\ j \in dom(g)\ x_i = y_j \rightarrow \varphi_i = \psi_j$.

Since $\eta \sqsubseteq \rho$ we can apply lemma 3.5 to also show that if $\eta = (w,\mu)$ then for each $i \in dom(\sigma),\ j \in dom(\eta)\ x_i = w_j \rightarrow z_i = \mu_j$.

We can apply the inductive hypothesis and obtain that for each $i = 1\ldots u$ $\#(k,\psi_i,\sigma) = \#(g,\psi_i,\eta)$.

Finally it follows that

$$\begin{aligned}\#(k,t,\sigma) &= \#(c_1)(\#(k,\psi_1,\sigma),\ldots,\#(k,\psi_u,\sigma))\\ &= \#(c_2)(\#(g,\psi_1,\eta),\ldots,\#(g,\psi_u,\eta)) = \#(g,t,\eta) = \#(h,t,\rho)\end{aligned}$$

◇

We now consider the subcase $t \in \bigcup_{f\in\mathcal{F}} E^f(n+1,k)$. This implies there exists $f_1 \in \mathcal{F}$ such that $t \in E^{f_1}(n+1,k)$. This also implies $k \in K(n)$ and we have $t \in H_{f_1}(n+1,k)$.

Clearly $t \notin E_a(\nu,g) \cup E_e(\nu,g) \cup \bigcup_{c\in\mathcal{C}'} E^c(\nu,g)$ so $t \in \bigcup_{f\in\mathcal{F}} E^f(\nu,g)$. This implies there exists $f_2 \in \mathcal{F}$ such that $t \in E^{f_2}(\nu,g)$. This also implies $g \in K(\nu-1)$ and we have $t \in H_{f_2}(\nu, g)$.

Since $t \in H_{f_1}(n+1, k)$ then $t = f_1(\psi)$ where $\psi \in \Sigma^*$. Since $t \in H_{f_2}(\nu, g)$ then $t = f_2(\varphi)$ where $\varphi \in \Sigma^*$. It follows that $f_2 = f_1$.

Let $\eta = \rho_{/dom(g)} \in \Xi(g)$. By lemma 8.13 we have that $k = \epsilon$ or $g = \epsilon$ or

- $k, g \neq \epsilon$ and so $k, h \neq \epsilon$,
- there exist p positive integer such that $p \leqslant q$, $g =<< y_1, \psi_1 > \cdots < y_r, \psi_p >>$ and for each $i \in dom(k)$, $j \in dom(g)$ $x_i = y_j \rightarrow \varphi_i = \psi_j$.

Since $\eta \sqsubseteq \rho$ we can apply lemma 3.5 to also show that if $\eta = (w, \mu)$ then for each $i \in dom(\sigma)$, $j \in dom(\eta)$ $x_i = w_j \rightarrow z_i = \mu_j$.

If f_1 has multiplicity 1 then there exists $\psi \in E(n, k)$ such that $t = f_1(\psi)$ and there exists $\varphi \in E(\nu - 1, g)$ such that $t = f_2(\varphi)$. It follows that $\varphi = \psi$. Clearly $g \in K(n)$ and $\varphi \in E(n, g)$.

So we can apply the inductive hypothesis and obtain that $\#(k, \psi, \sigma) = \#(g, \psi, \eta)$. It also follows that

$$\#(k, t, \sigma) = P_{f_1}(\#(k, \psi, \sigma)) = P_{f_2}(\#(g, \varphi, \eta)) = \#(g, t, \eta) = \#(h, t, \rho).$$

If f_1 has multiplicity 2 we can consider that $t = f_1(\psi)$ where $\psi \in \Sigma^*$ and that $t \in H_{f_1}(n+1, k)$, so by lemma 6.1.18 we can determine $\psi_1, \psi_2 \in E(n, k)$ such that $t = f_1(\psi_1, \psi_2)$.

We have also $t = f_2(\psi)$ where $\psi \in \Sigma^*$ and $t \in H_{f_2}(\nu, g)$, so using the same lemma we can determine that $\psi_1, \psi_2 \in E(\nu - 1, g)$ and $t = f_2(\psi_1, \psi_2)$. Clearly $g \in K(n)$ and $\psi_1, \psi_2 \in E(n, g)$.

By the inductive hypotesis $\#(k, \psi_1, \sigma) = \#(g, \psi_1, \eta)$ and $\#(k, \psi_2, \sigma) = \#(g, \psi_2, \eta)$, so

$$\#(k, t, \sigma) = P_{f_1}(\#(k, \psi_1, \sigma), \#(k, \psi_2, \sigma)) = P_{f_2}(\#(g, \psi_1, \eta), \#(g, \psi_2, \eta)) = \#(g, t, \eta),$$

$$\#(k, t, \sigma) = \#(g, t, \eta) = \#(h, t, \rho).$$

$\diamond$

We now consider the subcase $t \in E_e(n+1, k)$ (which implies $k \in K(n)$). Clearly $t \notin E_a(\nu, g) \cup \bigcup_{c \in \mathcal{C}'} E^c(\nu, g) \cup \bigcup_{f \in \mathcal{F}} E^f(\nu, g)$ so $t \in E_e(\nu, g)$. This implies $g \in K(\nu - 1)$.

We have $t \in H_e(n+1, k)$ and $t \in H_e(\nu, g)$.

Since $t = \{\}(\psi)$ for some $\psi \in \Sigma^*$ we can apply lemma 6.1.25.

Consider the set of the positive integers r such that $2 < r < \ell(t)$, $t[r] =$ ‘,’ and $d(t, r) = 1$. Since $t \in H_e(n+1, k)$ this set is not empty and let's name $r_1, \ldots, r_p$ its members (in increasing order).

Let's also define $\psi_1 = t[3, r_1 - 1]$ (if $r_1 - 1 < 3$ then $\psi_1 = \epsilon$ where ϵ is the empty string over the alphabet Σ).
If $p > 1$ then for each $i = 1 \ldots p - 1$ we define $\psi_{i+1} = t[r_i + 1, r_{i+1} - 1]$ (if $r_{i+1} - 1 < r_i + 1$ then $\psi_{i+1} = \epsilon$.
Finally we define $\psi_{p+1} = t[r_p + 1, \ell(t) - 1]$ (if $\ell(t) - 1 < r_p + 1$ then $\psi_{p+1} = \epsilon$).

Since $t \in H_e(n + 1, k)$ we have that for each $i = 1 \ldots p$ $\ell(\psi_i) \geqslant 3$, $\psi_i[2] =$ ':'; $\ell(\psi_{p+1}) \geqslant 1$ and we can define a function u over the domain $\{1, \ldots, p\}$ by setting $u(i) = \psi_i[1]$; we can define a function ϑ over the domain $\{1, \ldots, p\}$ by setting $\vartheta(i) = \psi_i[3, \ell(\psi_i)]$; let's also define $\theta = \psi_{p+1}$. With those definitions we have the following

- for each $i = 1 \ldots p$ $u_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \ldots p$ $i \neq j \rightarrow u_i \neq u_j$,
- for each $i = 1 \ldots p$ $\vartheta_i \in E(n)$,
- $\theta \in E(n)$,
- $\mathcal{E}(n, k, p, u, \vartheta, \theta)$.

Since $t \in H_e(\nu, g)$ we have also the following:

- for each $i = 1 \ldots p$ $u_i \in \mathcal{V} - var(g)$,
- for each $i = 1 \ldots p$ $\vartheta_i \in E(\nu - 1)$,
- $\theta \in E(\nu - 1)$,
- $\mathcal{E}(\nu - 1, g, p, u, \vartheta, \theta)$.

We have also $t = \{\}(\psi_1, \ldots, \psi_p, \psi_{p+1}) = \{\}(u_1 : \vartheta_1, \ldots, u_p : \vartheta_p, \theta)$, so as suggested by lemma 6.1.27 we have identified the elements p, u, ϑ, θ such that $t = \{\}(u_1 : \vartheta_1, \ldots, u_p : \vartheta_p, \theta)$.

We have

$$\#(k, t, \sigma) = \{\#(k'_p, \theta, \sigma'_p) |\ \sigma'_p \in \Xi(k'_p), \sigma \sqsubseteq \sigma'_p\} \ ,$$

where $k'_1 = k + < u_1, \vartheta_1 >$, and if $p > 1$ for each $i = 1 \ldots p - 1$
$k'_{i+1} = k'_i + < u_{i+1}, \vartheta_{i+1} >$.

Let $\eta = \rho_{/dom(g)} \in \Xi(g)$. We have also

$$\#(g, t, \eta) = \{\#(g'_p, \theta, \eta'_p) |\ \eta'_p \in \Xi(g'_p), \eta \sqsubseteq \eta'_p\} \ ,$$

where $g'_1 = g + < u_1, \vartheta_1 >$, and if $p > 1$ for each $i = 1 \ldots p - 1$
$g'_{i+1} = g'_i + < u_{i+1}, \vartheta_{i+1} >$.

We want to show that $\#(k, t, \sigma) = \#(h, t, \rho)$, but $\#(h, t, \rho) = \#(g, t, \eta)$ thus we have to show

$$\{\#(k'_p, \theta, \sigma'_p) |\ \sigma'_p \in \Xi(k'_p), \sigma \sqsubseteq \sigma'_p\} = \{\#(g'_p, \theta, \eta'_p) |\ \eta'_p \in \Xi(g'_p), \eta \sqsubseteq \eta'_p\} \ .$$

To prove this we just need to prove the following two assertions:

- for each $\sigma'_p \in \Xi(k'_p)$ such that $\sigma \sqsubseteq \sigma'_p$ there exists $\eta'_p \in \Xi(g'_p)$ such that $\eta \sqsubseteq \eta'_p$ and $\#(g'_p, \theta, \eta'_p) = \#(k'_p, \theta, \sigma'_p)$;

- for each $\eta'_p \in \Xi(g'_p)$ such that $\eta \sqsubseteq \eta'_p$ there exists $\sigma'_p \in \Xi(k'_p)$ such that $\sigma \sqsubseteq \sigma'_p$ and $\#(k'_p, \theta, \sigma'_p) = \#(g'_p, \theta, \eta'_p)$.

We begin with the first one.

Let $\sigma'_p \in \Xi(k'_p)$ such that $\sigma \sqsubseteq \sigma'_p$, we want to find $\eta'_p \in \Xi(g'_p)$ such that $\eta \sqsubseteq \eta'_p$ and $\#(g'_p, \theta, \eta'_p) = \#(k'_p, \theta, \sigma'_p)$.

If $p > 1$ we define $\sigma'_1 = (\sigma'_p)_{/dom(k'_1)}$, otherwise $\sigma'_1 = (\sigma'_p)_{/dom(k'_1)}$ holds all the same. We should be able to prove that:

- $\sigma'_1 \in \Xi(k'_1)$
- there exists $s_1 \in \#(k, \vartheta_1, \sigma)$ such that $\sigma'_1 = \sigma + (u_1, s_1)$.

If $p = 1$ then $\sigma'_1 \in \Xi(k'_1)$ clearly holds, else we have $k'_p \neq \epsilon$, $\sigma'_p \in \Xi(k'_p)$, $k'_1 \in \mathcal{R}(k'_p)$, $k'_1 \neq k'_p$, so by lemma 8.4 $\sigma'_1 = (\sigma'_p)_{/dom(k'_1)} \in \Xi(k'_1)$.

We have $k'_1 = k + < u_1, \vartheta_1 >$ and $k'_1 \in K(n)$, clearly $k'_1 \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $k \in K(n)$, $\vartheta_1 \in E_s(n, k)$, $u_1 \in \mathcal{V} - var(k)$, so by lemma 8.2

$$\Xi(k'_1) = \{\xi + (u_1, s) |\, \xi \in \Xi(k), s \in \#(k, \vartheta_1, \xi)\}.$$

Then there exist $\xi \in \Xi(k), s \in \#(k, \vartheta_1, \xi)$ such that $\sigma'_1 = \xi + (u_1, s)$. Here we can see that

$$(\sigma'_1)_{/dom(k)} = (\sigma'_1)_{/dom(\xi)} = \xi$$

and at the same time, since $dom(k) \subseteq dom(k'_1) \subseteq dom(k'_p) = dom(\sigma'_p)$,

$$(\sigma'_1)_{/dom(k)} = ((\sigma'_p)_{/dom(k'_1)})_{/dom(k)} = (\sigma'_p)_{/dom(k)} = (\sigma'_p)_{/dom(\sigma)} = \sigma.$$

Therefore $\xi = \sigma$ and there exists $s \in \#(k, \vartheta_1, \sigma)$ such that $\sigma'_1 = \sigma + (u_1, s)$.

If $p > 1$ then for each $i = 1 \dots p-1$ if $i + 1 < p$ we can define $\sigma'_{i+1} = (\sigma'_p)_{/dom(k'_{i+1})}$, otherwise $\sigma'_{i+1} = \sigma'_p = (\sigma'_p)_{/dom(k'_p)} = (\sigma'_p)_{/dom(k'_{i+1})}$ is equally true. We can observe that with the definitions we have provided for each $i = 1 \dots p$ $\sigma'_i = (\sigma'_p)_{/dom(k'_i)}$.

We should also be able to prove that for each $i = 1 \dots p-1$

- $\sigma'_{i+1} \in \Xi(k'_{i+1})$,
- there exists $s_{i+1} \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$ such that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s_{i+1})$.

If $i + 1 = p$ then $\sigma'_{i+1} \in \Xi(k'_{i+1})$ clearly holds, else $i + 1 < p$ and $k'_p \neq \epsilon$, $\sigma'_p \in \Xi(k'_p)$, $k'_{i+1} \in \mathcal{R}(k'_p)$, $k'_{i+1} \neq k'_p$, so by lemma 8.4 $\sigma'_{i+1} = (\sigma'_p)_{/dom(k'_{i+1})} \in \Xi(k'_{i+1})$.

We have $k'_{i+1} = k'_i + < u_{i+1}, \vartheta_{i+1} >$ and $k'_{i+1} \in K(n)$, clearly $k'_{i+1} \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $k'_i \in K(n)$, $\vartheta_{i+1} \in E_s(n, k'_i)$, $var(k'_i) = var(k) \cup \{u_1, \dots, u_i\}$, $u_{i+1} \in \mathcal{V} - var(k'_i)$, so by lemma 8.2

$$\Xi(k'_{i+1}) = \{\xi + (u_{i+1}, s) |\, \xi \in \Xi(k'_i), s \in \#(k'_i, \vartheta_{i+1}, \xi)\}.$$

Then there exist $\xi \in \Xi(k'_i), s \in \#(k'_i, \vartheta_{i+1}, \xi)$ such that $\sigma'_{i+1} = \xi + (u_{i+1}, s)$. Here we can see that

$$(\sigma'_{i+1})_{/dom(k'_i)} = (\sigma'_{i+1})_{/dom(\xi)} = \xi$$

and at the same time, since $dom(k'_i) \subseteq dom(k'_{i+1}) \subseteq dom(k'_p) = dom(\sigma'_p)$,

$$(\sigma'_{i+1})_{/dom(k'_i)} = ((\sigma'_p)_{/dom(k'_{i+1})})_{/dom(k'_i)} = (\sigma'_p)_{/dom(k'_i)} = \sigma'_i.$$

Therefore $\xi = \sigma'_i$ and there exists $s \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$ such that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s)$.

Then we define $\eta'_1 = \eta + (u_1, s_1)$, and we should be able to prove that $\eta'_1 \in \Xi(g'_1)$.

We have $\mathcal{E}(\nu - 1, g, p, u, \vartheta, \theta)$. This implies $\vartheta_1 \in E_s(\nu - 1, g) \subseteq E_s(\nu, g)$. We have $g'_1 = g+ < u_1, \vartheta_1 >$ and $g'_1 \in K(\nu - 1) \subseteq K(\nu)$, $g'_1 \neq \epsilon$, $\nu - 1 \geqslant 2$, moreover $g \in K(\nu)$, $u_1 \in \mathcal{V} - var(g)$ and therefore

$$\Xi(g'_1) = \{\xi + (u_1, s) |\, \xi \in \Xi(g), s \in \#(g, \vartheta_1, \xi)\}.$$

Since $\eta \in \Xi(g)$, to prove that $\eta'_1 \in \Xi(g'_1)$ we just need to prove that $s_1 \in \#(g, \vartheta_1, \eta)$. We know that $s_1 \in \#(k, \vartheta_1, \sigma)$. We have $\vartheta_1 \in E(n, k)$, $\vartheta_1 \in E(\nu - 1, g) \subseteq E(n, g)$.

We also notice that by lemma 8.13, since $g \sqsubseteq h$, $k = \epsilon$ or $g = \epsilon$ or

- $k, g \neq \epsilon$ and so $h \neq \epsilon$, $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>$, $h =<< y_1, \psi_1 > \cdots < y_q, \psi_q >>$ and for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \rightarrow \varphi_i = \psi_j$;
- there exists v positive integer such that $v \leqslant q$, $g =<< y_1, \psi_1 > \cdots < y_v, \psi_v >>$ and for each $i \in dom(k)$, $j \in dom(g)$ $x_i = y_j \rightarrow \varphi_i = \psi_j$.

Since $\eta \sqsubseteq \rho$ we can apply lemma 3.5 to also show that if $\eta = (w, \mu)$ then for each $i \in dom(\sigma)$, $j \in dom(\eta)$ $x_i = w_j \rightarrow z_i = \mu_j$.

With this we can apply the inductive hypothesis and obtain that $\#(k, \vartheta_1, \sigma) = \#(g, \vartheta_1, \eta)$, therefore $s_1 \in \#(g, \vartheta_1, \eta)$ and $\eta'_1 \in \Xi(g'_1)$.

We also notice that $k'_1 = k+ < u_1, \vartheta_1 >$, $g'_1 = g+ < u_1, \vartheta_1 >$, so if we set $k'_1 =<< x'_1, \varphi'_1 > \cdots < x'_{m'}, \varphi'_{m'} >>$ and $g'_1 =<< y'_1, \psi'_1 > \cdots < y'_{q'}, \psi'_{q'} >>$ then by lemma 8.12 for each $\alpha \in dom(k'_1)$, $\beta \in dom(g'_1)$ $x'_\alpha = y'_\beta \rightarrow \varphi'_\alpha = \psi'_\beta$.
Moreover we notice that $\sigma'_1 = \sigma + (u_1, s_1)$, $\eta'_1 = \eta + (u_1, s_1)$, and if we set $\sigma'_1 = (x', z')$, $\eta'_1 = (y', r')$ then by lemma 3.4 for each $\alpha \in dom(\sigma'_1)$, $\beta \in dom(\eta'_1)$ $x'_\alpha = y'_\beta \rightarrow z'_\alpha = r'_\beta$.

If $p > 1$ then for each $i = 1 \ldots p - 1$ we can define $\eta'_{i+1} = \eta'_i + (u_{i+1}, s_{i+1})$ and we expect to be able to prove that $\eta'_{i+1} \in \Xi(g'_{i+1})$.

We have $\mathcal{E}(\nu - 1, g, p, u, \vartheta, \theta)$ and $g'_{i+1} = g'_i + (u_{i+1}, \vartheta_{i+1})$. This implies $g'_{i+1} \in K(\nu - 1) \subseteq K(\nu)$, $g'_{i+1} \neq \epsilon$ and $\nu - 1 \geqslant 2$ holds too. Moreover $g'_i \in K(\nu)$, $\vartheta_{i+1} \in E_s(\nu - 1, g'_i) \subseteq E_s(\nu, g'_i)$, and, since $var(g'_i) = var(g) \cup \{u_1, \ldots, u_i\}$, $u_{i+1} \in \mathcal{V} - var(g'_i)$. Therefore

$$\Xi(g'_{i+1}) = \{\xi + (u_{i+1}, s) |\, \xi \in \Xi(g'_i), s \in \#(g'_i, \vartheta_{i+1}, \xi)\}.$$

By inductive hypothesis we can assume that $\eta'_i \in \Xi(g'_i)$, therefore to prove $\eta'_{i+1} \in \Xi(g'_{i+1})$ we just need to prove $s_{i+1} \in \#(g'_i, \vartheta_{i+1}, \eta'_i)$. We know that $s_{i+1} \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$.

By inductive hypothesis we can also assume that

- if we set $k'_i =<< x'_1, \varphi'_1 > \cdots < x'_{m'}, \varphi'_{m'} >>$ and $g'_i =<< y'_1, \psi'_1 > \cdots < y'_{q'}, \psi'_{q'} >>$ then for each $\alpha \in dom(k'_i)$, $\beta \in dom(g'_i)$ $x'_\alpha = y'_\beta \to \varphi'_\alpha = \psi'_\beta$.
- if we set $\sigma'_i = (x'_i, z'_i)$ and $\eta'_i = (w'_i, \mu'_i)$ then for each $\alpha \in dom(\sigma'_i)$, $\beta \in dom(\eta'_i)$ $(x'_i)_\alpha = (w'_i)_\beta \to (z'_i)_\alpha = (\mu'_i)_\beta$.

We have $k'_i \in K(n)$, $g'_i \in K(\nu) \subseteq K(n)$, $\vartheta_{i+1} \in E_s(n, k'_i)$, $\vartheta_{i+1} \in E_s(\nu, g'_i) \subseteq E_s(n, g'_i)$, $\sigma'_i \in \Xi(k'_i)$, $\eta'_i \in \Xi(g'_i)$, so we can apply the inductive hypothesis and obtain that $\#(k'_i, \vartheta_{i+1}, \sigma'_i) = \#(g'_i, \vartheta_{i+1}, \eta'_i)$. Therefore $s_{i+1} \in \#(g'_i, \vartheta_{i+1}, \eta'_i)$ and we have proved $\eta'_{i+1} \in \Xi(g'_{i+1})$.

In this proof that $\eta'_{i+1} \in \Xi(g'_{i+1})$ we have used an inductive hypothesis which we still haven't proved, so we need to prove it now. What we need to prove is the following:

- if we set $k'_{i+1} =<< x'_1, \varphi'_1 > \cdots < x'_{m'}, \varphi'_{m'} >>$ and $g'_{i+1} =<< y'_1, \psi'_1 > \cdots < y'_{q'}, \psi'_{q'} >>$ then for each $\alpha \in dom(k'_{i+1})$, $\beta \in dom(g'_{i+1})$ $x'_\alpha = y'_\beta \to \varphi'_\alpha = \psi'_\beta$.
- if we set $\sigma'_{i+1} = (x', z')$, $\eta'_{i+1} = (y', r')$ then for each $\alpha \in dom(\sigma'_{i+1})$, $\beta \in dom(\eta'_{i+1})$ $x'_\alpha = y'_\beta \to z'_\alpha = r'_\beta$.

To prove the first item we consider that $k'_{i+1} = k'_i + < u_{i+1}, \vartheta_{i+1} >$, $g'_{i+1} = g'_i + < u_{i+1}, \vartheta_{i+1} >$, $u_{i+1} \in \mathcal{V} - var(k'_i)$, $u_{i+1} \in \mathcal{V} - var(g'_i)$, $\vartheta_{i+1} \in E(n)$. So we can apply lemma 8.12 and the first condition is proved.

To prove the second item we consider that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s_{i+1})$, $\eta'_{i+1} = \eta'_i + (u_{i+1}, s_{i+1})$, $u_{i+1} \in \mathcal{V} - var(\sigma'_i)$, $u_{i+1} \in \mathcal{V} - var(\eta'_i)$. So we can apply lemma 3.4 and the second condition is proved.

At this point we have defined η'_p such that $\eta \sqsubseteq \eta'_p$ and proved that $\eta'_p \in \Xi(g'_p)$. We have also that $k'_p \in K(n)$, $\theta \in E(n, k'_p)$, $g'_p \in K(n)$, $\theta \in E(n, g'_p)$, $\sigma'_p \in \Xi(k'_p)$. Moreover

- if we set $k'_p =<< x'_1, \varphi'_1 > \cdots < x'_{m'}, \varphi'_{m'} >>$ and $g'_p =<< y'_1, \psi'_1 > \cdots < y'_{q'}, \psi'_{q'} >>$ then for each $\alpha \in dom(k'_p)$, $\beta \in dom(g'_p)$ $x'_\alpha = y'_\beta \to \varphi'_\alpha = \psi'_\beta$.
- if we set $\sigma'_p = (x', z')$, $\eta'_p = (y', r')$ then for each $\alpha \in dom(\sigma'_p)$, $\beta \in dom(\eta'_p)$ $x'_\alpha = y'_\beta \to z'_\alpha = r'_\beta$.

With that, $\#(g'_p, \theta, \eta'_p) = \#(k'_p, \theta, \sigma'_p)$ follows by inductive hypothesis.

We now examine the other side of the proof. Let $\eta'_p \in \Xi(g'_p)$ such that $\eta \sqsubseteq \eta'_p$, we want to find $\sigma'_p \in \Xi(k'_p)$ such that $\sigma \sqsubseteq \sigma'_p$ and $\#(k'_p, \theta, \sigma'_p) = \#(g'_p, \theta, \eta'_p)$. Remember that $\eta = \rho_{/dom(g)}$.

We notice that $dom(g'_1) \subseteq dom(g'_p) = dom(\eta'_p)$. So if $p > 1$ we can define $\eta'_1 = (\eta'_p)_{/dom(g'_1)}$, otherwise $\eta'_1 = (\eta'_p)_{/dom(g'_1)}$ holds all the same. We should be able to prove that:

- $\eta'_1 \in \Xi(g'_1)$
- there exists $s_1 \in \#(g, \vartheta_1, \eta)$ such that $\eta'_1 = \eta + (u_1, s_1)$.

If $p = 1$ then $\eta'_1 \in \Xi(g'_1)$ clearly holds, else we have $g'_p \neq \epsilon$, $\eta'_p \in \Xi(g'_p)$, $g'_1 \in \mathcal{R}(g'_p)$, $g'_1 \neq g'_p$, so by lemma 8.4 $\eta'_1 = (\eta'_p)_{/dom(g'_1)} \in \Xi(g'_1)$.

We have $g'_1 = g+ < u_1, \vartheta_1 >$ and $g'_1 \in K(n)$, clearly $g'_1 \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $g \in K(n)$, $\vartheta_1 \in E_s(n, g)$, $u_1 \in \mathcal{V} - var(g)$, so by lemma 8.2

$$\Xi(g'_1) = \{\xi + (u_1, s) |\, \xi \in \Xi(g), s \in \#(g, \vartheta_1, \xi)\}.$$

Then there exist $\xi \in \Xi(g), s \in \#(g, \vartheta_1, \xi)$ such that $\eta'_1 = \xi + (u_1, s)$. Here we can see that

$$(\eta'_1)_{/dom(g)} = (\eta'_1)_{/dom(\xi)} = \xi$$

and at the same time, since $dom(g) \subseteq dom(g'_1) \subseteq dom(g'_p) = dom(\eta'_p)$,

$$(\eta'_1)_{/dom(g)} = ((\eta'_p)_{/dom(g'_1)})_{/dom(g)} = (\eta'_p)_{/dom(g)} = (\eta'_p)_{/dom(\eta)} = \eta.$$

Therefore $\xi = \eta$ and there exists $s \in \#(g, \vartheta_1, \eta)$ such that $\eta'_1 = \eta + (u_1, s)$.

We notice that $dom(g'_{i+1}) \subseteq dom(g'_p) = dom(\eta'_p)$. So if $p > 1$ then for each $i = 1 \dots p-1$: if $i+1 < p$ we can define $\eta'_{i+1} = (\eta'_p)_{/dom(g'_{i+1})}$, otherwise $\eta'_{i+1} = \eta'_p = (\eta'_p)_{/dom(\eta'_p)} = \eta'_p)_{/dom(g'_p)} = (\eta'_p)_{/dom(g'_{i+1})}$ is equally true. We can observe that with the definitions we have provided for each $i = 1 \dots p$ $\eta'_i = (\eta'_p)_{/dom(g'_i)}$.

We should also be able to prove that for each $i = 1 \dots p-1$

- $\eta'_{i+1} \in \Xi(g'_{i+1})$,
- there exists $s_{i+1} \in \#(g'_i, \vartheta_{i+1}, \eta'_i)$ such that $\eta'_{i+1} = \eta'_i + (u_{i+1}, s_{i+1})$.

If $i+1 = p$ then $\eta'_{i+1} \in \Xi(g'_{i+1})$ clearly holds, else $i+1 < p$ and $g'_p \neq \epsilon$, $\eta'_p \in \Xi(g'_p)$, $g'_{i+1} \in \mathcal{R}(g'_p)$, $g'_{i+1} \neq g'_p$, so by lemma 8.4 $\eta'_{i+1} = (\eta'_p)_{/dom(g'_{i+1})} \in \Xi(g'_{i+1})$.

We have $g'_{i+1} = g'_i + < u_{i+1}, \vartheta_{i+1} >$ and $g'_{i+1} \in K(n)$, clearly $g'_{i+1} \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $g'_i \in K(n)$, $\vartheta_{i+1} \in E_s(n, g'_i)$, $var(g'_i) = var(g) \cup \{u_1, \dots, u_i\}$, $u_{i+1} \in \mathcal{V} - var(g'_i)$, so by lemma 8.2

$$\Xi(g'_{i+1}) = \{\xi + (u_{i+1}, s) |\, \xi \in \Xi(g'_i), s \in \#(g'_i, \vartheta_{i+1}, \xi)\}.$$

Then there exist $\xi \in \Xi(g'_i), s \in \#(g'_i, \vartheta_{i+1}, \xi)$ such that $\eta'_{i+1} = \xi + (u_{i+1}, s)$. Here we can see that

$$(\eta'_{i+1})_{/dom(g'_i)} = (\eta'_{i+1})_{/dom(\xi)} = \xi$$

and at the same time, since $dom(g'_i) \subseteq dom(g'_{i+1}) \subseteq dom(g'_p) = dom(\eta'_p)$,

$$(\eta'_{i+1})_{/dom(g'_i)} = ((\eta'_p)_{/dom(g'_{i+1})})_{/dom(g'_i)} = (\eta'_p)_{/dom(g'_i)} = \eta'_i.$$

Therefore $\xi = \eta'_i$ and there exists $s \in \#(g'_i, \vartheta_{i+1}, \eta'_i)$ such that $\eta'_{i+1} = \eta'_i + (u_{i+1}, s)$.

Then we define $\sigma'_1 = \sigma + (u_1, s_1)$, and we should be able to prove that $\sigma'_1 \in \Xi(k'_1)$.

We have $\mathcal{E}(n, k, p, u, \vartheta, \theta)$. This implies $\vartheta_1 \in E_s(n, k)$. We have $k'_1 = k+ < u_1, \vartheta_1 >$ and $k'_1 \in K(n)$, $k'_1 \neq \epsilon$, $n \geqslant 2$, moreover $k \in K(n)$, $u_1 \in \mathcal{V} - var(k)$ and therefore

$$\Xi(k'_1) = \{\xi + (u_1, s) |\, \xi \in \Xi(k), s \in \#(k, \vartheta_1, \xi)\}.$$

Since $\sigma \in \Xi(k)$, to prove that $\sigma'_1 \in \Xi(k'_1)$ we just need to prove that $s_1 \in \#(k, \vartheta_1, \sigma)$. We know that $s_1 \in \#(g, \vartheta_1, \eta)$. We have $\vartheta_1 \in E(n, k)$, $\vartheta_1 \in E(\nu - 1, g) \subseteq E(n, g)$.

We also notice that by lemma 8.13, since $g \sqsubseteq h$, $k = \epsilon$ or $g = \epsilon$ or

- $k, g \neq \epsilon$ and so $h \neq \epsilon$, $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>$, $h =<< y_1, \psi_1 > \cdots < y_q, \psi_q >>$ and for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$;
- there exists v positive integer such that $v \leqslant q$, $g =<< y_1, \psi_1 > \cdots < y_v, \psi_v >>$ and for each $i \in dom(k)$, $j \in dom(g)$ $x_i = y_j \to \varphi_i = \psi_j$.

Since $\eta \sqsubseteq \rho$ we can apply lemma 3.5 to also show that if $\eta = (w, \mu)$ then for each $i \in dom(\sigma)$, $j \in dom(\eta)$ $x_i = w_j \to z_i = \mu_j$.

With this we can apply the inductive hypothesis and obtain that $\#(k, \vartheta_1, \sigma) = \#(g, \vartheta_1, \eta)$, therefore $s_1 \in \#(k, \vartheta_1, \sigma)$ and $\sigma'_1 \in \Xi(k'_1)$.

We also notice that $k'_1 = k+ < u_1, \vartheta_1 >$, $g'_1 = g+ < u_1, \vartheta_1 >$, so if we set $k'_1 =<< x'_1, \varphi'_1 > \cdots < x'_{m'}, \varphi'_{m'} >>$ and $g'_1 =<< y'_1, \psi'_1 > \cdots < y'_{q'}, \psi'_{q'} >>$ then by lemma 8.12 for each $\alpha \in dom(k'_1)$, $\beta \in dom(g'_1)$ $x'_\alpha = y'_\beta \to \varphi'_\alpha = \psi'_\beta$.
Moreover we notice that $\sigma'_1 = \sigma + (u_1, s_1)$, $\eta'_1 = \eta + (u_1, s_1)$, and if we set $\sigma'_1 = (x', z')$, $\eta'_1 = (y', r')$ then by lemma 3.4 for each $\alpha \in dom(\sigma'_1)$, $\beta \in dom(\eta'_1)$ $x'_\alpha = y'_\beta \to z'_\alpha = r'_\beta$.

If $p > 1$ then for each $i = 1 \ldots p-1$ we can define $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s_{i+1})$ and we expect to be able to prove that $\sigma'_{i+1} \in \Xi(k'_{i+1})$.

We have $\mathcal{E}(n, k, p, u, \vartheta, \theta)$ and $k'_{i+1} = k'_i + (u_{i+1}, \vartheta_{i+1})$. This implies $k'_{i+1} \in K(n)$, $k'_{i+1} \neq \epsilon$ and $n \geqslant 2$ holds too. Moreover $k'_i \in K(n)$, $\vartheta_{i+1} \in E_s(n, k'_i)$ and, since $var(k'_i) = var(k) \cup \{u_1, \ldots, u_i\}$, $u_{i+1} \in \mathcal{V} - var(k'_i)$. Therefore

$$\Xi(k'_{i+1}) = \{\xi + (u_{i+1}, s) |\, \xi \in \Xi(k'_i), s \in \#(k'_i, \vartheta_{i+1}, \xi)\}.$$

By inductive hypothesis we can assume that $\sigma'_i \in \Xi(k'_i)$, therefore to prove $\sigma'_{i+1} \in \Xi(k'_{i+1})$ we just need to prove $s_{i+1} \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$. We know that $s_{i+1} \in \#(g'_i, \vartheta_{i+1}, \eta'_i)$.

By inductive hypothesis we can also assume that

- if we set $k'_i = (x'_i, \varphi'_i)$ and $g'_i = (w'_i, \phi'_i)$ then for each $\alpha \in dom(k'_i)$, $\beta \in dom(g'_i)$ $(x'_i)_\alpha = (w'_i)_\beta \to (\varphi'_i)_\alpha = (\phi'_i)_\beta$;
- if we set $\sigma'_i = (x'_i, z'_i)$ and $\eta'_i = (w'_i, \mu'_i)$ then for each $\alpha \in dom(\sigma'_i)$, $\beta \in dom(\eta'_i)$ $(x'_i)_\alpha = (w'_i)_\beta \to (z'_i)_\alpha = (\mu'_i)_\beta$.

We have $k'_i \in K(n)$, $g'_i \in K(p) \subseteq K(n)$, $\vartheta_{i+1} \in E_s(n, k'_i)$, $\vartheta_{i+1} \in E_s(p, g'_i) \subseteq E_s(n, g'_i)$, $\sigma'_i \in \Xi(k'_i)$, $\eta'_i \in \Xi(g'_i)$, so we can apply the inductive hypothesis and obtain that $\#(g'_i, \vartheta_{i+1}, \eta'_i) = \#(k'_i, \vartheta_{i+1}, \sigma'_i)$. Therefore $s_{i+1} \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$ and we have proved $\sigma'_{i+1} \in \Xi(k'_{i+1})$.

In this proof that $\sigma'_{i+1} \in \Xi(k'_{i+1})$ we have used an inductive hypothesis which we still haven't proved, so we need to prove it now. What we need to prove is the following:

- if we set $k'_{i+1} =<< x'_1, \varphi'_1 > \cdots < x'_{m'}, \varphi'_{m'} >>$ and $g'_{i+1} =<< y'_1, \psi'_1 > \cdots < y'_{q'}, \psi'_{q'} >>$ then for each $\alpha \in dom(k'_{i+1})$, $\beta \in dom(g'_{i+1})$ $x'_\alpha = y'_\beta \to \varphi'_\alpha = \psi'_\beta$.
- if we set $\sigma'_{i+1} = (x', z')$, $\eta'_{i+1} = (y', r')$ then for each $\alpha \in dom(\sigma'_{i+1})$, $\beta \in dom(\eta'_{i+1})$ $x'_\alpha = y'_\beta \to z'_\alpha = r'_\beta$.

To prove the first item we consider that $k'_{i+1} = k'_i + < u_{i+1}, \vartheta_{i+1} >$, $g'_{i+1} = g'_i + < u_{i+1}, \vartheta_{i+1} >$, $u_{i+1} \in \mathcal{V} - var(k'_i)$, $u_{i+1} \in \mathcal{V} - var(g'_i)$, $\vartheta_{i+1} \in E(n)$. So we can apply lemma 8.12 and the first condition is proved.

To prove the second item we consider that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s_{i+1})$, $\eta'_{i+1} = \eta'_i + (u_{i+1}, s_{i+1})$, $u_{i+1} \in \mathcal{V} - var(\sigma'_i)$, $u_{i+1} \in \mathcal{V} - var(\eta'_i)$. So we can apply lemma 3.4 and the second condition is proved.

At this point we have defined σ'_p such that $\sigma \sqsubseteq \sigma'_p$ and proved that $\sigma'_p \in \Xi(k'_p)$. We have also that $k'_p \in K(n)$, $\theta \in E(n, k'_p)$, $g'_p \in K(n)$, $\theta \in E(n, g'_p)$, $\eta'_p \in \Xi(g'_p)$. Moreover

- if we set $k'_p =<< x'_1, \varphi'_1 > \cdots < x'_{m'}, \varphi'_{m'} >>$ and $g'_p =<< y'_1, \psi'_1 > \cdots < y'_{q'}, \psi'_{q'} >>$ then for each $\alpha \in dom(k'_p)$, $\beta \in dom(g'_p)$ $x'_\alpha = y'_\beta \to \varphi'_\alpha = \psi'_\beta$.
- if we set $\sigma'_p = (x', z')$, $\eta'_p = (y', r')$ then for each $\alpha \in dom(\sigma'_p)$, $\beta \in dom(\eta'_p)$ $x'_\alpha = y'_\beta \to z'_\alpha = r'_\beta$.

With this, $\#(k'_q, \theta, \sigma'_q) = \#(g'_q, \theta, \eta'_q)$ follows by inductive hypothesis.

$\diamond$

Finally, let's consider the third case, which, we recall, is the following.

- $n+1 > 2$ and there exist μ positive integer such that $2 \leqslant \mu < n+1$, $\kappa \in K(\mu)$ such that $\kappa \sqsubseteq k$, $t \in E_a(\mu, \kappa) \cup E_e(\mu, \kappa) \cup \bigcup_{c \in \mathcal{C}'} E^c(\mu, \kappa) \cup \bigcup_{f \in \mathcal{F}} E^f(\mu, \kappa)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(\kappa)} \in \Xi(\kappa)$ and $\#(k, t, \sigma) = \#(\kappa, t, \sigma_{/dom(\kappa)})$ and
- $n+1 > 2$ and there exist ν positive integer such that $2 \leqslant \nu < n+1$, $g \in K(\nu)$ such that $g \sqsubseteq h$, $t \in E_a(\nu, g) \cup E_e(\nu, g) \cup \bigcup_{c \in \mathcal{C}'} E^c(\nu, g) \cup \bigcup_{f \in \mathcal{F}} E^f(\nu, g)$ and for each $\rho \in \Xi(h)$ $\rho_{/dom(g)} \in \Xi(g)$ and $\#(h, t, \rho) = \#(g, t, \rho_{/dom(g)})$.

We have $t \in E(\mu, \kappa) \cap E(\nu, g)$, with $\mu, \nu < n+1$.

We have also $\sigma = (x, z) \in \Xi(k)$, $\rho = (y, r) \in \Xi(h)$ such that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \to z_i = r_j$ and we want to show that $\#(k, t, \sigma) = \#(h, t, \rho)$. So we just need to show that $\#(\kappa, t, \sigma_{/dom(\kappa)}) = \#(g, t, \rho_{/dom(g)})$.

We can have $\kappa = \epsilon$ or $g = \epsilon$. Otherwise $\kappa, g \neq \epsilon$, $k, h \neq \epsilon$, $k =<< x_1, \varphi_1 > \cdots < x_m, \varphi_m >>$, $h =<< y_1, \psi_1 > \cdots < y_q, \psi_q >>$ and for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$. By lemma 8.13 there exist p, v positive integers such that $p \leqslant m$, $v \leqslant q$, $\kappa =<< x_1, \varphi_1 > \cdots < x_p, \varphi_p >>$, $g =<< y_1, \psi_1 > \cdots < y_v, \psi_v >>$ and for each $i \in dom(\kappa)$, $j \in dom(g)$ $x_i = y_j \to \varphi_i = \psi_j$.

If we define $u = max\{\mu, \nu\}$ then $\kappa, g \in K(u)$, $t \in E(u, \kappa) \cap E(u, g)$ and $u < n+1$. Moreover let $\sigma' = \sigma_{/dom(\kappa)}$, $\sigma' = (x', z')$, $\rho' = \rho_{/dom(g)}$, $\rho' = (y', r')$. Since $\sigma' \sqsubseteq \sigma$ and $\rho' \sqsubseteq \rho$ by lemma 3.5 we obtain that for each $i \in dom(\sigma')$, $j \in dom(\rho')$ $x'_i = y'_j \to z'_i = r'_j$.
By the inductive hypothesis we then obtain $\#(\kappa, t, \sigma') = \#(g, t, \rho')$, and so we have proved $\#(k, t, \sigma) = \#(h, t, \rho)$. $\square$

Lemma 8.15. *Given*

- *a positive integer n;*

- *$k \in K(n)$;*
- *$f \in \mathcal{F}$ such that f has multplicity 2;*
- *$\varphi_1, \varphi_2 \in E(n,k)$;*

such that for each $\sigma \in \Xi(k)$ $A_f(\#(k,\varphi_1,\sigma), \#(k,\varphi_2,\sigma))$ is true, we have that $t = f(\varphi_1,\varphi_2) \in E(n+1,k)$.

Given $\sigma \in \Xi(k)$ we have also

$$\#(k,t,\sigma) = P_f(\#(k,\varphi_1,\sigma), \#(k,\varphi_2,\sigma)) \ .$$

Proof. If $t \in E(n,k) \cup E_b(n+1,k)$ then $t \in E(n+1,k)$, else $t \in E^f(n+1,k) \subseteq E(n+1,k)$.

Using lemma 8.10 we have that one of the following alternatives holds:

- $t \in E_a(n+1,k) \cup E_e(n+1,k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1,k) \cup \bigcup_{g \in \mathcal{F}} E^g(n+1,k)$;
- there exist m positive integer such that $2 \leqslant m < n+1$, $h \in K(m)$ such that $h \sqsubseteq k$, $t \in E_a(m,h) \cup E_e(m,h) \cup \bigcup_{c \in \mathcal{C}'} E^c(m,h) \cup \bigcup_{g \in \mathcal{F}} E^g(m,h)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k,t,\sigma) = \#(h,t,\sigma_{/dom(h)})$.

If the first alternative holds, that is $t \in E_a(n+1,k) \cup E_e(n+1,k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1,k) \cup \bigcup_{g \in \mathcal{F}} E^g(n+1,k)$, then clearly $t \in E^f(n+1,k)$. This implies that $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,<f>)}$, so in this case our proof is finished.

Otherwise it must be $t \in E^f(m,h)$. This implies that there exist $\psi_1, \psi_2 \in E(m-1,h)$ such that $t = f(\psi_1,\psi_2)$, for each $\rho \in \Xi(h)$

- $A_f(\#(h,\psi_1,\rho), \#(h,\psi_2,\rho))$ is true;
- $\#(h,t,\rho) = P_f(\#(h,\psi_1,\rho), \#(h,\psi_2,\rho))$.

We now consider what we have seen in lemma 6.1.18. We have $t = f(\psi)$ with $\psi \in \Sigma^*$. Since $t \in H_f(n+1,k)$ the set of the positive integers r such that $2 < r < \ell(t)$, $t[r] =$ ',' and $d(t,r) = 1$ has just one member r_1. We can define $\chi_1 = t[3, r_1 - 1]$ (if $r_1 - 1 < 3$ then $\chi_1 = \epsilon$ where ϵ is the empty string over the alphabet Σ). We also define $\chi_2 = t[r_1 + 1, \ell(t) - 1]$ (if $\ell(t) - 1 < r_1 + 1$ then $\chi_2 = \epsilon$). The lemma tells us that $\chi_1, \chi_2 \in E(n,k)$, and we can notice that $t = f(\chi_1,\chi_2)$.

Using lemma 6.1.21 we obtain that $\varphi_1 = \chi_1$ and $\varphi_2 = \chi_2$.

We can apply again lemma 6.1.18 using the fact that $t \in H_f(m,h)$, to obtain that $\chi_1, \chi_2 \in E(m-1,h)$ and $t = f(\chi_1,\chi_2)$ still holds. Using lemma 6.1.21 we obtain that $\psi_1 = \chi_1 = \varphi_1$ and $\psi_2 = \chi_2 = \varphi_2$. Therefore $\varphi_1, \varphi_2 \in E(m-1,h) \subseteq E(n,h)$ and for each $\rho \in \Xi(h)$ $\#(h,t,\rho) = P_f(\#(h,\varphi_1,\rho), \#(h,\varphi_2,\rho))$.

So given $\sigma \in \Xi(k)$ if we define $\rho = \sigma_{/dom(h)} \in \Xi(h)$ then

$$\#(k,t,\sigma) = \#(h,t,\rho) = P_f(\#(h,\varphi_1,\rho), \#(h,\varphi_2,\rho)) \ .$$

So we want to prove that

$$P_f(\#(h,\varphi_1,\rho), \#(h,\varphi_2,\rho)) = P_f(\#(k,\varphi_1,\sigma), \#(k,\varphi_2,\sigma)) \ ,$$

and to prove this it is enough to prove that for each $\alpha \in \{1,2\}$

$$\#(h,\varphi_i,\rho) = \#(k,\varphi_i,\sigma) \ .$$

It is not difficult to prove this. In fact, by lemma 8.8, if $k =<< x_1, \theta_1 > \cdots < x_u, \theta_u >>$, $h =<< y_1, \vartheta_1 > \cdots < y_q, \vartheta_q >> \in K(n) - \{\epsilon\}$ since $h \sqsubseteq k$ then for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \rightarrow \theta_i = \vartheta_j$. If $\sigma = (x,z)$, $\rho = (y,r)$ then using lemma 8.9 we obtain that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \rightarrow z_i = r_j$. With this we can apply lemma 8.14 and obtain that $\#(h,\varphi_\alpha,\rho) = \#(k,\varphi_\alpha,\sigma)$. □

Lemma 8.16. *Given*

- *a positive integer* n*;*
- $k \in K(n)$*;*
- $f \in \mathcal{F}$ *such that* f *has multplicity* 1*;*
- $\varphi_1 \in E(n,k)$*;*

such that for each $\sigma \in \Xi(k)$ $A_f(\#(k,\varphi_1,\sigma))$ *is true,*
we have that $t = f(\varphi_1) \in E(n+1,k)$*.*

Given $\sigma \in \Xi(k)$ *we have also*

$$\#(k,t,\sigma) = P_f(\#(k,\varphi_1,\sigma)) \ .$$

Proof. If $t \in E(n,k) \cup E_b(n+1,k)$ then $t \in E(n+1,k)$, else $t \in E^f(n+1,k) \subseteq E(n+1,k)$.

Using lemma 8.10 we have that one of the following alternatives holds:

- $t \in E_a(n+1,k) \cup E_e(n+1,k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1,k) \cup \bigcup_{g \in \mathcal{F}} E^g(n+1,k)$;
- there exist m positive integer such that $2 \leqslant m < n+1$, $h \in K(m)$ such that $h \sqsubseteq k$, $t \in E_a(m,h) \cup E_e(m,h) \cup \bigcup_{c \in \mathcal{C}'} E^c(m,h) \cup \bigcup_{g \in \mathcal{F}} E^g(m,h)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k,t,\sigma) = \#(h,t,\sigma_{/dom(h)})$.

If the first alternative holds, that is $t \in E_a(n+1,k) \cup E_e(n+1,k) \cup \bigcup_{c \in \mathcal{C}'} E^c(n+1,k) \cup \bigcup_{g \in \mathcal{F}} E^g(n+1,k)$, then clearly $t \in E^f(n+1,k)$. This implies that $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,<f>)}$, so in this case our proof is finished.

Otherwise it must be $t \in E^f(m,h)$. This implies that there exist $\psi_1 \in E(m-1,h)$ such that $t = f(\psi_1)$, for each $\rho \in \Xi(h)$

- $A_f(\#(h,\psi_1,\rho))$ is true;
- $\#(h,t,\rho) = \#(h,t,\rho)_{(m,h,<f>)} = P_f(\#(h,\psi_1,\rho))$.

Clearly $\psi_1 = \varphi_1$ so for each $\rho \in \Xi(h)$ $\#(h,t,\rho) = P_f(\#(h,\varphi_1,\rho))$.

Moreover given $\sigma \in \Xi(k)$ if we define $\rho = \sigma_{/dom(h)} \in \Xi(h)$ then

$$\#(k,t,\sigma) = \#(h,t,\rho) = P_f(\#(h,\varphi_1,\rho)) \ .$$

So we want to prove that

$$P_f(\#(h,\varphi_1,\rho)) = P_f(\#(k,\varphi_1,\sigma)) \ ,$$

and to prove this it is enough to prove that

$$\#(h,\varphi_1,\rho) = \#(k,\varphi_1,\sigma) \ .$$

It is not difficult to prove this. In fact, by lemma 8.8, if $k =<< x_1, \theta_1 > \cdots < x_u, \theta_u >>$, $h =<< y_1, \vartheta_1 > \cdots < y_q, \vartheta_q >> \in K(n) - \{\epsilon\}$ since $h \sqsubseteq k$ then for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \theta_i = \vartheta_j$. If $\sigma = (x,z)$, $\rho = (y,r)$ then using lemma 8.9 we obtain that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \to z_i = r_j$. With this we can apply lemma 8.14 and obtain that $\#(h,\varphi_1,\rho) = \#(k,\varphi_1,\sigma)$. □

Lemma 8.17. *Given*

- *a positive integer n;*
- *$k \in K(n)$;*
- *$c \in \mathcal{C}$ such that there exist $i = 1 \dots p$ and a positive integer m such that $\#(c)$ is a function whose domain is $(D_i)^m$ and whose range is D_i;*
- *$\varphi_1, \dots, \varphi_m \in E(n,k)$;*

such that for each $j = 1 \dots m$, $\sigma \in \Xi(k)$ $\#(k,\varphi_j,\sigma) \in D_i$, we have that $t = (c)(\varphi_1, \dots, \varphi_m) \in E(n+1,k)$.

Given $\sigma \in \Xi(k)$ we have also

$$\#(k,t,\sigma) = \#(c)(\#(k,\varphi_1,\sigma), \dots, \#(k,\varphi_m,\sigma)) \ .$$

Proof. If $t \in E(n,k) \cup E_b(n+1,k)$ then $t \in E(n+1,k)$, else $t \in E^c(n+1,k) \subseteq E(n+1,k)$.

Using lemma 8.10 we have that one of the following alternatives holds:

- $t \in E_a(n+1,k) \cup E_e(n+1,k) \cup \bigcup_{d \in \mathcal{C}'} E^d(n+1,k) \cup \bigcup_{g \in \mathcal{F}} E^g(n+1,k)$;
- there exist ν positive integer such that $2 \leqslant \nu < n+1$, $h \in K(p)$ such that $h \sqsubseteq k$, $t \in E_a(\nu,h) \cup E_e(\nu,h) \cup \bigcup_{d \in \mathcal{C}'} E^d(\nu,h) \cup \bigcup_{g \in \mathcal{F}} E^g(\nu,h)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k,t,\sigma) = \#(h,t,\sigma_{/dom(h)})$.

If the first alternative holds, that is $t \in E_a(n+1,k) \cup E_e(n+1,k) \cup \bigcup_{d \in \mathcal{C}'} E^d(n+1,k) \cup \bigcup_{g \in \mathcal{F}} E^g(n+1,k)$, then clearly $t \in E^c(n+1,k)$. This implies that $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,<c>)}$, so in this case our proof is finished.

Otherwise it must be $t \in E^c(\nu,h)$. This implies that there exist $\psi_1, \dots, \psi_m \in E(\nu-1,h)$ such that $t = (c)(\psi_1, \dots, \psi_m)$, for each $\rho \in \Xi(h)$

- for each $j = 1 \dots m$ $\#(h,\psi_j,\rho) \in D_i$;
- $\#(h,t,\rho) = \#(c)(\#(h,\psi_1,\rho), \dots, \#(h,\psi_m,\rho))$.

We have $t = (c)(\chi)$ with $\chi \in \Sigma^*$. Consider the set of the positive integers r such that $4 < r < \ell(t)$, $t[r] =$ ‘,’ and $d(t,r) = 1$. If this set is empty we can call $\chi_1 = \chi$

and using lemma 6.1.12, given that $t \in H_c(n+1,k)$ and $t \in H_c(\nu,h)$ we obtain that $\chi_1 \in E(n,k)$ and $\chi_1 \in E(\nu-1,h)$.
If the mentioned set is not empty let's name $r_1,\ldots,r_u$ its members (in increasing order).
Let's also define $\chi_1 = t[5, r_1-1]$ (if $r_1 - 1 < 5$ then $\chi_1 = \epsilon$ where ϵ is the empty string over the alphabet Σ).
If $u > 1$ then for each $i = 1\ldots u-1$ we define $\chi_{i+1} = t[r_i+1, r_{i+1}-1]$ (if $r_{i+1}-1 < r_i+1$ then $\chi_{i+1} = \epsilon$).
Finally we define $\chi_{u+1} = t[r_u+1, \ell(t)-1]$ (if $\ell(t)-1 < r_u+1$ then $\chi_{u+1} = \epsilon$).
Using lemma 6.1.13 we obtain $t = (c)(\psi_1,\ldots,\psi_{u+1})$ and for each $i = 1\ldots u+1$ $\chi_i \in E(n,k)$, $\chi_i \in E(\nu-1,h)$.

Using lemma 6.1.16 we obtain that in the first case $m = 1$ and $\chi_1 = \varphi_1$, $\chi_1 = \psi_1$, in the second case $m = u+1$ and for each $j = 1\ldots u+1$ $\chi_j = \varphi_j$, $\chi_j = \psi_j$. In both cases for each $j = 1\ldots m$ $\varphi_j = \psi_j$.

Moreover given $\sigma \in \Xi(k)$ if we define $\rho = \sigma_{/dom(h)} \in \Xi(h)$ then

$$\#(k,t,\sigma) = \#(h,t,\rho) = \#(c)(\#(h,\psi_1,\rho),\ldots,\#(h,\psi_m,\rho)) \ .$$

In order to prove that $\#(k,t,\sigma) = \#(c)(\#(k,\varphi_1,\sigma),\ldots,\#(k,\varphi_m,\sigma))$ we just need to prove that for each $j = 1\ldots m$ $\#(k,\varphi_j,\sigma) = \#(h,\psi_j,\rho)$.

It is not difficult to prove this. In fact, by lemma 8.8, if $k =<< x_1,\theta_1 > \cdots < x_u,\theta_u >>$, $h =<< y_1,\vartheta_1 > \cdots < y_q,\vartheta_q >> \in K(n) - \{\epsilon\}$ since $h \sqsubseteq k$ then for each $i \in dom(k)$, $\alpha \in dom(h)$ $x_i = y_\alpha \to \theta_i = \vartheta_\alpha$. If $\sigma = (x,z)$, $\rho = (y,r)$ then using lemma 8.9 we obtain that for each $i \in dom(\sigma)$, $\alpha \in dom(\rho)$ $x_i = y_\alpha \to z_i = r_\alpha$. Moreover $\varphi_j = \psi_j \in E(\nu-1,h) \subseteq E(n,h)$. With this we can apply lemma 8.14 and obtain that $\#(h,\psi_j,\rho) = \#(h,\varphi_j,\rho) = \#(k,\varphi_j,\sigma)$. □

Lemma 8.18. *Assume $\Pi \in \mathcal{C}$. Given*

- *a positive integer n;*
- *$k \in K(n)$;*
- *$\varphi_1 \in E(n,k)$;*

such that for each $\sigma \in \Xi(k)$ $\#(k,\varphi_1,\sigma) \in \mathcal{P}(D_1)$,
we have that $t = (\Pi)(\varphi_1) \in E(n+1,k)$.

Given $\sigma \in \Xi(k)$ we have also

$$\#(k,t,\sigma) = \#(\Pi)(\#(k,\varphi_1,\sigma)) \ .$$

Proof. If $t \in E(n,k) \cup E_b(n+1,k)$ then $t \in E(n+1,k)$, else $t \in E^{\Pi}(n+1,k) \subseteq E(n+1,k)$.

Using lemma 8.10 we have that one of the following alternatives holds:

- $t \in E_a(n+1,k) \cup E_e(n+1,k) \cup \bigcup_{d\in\mathcal{C}'} E^d(n+1,k) \cup \bigcup_{g\in\mathcal{F}} E^g(n+1,k)$;

- there exist ν positive integer such that $2 \leqslant \nu < n+1$, $h \in K(p)$ such that $h \sqsubseteq k$, $t \in E_a(\nu, h) \cup E_e(\nu, h) \cup \bigcup_{d \in \mathcal{C}'} E^d(\nu, h) \cup \bigcup_{g \in \mathcal{F}} E^g(\nu, h)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k, t, \sigma) = \#(h, t, \sigma_{/dom(h)})$.

If the first alternative holds, that is $t \in E_a(n+1, k) \cup E_e(n+1, k) \cup \bigcup_{d \in \mathcal{C}'} E^d(n+1, k) \cup \bigcup_{g \in \mathcal{F}} E^g(n+1, k)$, then clearly $t \in E^{\Pi}(n+1, k)$. This implies that $\#(k, t, \sigma) = \#(k, t, \sigma)_{(n+1, k, <\Pi>)}$, so in this case our proof is finished.

Otherwise it must be $t \in E^{\Pi}(\nu, h)$. This implies that there exists $\psi_1 \in E(\nu - 1, h)$ such that $t = (\Pi)(\psi_1)$

- there exist $i = 1 \ldots p$, $q = 1 \ldots q_{max}$ such that for each $\rho \in \Xi(h)$ $\#(h, \psi_1, \rho) \in \mathcal{P}^q(D_i)$;
- for each $\rho \in \Xi(h)$ $\#(h, t, \rho) = \#(\Pi)(\#(h, \psi_1, \rho))$.

Moreover given $\sigma \in \Xi(k)$ if we define $\rho = \sigma_{/dom(h)} \in \Xi(h)$ then

$$\#(k, t, \sigma) = \#(h, t, \rho) = \#(\Pi)(\#(h, \psi_1, \rho)) \ .$$

In order to prove that $\#(k, t, \sigma) = \#(\Pi)(\#(k, \varphi_1, \sigma))$ we just need to prove that $\#(k, \varphi_1, \sigma) = \#(h, \psi_1, \rho)$.

It is not difficult to prove this. In fact, by lemma 8.8, if $k =<< x_1, \theta_1 > \cdots < x_u, \theta_u >>$, $h =<< y_1, \vartheta_1 > \cdots < y_q, \vartheta_q >> \in K(n) - \{\epsilon\}$ since $h \sqsubseteq k$ then for each $i \in dom(k)$, $\alpha \in dom(h)$ $x_i = y_\alpha \to \theta_i = \vartheta_\alpha$. If $\sigma = (x, z)$, $\rho = (y, r)$ then using lemma 8.9 we obtain that for each $i \in dom(\sigma)$, $\alpha \in dom(\rho)$ $x_i = y_\alpha \to z_i = r_\alpha$. Moreover $\varphi_1 = \psi_1 \in E(\nu - 1, h) \subseteq E(n, h)$. With this we can apply lemma 8.14 and obtain that $\#(h, \psi_1, \rho) = \#(h, \varphi_1, \rho) = \#(k, \varphi_1, \sigma)$. □

Lemma 8.19. *Given*

- *a positive integer n;*
- *$k, h \in K(n)$ such that $h = \epsilon$ or $k = \epsilon$ or ($h, k \neq \epsilon$ and $k =<< u_1, \eta_1 > \cdots < u_w, \eta_w >>$, $h =<< v_1, \vartheta_1 > \cdots < v_q, \vartheta_q >>$ and for each $i \in dom(k)$, $j \in dom(h)$ $u_i = v_j \to \eta_i = \vartheta_j$)*
- *$\rho = (v, \nu) \in \Xi(h)$, $\sigma = (u, \mu) \in \Xi(k)$ such that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $u_i = v_j \to \mu_i = \nu_j$;*
- *a positive integer m;*
- *a function x whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $x_i \in \mathcal{V} - var(k)$, $x_i \in \mathcal{V} - var(h)$, and for each $i, j = 1 \ldots m$ $i \neq j \to x_i \neq x_j$;*
- *a function φ whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $\varphi_i \in E(n)$;*
- *$\phi \in E(n)$;*

such that

- *$\mathcal{E}(n, k, m, x, \varphi, \phi)$;*
- *$\mathcal{E}(n, h, m, x, \varphi, \phi)$;*

and given $t = \{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi)$, we have that

for each $\sigma'_m \in \Xi(k'_m)$ such that $\sigma \sqsubseteq \sigma'_m$ there exists $\rho'_m \in \Xi(h'_m)$ such that $\rho \sqsubseteq \rho'_m$ and $\#(h'_m, \phi, \rho'_m) = \#(k'_m, \phi, \sigma'_m)$,

where of course

- $k'_1 = k+ < x_1, \varphi_1 >$, *and if $m > 1$ for each $i = 1 \ldots m-1$ $k'_{i+1} = k'_i + < x_{i+1}, \varphi_{i+1} >$,*
- $h'_1 = h+ < x_1, \varphi_1 >$, *and if $m > 1$ for each $i = 1 \ldots m-1$ $h'_{i+1} = h'_i + < x_{i+1}, \varphi_{i+1} >$.*

Proof. Let $\sigma'_m \in \Xi(k'_m)$ such that $\sigma \sqsubseteq \sigma'_m$, we want to find $\rho'_m \in \Xi(h'_m)$ such that $\rho \sqsubseteq \rho'_m$ and $\#(h'_m, \phi, \rho'_m) = \#(k'_m, \phi, \sigma'_m)$.

If $m = 1$ then σ'_1 is defined, else let $\sigma'_1 = (\sigma'_m)_{/dom(k'_1)}$. We should be able to prove that:

- $\sigma'_1 \in \Xi(k'_1)$
- there exists $s_1 \in \#(k, \varphi_1, \sigma)$ such that $\sigma'_1 = \sigma + (x_1, s_1)$.

If $m = 1$ then $\sigma'_1 \in \Xi(k'_1)$ clearly holds, else we have $k'_m \neq \epsilon$, $\sigma'_m \in \Xi(k'_m)$, $k'_1 \in \mathcal{R}(k'_m)$, $k'_1 \neq k'_m$, so by lemma 8.4 $\sigma'_1 = (\sigma'_m)_{/dom(k'_1)} \in \Xi(k'_1)$.

We have $k'_1 = k+ < x_1, \varphi_1 >$ and $k'_1 \in K(n)$, clearly $k'_1 \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $k \in K(n)$, $\varphi_1 \in E_s(n, k)$, $x_1 \in \mathcal{V} - var(k)$, so by lemma 8.2

$$\Xi(k'_1) = \{\xi + (x_1, s) | \, \xi \in \Xi(k), s \in \#(k, \varphi_1, \xi)\}.$$

Then there exist $\xi \in \Xi(k), s \in \#(k, \varphi_1, \xi)$ such that $\sigma'_1 = \xi + (x_1, s)$. Here we can see that

$$(\sigma'_1)_{/dom(k)} = (\sigma'_1)_{/dom(\xi)} = \xi$$

and at the same time, since $dom(k) \subseteq dom(k'_1) \subseteq dom(k'_m) = dom(\sigma'_m)$,

$$(\sigma'_1)_{/dom(k)} = ((\sigma'_m)_{/dom(k'_1)})_{/dom(k)} = (\sigma'_m)_{/dom(k)} = (\sigma'_m)_{/dom(\sigma)} = \sigma.$$

Therefore $\xi = \sigma$ and there exists $s \in \#(k, \varphi_1, \sigma)$ such that $\sigma'_1 = \sigma + (x_1, s)$.

If $m > 1$ let $i = 1 \ldots m-1$, if $i+1 = m$ then $\sigma'_{i+1} = \sigma'_m$ is defined, else we can define $\sigma'_{i+1} = (\sigma'_m)_{/dom(k'_{i+1})}$. If $i+1 = m$ $\sigma'_{i+1} = \sigma'_m = (\sigma'_m)_{/dom(k'_m)} = (\sigma'_m)_{/dom(k'_{i+1})}$ is equally true. We can observe that with the definitions we have provided for each $i = 1 \ldots m$ $\sigma'_i = (\sigma'_m)_{/dom(k'_i)}$.

We should also be able to prove that

- $\sigma'_{i+1} \in \Xi(k'_{i+1})$,
- there exists $s_{i+1} \in \#(k'_i, \varphi_{i+1}, \sigma'_i)$ such that $\sigma'_{i+1} = \sigma'_i + (x_{i+1}, s_{i+1})$.

If $i+1 = m$ then $\sigma'_{i+1} \in \Xi(k'_{i+1})$ clearly holds, else $i+1 < m$ and $k'_m \neq \epsilon$, $\sigma'_m \in \Xi(k'_m)$, $k'_{i+1} \in \mathcal{R}(k'_m)$, $k'_{i+1} \neq k'_m$, so by lemma 8.4 $\sigma'_{i+1} = (\sigma'_m)_{/dom(k'_{i+1})} \in \Xi(k'_{i+1})$.

We have $k'_{i+1} = k'_i + < x_{i+1}, \varphi_{i+1} >$ and $k'_{i+1} \in K(n)$, clearly $k'_{i+1} \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $k'_i \in K(n)$, $\varphi_{i+1} \in E_s(n, k'_i)$, $var(k'_i) = var(k) \cup \{x_1, \ldots, x_i\}$,

$x_{i+1} \in \mathcal{V} - var(k'_i)$, so by lemma 8.2

$$\Xi(k'_{i+1}) = \{\xi + (x_{i+1}, s) |\, \xi \in \Xi(k'_i), s \in \#(k'_i, \varphi_{i+1}, \xi)\}.$$

Then there exist $\xi \in \Xi(k'_i), s \in \#(k'_i, \varphi_{i+1}, \xi)$ such that $\sigma'_{i+1} = \xi + (x_{i+1}, s)$. Here we can see that

$$(\sigma'_{i+1})_{/dom(k'_i)} = (\sigma'_{i+1})_{/dom(\xi)} = \xi.$$

At the same time, if $i + 1 = m$ then

$$(\sigma'_{i+1})_{/dom(k'_i)} = (\sigma'_m)_{/dom(k'_i)} = \sigma'_i.$$

Else since $dom(k'_i) \subseteq dom(k'_{i+1}) \subseteq dom(k'_m) = dom(\sigma'_m)$,

$$(\sigma'_{i+1})_{/dom(k'_i)} = ((\sigma'_m)_{/dom(k'_{i+1})})_{/dom(k'_i)} = (\sigma'_m)_{/dom(k'_i)} = \sigma'_i.$$

Therefore $\xi = \sigma'_i$ and there exists $s \in \#(k'_i, \varphi_{i+1}, \sigma'_i)$ such that $\sigma'_{i+1} = \sigma'_i + (x_{i+1}, s)$.

Then we define $\rho'_1 = \rho + (x_1, s_1)$, and we should be able to prove that $\rho'_1 \in \Xi(h'_1)$.

We have $\mathcal{E}(n, h, m, x, \varphi, \phi)$. This implies $\varphi_1 \in E_s(n, h)$. We have $h'_1 = h+ < x_1, \varphi_1 >$ and $h'_1 \in K(n)$, $h'_1 \neq \epsilon$, $n \geqslant 2$, moreover $h \in K(n)$, $x_1 \in \mathcal{V} - var(h)$ and therefore

$$\Xi(h'_1) = \{\xi + (x_1, s) |\, \xi \in \Xi(h), s \in \#(h, \varphi_1, \xi)\}.$$

Since $\rho \in \Xi(h)$, to prove that $\rho'_1 \in \Xi(h'_1)$ we just need to prove that $s_1 \in \#(h, \varphi_1, \rho)$. We know that $s_1 \in \#(k, \varphi_1, \sigma)$. We have $\varphi_1 \in E(n, k)$, $\varphi_1 \in E(n, h)$. We have also that for each $i \in dom(k)$, $j \in dom(h)$ $u_i = v_j \to \eta_i = \vartheta_j$ and for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $u_i = v_j \to \mu_i = \nu_j$. With this we can apply lemma 8.14 and obtain that $\#(k, \varphi_1, \sigma) = \#(h, \varphi_1, \rho)$, therefore $s_1 \in \#(h, \varphi_1, \rho)$ and $\rho'_1 \in \Xi(h'_1)$.

We also notice that $k'_1 = k+ < x_1, \varphi_1 >$, $h'_1 = h+ < x_1, \varphi_1 >$, so if we set $k'_1 =<< u'_1, \eta'_1 > \cdots < u'_{w'}, \eta'_{w'} >>$, $h'_1 =<< v'_1, \vartheta'_1 > \cdots < v'_{q'}, \vartheta'_{q'} >>$ then by lemma 8.12 for each $\alpha \in dom(k'_1)$, $\beta \in dom(h'_1)$ $u'_\alpha = v'_\beta \to \eta'_\alpha = \vartheta'_\beta$.

Moreover we notice that $\sigma'_1 = \sigma + (x_1, s_1)$, $\rho'_1 = \rho + (x_1, s_1)$, and if we set $\sigma'_1 = (u', \mu')$, $\rho'_1 = (v', \nu')$ then by lemma 3.4 for each $\alpha \in dom(\sigma'_1)$, $\beta \in dom(\rho'_1)$ $u'_\alpha = v'_\beta \to \mu'_\alpha = \nu'_\beta$.

If $m > 1$ then for each $i = 1 \ldots m - 1$ we can define $\rho'_{i+1} = \rho'_i + (x_{i+1}, s_{i+1})$ and we expect to be able to prove that $\rho'_{i+1} \in \Xi(h'_{i+1})$.

We have $\mathcal{E}(n, h, m, x, \varphi, \phi)$ and $h'_{i+1} = h'_i+ < x_{i+1}, \varphi_{i+1} >$. This implies $h'_{i+1} \in K(n)$, $h'_{i+1} \neq \epsilon$ and $n \geqslant 2$ holds too. Moreover $h'_i \in K(n)$, $\varphi_{i+1} \in E_s(n, h'_i)$, and, since $var(h'_i) = var(h) \cup \{x_1, \ldots, x_i\}$, $x_{i+1} \in \mathcal{V} - var(h'_i)$. Therefore

$$\Xi(h'_{i+1}) = \{\xi + (x_{i+1}, s) |\, \xi \in \Xi(h'_i), s \in \#(h'_i, \varphi_{i+1}, \xi)\}.$$

By inductive hypothesis we can assume that $\rho'_i \in \Xi(h'_i)$, therefore to prove $\rho'_{i+1} \in \Xi(h'_{i+1})$ we just need to prove $s_{i+1} \in \#(h'_i, \varphi_{i+1}, \rho'_i)$. We know that

$s_{i+1} \in \#(k'_i, \varphi_{i+1}, \sigma'_i)$.

As an inductive hypothesis we can also assume that

- if we set $k'_i =<< u'_1, \eta'_1 > \cdots < u'_{w'}, \eta'_{w'} >>$, $h'_i =<< v'_1, \vartheta'_1 > \cdots < v'_{q'}, \vartheta'_{q'} >>$ then for each $\alpha \in dom(k'_i)$, $\beta \in dom(h'_i)$ $u'_\alpha = v'_\beta \rightarrow \eta'_\alpha = \vartheta'_\beta$.

- if we set $\sigma'_i = (u', \mu')$, $\rho'_i = (v', \nu')$ then for each $\alpha \in dom(\sigma'_i)$, $\beta \in dom(\rho'_i)$ $u'_\alpha = v'_\beta \rightarrow \mu'_\alpha = \nu'_\beta$.

We have $k'_i \in K(n)$, $h'_i \in K(n)$, $\varphi_{i+1} \in E_s(n, k'_i)$, $\varphi_{i+1} \in E_s(n, h'_i)$, $\sigma'_i \in \Xi(k'_i)$, $\rho'_i \in \Xi(h'_i)$, so we can apply lemma 8.14 and obtain that $\#(k'_i, \varphi_{i+1}, \sigma'_i) = \#(h'_i, \varphi_{i+1}, \rho'_i)$. Therefore $s_{i+1} \in \#(h'_i, \varphi_{i+1}, \rho'_i)$ and we have proved $\rho'_{i+1} \in \Xi(h'_{i+1})$.

In this proof that $\rho'_{i+1} \in \Xi(h'_{i+1})$ we have used an inductive hypothesis which we haven't proved, so we need to prove it now. What we need to prove is the following:

- if we set $k'_{i+1} =<< u'_1, \eta'_1 > \cdots < u'_{w'}, \eta'_{w'} >>$, $h'_{i+1} =<< v'_1, \vartheta'_1 > \cdots < v'_{q'}, \vartheta'_{q'} >>$ then for each $\alpha \in dom(k'_{i+1})$, $\beta \in dom(h'_{i+1})$ $u'_\alpha = v'_\beta \rightarrow \eta'_\alpha = \vartheta'_\beta$.
- if we set $\sigma'_{i+1} = (u', \mu')$, $\rho'_{i+1} = (v', \nu')$ then for each $\alpha \in dom(\sigma'_{i+1})$, $\beta \in dom(\rho'_{i+1})$ $u'_\alpha = v'_\beta \rightarrow \mu'_\alpha = \nu'_\beta$.

To prove the first item we consider that $k'_{i+1} = k'_i + < x_{i+1}, \varphi_{i+1} >$, $h'_{i+1} = h'_i + < x_{i+1}, \varphi_{i+1} >$, $x_{i+1} \in \mathcal{V} - var(k'_i)$, $x_{i+1} \in \mathcal{V} - var(h'_i)$. So we can apply lemma 8.12 and the first condition is proved.

To prove the second item we consider that $\sigma'_{i+1} = \sigma'_i + (x_{i+1}, s_{i+1})$, $\rho'_{i+1} = \rho'_i + (x_{i+1}, s_{i+1})$, $x_{i+1} \in \mathcal{V} - var(\sigma'_i)$, $x_{i+1} \in \mathcal{V} - var(\rho'_i)$. So we can apply lemma 3.4 and the second condition is proved.

At this point we have defined ρ'_m such that $\rho \sqsubseteq \rho'_m$ and proved that $\rho'_m \in \Xi(h'_m)$. We have also that $k'_m \in K(n)$, $\phi \in E(n, k'_m)$, $h'_m \in K(n)$, $\phi \in E(n, h'_m)$, $\sigma'_m \in \Xi(k'_m)$. Moreover

- if we set $k'_m =<< u'_1, \eta'_1 > \cdots < u'_{w'}, \eta'_{w'} >>$, $h'_m =<< v'_1, \vartheta'_1 > \cdots < v'_{q'}, \vartheta'_{q'} >>$ then for each $\alpha \in dom(k'_m)$, $\beta \in dom(h'_m)$ $u'_\alpha = v'_\beta \rightarrow \eta'_\alpha = \vartheta'_\beta$.
- if we set $\sigma'_m = (u', \mu')$, $\rho'_m = (v', \nu')$ then for each $\alpha \in dom(\sigma'_m)$, $\beta \in dom(\rho'_m)$ $u'_\alpha = v'_\beta \rightarrow \mu'_\alpha = \nu'_\beta$.

With this, $\#(h'_m, \phi, \rho'_m) = \#(k'_m, \phi, \sigma'_m)$ follows by lemma 8.14.

□

Lemma 8.20. *Given*

- *a positive integer* n*;*
- $k \in K(n)$*;*
- *a positive integer* m*;*
- *a function* x *whose domain is* $\{1, \ldots, m\}$ *such that for each* $i = 1 \ldots m$ $x_i \in \mathcal{V} - var(k)$*, and for each* $i, j = 1 \ldots m$ $i \neq j \rightarrow x_i \neq x_j$*;*

- *a function φ whose domain is $\{1,\ldots,m\}$ such that for each $i = 1\ldots m$ $\varphi_i \in E(n)$;*
- *$\phi \in E(n)$;*

such that $\mathcal{E}(n,k,m,x,\varphi,\phi)$,

we have that $t = \{\}(x_1 : \varphi_1,\ldots,x_m : \varphi_m,\phi) \in E(n+1,k)$.

Given $\sigma \in \Xi(k)$ we have also

$$\#(k,t,\sigma) = \{\#(k'_m,\phi,\sigma'_m)|\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\},$$

where $k'_1 = k+ < x_1,\varphi_1 >$, and if $m > 1$ for each $i = 1\ldots m-1$ $k'_{i+1} = k'_i + < x_{i+1},\varphi_{i+1} >$.

Proof. If $t \in E(n,k) \cup E_b(n+1,k)$ then $t \in E(n+1,k)$, else $t \in E_e(n+1,k) \subseteq E(n+1,k)$.

Using lemma 8.10 we have that one of the following alternatives holds:

- $t \in E_a(n+1,k) \cup E_e(n+1,k) \cup \bigcup_{c\in\mathcal{C}'} E^c(n+1,k) \cup \bigcup_{f\in\mathcal{F}} E^f(n+1,k)$;
- $n+1 > 2$ and there exist p positive integer such that $2 \leqslant p < n+1$, $h \in K(p)$ such that $h \sqsubseteq k$, $t \in E_a(p,h) \cup E_e(p,h) \cup \bigcup_{c\in\mathcal{C}'} E^c(p,h) \cup \bigcup_{f\in\mathcal{F}} E^f(p,h)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k,t,\sigma) = \#(h,t,\sigma_{/dom(h)})$.

If the first alternative holds, that is $t \in E_a(n+1,k) \cup E_e(n+1,k) \cup \bigcup_{c\in\mathcal{C}'} E^c(n+1,k) \cup \bigcup_{f\in\mathcal{F}} E^f(n+1,k)$, then clearly $t \in E_e(n+1,k)$. This implies that $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,e)}$, so in this case our proof is finished.

Otherwise it must be $t \in E_e(p,h)$ and $h \in K(p-1)$. This implies that there exist:

- a positive integer q;
- a function y whose domain is $\{1,\ldots,q\}$ such that for each $i = 1\ldots q$ $y_i \in \mathcal{V} - var(h)$, and for each $i,j = 1\ldots q$ $i \neq j \to y_i \neq y_j$;
- a function η whose domain is $\{1,\ldots,q\}$ such that for each $i = 1\ldots q$ $\eta_i \in E(p-1)$;
- $\theta \in E(p-1)$;

such that

- $\mathcal{E}(p-1,h,q,y,\eta,\theta)$;
- $\{\}(y_1 : \eta_1,\ldots,y_q : \eta_q,\theta) \notin E(p-1,h)$.
- $t = \{\}(y_1 : \eta_1,\ldots,y_q : \eta_q,\theta)$.

Clearly given $\rho \in \Xi(h)$ we have

$$\#(h,t,\rho) = \{\#(h'_q,\theta,\rho'_q)|\ \rho'_q \in \Xi(h'_q), \rho \sqsubseteq \rho'_q\},$$

where $h'_1 = h+ < y_1,\eta_1 >$, and if $q > 1$ for each $i = 1\ldots q-1$ $h'_{i+1} = h'_i + < y_{i+1},\eta_{i+1} >$.

Consider the set of the positive integers r such that $2 < r < \ell(t)$, $t[r] =$ ',' and $d(t,r) = 1$. Since $t \in H_e(n+1,k)$ then this set is not empty, let's name $r_1, \ldots, r_u$ its members (in increasing order).

Let's also define $\psi_1 = t[3, r_1 - 1]$ (if $r_1 - 1 < 3$ then $\psi_1 = \epsilon$ where ϵ is the empty string over the alphabet Σ).
If $u > 1$ then for each $i = 1 \ldots u-1$ we define $\psi_{i+1} = t[r_i + 1, r_{i+1} - 1]$ (if $r_{i+1} - 1 < r_i + 1$ then $\psi_{i+1} = \epsilon$.
Finally we define $\psi_{u+1} = t[r_u + 1, \ell(t) - 1]$ (if $\ell(t) - 1 < r_u + 1$ then $\psi_{u+1} = \epsilon$).

Using lemma 6.1.25, since $t \in H_e(n+1,k)$ then

- for each $i = 1 \ldots u$ $\ell(\psi_i) \geqslant 3$, $\psi_i[2] =$ ':'; $\ell(\psi_{u+1}) \geqslant 1$;
- let's define a function z over the domain $\{1, \ldots, u\}$ by setting $z(i) = \psi_i[1]$; let's define a function χ over the domain $\{1, \ldots, u\}$ by setting $\chi(i) = \psi_i[3, \ell(\psi_i)]$; let's define $\vartheta = \psi_{u+1}$ then
  - for each $i = 1 \ldots u$ $z_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \ldots u$ $i \neq j \to z_i \neq z_j$,
  - for each $i = 1 \ldots u$ $\chi_i \in E(n)$,
  - $\vartheta \in E(n)$;
  - $\mathcal{E}(n, k, u, z, \chi, \vartheta)$.

Using lemma 6.1.27 we obtain that $u = m$, $z = x$, $\chi = \varphi$, $\vartheta = \phi$.

We can use lemma 6.1.25 another time, in fact since $t \in E_e(p,h)$ we have the following

- for each $i = 1 \ldots u$ $z_i \in \mathcal{V} - var(h)$, and for each $i, j = 1 \ldots u$ $i \neq j \to z_i \neq z_j$,
- for each $i = 1 \ldots u$ $\chi_i \in E(p-1)$,
- $\vartheta \in E(p-1)$;
- $\mathcal{E}(p-1, h, u, z, \chi, \vartheta)$.

And again using lemma 6.1.27 we obtain that $u = q$, $z = y$, $\chi = \eta$, $\vartheta = \theta$.

It follows that $q = m$, $y = x$, $\eta = \varphi$, $\theta = \phi$ and so given $\rho \in \Xi(h)$ we have

$$\#(h, t, \rho) = \{\#(h'_m, \phi, \rho'_m) |\ \rho'_m \in \Xi(h'_m), \rho \sqsubseteq \rho'_m\},$$

where $h'_1 = h+ < x_1, \varphi_1 >$, and if $m > 1$ for each $i = 1 \ldots m-1$ $h'_{i+1} = h'_i + < x_{i+1}, \varphi_{i+1} >$.

Now given $\sigma \in \Xi(k)$ we want to prove that

$$\#(k, t, \sigma) = \{\#(k'_m, \phi, \sigma'_m) |\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\}.$$

If we define $\rho = \sigma_{/dom(h)} \in \Xi(h)$ then

$$\#(k, t, \sigma) = \#(h, t, \rho) = \{\#(h'_m, \phi, \rho'_m) |\ \rho'_m \in \Xi(h'_m), \rho \sqsubseteq \rho'_m\}.$$

So in the end what we need to prove is that

$$\{\#(k'_m, \phi, \sigma'_m)|\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\} = \{\#(h'_m, \phi, \rho'_m)|\ \rho'_m \in \Xi(h'_m), \rho \sqsubseteq \rho'_m\}.$$

To prove this we just need to prove the following two assertions:

- for each $\sigma'_m \in \Xi(k'_m)$ such that $\sigma \sqsubseteq \sigma'_m$ there exists $\rho'_m \in \Xi(h'_m)$ such that $\rho \sqsubseteq \rho'_m$ and $\#(h'_m, \phi, \rho'_m) = \#(k'_m, \phi, \sigma'_m)$;
- for each $\rho'_m \in \Xi(h'_m)$ such that $\rho \sqsubseteq \rho'_m$ there exists $\sigma'_m \in \Xi(k'_m)$ such that $\sigma \sqsubseteq \sigma'_m$ and $\#(k'_m, \phi, \sigma'_m) = \#(h'_m, \phi, \rho'_m)$.

Here we want to apply lemma 8.19. This is possible since

- $h, k \in K(n)$, since $h \sqsubseteq k$ we have $h = \epsilon$ or $(h, k \neq \epsilon$ and $k =<< u_1, \eta_1 > \cdots < u_w, \eta_w >>$, $h =<< v_1, \vartheta_1 > \cdots < v_q, \vartheta_q >>$ and for each $i \in dom(k)$, $j \in dom(h)$ $u_i = v_j \to \eta_i = \vartheta_j)$
- $\rho \in \Xi(h)$, $\sigma \in \Xi(k)$, since $\rho \sqsubseteq \sigma$ if $\rho = (v, \nu)$, $\sigma = (u, \mu)$ then by lemma 8.9 for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $u_i = v_j \to \mu_i = \nu_j$;
- x is a function whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $x_i \in \mathcal{V} - var(k)$, $x_i \in \mathcal{V} - var(h)$ and for each $i, j = 1 \ldots m$ $i \neq j \to x_i \neq x_j$;
- φ is a function whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $\varphi_i \in E(n)$;
- $\phi \in E(n)$;
- $\mathcal{E}(n, k, m, x, \varphi, \phi)$;
- $\mathcal{E}(n, h, m, x, \varphi, \phi)$;
- $t = \{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi)$.

Clearly $\mathcal{E}(n, h, m, x, \varphi, \phi)$ holds because of $\mathcal{E}(p-1, h, m, x, \varphi, \phi)$. Indeed $\mathcal{E}(p-1, h, m, x, \varphi, \phi)$ implies

- $\varphi_1 \in E_s(p-1, h) \subseteq E_s(n, h)$;
- if $m > 1$ then for each $i = 1 \ldots m-1$ $h'_i \in K(p-1) \subseteq K(n) \ \wedge\ \varphi_{i+1} \in E_s(p-1, h'_i) \subseteq E_s(n, h'_i)$;
- $h'_m \in K(p-1) \subseteq K(n) \wedge \phi \in E(p-1, h'_m) \subseteq E(n, h'_m)$.

Both of our statements hold because, while we can use lemma 8.19 to prove the first one, it is also clear that in the same lemma we could use the exact same reasoning to be able to prove the second result.

□

Lemma 8.21. *Let* $h \in K$, $\phi \in E_s(h)$, $y \in (\mathcal{V} - var(h))$, $k = h+ < y, \phi >$. *We have* $k \in K$, *and if* $\vartheta \in S(k)$ *then*

- $\{\}(y : \phi, \vartheta) \in E(h)$;
- $\forall(\{\}(y : \phi, \vartheta)) \in S(h)$, $\exists(\{\}(y : \phi, \vartheta)) \in S(h)$;
- $\forall\ \rho \in \Xi(h)$ $\#(h, \forall(\{\}(y : \phi, \vartheta)), \rho) = P_\forall(\{\#(k, \vartheta, \sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$;
- $\forall\ \rho \in \Xi(h)$ $\#(h, \exists(\{\}(y : \phi, \vartheta)), \rho) = P_\exists(\{\#(k, \vartheta, \sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$.

Proof. Since $\phi \in E_s(h)$ there is a positive integer n such that $h \in K(n)$, $\phi \in E_s(n, h)$. This implies that $k \in K(n)^+ \cup K(n) = K(n+1) \subseteq K$.

Let $\vartheta \in S(k)$. There is a positive integer m such that $k \in K(m)$ and $\vartheta \in E(m,k)$. We define $p = \max\{n+1, m\}$, then we have

- $h \in K(p)$
- $y \in (\mathcal{V} - var(h))$
- $\phi \in E_s(p,h)$
- $k \in K(p),\ \vartheta \in E(p,k)$.

Here we can apply lemma 8.20, in fact in the statement of the lemma we can replace n with p, k with h, m with 1, x with $(1,y)$, φ with $(1,\phi)$, ϕ with ϑ. Every required condition is satisfied, including the condition $\mathcal{E}(p,h,1,(1,y),(1,\phi),\vartheta)$.

So by lemma 8.20 we have that $\{\}(y:\phi,\vartheta) \in E(p+1,h)$ and for each $\rho \in \Xi(h)$

$$\#(h,\{\}(y:\phi,\vartheta),\rho) = \{\#(k,\vartheta,\sigma) |\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\}\ .$$

We want to show that $\forall(\{\}(y:\phi,\vartheta)) \in E(p+2,h)$. To obtain this we can use lemma 8.16, so we just need to show that for each $\rho \in \Xi(h)$ $A_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho))$ holds.
Now $A_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho))$ is equal to

$\#(h,\{\}(y:\phi,\vartheta),\rho)$ is a set and for each $u \in \#(h,\{\}(y:\phi,\vartheta),\rho)$ u is true or u is false.

Clearly $\#(h,\{\}(y:\phi,\vartheta),\rho)$ is a set, furthermore for each $u \in \#(h,\{\}(y:\phi,\vartheta),\rho)$ there is $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $u = \#(k,\vartheta,\sigma)$. Since $\vartheta \in S(k)$ u is true or u is false. So $A_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho))$ holds and $\forall(\{\}(y:\phi,\vartheta)) \in E(p+2,h)$.

Moreover for each $\rho \in \Xi(h)$

$$\begin{aligned}\#(h,\forall(\{\}(y:\phi,\vartheta)),\rho) &= P_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho)) = \\ &= P_\forall(\{\#(k,\vartheta,\sigma) |\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})\ .\end{aligned}$$

and $P_\forall(\{\#(k,\vartheta,\sigma) |\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$ is clearly true or false.

Hence $\forall(\{\}(y:\phi,\vartheta)) \in S(h)$.

Similarly we can show that $\exists(\{\}(y:\phi,\vartheta)) \in E(p+2,h)$. In fact to show this we just need to prove that for each $\rho \in \Xi(h)$ $A_\exists(\#(h,\{\}(y:\phi,\vartheta),\rho))$ holds, and this is proved since

$$A_\exists(\#(h,\{\}(y:\phi,\vartheta),\rho)) = A_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho))\ .$$

Moreover for each $\rho \in \Xi(h)$

$$\begin{aligned}\#(h,\exists(\{\}(y:\phi,\vartheta)),\rho) &= P_\exists(\#(h,\{\}(y:\phi,\vartheta),\rho)) = \\ &= P_\exists(\{\#(k,\vartheta,\sigma) |\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})\ .\end{aligned}$$

and $P_\exists(\{\#(k,\vartheta,\sigma) |\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$ is clearly true or false.

Hence $\exists(\{\}(y:\phi,\vartheta)) \in S(h)$.

□

Lemma 8.22. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $k_0 = \epsilon$ and for each $i = 1 \ldots m$ $k_i = k[x_1 : \varphi_1, \ldots, x_i : \varphi_i]$. Let $\varphi \in S(k_m)$. Then for each $i = 1 \ldots m$ $\gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi] \in S(k_{i-1})$.*

Proof. By definition we have $\gamma[x_m : \varphi_m, \varphi] = \forall(\{\}(x_m : \varphi_m, \varphi))$.

Moreover $k_{m-1} \in K$, $k_m = k_{m-1}+ < x_m, \varphi_m >$, $\varphi_m \in E_s(k_{m-1})$, $x_m \in \mathcal{V} - var(k_{m-1})$. So we can apply lemma 8.21 and obtain that $\gamma[x_m : \varphi_m, \varphi] \in S(k_{m-1})$.

If $m > 1$ for each $i = 2 \ldots m$ we have defined $\gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi]$ and we can assume it is a member of $S(k_{i-1})$, by our definitions we have also

$$\gamma[x_{i-1} : \varphi_{i-1}, \ldots, x_m : \varphi_m, \varphi] = \forall(\{\}(x_{i-1} : \varphi_{i-1}, \gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi])) \ .$$

We have also $k_{i-2} \in K$, $k_{i-1} = k_{i-2}+ < x_{i-1}, \varphi_{i-1} >$, $\varphi_{i-1} \in E_s(k_{i-2})$, $x_{i-1} \in \mathcal{V} - var(k_{i-2})$. So we can apply again lemma 8.21 and obtain that $\gamma[x_{i-1} : \varphi_{i-1}, \ldots, x_m : \varphi_m, \varphi] \in S(k_{i-2})$. □

Lemma 8.23. *Let X be a set, let f, g be functions whose domain is X. Then let $B = \{f(x)|\ x \in X\}$ and $C = \{g(x)|\ x \in X\}$. Suppose for each $x \in X$*

- *$f(x)$ is true or $f(x)$ is false,*
- *$g(x)$ is true or $g(x)$ is false,*
- *$f(x) \leftrightarrow g(x)$.*

Then the following hold

- *$A_\forall(B)$,*
- *$A_\forall(C)$,*
- *$P_\forall(B) \leftrightarrow P_\forall(C)$.*

Proof. Clearly B is a set and for each $b \in B$ there exists $x \in X$ such that $b = f(x)$, so b is true or false. So $A_\forall(B)$ holds and similarly $A_\forall(C)$ holds.

Moreover, if $P_\forall(B)$ holds this means that for each $b \in B$ b is true, so for each $x \in X$ $f(x)$ is true and for each $x \in X$ $g(x)$ is true. Let $c \in C$, there exists $x \in X$ such that $c = g(x)$, $g(x)$ is true and so c is true. So $P_\forall(C)$ holds. Conversely with the same reasoning we can prove that if $P_\forall(C)$ holds then $P_\forall(B)$ also holds. □

Theorem 8.24. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\varphi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$. Then $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi] \in S(\epsilon)$ and*

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi]) \leftrightarrow$$
$$\leftrightarrow P_\forall(\{\#(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m], \varphi, \sigma)|\ \sigma \in \Xi(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])\})$$

Proof. By lemma 8.22 $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi] \in S(\epsilon)$.

Let $k_0 = \epsilon$ and for each $i = 1 \dots m$ $k_i = k[x_1 : \varphi_1, \dots, x_i : \varphi_i]$ as in remark 7.4. What we need to show is:

$$\#(\gamma[x_1 : \varphi_1, \dots, x_m : \varphi_m, \varphi]) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m)\}) \ .$$

Let's consider that, by lemma 8.22, for each $i = 1 \dots m$ $\gamma[x_i : \varphi_i, \dots, x_m : \varphi_m, \varphi] \in S(k_{i-1})$.

In order to prove our result we try to show that for each $i = m \dots 1$ and for each $\rho \in \Xi(k_{i-1})$

$$\#(k_{i-1}, \gamma[x_i : \varphi_i, \dots, x_m : \varphi_m, \varphi], \rho) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\}) \ .$$

We prove this by induction on i, starting with the case where $i = m$. Here we need to show that for each $\rho \in \Xi(k_{m-1})$

$$\#(k_{m-1}, \gamma[x_m : \varphi_m, \varphi], \rho) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\}) \ .$$

Actually, by lemma 8.21,

$$\begin{aligned} \#(k_{m-1}, \gamma[x_m : \varphi_m, \varphi], \rho) &= \#(k_{m-1}, \forall(\{\}(x_m : \varphi_m, \varphi)), \rho) = \\ &= P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\}) \ . \end{aligned}$$

Now suppose $m > 1$, let $i = 2 \dots m$ and suppose the property holds for i, we show it also holds for $i - 1$. We need to prove that for each $\rho \in \Xi(k_{i-2})$

$$\#(k_{i-2}, \gamma[x_{i-1} : \varphi_{i-1}, \dots, x_m : \varphi_m, \varphi], \rho) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\}) \ .$$

By our definitions we have

$$\begin{aligned} &\#(k_{i-2}, \gamma[x_{i-1} : \varphi_{i-1}, \dots, x_m : \varphi_m, \varphi], \rho) = \\ &\quad = \#(k_{i-2}, \forall(\{\}(x_{i-1} : \varphi_{i-1}, \gamma[x_i : \varphi_i, \dots, x_m : \varphi_m, \varphi])), \rho) \end{aligned}$$

By lemma 8.21

$$\begin{aligned} &\#(k_{i-2}, \forall(\{\}(x_{i-1} : \varphi_{i-1}, \gamma[x_i : \varphi_i, \dots, x_m : \varphi_m, \varphi])), \rho) = \\ &\quad = P_\forall(\{\#(k_{i-1}, \gamma[x_i : \varphi_i, \dots, x_m : \varphi_m, \varphi], \delta) |\ \delta \in \Xi(k_{i-1}),\ \rho \sqsubseteq \delta\}) \end{aligned}$$

By the inductive hypothesis given $\delta \in \Sigma(k_{i-1})$ we have

$$\#(k_{i-1}, \gamma[x_i : \varphi_i, \dots, x_m : \varphi_m, \varphi], \delta) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \delta \sqsubseteq \sigma\}) \ .$$

and so

$$\begin{aligned} &\#(k_{i-2}, \forall(\{\}(x_{i-1} : \varphi_{i-1}, \gamma[x_i : \varphi_i, \dots, x_m : \varphi_m, \varphi])), \rho) \leftrightarrow \\ &\quad \leftrightarrow P_\forall(\{P_\forall(\{\#(k_m, \varphi, \sigma) |\ \sigma \in \Xi(k_m),\ \delta \sqsubseteq \sigma\}) |\ \delta \in \Xi(k_{i-1}),\ \rho \sqsubseteq \delta\}) \ . \end{aligned}$$

So it comes to showing that

$$P_\forall(\{P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \delta\sqsubseteq\sigma\})|\ \delta\in\Xi(k_{i-1}),\ \rho\sqsubseteq\delta\})\leftrightarrow$$
$$\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \rho\sqsubseteq\sigma\})\ .$$

Suppose $P_\forall(\{P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \delta\sqsubseteq\sigma\})|\ \delta\in\Xi(k_{i-1}),\ \rho\sqsubseteq\delta\})$.

This means that for each $\delta\in\Xi(k_{i-1})$ such that $\rho\sqsubseteq\delta$ and for each $\sigma\in\Xi(k_m):\ \delta\sqsubseteq\sigma$ $\#(k_m,\varphi,\sigma)$ holds.

Let $\sigma\in\Xi(k_m):\rho\sqsubseteq\sigma$, we need to prove $\#(k_m,\varphi,\sigma)$.

We define $\delta=\sigma_{/dom(k_{i-1})}$. By lemma 8.4 $\delta\in\Xi(k_{i-1})$. Moreover $\delta,\rho\in\mathcal{R}(\sigma)$ and $dom(\rho)=dom(k_{i-2})\subseteq dom(k_{i-1})=dom(\delta)$. By lemma 3.10 we obtain $\rho\sqsubseteq\delta$. Therefore $\#(k_m,\varphi,\sigma)$ holds.

Conversely suppose $P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \rho\sqsubseteq\sigma\})$, so that for each $\sigma\in\Xi(k_m):\rho\sqsubseteq\sigma$ $\#(k_m,\varphi,\sigma)$ is true. Let $\delta\in\Xi(k_{i-1})$ be such that $\rho\sqsubseteq\delta$ and let $\sigma\in\Xi(k_m)$ be such that $\delta\sqsubseteq\sigma$. Since $\sigma\in\Xi(k_m)$ and $\rho\sqsubseteq\sigma$ we have $\#(k_m,\varphi,\sigma)$.

This completes the proof that for each $\rho\in\Xi(k_{i-2})$

$$\#(k_{i-2},\gamma[x_{i-1}:\varphi_{i-1},\ldots,x_m:\varphi_m,\varphi],\rho)\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \rho\sqsubseteq\sigma\})\ .$$

We have also finished the proof that for each $i=m\ldots1$ and for each $\rho\in\Xi(k_{i-1})$

$$\#(k_{i-1},\gamma[x_i:\varphi_i,\ldots,x_m:\varphi_m,\varphi],\rho)\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \rho\sqsubseteq\sigma\})\ .$$

It follows that for each $\rho\in\Xi(k_0)$

$$\#(k_0,\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\varphi],\rho)\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \rho\sqsubseteq\sigma\})\ .$$

and clearly this can be rewritten

$$\#(\epsilon,\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\varphi],\epsilon)\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \epsilon\sqsubseteq\sigma\})\ ,$$
$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\varphi])\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m)\})\ .$$

□

We now need to prove the following result, which is in some way similar to 8.21 but involves the other logical connectives. After that we will be able to discuss the consistency and the completeness of our system.

Lemma 8.25. *Let $h\in K$, $\varphi_1,\varphi_2\in S(h)$. Then*

- $\wedge(\varphi_1,\varphi_2),\vee(\varphi_1,\varphi_2),\rightarrow(\varphi_1,\varphi_2),\leftrightarrow(\varphi_1,\varphi_2),\neg(\varphi_1)\in S(h)$;
- *for each* $\rho\in\Xi(h)$ $\#(h,\wedge(\varphi_1,\varphi_2),\rho)=P_\wedge(\#(h,\varphi_1,\rho),\#(h,\varphi_2,\rho))$;
- *for each* $\rho\in\Xi(h)$ $\#(h,\vee(\varphi_1,\varphi_2),\rho)=P_\vee(\#(h,\varphi_1,\rho),\#(h,\varphi_2,\rho))$;
- *for each* $\rho\in\Xi(h)$ $\#(h,\rightarrow(\varphi_1,\varphi_2),\rho)=P_\rightarrow(\#(h,\varphi_1,\rho),\#(h,\varphi_2,\rho))$;
- *for each* $\rho\in\Xi(h)$ $\#(h,\leftrightarrow(\varphi_1,\varphi_2),\rho)=P_\leftrightarrow(\#(h,\varphi_1,\rho),\#(h,\varphi_2,\rho))$;

- *for each* $\rho \in \Xi(h)$ $\#(h, \neg(\varphi_1), \rho) = P_\neg(\#(h, \varphi_1, \rho))$.

Proof. For each $\rho \in \Xi(h)$ $\#(h, \varphi_1, \rho)$ is true or $\#(h, \varphi_1, \rho)$ is false; $\#(h, \varphi_2, \rho)$ is true or $\#(h, \varphi_2, \rho)$ is false.

We recall that for each $\rho \in \Xi(h)$ $A_\wedge(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\vee(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\rightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\leftrightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$ are all defined as
($\#(h, \varphi_1, \rho)$ is true or $\#(h, \varphi_1, \rho)$ is false) and ($\#(h, \varphi_2, \rho)$ is true or $\#(h, \varphi_2, \rho)$ is false).

Therefore $A_\wedge(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\vee(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\rightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\leftrightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$ are all true.

And for each $\rho \in \Xi(h)$ $A_\neg(\#(h, \varphi_1, \rho))$ is true.

Then by lemmas 8.15 and 8.16

$$\wedge(\varphi_1, \varphi_2), \vee(\varphi_1, \varphi_2), \rightarrow(\varphi_1, \varphi_2), \leftrightarrow(\varphi_1, \varphi_2), \neg(\varphi_1) \in E(h) .$$

Moreover for each $\rho \in \Xi(h)$

$$\begin{aligned}
\#(h, \wedge(\varphi_1, \varphi_2), \rho) &= P_\wedge(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho)); \\
\#(h, \vee(\varphi_1, \varphi_2), \rho) &= P_\vee(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho)); \\
\#(h, \rightarrow(\varphi_1, \varphi_2), \rho) &= P_\rightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho)); \\
\#(h, \leftrightarrow(\varphi_1, \varphi_2), \rho) &= P_\leftrightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho)); \\
\#(h, \neg(\varphi_1), \rho) &= P_\neg(\#(h, \varphi_1, \rho)) .
\end{aligned}$$

so

$$\begin{aligned}
&\#(h, \wedge(\varphi_1, \varphi_2), \rho) \text{ is true or false;} \\
&\#(h, \vee(\varphi_1, \varphi_2), \rho) \text{ is true or false;} \\
&\#(h, \rightarrow(\varphi_1, \varphi_2), \rho) \text{ is true or false;} \\
&\#(h, \leftrightarrow(\varphi_1, \varphi_2), \rho) \text{ is true or false;} \\
&\#(h, \neg(\varphi_1), \rho) \text{ is true or false .}
\end{aligned}$$

Therefore we get

$$\wedge(\varphi_1, \varphi_2), \vee(\varphi_1, \varphi_2), \rightarrow(\varphi_1, \varphi_2), \leftrightarrow(\varphi_1, \varphi_2), \neg(\varphi_1) \in S(h) .$$

□

8.1. Consistency

We have proved that a deductive system is sound, i.e. if we can derive a sentence φ in our system then $\#(\varphi)$ holds. We now discuss the consistency of a deductive system.

Our definition of consistency implies that the symbol $\neg$ with the meaning we have associated to it in section 3 is in the set $\mathcal{F}$ of our language. Actually in this section we have assumed that all of these symbols: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists$ (with their meaning defined in section 3) are in our set $\mathcal{F}$, and we usually assume this since we expect in our deductions we'll frequently need these symbols.

A deductive system $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ is said to be *consistent* if and only if for each φ sentence in $\mathcal{L}$ $(\vdash_{\mathcal{D}} \varphi)$ and $(\vdash_{\mathcal{D}} \neg(\varphi))$ aren't both true.

Lemma 8.26. *Let $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ be a deductive system in $\mathcal{L}$. Then $\mathcal{D}$ is consistent.*

Proof. Suppose there exists a sentence φ such that $\vdash_{\mathcal{D}} \varphi$ and $\vdash_{\mathcal{D}} \neg(\varphi)$ both hold. By the soundness property we have $\#(\varphi)$ and $\#(\neg(\varphi))$. Clearly by lemma 8.25

$$\#(\neg(\varphi)) = \#(\epsilon, \neg(\varphi), \epsilon) = P_{\neg}(\#(\varphi)) = (\#(\varphi) \text{ is false}) \ .$$

So $\#(\varphi)$ would be true and false at the same time, a plain contradiction. □

8.2. Completeness

Let's now define the *completeness* of a deductive system and talk a bit about this. Completeness is the converse property of soundness. A deductive system $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ is said to be *complete* if and only if for each φ sentence in $\mathcal{L}$ if $\#(\varphi)$ holds then $\vdash_{\mathcal{D}} \varphi$. It was easy to prove the soundness of our system, unfortunately the topic of completeness is not as easy. Clearly, if we have defined a deductive system, there is no obvious reason to expect it is also complete.

Anyway, let's define a set A as the set of all sentences φ such that $\#(\varphi)$ holds. Assume A is an axiom in $\mathcal{L}$ (this is a wrong assumption, but let's accept it for a moment). If we define $\mathcal{D} = (\{A\}, \emptyset)$ then $\mathcal{D}$ is a deductive system in $\mathcal{L}$. For each φ sentence in $\mathcal{L}$ if $\#(\varphi)$ holds then $\varphi \in A$ and so $\vdash_{\mathcal{D}} \varphi$. In other words $\mathcal{D}$ is a complete deductive system. So, in the assumptions we made, a complete deductive system exists. Anyway as we said earlier, the assumption that A is an axiom is clearly wrong, and it is wrong because there is no proof or evidence that A is r.e..

Another trivial attempt we could make to define a complete system is the following. For each sentence φ such that $\#(\varphi)$ holds let $\{\varphi\}$ be an axiom in our deductive system $\mathcal{D}$. In this case for each φ sentence in $\mathcal{L}$ if $\#(\varphi)$ holds then $\vdash_{\mathcal{D}} \varphi$ and so the system is complete. However, even in this case we have violated a requirement in the definition of a deductive system. In fact, there is no proof or evidence that our set of axioms is finite.

So we cannot trivially define a complete deductive system. It seems Cutland's book [1] has interesting material with respect to the completeness or incompleteness of deductive systems, in chapter 8. Actually Cutland introduces a notion of 'recursively axiomatised formal system' and what he names a 'simplified version of Goedel incompleteness theorem'. This theorem states that, given a recursively axiomatised formal system in which all provable statements are true, in this system there is a statement which is true but not provable (and so this system is not complete). The proof of this theorem is based on the fact that the set $\mathcal{P}$ of the provable statements of the system is recursively enumerable (r.e.) while the set $\mathcal{T}$ of the true statements of the system is not r.e.. Actually it seems to understand that Cutland refers to recursively axiomatised formal systems 'of arithmetic' i.e. systems that are 'adequate for making statements of ordinary arithmetic' and so include symbols like $0, 1, +, *, =$ and the logical connectives and quantifiers.

So, given a deductive system within our logic system, if we could describe it as a recursively axiomatised formal system of arithmetic, we would have proved that this same system is not complete. From another point of view, given a deductive system within our logic system, if one of the following conditions holds

- the system cannot be described as a recursively axiomatised formal system
- the language does not include arithmetic

we cannot state the incompleteness of the system.

This suggests two questions:

- can we describe a deductive system within our logic system as a recursively axiomatised formal system?
- given a language that does not include arithmetic, under which conditions, if any, a deductive system within our logic system is complete?

However these are non-trivial questions that I do not want to discuss in this manuscript, they are obviously of interest in further investigation of this approach.

In the next section we will build a deductive system and then use it to prove a given statement. This example system has many interesting and general features that can be applied also in other contexts in proving many statements. With our logic system we can certainly use many ideas to build powerful deductive systems and the example helps us to understand this. Anyway, looking at this single system, we just prove one single statement with it. We may want to prove other true statements in the same language, we may be able to do this with the axioms and rules we have provided or, to be able to do this, we may need to add other axioms or rules. However we will not make any statement about the completeness or incompleteness of the system.

We can also think to an alternative definition of completeness, let's call it *d-completeness*. Given a sentence φ in $\mathcal{L}$ we say that φ is *derivable* in $\mathcal{L}$ if there exists a deductive system $\mathcal{D}$ in $\mathcal{L}$ such that $\vdash_{\mathcal{D}} \varphi$. We define the d-completeness of a deductive system $\mathcal{D}$ as follows: $\mathcal{D}$ is *d-complete* if and only if for each φ sentence in $\mathcal{L}$ if φ is derivable in $\mathcal{L}$ then $\vdash_{\mathcal{D}} \varphi$.

Here we notice that if $\#(\varphi)$ holds then we can define $A = \{\varphi\}$ and A is clearly an axiom in $\mathcal{L}$. If we define $\mathcal{D} = (\{A\}, \emptyset)$ then $\mathcal{D}$ is a deductive system and $\vdash_{\mathcal{D}} \varphi$, so φ is derivable in $\mathcal{L}$. Conversely is φ is derivable in $\mathcal{L}$ then by soundess $\#(\varphi)$ holds. Therefore φ is derivable in $\mathcal{L}$ if and only if $\#(\varphi)$ holds, so the notion of d-completeness is actually equivalent to the notion of completeness.

9. Deductive methodology: further results

In this section we show some additional results, which can be referred to any language $\mathcal{L} = (\mathcal{V}, \mathcal{F}, \mathcal{C}, \#, \{D_1, \ldots, D_n\}, q_{max})$ such that all of these symbols: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in, =$ are in our set $\mathcal{F}$. For each of these operators f $A_f(x_1, \ldots, x_n)$ and $P_f(x_1, \ldots, x_n)$ are defined as specified at the beginning of section 3.

Lemma 9.1. *Let m be a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$, assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$, define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and as usual $k_0 = \epsilon$ and for each $i = 1 \ldots m$ $k_i = k[x_1 : \varphi_1, \ldots, x_i : \varphi_i]$.*
Then for each $i = 1 \ldots m$, $j = i \ldots m$

- *$x_i \in E(k_j)$,*
- *$\varphi_i \in E(k_j)$,*
- *for each $\sigma \in \Xi(k_j)$*
  - *$\sigma_{/dom(k_{i-1})} \in \Xi(k_{i-1})$,*
  - *$\#(k_j, x_i, \sigma) \in \#(k_{i-1}, \varphi_i, \sigma_{/dom(k_{i-1})})$,*
  - *$\#(k_j, \varphi_i, \sigma) = \#(k_{i-1}, \varphi_i, \sigma_{/dom(k_{i-1})})$,*
  - *$\#(k_j, x_i, \sigma) \in \#(k_j, \varphi_i, \sigma)$.*

Proof. We prove our assertion by induction on j, so we begin by proving it at level i.

Since $k_i \in K$ there exists a positive integer n such that $k_i \in K(n)$, and since $k_i \neq \epsilon$ we have $n \geqslant 2$. By lemma 8.1 there exists a positive integer $q < n$ such that $k_i \in K(q)^+$. So there exist $h \in K(q), \phi \in E_s(q, h), y \in (\mathcal{V} - var(h))$ such that $k_i = h+ < y, \phi >$. We have also $k_i = k_{i-1}+ < x_i, \varphi_i >$ so

$$x_i = y \in E_a(q+1, k_i) \subseteq E(q+1, k_i) \subseteq E(k_i) \ .$$

For each $\sigma = \rho + (y, s) \in \Xi(k_i)$

$$\#(k_i, x_i, \sigma)_{(n+1,k,a)} = s \in \#(h, \phi, \rho) = \#(k_{i-1}, \varphi_i, \rho).$$

Clearly $\sigma_{/dom(\rho)} = \rho$, $dom(\rho) = dom(h) = dom(k_{i-1})$, therefore $\sigma_{/dom(k_{i-1})} = \rho$ and finally

$$\#(k_i, x_i, \sigma) = \#(k_i, x_i, \sigma)_{(n+1,k,a)} = s \in \#(k_{i-1}, \varphi_i, \sigma_{/dom(k_{i-1})}).$$

Since $\varphi_i \in E(k_{i-1})$ there exists a positive integer q such that $k_{i-1} \in K(q)$ and $\varphi_i \in E(q, k_{i-1})$. Since $k_i \in K$ there also exists a positive integer n such that $k_i \in K(n)$. Let $p = max\{q, n\}$, then $\varphi_i \in E(p, k_{i-1})$ and $k_i \in K(p)$.

If $\varphi_i \in E(p, k_i)$ then clearly $\varphi_i \in E(k_i)$. Otherwise, since $k_i = k_{i-1}+ < x_i, \varphi_i >$, $\varphi_i \in E_b(p+1, k_i) \subseteq E(p+1, k_i) \subseteq E(k_i)$.

At this point, given $\sigma \in \Xi(k_i)$, we observe that $k_{i-1}, k_i \in K(p+1)$, $k_{i-1} \sqsubseteq k_i$, $\varphi_i \in E(p+1, k_{i-1}) \cap E(p+1, k_i)$, $\sigma_{/dom(k_{i-1})} \in \Xi(k_{i-1})$, $\sigma_{/dom(k_{i-1})} \sqsubseteq \sigma$. Here we can apply lemma 8.14 and obtain that $\#(k_i, \varphi_i, \sigma) = \#(k_{i-1}, \varphi_i, \sigma_{/dom(k_{i-1})})$.

Now, in the case $i < m$, let $j = i \ldots m - 1$, we assume all of the following hold:

- $x_i \in E(k_j)$,
- $\varphi_i \in E(k_j)$,
- for each $\sigma \in \Xi(k_j)$
  - $\sigma_{/dom(k_{i-1})} \in \Xi(k_{i-1})$,
  - $\#(k_j, x_i, \sigma) \in \#(k_{i-1}, \varphi_i, \sigma_{/dom(k_{i-1})})$,
  - $\#(k_j, \varphi_i, \sigma) = \#(k_{i-1}, \varphi_i, \sigma_{/dom(k_{i-1})})$,
  - $\#(k_j, x_i, \sigma) \in \#(k_j, \varphi_i, \sigma)$,

and we try to prove the same statements for $j+1$.

Since $k_{j+1} \in K$ there exists a positive integer n such that $k_{j+1} \in K(n)$. There exists a positive integer q such that $x_i \in E(q, k_j)$. Let $p = max\{q, n\}$, then $k_{j+1} \in K(p)$ and $x_i \in E(p, k_j)$.

We can also observe that $k_{j+1} = k_j + (x_{j+1}, \varphi_{j+1}) \in K(p) - \{\epsilon\}$, so

$$E_b(p+1, k_{j+1}) = \{t|\ t \in E(p, k_j),\ t \notin E(p, k_{j+1})\}.$$

Clearly if $x_i \in E(p, k_{j+1})$ then $x_i \in E(k_{j+1})$, otherwise $x_i \in E(p, k_j)$ and $x_i \notin E(p, k_{j+1})$, so $x_i \in E_b(p+1, k_{j+1}) \subseteq E(k_{j+1})$.

We now want to show that for each $\sigma \in \Xi(k_{j+1})$ $\sigma_{/dom(k_{i-1})} \in \Xi(k_{i-1})$ and

$$\#(k_{j+1}, x_i, \sigma) \in \#(k_{i-1}, \varphi_i, \sigma_{/dom(k_{i-1})})\ .$$

We define $\rho = \sigma_{/dom(k_j)}$, so (by lemma 8.4) $\rho \in \Xi(k_j)$ and by the inductive hypothesis $\rho_{/dom(k_{i-1})} \in \Xi(k_{i-1})$ $\#(k_j, x_i, \rho) \in \#(k_{i-1}, \varphi_i, \rho_{/dom(k_{i-1})})$.

It is also clear that $dom(k_{i-1}) \subseteq dom(k_j) \subseteq dom(k_{j+1}) = dom(\sigma)$ and therefore $\sigma_{/dom(k_{i-1})} = (\sigma_{/dom(k_j)})_{/dom(k_{i-1})} = \rho_{/dom(k_{i-1})} \in \Xi(k_{i-1})$.

Hence $\#(k_j, x_i, \rho) \in \#(k_{i-1}, \varphi_i, \sigma_{/dom(k_{i-1})})$ and to complete our proof that $\#(k_{j+1}, x_i, \sigma) \in \#(k_{i-1}, \varphi_i, \sigma_{/dom(k_{i-1})})$ we just need to show that $\#(k_{j+1}, x_i, \sigma) = \#(k_j, x_i, \rho)$.

In order to prove this we can use lemma 8.14. In fact $k_j, k_{j+1} \in K(p+1)$, $k_j \sqsubseteq k_{j+1}$, $x_i \in E(p+1, k_j) \cap E(p+1, k_{j+1})$, $\sigma \in \Xi(k_{j+1})$, $\rho \in \Xi(k_j)$, $\rho \sqsubseteq \sigma$.

Since $\varphi_i \in E(k_j)$ there exists a positive integer q such that $k_j \in K(q)$ and $\varphi_i \in E(q, k_j)$. Since $k_{j+1} \in K$ there also exists a positive integer n such that $k_{j+1} \in K(n)$. Let $p = max\{q, n\}$, then $\varphi_i \in E(p, k_j)$ and $k_{j+1} \in K(p)$.

If $\varphi_i \in E(p, k_{j+1})$ then clearly $\varphi_i \in E(k_{j+1})$. Otherwise, since $k_{j+1} = k_j + (x_{j+1}, \varphi_{j+1})$, $\varphi_i \in E_b(p+1, k_{j+1}) \subseteq E(p+1, k_{j+1}) \subseteq E(k_{j+1})$.

At this point we observe that $k_{i-1} \sqsubseteq k_{j+1}$, $\varphi_i \in E(k_{i-1}) \cap E(k_{j+1})$, $\sigma \in \Xi(k_{j+1})$, $\sigma_{/dom(k_{i-1})} \in \Xi(k_{i-1})$, $\sigma_{/dom(k_{i-1})} \sqsubseteq \sigma$. Here we can apply lemma 8.14 and obtain that $\#(k_{j+1}, \varphi_i, \sigma) = \#(k_{i-1}, \varphi_i, \sigma_{/dom(k_{i-1})})$.

□

Lemma 9.2. *Suppose $k \in K$, $t, \varphi \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma)$ is a set. Then*

- $\in (t, \varphi) \in S(k)$;
- *for each* $\sigma \in \Xi(k)$ $\#(k, \in (t, \varphi), \sigma) = P_{\in}(\#(k, t, \sigma), \#(k, \varphi, \sigma))$.

Proof. This is a trivial consequence of lemma 8.15 . □

Lemma 9.3. *Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ also holds, we define $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\varphi \in E_s(h)$.

Then $\varphi \in E_s(k)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$, $\#(k, \varphi, \sigma) = \#(h, \varphi, \sigma_{/dom(h)})$.

Proof. Since $\varphi \in E(h)$ there exists a positive integer q such that $h \in K(q)$ and $\varphi \in E(q, h)$. Since $k \in K$ there also exists a positive integer n such that $k \in K(n)$. Let $p = max\{q, n\}$, then $\varphi \in E(p, h)$ and $k \in K(p)$.

If $\varphi \in E(p, k)$ then clearly $\varphi \in E(k)$. Otherwise, since $k = k_{m+1} = k_m + < x_{m+1}, \varphi_{m+1} >= h + < x_{m+1}, \varphi_{m+1} >$, $\varphi \in E_b(p+1, k) \subseteq E(p+1, k) \subseteq E(k)$.

Let now $\sigma \in \Xi(k)$, by lemma 8.4 we have $\sigma_{/dom(h)} \in \Xi(h)$, moreover $h \sqsubseteq k$, $\varphi \in E(h) \cap E(k)$, $\sigma_{/dom(h)} \sqsubseteq \sigma$. Here we can apply lemma 8.14 and obtain that $\#(k, \varphi, \sigma) = \#(h, \varphi, \sigma_{/dom(h)})$. □

Lemma 9.4. *Let $c \in \mathcal{C}$. For each positive integer n and $k \in K(n)$*

- $c \in E(n+1, k)$;
- *for each* $\sigma \in \Xi(k)$ $\#(k, c, \sigma) = \#(c)$.

Proof. The proof is by induction on n.

For $n = 1$ we have $k = \epsilon$ so $c \in E(1, \epsilon) = E(n, k) \subseteq E(n+1, k)$ and for each $\sigma \in \Xi(k)$ $\sigma = \epsilon$, so $\#(k, c, \sigma) = \#(\epsilon, c, \epsilon) = \#(c)$.

Let n be a positive integer and $k \in K(n+1) = K(n) \cup K(n)^+$.

If $k \in K(n)$ then

- $c \in E(n+1, k) \subseteq E(n+2, k)$;
- for each $\sigma \in \Xi(k)$ $\#(k, c, \sigma) = \#(c)$.

Otherwise $k \in K(n)^+$, so there exist $h \in K(n), \phi \in E_s(n, h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y, \phi)$. By the inductive hypothesis

- $c \in E(n+1, h)$;
- for each $\rho \in \Xi(h)$ $\#(h, c, \rho) = \#(c)$.

We have $c \notin E(n+1,k)$, $k = h + (y,\phi) \in K(n+1) - \{\epsilon\}$, so $c \in E_b(n+2,k) \subseteq E(n+2,k)$ and for each $\sigma = \rho + (y,s) \in \Xi(k)$

$$\#(k,c,\sigma) = \#(k,c,\sigma)_{(n+2,k,b)} = \#(h,c,\rho) = \#(c)\ .$$

□

10. Building a deductive system

In this section we will build a deductive sytem $\mathcal{D} = (\mathcal{A},\mathcal{R})$, in order to be able to show an example of proof in the next section. The deductive system we are building can refer to any language $\mathcal{L} = (\mathcal{V},\mathcal{F},\mathcal{C},\#,\{D_1,\dots,D_p\},q_{max})$ such that all of these symbols: $\neg,\wedge,\vee,\rightarrow,\leftrightarrow,\forall,\exists,\in,=$ are in our set $\mathcal{F}$. For each of these operators f $A_f(x_1,\dots,x_n)$ and $P_f(x_1,\dots,x_n)$ are defined as specified at the beginning of section 3.

We'll now list the set of axioms and rules of our deductive system. For every axiom/rule we first prove a result which ensures the soundness of the axiom/rule and then define properly the axiom/rule itself.

In our proofs we'll frequently use the following simple result.

Lemma 10.1. *Let S be a set and q,r be functions over S such that for each $\sigma \in S$ $q(\sigma)$ and $r(\sigma)$ are true or false (in these assumptions q,r can be called 'predicates over S'). Then*

$$A_\forall(\{q(\sigma)|\ \sigma \in S\}),\ A_\exists(\{q(\sigma)|\ \sigma \in S\})$$

$$P_\forall(\{q(\sigma)|\ \sigma \in S\}) \leftrightarrow \textit{ for each } \sigma \in S\ q(\sigma),$$

$$P_\exists(\{q(\sigma)|\ \sigma \in S\}) \leftrightarrow \textit{ there exists } \sigma \in S:\ q(\sigma),$$

$$A_\forall(\{q(\sigma)|\ \sigma \in S, r(\sigma)\}),\ A_\exists(\{q(\sigma)|\ \sigma \in S, r(\sigma)\})$$

$$P_\forall(\{q(\sigma)|\ \sigma \in S, r(\sigma)\}) \leftrightarrow \textit{ for each } \sigma \in S \textit{ if } r(\sigma) \textit{ then } q(\sigma),$$

$$P_\exists(\{q(\sigma)|\ \sigma \in S, r(\sigma)\}) \leftrightarrow \textit{ there exists } \sigma \in S : r(\sigma) \textit{ and } q(\sigma).$$

Proof. Let $x_1 = \{q(\sigma)|\ \sigma \in S\}$.

Clearly x_1 is a set and for each $x \in x_1$ there exists $\sigma \in S$ such that $x = q(\sigma)$, so x is true or false. So $A_\forall(x_1)$ and $A_\exists(x_1)$ both hold.

We suppose $P_\forall(x_1)$ and try to prove for each $\sigma \in S$ $q(\sigma)$.
Let $\sigma \in S$, clearly $q(\sigma) \in x_1$, so $q(\sigma)$ is true.

Conversely we suppose for each $\sigma \in S$ $q(\sigma)$ and try to prove $P_\forall(x_1)$.
Let $x \in x_1$, there exists $\sigma \in S$ such that $x = q(\sigma)$ is true.

We suppose $P_\exists(x_1)$ and try to prove there exists $\sigma \in S$ $q(\sigma)$.
There exists x in x_1 such that (x is true). There exists $\sigma \in S$ such that $x = q(\sigma)$, therefore $q(\sigma)$ is true.

Conversely we suppose there exists $\sigma \in S$ $q(\sigma)$ and try to prove $P_\exists(x_1)$.
Clearly $q(\sigma) \in x_1$ and $q(\sigma)$ is true, so $P_\exists(x_1)$ is proved.

Now, to prove the remaining results, let $x_1 = \{q(\sigma) |\ \sigma \in S,\ r(\sigma)\}$.

Clearly x_1 is a set and for each $x \in x_1$ there exists $\sigma \in S$ such that ($r(\sigma)$ and) $x = q(\sigma)$, so x is true or false. So $A_\forall(x_1)$ and $A_\exists(x_1)$ both hold.

We suppose $P_\forall(x_1)$ and try to prove for each $\sigma \in S$ if $r(\sigma)$ then $q(\sigma)$.
Let $\sigma \in S$ and assume $r(\sigma)$, clearly $q(\sigma) \in x_1$, so $q(\sigma)$ is true.

Conversely we suppose for each $\sigma \in S$ if $r(\sigma)$ then $q(\sigma)$ and try to prove $P_\forall(x_1)$.
Let $x \in x_1$, there exists $\sigma \in S$ such that $r(\sigma)$ and $x = q(\sigma)$ is true.

We suppose $P_\exists(x_1)$ and try to prove there exists $\sigma \in S : r(\sigma)$ and $q(\sigma)$.
There exists x in x_1 such that x is true. So there exists $\sigma \in S$ such that $r(\sigma)$ and $x = q(\sigma)$, therefore $q(\sigma)$ is true.

Conversely we suppose there exists $\sigma \in S : r(\sigma)$ and $q(\sigma)$ and try to prove $P_\exists(x_1)$.
Clearly $q(\sigma) \in x_1$ and $q(\sigma)$ is true, so $P_\exists(x_1)$ is proved. □

Lemma 10.2. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi \in S(k)$.*

Under these assumptions we have

- $\wedge(\varphi, \psi), \rightarrow (\wedge(\varphi, \psi), \varphi), \rightarrow (\wedge(\varphi, \psi), \psi) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \varphi)] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \psi)] \in S(\epsilon)$.

Moreover $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \varphi)])$ and $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \psi)])$ are both true.

Proof. Using theorem 8.24 and lemma 8.25 we can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \varphi)])$ as follows:

$$P_\forall(\{\#(k, \rightarrow (\wedge(\varphi, \psi), \varphi), \sigma) |\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_\rightarrow (\#(k, \wedge(\varphi, \psi), \sigma), \#(k, \varphi, \sigma)) \mid \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_\rightarrow (P_\wedge (\#(k, \varphi, \sigma), \#(k, \psi, \sigma)), \#(k, \varphi, \sigma)) \mid \sigma \in \Xi(k)\}).$$

This can be expressed as

$$\text{for each } \sigma \in \Xi(k) \text{ if } \#(k, \varphi, \sigma) \text{ and } \#(k, \psi, \sigma) \text{ then } \#(k, \varphi, \sigma),$$

which is clearly true.

In the same way we can prove the truth of

$$\#\left(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \psi)]\right).$$

□

We can create a set $A_{10.2}$ which is the union of two sets of sentences.

Let G_1 be the set of all the sentences $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \varphi)]$ such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Let G_2 be the set of all the sentences $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \psi)]$ such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Then $A_{10.2}$ is the union of G_1 and G_2. Lemma 10.2 shows us that this set of sentences (which is a potential axiom) is 'sound'. In order to use $A_{10.2}$ as an axiom in our system we also need to show that $A_{10.2}$ is r.e..

Lemma 10.3. *$A_{10.2}$ is r.e. .*

Proof. Given a positive integer m and $(x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m$ we can notice the following:

- $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \in K$;
- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ is r.e.;
- $\{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^2$ is r.e..

So we can define the following

$$Q_{m,2} = \bigcup_{(x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m} \{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^2 .$$

Clearly $Q_{m,2} \subseteq (\Sigma^*)^{2m} \times (\Sigma^*)^2$ is r.e..

We can define a function χ over $(\Sigma^*)^{2m} \times (\Sigma^*)^2$ such that for each $((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) \in (\Sigma^*)^{2m} \times (\Sigma^*)^2$

$$\chi(((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi))) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \varphi)] .$$

Now χ clearly is a computable function and so the set $\{\chi((x_1,\varphi_1,\ldots,x_m,\varphi_m),(\varphi,\psi))|\,((x_1,\varphi_1,\ldots,x_m,\varphi_m),(\varphi,\psi)) \in Q_{m,2}\}$ is a r.e. subset of Σ^*. And finally the set

$$\bigcup_{m\geqslant 1}\{\chi(((x_1,\varphi_1,\ldots,x_m,\varphi_m),(\varphi,\psi)))|\,(((x_1,\varphi_1,\ldots,x_m,\varphi_m),(\varphi,\psi))) \in Q_{m,2}\}$$

is itself a r.e. set. This set can obvioulsy be rewritten as follows

$$\bigcup_{m\geqslant 1}\{\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\wedge(\varphi,\psi),\varphi)]|\,(((x_1,\varphi_1,\ldots,x_m,\varphi_m),(\varphi,\psi))) \in Q_{m,2}\}$$

and it should be clear at this point that this set is actually our axiom G_1, and so that G_1 is r.e..

In fact if $\xi \in G_1$ then there exist a positive integer m, $x_1,\ldots,x_m \in \mathcal{V}$ such that $x_i \neq x_j$ for $i \neq j$, $\varphi_1,\ldots,\varphi_m \in E$ such that $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$, $\varphi,\psi \in S(k[x_1:\varphi_1,\ldots,x_m:\varphi_m])$ such that $\xi = \gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\wedge(\varphi,\psi),\varphi)]$.

It follows that $(x_1,\varphi_1,\ldots,x_m,\varphi_m) \in R_m$ and $((x_1,\varphi_1,\ldots,x_m,\varphi_m),(\varphi,\psi)) \in Q_{m,2}$, so $\xi \in \{\chi((y_1,\psi_1,\ldots,y_m,\psi_m),(\phi,\theta))|\,((y_1,\psi_1,\ldots,y_m,\psi_m),(\phi,\theta)) \in Q_{m,2}\}$, and so

$$\xi \in \bigcup_{p\geqslant 1}\{\chi(((y_1,\psi_1,\ldots,y_p,\psi_p),(\phi,\theta)))|\,((y_1,\psi_1,\ldots,y_p,\psi_p),(\phi,\theta)) \in Q_{p,2}\}\ .$$

Conversely if

$$\xi \in \bigcup_{p\geqslant 1}\{\chi(((y_1,\psi_1,\ldots,y_p,\psi_p),(\phi,\theta)))|\,((y_1,\psi_1,\ldots,y_p,\psi_p),(\phi,\theta)) \in Q_{p,2}\}\ ,$$

there exist $p \geqslant 1$, $((y_1,\psi_1,\ldots,y_p,\psi_p),(\phi,\theta)) \in Q_{p,2}$ such that $\xi = \chi(((y_1,\psi_1,\ldots,y_p,\psi_p),(\phi,\theta))) = \gamma[y_1:\psi_1,\ldots,y_p:\psi_p,\rightarrow(\wedge(\phi,\theta),\phi)]$, so $(y_1,\psi_1,\ldots,y_p,\psi_p) \in R_p$, $\phi,\theta \in S(k[y_1:\psi_1,\ldots,y_p:\psi_p])$.

So $y_1,\ldots,y_p \in \mathcal{V}$ with $y_i \neq y_j$ for $i \neq j$, $\psi_1,\ldots,\psi_p \in E, H[y_1:\psi_1,\ldots,y_p:\psi_p]$.

And this implies $\xi \in G_1$.

Similarly G_2 is r.e. and so $A_{10.2}$ is r.e.. □

Then let $A_{10.2} \in \mathcal{A}$.

Lemma 10.4. *Let m be a positive integer. Let $x_1,\ldots,x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1,\ldots,\varphi_m \in E$ and assume $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$. Define $k = k[x_1:\varphi_1,\ldots,x_m:\varphi_m]$ and let $\varphi,\psi,\chi \in S(k)$.*

Under these assumptions we have

- $\to(\varphi,\psi), \to(\psi,\chi), \to(\varphi,\chi) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to(\varphi,\psi)] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to(\psi,\chi)] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to(\varphi,\chi)] \in S(\epsilon)$.

Moreover if

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to(\varphi,\psi)])$,
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to(\psi,\chi)])$

then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to(\varphi,\chi)])$.

Proof. We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to(\varphi,\psi)])$ as follows:

$$P_\forall(\{\#(k, \to(\varphi,\psi), \sigma) |\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_\to(\#(k,\varphi,\sigma), \#(k,\psi,\sigma)) \mid \sigma \in \Xi(k)\}).$$

And we can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to(\psi,\chi)])$ as follows:

$$P_\forall(\{\#(k, \to(\psi,\chi), \sigma) |\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_\to(\#(k,\psi,\sigma), \#(k,\chi,\sigma)) \mid \sigma \in \Xi(k)\}).$$

In other words for each $\sigma \in \Xi(k)$ if $\#(k,\varphi,\sigma)$ then $\#(k,\psi,\sigma)$, and if $\#(k,\psi,\sigma)$ then $\#(k,\chi,\sigma)$. So, for each $\sigma \in \Xi(k)$, if $\#(k,\varphi,\sigma)$ then $\#(k,\chi,\sigma)$. This can be written as follows:

$$P_\forall(\{P_\to(\#(k,\varphi,\sigma), \#(k,\chi,\sigma)) \mid \sigma \in \Xi(k)\})$$

$$P_\forall(\{\#(k, \to(\varphi,\chi), \sigma) |\ \sigma \in \Xi(k)\}),$$

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to(\varphi,\chi)]).$$

□

We can create a set $R_{10.4}$ as the set of all 3-tuples

$$\begin{pmatrix} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to(\varphi,\psi)], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to(\psi,\chi)], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to(\varphi,\chi)] \end{pmatrix}$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi, \chi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Lemma 10.4 shows us that this set (which is a potential 2-ary rule) is 'sound'. In order to use $R_{10.4}$ as a rule in our system we also need to show that $R_{10.4}$ is r.e..

Lemma 10.5. *$R_{10.4}$ is r.e. .*

Proof. Given a positive integer m and $(x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m$ we can notice the following:

- $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \in K$;
- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ is r.e.;
- $\{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^3$ is r.e..

Let's define

$$Q_{m,3} = \bigcup_{(x_1,\varphi_1,\ldots,x_m,\varphi_m)\in R_m} \{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^3 \ .$$

Clearly $Q_{m,3} \subseteq (\Sigma^*)^{2m} \times (\Sigma^*)^3$ is also r.e..

We now define three functions $\delta_{1,m},\ \delta_{2,m},\ \delta_{3,m}$ over $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ as follows. Given $((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) \in (\Sigma^*)^{2m} \times (\Sigma^*)^3$

$$\delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \rightarrow (\varphi, \psi)] \ .$$

$$\delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \rightarrow (\psi, \chi)] \ .$$

$$\delta_{3,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \rightarrow (\varphi, \chi)] \ .$$

All of the three functions we have defined are computable functions from $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ to Σ^*. If we define a function δ_m over $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ as follows:

$$\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \begin{pmatrix} \delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)), \\ \delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)), \\ \delta_{3,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) \end{pmatrix}$$

then δ_m is a computable function from $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ to $(\Sigma^*)^3$, therefore the set

$$D_m = \{\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) |\ ((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) \in Q_{m,3}\}$$

is a r.e. subset of $(\Sigma^*)^3$.

If we now consider the set $\bigcup_{m\geqslant 1} D_m$ then this is a r.e. subset of $(\Sigma^*)^3$ and actually this set is equal to our rule $R_{10.4}$ which so is r.e. itself.

If $\xi = (\xi_1, \xi_2, \xi_3) \in R_{10.4}$ then there exist a positive integer m, $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$ such that $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$; if we define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ there also exist $\varphi, \psi, \chi \in S(k)$ such that

- $\xi_1 = \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \psi)]$,
- $\xi_2 = \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\psi, \chi)]$,
- $\xi_3 = \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \chi)]$.

This means $(x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m$, $\varphi, \psi, \chi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$, so $((x_1, \varphi_1, \ldots, x_m, \varphi_m), (\varphi, \psi, \chi)) \in Q_{m,3}$

- $\xi_1 = \delta_{1,m}((x_1, \varphi_1, \ldots, x_m, \varphi_m), (\varphi, \psi, \chi))$,
- $\xi_2 = \delta_{2,m}((x_1, \varphi_1, \ldots, x_m, \varphi_m), (\varphi, \psi, \chi))$,
- $\xi_3 = \delta_{3,m}((x_1, \varphi_1, \ldots, x_m, \varphi_m), (\varphi, \psi, \chi))$.

i.e. $\xi = \delta_m((x_1, \varphi_1, \ldots, x_m, \varphi_m), (\varphi, \psi, \chi)) \in D_m$.

Conversely if there exists $p \geqslant 1$ such that $\xi \in D_p$ then there exists $((\psi_1, \varphi_1, \ldots, \psi_p, \varphi_p), (\varphi, \psi, \chi)) \in Q_{p,3}$ such that $\xi = \delta_p((\psi_1, \varphi_1, \ldots, \psi_p, \varphi_p), (\varphi, \psi, \chi))$.

It follows that $(\psi_1, \varphi_1, \ldots, \psi_p, \varphi_p) \in R_p$, $\varphi, \psi, \chi \in S(k[x_1 : \varphi_1, \ldots, x_p : \varphi_p])$, so $\psi_1, \ldots, \psi_p \in \mathcal{V}$, $\psi_i \neq \psi_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_p \in E$, $H[\psi_1 : \varphi_1, \ldots, \psi_p : \varphi_p]$.

Moreover

$$\xi = \delta_p((\psi_1, \varphi_1, \ldots, \psi_p, \varphi_p), \varphi, \psi, \chi) = \begin{pmatrix} \delta_{1,p}((\psi_1, \varphi_1, \ldots, \psi_p, \varphi_p), \varphi, \psi, \chi), \\ \delta_{2,p}((\psi_1, \varphi_1, \ldots, \psi_p, \varphi_p), \varphi, \psi, \chi), \\ \delta_{3,p}((\psi_1, \varphi_1, \ldots, \psi_p, \varphi_p), \varphi, \psi, \chi) \end{pmatrix} =$$
$$= \begin{pmatrix} \gamma[\psi_1 : \varphi_1, \ldots, \psi_p : \varphi_p, \rightarrow (\varphi, \psi)], \\ \gamma[\psi_1 : \varphi_1, \ldots, \psi_p : \varphi_p, \rightarrow (\psi, \chi)], \\ \gamma[\psi_1 : \varphi_1, \ldots, \psi_p : \varphi_p, \rightarrow (\varphi, \chi)] \end{pmatrix}$$

and so $\xi \in R_{10.4}$.

□

Then let $R_{10.4} \in \mathcal{R}$.

Lemma 10.6. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$.*

Let $i = 1 \ldots m$, then

- $\in (x_i, \varphi_i) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \in (x_i, \varphi_i)] \in S(\epsilon)$,
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \in (x_i, \varphi_i)])$.

Proof. Using lemma 9.1 we obtain

- $x_i \in E(k)$,
- $\varphi_i \in E(k)$,
- for each $\sigma \in \Xi(k)$
  - $\sigma_{/dom(k_{i-1})} \in \Xi(k_{i-1})$,
  - $\#(k, \varphi_i, \sigma) = \#(k_{i-1}, \varphi_i, \sigma_{/dom(k_{i-1})})$,
  - $\#(k, x_i, \sigma) \in \#(k, \varphi_i, \sigma)$.

We have also that $\varphi_i \in E_s(k_{i-1})$, so for each $\sigma \in \Xi(k)$ $\#(k, \varphi_i, \sigma) = \#(k_{i-1}, \varphi_i, \sigma_{/dom(k_{i-1})})$ is a set. Therefore we can apply lemma 9.2 and obtain that $\in (x_i, \varphi_i) \in S(k)$. Consequently

$$\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\in)(x_i, \varphi_i)] \in S(\epsilon) \ .$$

Moreover we can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\in)(x_i, \varphi_i)])$ as follows

$$P_\forall(\{\#(k, (\in)(x_i, \varphi_i), \sigma) |\ \sigma \in \Xi(k)\}) \ ,$$

$$P_\forall(\{P_\in(\#(k, x_i, \sigma), \#(k, \varphi_i, \sigma)) |\ \sigma \in \Xi(k)\}) \ .$$

To show this we have to prove that for each $\sigma \in \Xi(k)$ $\#(k, x_i, \sigma)$ belongs to $\#(k, \varphi_i, \sigma)$. But we have just seen this is true. □

We can create a set $A_{10.6}$ which is the set of all sentences $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \in (x_i, \varphi_i)]$ such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $i = 1 \ldots m$.

Lemma 10.6 shows us that this set of sentences (which is a potential axiom) is 'sound'. In order to use $A_{10.6}$ as an axiom in our system we also need to show that $A_{10.6}$ is r.e..

Lemma 10.7. *$A_{10.6}$ is r.e. .*

Proof. Let m be a positive integer and let $i = 1 \ldots m$. We define a function χ_i over $(\Sigma^*)^{2m}$ such that for each $(\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m) \in (\Sigma^*)^{2m}$

$$\chi_i(\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \in (\psi_i, \varphi_i)] \ .$$

Now χ_i clearly is a computable function and so the set $\{\chi_i(x_1, \varphi_1, \ldots, x_m, \varphi_m) | (x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m\}$ is a r.e. subset of Σ^*. And moreover the set

$$\bigcup_{i=1\ldots m} \{\chi_i(x_1, \varphi_1, \ldots, x_m, \varphi_m) | (x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m\}$$

is itself a r.e. set. And finally the set

$$\bigcup_{m \geqslant 1} (\bigcup_{i=1\ldots m} \{\chi_i(x_1, \varphi_1, \ldots, x_m, \varphi_m) | (x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m\})$$

is itself a r.e. set. This set can obviously be rewritten as follows:

$$\bigcup_{m\geqslant 1}(\bigcup_{i=1\ldots m}\{\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\in(x_i,\varphi_i)]|\,(x_1,\varphi_1,\ldots,x_m,\varphi_m)\in R_m\})$$

and it should be clear at this point that this set is actually our set $A_{10.6}$.

In fact if $\xi\in A_{10.6}$ then there exist a positive integer m, $x_1,\ldots,x_m\in\mathcal{V}$ such that $x_\alpha\neq x_\beta$ for $\alpha\neq\beta$, $\varphi_1,\ldots,\varphi_m\in E$ such that $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$, $i=1\ldots m$ such that $\xi=\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\in(x_i,\varphi_i)]$.

Of course this implies $(x_1,\varphi_1,\ldots,x_m,\varphi_m)\in R_m$, so

$$\xi\in\{\chi_i(x_1,\varphi_1,\ldots,x_m,\varphi_m)|\,(x_1,\varphi_1,\ldots,x_m,\varphi_m)\in R_m\}\ .$$

And then

$$\xi\in\bigcup_{j=1\ldots m}\{\chi_j(x_1,\varphi_1,\ldots,x_m,\varphi_m)|\,(x_1,\varphi_1,\ldots,x_m,\varphi_m)\in R_m\}\ ;$$

$$\xi\in\bigcup_{p\geqslant 1}(\bigcup_{j=1\ldots p}\{\chi_j(x_1,\varphi_1,\ldots,x_p,\varphi_p)|\,(x_1,\varphi_1,\ldots,x_p,\varphi_p)\in R_p\})\ .$$

Conversely if

$$\xi\in\bigcup_{p\geqslant 1}(\bigcup_{j=1\ldots p}\{\chi_j(x_1,\varphi_1,\ldots,x_p,\varphi_p)|\,(x_1,\varphi_1,\ldots,x_p,\varphi_p)\in R_p\})$$

then there exists p positive integer, $j=1\ldots p$, $(x_1,\varphi_1,\ldots,x_p,\varphi_p)\in R_p$ such that

$$\xi=\chi_j(x_1,\varphi_1,\ldots,x_p,\varphi_p)=\gamma[x_1:\varphi_1,\ldots,x_p:\varphi_p,\in(x_j,\varphi_j)]\ .$$

Clearly we have $x_1,\ldots,x_p\in\mathcal{V}$ such that $x_\alpha\neq x_\beta$ for $\alpha\neq\beta$, $\varphi_1,\ldots,\varphi_p\in E$ such that $H[x_1:\varphi_1,\ldots,x_p:\varphi_p]$, so $\xi\in A_{10.6}$. □

At this point let $A_{10.6}\in\mathcal{A}$.

Lemma 10.8. *Let m be a positive integer. Let $x_1,\ldots,x_m\in\mathcal{V}$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1,\ldots,\varphi_m\in E$ and assume $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$. Define $k=k[x_1:\varphi_1,\ldots,x_m:\varphi_m]$ and let $\varphi,\psi\in S(k)$.*

Under these assumptions we have

- $\rightarrow(\psi,\varphi)\in S(k)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\varphi]\in S(\epsilon)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\psi,\varphi)]\in S(\epsilon)$.

Moreover if $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\varphi])$ then $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\psi,\varphi)])$ also holds.

Proof. Suppose $\#\,(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi])$ holds. It can be rewritten as

$$P_\forall(\{\#(k, \varphi, \sigma)|\ \sigma \in \Xi(k)\})\ .$$

We can rewrite $\#\,(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\psi, \varphi)])$ as

$$P_\forall(\{\#(k, \rightarrow(\psi, \varphi), \sigma)|\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\rightarrow(\#(k, \psi, \sigma), \#(k, \varphi, \sigma))|\ \sigma \in \Xi(k)\})\ .$$

For each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma)$ holds, this implies that

$$P_\rightarrow(\#(k, \psi, \sigma), \#(k, \varphi, \sigma))$$

holds too, therefore

$$P_\forall(\{P_\rightarrow(\#(k, \psi, \sigma), \#(k, \varphi, \sigma))|\ \sigma \in \Xi(k)\})$$

also holds and this completes the proof. □

We can create a set $R_{10.8}$ as the set of all pairs

$$(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi], \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\psi, \varphi)])$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Lemma 10.8 shows us that this set (which is a potential 1-ary rule) is 'sound'. In order to use $R_{10.8}$ as a rule in our system we also need to show that $R_{10.8}$ is r.e..

Lemma 10.9. *$R_{10.8}$ is r.e. .*

Proof. Given a positive integer m and $(x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m$ we can notice the following:

- $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \in K$;
- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ is r.e.;
- $\{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^2$ is r.e..

So we can define the following

$$Q_{m,2} = \bigcup_{(x_1,\varphi_1,\ldots,x_m,\varphi_m) \in R_m} \{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^2\ .$$

Clearly $Q_{m,2} \subseteq (\Sigma^*)^{2m} \times (\Sigma^*)^2$ is r.e..

We now define two functions $\delta_{1,m}$, $\delta_{2,m}$ over $(\Sigma^*)^{2m} \times (\Sigma^*)^2$ as follows. Given $((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) \in (\Sigma^*)^{2m} \times (\Sigma^*)^2$

$$\delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \varphi] \ .$$

$$\delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \rightarrow (\psi, \varphi)] \ .$$

All of the two functions we have defined are computable functions from $(\Sigma^*)^{2m} \times (\Sigma^*)^2$ to Σ^*. If we define a function δ_m over $(\Sigma^*)^{2m} \times (\Sigma^*)^2$ as follows:

$$\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) = \begin{pmatrix} \delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)), \\ \delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)), \end{pmatrix}$$

then δ_m is a computable function from $(\Sigma^*)^{2m} \times (\Sigma^*)^2$ to $(\Sigma^*)^2$, therefore the set

$$D_m = \{\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) |\ ((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) \in Q_{m,2}\}$$

is a r.e. subset of $(\Sigma^*)^2$.

If we now consider the set $\bigcup_{m \geqslant 1} D_m$ then this is a r.e. subset of $(\Sigma^*)^2$ and actually this set is equal to our rule $R_{10.8}$ which so is r.e. itself. □

Then let $R_{10.8} \in \mathcal{R}$.

Lemma 10.10. *Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ also holds, we define $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\chi \in S(h)$, $t \in E(h)$, $\varphi \in E_s(h)$.

Under these assumptions

- $\in (x_{m+1}, \varphi) \in S(k)$,
- $\forall(\{\} \, (x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \forall(\{\} \, (x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))))] \in S(\epsilon)$,
- $\in (t, \varphi_{m+1}) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \in (t, \varphi_{m+1}))] \in S(\epsilon)$,
- $\in (t, \varphi) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \in (t, \varphi))] \in S(\epsilon)$.

Moreover if

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \forall(\{\} \, (x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))))])$ *and*
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \in (t, \varphi_{m+1}))])$

then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \in (t, \varphi))])$.

Proof. By lemma 9.1 we obtain that $x_{m+1} \in E(k)$.

By lemma 9.3, since $\varphi \in E_s(h)$, we obtain that $\varphi \in E_s(k)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$, $\#(k, \varphi, \sigma) = \#(h, \varphi, \sigma_{/dom(h)})$.

By lemma 9.2 we obtain that $\in (x_{m+1}, \varphi) \in S(k)$.

By lemma 8.21 we obtain $\forall(\{\}\,(x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))) \in S(h)$.

Clearly this implies that

$$\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \forall(\{\}\,(x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))))] \in S(\epsilon).$$

Furthermore we have $t \in E(h)$, $\varphi_{m+1} \in E_s(h)$, so $\in (t, \varphi_{m+1}) \in S(h)$. It clearly follows that $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \in (t, \varphi_{m+1}))] \in S(\epsilon)$.

We have also $\varphi \in E_s(h)$, so $\in (t, \varphi) \in S(h)$. It follows that

$$\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \in (t, \varphi))] \in S(\epsilon).$$

We now assume

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \forall(\{\}\,(x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))))])$ and
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \in (t, \varphi_{m+1}))])$

both hold and we try to prove $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \in (t, \varphi))])$.

We can rewrite

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \forall(\{\}\,(x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))))])$$

as

$$P_\forall(\{\#\,(h, \to (\chi, \forall\,(\{\}\,(x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi)))), \rho) \mid \rho \in \Xi(h)\})\ ,$$

$$P_\forall(\{P_\to\,(\#\,(h, \chi, \rho)\,, \#\,(h, \forall\,(\{\}\,(x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi)))\,, \rho)) \mid \rho \in \Xi(h)\})\ ,$$

$$P_\forall(\{P_\to\,(\#\,(h, \chi, \rho)\,, P_\forall\,(\{\#\,(k, \in (x_{m+1}, \varphi), \sigma) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})) \mid \rho \in \Xi(h)\})\ ,$$

$$P_\forall(\{P_\to\,(\#\,(h, \chi, \rho)\,, P_\forall\,(\{P_\in\,(\#(k, x_{m+1}, \sigma), \#(k, \varphi, \sigma)) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})) \mid \rho \in \Xi(h)\})\ .$$

We can rewrite

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \in (t, \varphi_{m+1}))])$$

as

$$P_\forall(\{\#(h, \to (\chi, \in (t, \varphi_{m+1})), \rho) \mid \rho \in \Xi(h)\})\ ,$$

$$P_\forall(\{P_\to(\#(h, \chi, \rho), \#(h, \in (t, \varphi_{m+1}), \rho)) \mid \rho \in \Xi(h)\})\ ,$$

$$P_\forall(\{P_\to(\#(h, \chi, \rho), P_\in(\#(h, t, \rho), \#(h, \varphi_{m+1}, \rho))) \mid \rho \in \Xi(h)\})\ .$$

We can rewrite

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\chi,\in(t,\varphi))])$$

as

$$P_\forall(\{\#(h,\rightarrow(\chi,\in(t,\varphi)),\rho)|\ \rho\in\Xi(h)\})\ ,$$

$$P_\forall(\{P_\rightarrow(\#(h,\chi,\rho),\#(h,\in(t,\varphi),\rho))|\ \rho\in\Xi(h)\})\ ,$$

$$P_\forall(\{P_\rightarrow(\#(h,\chi,\rho),P_\in(\#(h,t,\rho),\#(h,\varphi,\rho)))|\ \rho\in\Xi(h)\})\ .$$

Let $\rho\in\Xi(h)$ and let $\#(h,\chi,\rho)$. We need to show that $\#(h,t,\rho)$ belongs to $\#(h,\varphi,\rho)$.

There exists a positive integer q such that $k\in K(q)^+$. So there exist $g\in K(q),\phi\in E_s(q,g),y\in(\mathcal{V}-var(g))$ such that $k=g+<y,\phi>$. At the same time

$$k=k_{m+1}=k_m+<x_{m+1},\varphi_{m+1}>=h+<x_{m+1},\varphi_{m+1}>\ \ .$$

Therefore

$$\begin{aligned}\Xi(k)&=\{\delta+(y,s)|\,\delta\in\Xi(g),s\in\#(g,\phi,\delta)\}=\\&=\{\delta+(x_{m+1},s)|\,\delta\in\Xi(h),s\in\#(h,\varphi_{m+1},\delta)\}\,.\end{aligned}$$

We have $\rho\in\Xi(h)$, $\#(h,t,\rho)\in\#(h,\varphi_{m+1},\rho)$, so $\rho+(x_{m+1},\#(h,t,\rho))\in\Xi(k)$.

Let $\sigma=\rho+(x_{m+1},\#(h,t,\rho))\in\Xi(k)$, clearly $\rho\sqsubseteq\sigma$, so $\#(k,x_{m+1},\sigma)$ belongs to $\#(k,\varphi,\sigma)$. And we have also

$$x_{m+1}=y\in E_a(q+1,k)\subseteq E(q+1,k)\,,$$

$$\#(k,x_{m+1},\sigma)=\#(k,x_{m+1},\sigma)_{(q+1,k,a)}=\#(h,t,\rho)\,,$$

$$\#(k,\varphi,\sigma)=\#(h,\varphi,\sigma_{/dom(h)})=\#(h,\varphi,\sigma_{/dom(\rho)})=\#(h,\varphi,\rho)\,.$$

Finally we obtain $\#(h,t,\rho)=\#(k,x_{m+1},\sigma)$ belongs to $\#(k,\varphi,\sigma)=\#(h,\varphi,\rho)$. $\Box$

We can create a set $R_{10.10}$ which is the set of all 3-tuples

$$\left(\begin{array}{l}\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\chi,\forall(\{\}\,(x_{m+1}:\varphi_{m+1},\in(x_{m+1},\varphi))))],\\\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\chi,\in(t,\varphi_{m+1}))],\\\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\chi,\in(t,\varphi))]\end{array}\right)$$

such that

- m is a positive integer, $x_1, \dots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \dots, \varphi_{m+1} \in E$, $H[x_1 : \varphi_1, \dots, x_{m+1} : \varphi_{m+1}]$;
- if we define $k = k[x_1 : \varphi_1, \dots, x_{m+1} : \varphi_{m+1}]$ and $h = k[x_1 : \varphi_1, \dots, x_m : \varphi_m]$ then
  - $\chi \in S(h)$,
  - $t \in E(h)$,
  - $\varphi \in E_s(h)$.

Lemma 10.10 shows us that this set (which is a potential 2-ary rule) is 'sound'. In order to use $R_{10.10}$ as a rule in our system we also need to show that $R_{10.10}$ is r.e..

Lemma 10.11. *$R_{10.10}$ is r.e. .*

Proof. Given a positive integer m and $(x_1, \varphi_1, \dots, x_{m+1}, \varphi_{m+1}) \in R_{m+1}$ all of the following sets are r.e.:

- $E(k[x_1 : \varphi_1, \dots, x_m : \varphi_m])$,
- $S(k[x_1 : \varphi_1, \dots, x_m : \varphi_m])$,
- $E_s(k[x_1 : \varphi_1, \dots, x_m : \varphi_m])$.

Therefore the following set is also r.e.:

$$\{(x_1, \varphi_1, \dots, x_{m+1}, \varphi_{m+1})\} \times S(k[x_1 : \varphi_1, \dots, x_m : \varphi_m]) \times E(k[x_1 : \varphi_1, \dots, x_m : \varphi_m]) \times E_s(k[x_1 : \varphi_1, \dots, x_m : \varphi_m]).$$

Let's use this temporary definition

$$Q'_{m+1,3} = \bigcup_{(x_1,\varphi_1,\dots,x_{m+1},\varphi_{m+1}) \in R_{m+1}} \{(x_1, \varphi_1, \dots, x_{m+1}, \varphi_{m+1})\} \times S(k[x_1 : \varphi_1, \dots, x_m : \varphi_m]) \times E(k[x_1 : \varphi_1, \dots, x_m : \varphi_m]) \times E_s(k[x_1 : \varphi_1, \dots, x_m : \varphi_m]).$$

With this $Q'_{m+1,3}$ is a r.e. subset of $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^* \times \Sigma^*$.

We now define three functions $\delta_{1,m}$, $\delta_{2,m}$, $\delta_{3,m}$ over $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^* \times \Sigma^*$ as follows. Given $((\psi_1, \varphi_1, \dots, \psi_{m+1}, \varphi_{m+1}), \chi, t, \varphi) \in (\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^* \times \Sigma^*$

$$\delta_{1,m}((\psi_1, \varphi_1, \dots, \psi_{m+1}, \varphi_{m+1}), \chi, t, \varphi) = \gamma[\psi_1 : \varphi_1, \dots, \psi_m : \varphi_m, \rightarrow (\chi, \forall(\{\} (\psi_{m+1} : \varphi_{m+1}, \in (\psi_{m+1}, \varphi)))))] \ .$$

$$\delta_{2,m}((\psi_1, \varphi_1, \dots, \psi_{m+1}, \varphi_{m+1}), \chi, t, \varphi) = \gamma[\psi_1 : \varphi_1, \dots, \psi_m : \varphi_m, \rightarrow (\chi, \in (t, \varphi_{m+1}))] \ .$$

$$\delta_{3,m}((\psi_1, \varphi_1, \dots, \psi_{m+1}, \varphi_{m+1}), \chi, t, \varphi) = \gamma[\psi_1 : \varphi_1, \dots, \psi_m : \varphi_m, \rightarrow (\chi, \in (t, \varphi))] \ .$$

All of the three functions we have defined are computable functions from $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^* \times \Sigma^*$ to Σ^*. If we define a function δ_m over $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^* \times \Sigma^*$

as follows:

$$\delta_m((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\chi,t,\varphi) = \begin{pmatrix} \delta_{1,m}((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\chi,t,\varphi), \\ \delta_{2,m}((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\chi,t,\varphi), \\ \delta_{3,m}((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\chi,t,\varphi) \end{pmatrix}$$

then δ_m is a computable function from $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^* \times \Sigma^*$ to $(\Sigma^*)^3$, therefore the set

$$D_m = \{\delta_m((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\chi,t,\varphi) | ((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\chi,t,\varphi) \in Q'_{m+1,3}\}$$

is a r.e. subset of $(\Sigma^*)^3$.

If we now consider the set $\bigcup_{m\geqslant 1} D_m$ then this is a r.e. subset of $(\Sigma^*)^3$ and actually this set is equal to our rule $R_{10.10}$ which so is r.e. itself.

If $\xi \in R_{10.10}$ then there exist a positive integer m, $x_1,\ldots,x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1,\ldots,\varphi_{m+1} \in E$ such that $H[x_1 : \varphi_1,\ldots,x_{m+1} : \varphi_{m+1}]$; if we define $k = k[x_1 : \varphi_1,\ldots,x_{m+1} : \varphi_{m+1}]$ and $h = k[x_1 : \varphi_1,\ldots,x_m : \varphi_m]$ there also exist $\chi \in S(h)$, $t \in E(h)$, $\varphi \in E_s(h)$, $\xi_1,\xi_2,\xi_3 \in \Sigma^*$ such that

- $\xi = (\xi_1,\xi_2,\xi_3)$
- $\xi_1 = \gamma[x_1 : \varphi_1,\ldots,x_m : \varphi_m, \rightarrow (\chi, \forall(\{\} (x_{m+1} : \varphi_{m+1}, \in (x_{m+1},\varphi))))]$,
- $\xi_2 = \gamma[x_1 : \varphi_1,\ldots,x_m : \varphi_m, \rightarrow (\chi, \in (t,\varphi_{m+1}))]$,
- $\xi_3 = \gamma[x_1 : \varphi_1,\ldots,x_m : \varphi_m, \rightarrow (\chi, \in (t,\varphi))]$.

This means that $(x_1,\varphi_1,\ldots,x_{m+1},\varphi_{m+1}) \in R_{m+1}$, $\chi \in S(k[x_1 : \varphi_1,\ldots,x_m : \varphi_m])$, $t \in E(k[x_1 : \varphi_1,\ldots,x_m : \varphi_m])$, $\varphi \in E_s(k[x_1 : \varphi_1,\ldots,x_m : \varphi_m])$, so $((x_1,\varphi_1,\ldots,x_{m+1},\varphi_{m+1}),\chi,t,\varphi) \in Q'_{m+1,3}$.

Moreover

- $\xi_1 = \delta_{1,m}((x_1,\varphi_1,\ldots,x_{m+1},\varphi_{m+1}),\chi,t,\varphi)$,
- $\xi_2 = \delta_{2,m}((x_1,\varphi_1,\ldots,x_{m+1},\varphi_{m+1}),\chi,t,\varphi)$,
- $\xi_3 = \delta_{3,m}((x_1,\varphi_1,\ldots,x_{m+1},\varphi_{m+1}),\chi,t,\varphi)$.

i.e. $\xi = \delta_m((x_1,\varphi_1,\ldots,x_{m+1},\varphi_{m+1}),\chi,t,\varphi) \in D_m$.

Conversely if there exists $p \geqslant 1$ such that $\xi \in D_p$ then there exists $((\psi_1,\varphi_1,\ldots,\psi_{p+1},\varphi_{p+1}),\chi,t,\varphi) \in Q'_{p+1,3}$ such that
$\xi = \delta_p((\psi_1,\varphi_1,\ldots,\psi_{p+1},\varphi_{p+1}),\chi,t,\varphi)$.

Since $((\psi_1,\varphi_1,\ldots,\psi_{p+1},\varphi_{p+1}),\chi,t,\varphi) \in Q'_{p+1,3}$ we have $(\psi_1,\varphi_1,\ldots,\psi_{p+1},\varphi_{p+1}) \in R_{p+1}$, $\chi \in S(k[\psi_1 : \varphi_1,\ldots,\psi_p : \varphi_p])$, $t \in E(k[\psi_1 : \varphi_1,\ldots,\psi_p : \varphi_p])$, $\varphi \in E_s(k[\psi_1 : \varphi_1,\ldots,\psi_p : \varphi_p])$.

It follows that $\psi_1,\ldots,\psi_{p+1} \in \mathcal{V}$, $\psi_i \neq \psi_j$ for $i \neq j$, $\varphi_1,\ldots,\varphi_{p+1} \in E$, $H[\psi_1 : \varphi_1,\ldots,\psi_{p+1} : \varphi_{p+1}]$.

Moreover

$$\xi = \delta_p((\psi_1, \varphi_1, \ldots, \psi_{p+1}, \varphi_{p+1}), \chi, t, \varphi) = \begin{pmatrix} \delta_{1,p}((\psi_1, \varphi_1, \ldots, \psi_{p+1}, \varphi_{p+1}), \chi, t, \varphi), \\ \delta_{2,p}((\psi_1, \varphi_1, \ldots, \psi_{p+1}, \varphi_{p+1}), \chi, t, \varphi), \\ \delta_{3,p}((\psi_1, \varphi_1, \ldots, \psi_{p+1}, \varphi_{p+1}), \chi, t, \varphi) \end{pmatrix} =$$

$$= \begin{pmatrix} \gamma[\psi_1 : \varphi_1, \ldots, \psi_p : \varphi_p, \to (\chi, \forall(\{\} (\psi_{p+1} : \varphi_{p+1}, \in (\psi_{p+1}, \varphi))))], \\ \gamma[\psi_1 : \varphi_1, \ldots, \psi_p : \varphi_p, \to (\chi, \in (t, \varphi_{p+1}))], \\ \gamma[\psi_1 : \varphi_1, \ldots, \psi_p : \varphi_p, \to (\chi, \in (t, \varphi))] \end{pmatrix}$$

and so $\xi \in R_{10.10}$.

□

Then let $R_{10.10} \in \mathcal{R}$.

Lemma 10.12. *Let $x_1 \in \mathcal{V}$, $\varphi_1 \in E$ and assume $H[x_1 : \varphi_1]$. Define $k = k[x_1 : \varphi_1]$. Let $\psi \in S(k)$ and $\varphi \in S(k) \cap S(\epsilon)$. Under these assumptions we have*

- $\to (\psi, \varphi) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \to (\psi, \varphi)] \in S(\epsilon)$,
- $\exists (\{\}(x_1 : \varphi_1, \psi)) \in S(\epsilon)$,
- $\to (\exists (\{\}(x_1 : \varphi_1, \psi)), \varphi) \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \to (\psi, \varphi)])$ then $\#(\to (\exists (\{\}(x_1 : \varphi_1, \psi)), \varphi))$.

Proof. Suppose $\#(\gamma[x_1 : \varphi_1, (\to)(\psi, \varphi)])$. We have

$$P_\forall(\{\#(k, \to (\psi, \varphi), \sigma)|\ \sigma \in \Xi(k)\}) \ ,$$

$$P_\forall(\{P_\to(\#(k, \psi, \sigma), \#(k, \varphi, \sigma))|\ \sigma \in \Xi(k)\}) \ .$$

In turn $\#(\to (\exists (\{\}(x_1 : \varphi_1, \psi)), \varphi))$ can be rewritten as

$$\#(\epsilon, \to (\exists (\{\}(x_1 : \varphi_1, \psi)), \varphi), \epsilon) \ ,$$

$$P_\to(\#(\epsilon, \exists (\{\}(x_1 : \varphi_1, \psi)), \epsilon), \#(\epsilon, \varphi, \epsilon)) \ ,$$

$$P_\to(\#(\exists (\{\}(x_1 : \varphi_1, \psi))), \#(\varphi)) \ ,$$

$$P_\to(P_\exists(\{\#(k, \psi, \sigma)|\ \sigma \in \Xi(k)\}), \#(\varphi)) \ .$$

In order to prove the last statement, we suppose there exists $\sigma \in \Xi(k)$ such that $\#(k, \psi, \sigma)$. This implies $\#(k, \varphi, \sigma)$, but we need to show that $\#(\varphi)$ holds.

To perform this step we can use lemma 8.14. In fact there exists a positive integer q such that $\epsilon, k \in K(q)$, $\varphi \in E(q, \epsilon) \cap E(q, k)$. Moreover $\epsilon \sqsubseteq k$, $\epsilon \in \Xi(\epsilon)$, $\sigma \in \Xi(k)$, $\epsilon \sqsubseteq \sigma$ so by lemma 8.14 $\#(k, \varphi, \sigma) = \#(\epsilon, \varphi, \epsilon) = \#(\varphi)$. □

We can create a set $R_{10.12}$ as the set of all pairs

$$\left(\begin{array}{l} \gamma[x_1 : \varphi_1, \rightarrow (\psi, \varphi)], \\ \rightarrow (\exists (\{\}(x_1 : \varphi_1, \psi)), \varphi) \end{array} \right)$$

such that $x_1 \in \mathcal{V}$, $\varphi_1 \in E$, $H[x_1 : \varphi_1]$, $\psi \in S(k[x_1 : \varphi_1])$ and $\varphi \in S(k[x_1 : \varphi_1]) \cap S(\epsilon)$.

Lemma 10.12 shows us that this set (which is a potential 1-ary rule) is 'sound'. In order to use $R_{10.12}$ as a rule in our system we also need to show that $R_{10.12}$ is r.e..

Lemma 10.13. *$R_{10.12}$ is r.e. .*

Proof. Given $(x_1, \varphi_1) \in R_1$ all of the following sets are r.e.:

- $S(k[x_1 : \varphi_1])$,
- $S(k[x_1 : \varphi_1]) \cap S(\epsilon)$,
- $\{(x_1, \varphi_1)\} \times S(k[x_1 : \varphi_1]) \times (S(k[x_1 : \varphi_1]) \cap S(\epsilon))$.

Let's use this temporary definition

$$Q'_{1,2} = \bigcup_{(x_1,\varphi_1)\in R_1} \{(x_1, \varphi_1)\} \times S(k[x_1 : \varphi_1]) \times (S(k[x_1 : \varphi_1]) \cap S(\epsilon)) \ .$$

With this $Q'_{1,2}$ is a r.e. subset of $(\Sigma^*)^2 \times \Sigma^* \times \Sigma^*$.

We now define two functions $\delta_{1,1},\ \delta_{2,1}$ over $(\Sigma^*)^2 \times \Sigma^* \times \Sigma^*$ as follows:

$$\delta_{1,1}((\psi_1, \varphi_1), \psi, \varphi) = \gamma[\psi_1 : \varphi_1, \rightarrow (\psi, \varphi)] \ ,$$

$$\delta_{2,1}((\psi_1, \varphi_1), \psi, \varphi) = \rightarrow (\exists (\{\}(\psi_1 : \varphi_1, \psi)), \varphi) \ .$$

The two functions we have defined are both computable functions from $(\Sigma^*)^2 \times \Sigma^* \times \Sigma^*$ to Σ^*. If we define a function δ_1 over $(\Sigma^*)^2 \times \Sigma^* \times \Sigma^*$ as follows

$$\delta_1((\psi_1, \varphi_1), \psi, \varphi) = \left(\begin{array}{l} \delta_{1,1}((\psi_1, \varphi_1), \psi, \varphi), \\ \delta_{2,1}((\psi_1, \varphi_1), \psi, \varphi) \end{array} \right) \ ,$$

then δ_1 is a computable function from $(\Sigma^*)^2 \times \Sigma^* \times \Sigma^*$ to $(\Sigma^*)^2$, therefore the set

$$D_1 = \{\delta_1((\psi_1, \varphi_1), \psi, \varphi) | ((\psi_1, \varphi_1), \psi, \varphi) \in Q'_{1,2}\}$$

is a r.e. subset of $(\Sigma^*)^2$, and D_1 is equal to our set $R_{10.12}$ which so is r.e. itself. □

Then let $R_{10.12} \in \mathcal{R}$.

Lemma 10.14. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi_1, \psi_2 \in S(k)$.*

Under these assumptions we have $\to(\varphi,\psi_1), \to(\varphi,\psi_2), \to(\varphi,\wedge(\psi_1,\psi_2)) \in S(k)$*.* *Moreover, if*

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\varphi,\psi_1)]),\ \#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\varphi,\psi_2)])$$

then

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\varphi,\wedge(\psi_1,\psi_2))])\ .$$

Proof. We need to show

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\varphi,\wedge(\psi_1,\psi_2))])\ ,$$

that is

$$\begin{aligned}
&P_\forall(\{\#(k,\to(\varphi,\wedge(\psi_1,\psi_2)),\sigma)|\ \sigma\in\Xi(k)\})\ ,\\
&P_\forall(\{P_\to(\#(k,\varphi,\sigma),\#(k,\wedge(\psi_1,\psi_2),\sigma))|\ \sigma\in\Xi(k)\})\ ,\\
&P_\forall(\{P_\to(\#(k,\varphi,\sigma),P_\wedge(\#(k,\psi_1,\sigma),\#(k,\psi_2,\sigma)))|\ \sigma\in\Xi(k)\})\ . \qquad (10.1)
\end{aligned}$$

But we have

$$\begin{aligned}
&\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\varphi,\psi_1)])\ ,\\
&P_\forall(\{\#(k,\to(\varphi,\psi_1),\sigma)|\ \sigma\in\Xi(k)\})\ ,\\
&P_\forall(\{P_\to(\#(k,\varphi,\sigma),\#(k,\psi_1,\sigma))|\ \sigma\in\Xi(k)\})\ .
\end{aligned}$$

And we have

$$\begin{aligned}
&\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\varphi,\psi_2)])\ ,\\
&P_\forall(\{\#(k,\to(\varphi,\psi_2),\sigma)|\ \sigma\in\Xi(k)\})\ ,\\
&P_\forall(\{P_\to(\#(k,\varphi,\sigma),\#(k,\psi_2,\sigma))|\ \sigma\in\Xi(k)\})\ .
\end{aligned}$$

So for each $\sigma\in\Xi(k)$ if $\#(k,\varphi,\sigma)$ holds true then both $\#(k,\psi_1,\sigma)$ and $\#(k,\psi_2,\sigma)$ hold. This implies 10.1 holds true in turn. □

We can create a set $R_{10.14}$ which is the set of all 3-tuples

$$\left(\begin{array}{l}
\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\varphi,\psi_1)],\\
\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\varphi,\psi_2)],\\
\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\varphi,\wedge(\psi_1,\psi_2))]
\end{array}\right)$$

such that

- m is a positive integer, $x_1,\ldots,x_m\in\mathcal{V}$, $x_i\neq x_j$ for $i\neq j$, $\varphi_1,\ldots,\varphi_m\in E$, $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$,
- $\varphi,\psi_1,\psi_2\in S(k[x_1:\varphi_1,\ldots,x_m:\varphi_m])$.

Lemma 10.14 shows us that this set (which is a potential 2-ary rule) is 'sound'. In order to use $R_{10.14}$ as a rule in our system we also need to show that $R_{10.14}$ is r.e..

Lemma 10.15. *$R_{10.14}$ is r.e. .*

Proof. Given a positive integer m and $(x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m$ we can notice the following:

- $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \in K$;
- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ is r.e.;
- $\{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^3$ is r.e..

Let's define

$$Q_{m,3} = \bigcup_{(x_1,\varphi_1,\ldots,x_m,\varphi_m)\in R_m} \{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^3 \ .$$

Clearly $Q_{m,3} \subseteq (\Sigma^*)^{2m} \times (\Sigma^*)^3$ is also r.e..

We now define three functions $\delta_{1,m},\ \delta_{2,m},\ \delta_{3,m}$ over $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ as follows. Given $((\theta_1, \varphi_1, \ldots, \theta_m, \varphi_m), (\varphi, \psi_1, \psi_2)) \in (\Sigma^*)^{2m} \times (\Sigma^*)^3$

$$\delta_{1,m}((\theta_1, \varphi_1, \ldots, \theta_m, \varphi_m), (\varphi, \psi_1, \psi_2)) = \gamma[\theta_1 : \varphi_1, \ldots, \theta_m : \varphi_m, \rightarrow (\varphi, \psi_1)] \ .$$

$$\delta_{2,m}((\theta_1, \varphi_1, \ldots, \theta_m, \varphi_m), (\varphi, \psi_1, \psi_2)) = \gamma[\theta_1 : \varphi_1, \ldots, \theta_m : \varphi_m, \rightarrow (\varphi, \psi_2)] \ .$$

$$\delta_{3,m}((\theta_1, \varphi_1, \ldots, \theta_m, \varphi_m), (\varphi, \psi_1, \psi_2)) = \gamma[\theta_1 : \varphi_1, \ldots, \theta_m : \varphi_m, \rightarrow (\varphi, \wedge(\psi_1, \psi_2))] \ .$$

All of the three functions we have defined are computable functions from $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ to Σ^*. If we define a function δ_m over $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ as follows:

$$\delta_m((\theta_1, \varphi_1, \ldots, \theta_m, \varphi_m), (\varphi, \psi_1, \psi_2)) = \begin{pmatrix} \delta_{1,m}((\theta_1, \varphi_1, \ldots, \theta_m, \varphi_m), (\varphi, \psi_1, \psi_2)), \\ \delta_{2,m}((\theta_1, \varphi_1, \ldots, \theta_m, \varphi_m), (\varphi, \psi_1, \psi_2)), \\ \delta_{3,m}((\theta_1, \varphi_1, \ldots, \theta_m, \varphi_m), (\varphi, \psi_1, \psi_2)) \end{pmatrix}$$

then δ_m is a computable function from $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ to $(\Sigma^*)^3$, therefore the set

$$D_m = \{\delta_m((\theta_1, \varphi_1, \ldots, \theta_m, \varphi_m), (\varphi, \psi_1, \psi_2)) |\ ((\theta_1, \varphi_1, \ldots, \theta_m, \varphi_m), (\varphi, \psi_1, \psi_2)) \in Q_{m,3}\}$$

is a r.e. subset of $(\Sigma^*)^3$.

If we now consider the set $\bigcup_{m\geqslant 1} D_m$ then this is a r.e. subset of $(\Sigma^*)^3$ and actually this set is equal to our set $R_{10.14}$ which so is r.e. itself. □

Then let $R_{10.14} \in \mathcal{R}$.

Lemma 10.16. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi \in S(k)$.*

Under these assumptions we have

- $\rightarrow(\varphi, \wedge(\psi, \neg(\psi))), \neg(\varphi) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\varphi, \wedge(\psi, \neg(\psi)))] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg(\varphi)] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\varphi, \wedge(\psi, \neg(\psi)))])$ then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg(\varphi)])$.

Proof. We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\varphi, \wedge(\psi, \neg(\psi)))])$ as

$$P_\forall(\{\#(k, \rightarrow(\varphi, \wedge(\psi, \neg(\psi))), \sigma) |\ \sigma \in \Xi(k)\}) ,$$

$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), \#(k, \wedge(\psi, \neg(\psi)), \sigma)) \mid \sigma \in \Xi(k)\}) ,$$

$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), P_\wedge(\#(k, \psi, \sigma), \#(k, \neg(\psi), \sigma))) \mid \sigma \in \Xi(k)\}) ,$$

$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), P_\wedge(\#(k, \psi, \sigma), P_\neg(\#(k, \psi, \sigma)))) \mid \sigma \in \Xi(k)\}) .$$

This can be expressed as 'for each $\sigma \in \Xi(k)$ either $\#(k, \varphi, \sigma)$ is false or both $\#(k, \psi, \sigma)$ and ($\#(k, \psi, \sigma)$ is false) are true'.

Since $\#(k, \psi, \sigma)$ cannot be both true and false at the same time we have that 'for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma)$ is false'. This is formally expressed as

$$P_\forall(\{P_\neg(\#(k, \varphi, \sigma)) |\ \sigma \in \Xi(k)\}) ,$$

$$P_\forall(\{\#(k, \neg(\varphi), \sigma) |\ \sigma \in \Xi(k)\}) ,$$

which we can finally rewrite as $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg(\varphi)])$. □

We can create a set $R_{10.16}$ which is the set of all pairs

$$(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\varphi, \wedge(\psi, \neg(\psi)))], \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg(\varphi)])$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Lemma 10.16 shows us that this set (which is a potential 1-ary rule) is 'sound'. In order to use $R_{10.16}$ as a rule in our system we also need to show that $R_{10.16}$ is r.e..

Lemma 10.17. *$R_{10.16}$ is r.e..*

Proof. Given a positive integer m and $(x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m$ we can notice the following:

- $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \in K$;
- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ is r.e.;
- $\{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^2$ is r.e..

Let's define

$$Q_{m,2} = \bigcup_{(x_1,\varphi_1,\ldots,x_m,\varphi_m)\in R_m} \{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^2 \ .$$

Clearly $Q_{m,2} \subseteq (\Sigma^*)^{2m} \times (\Sigma^*)^2$ is also r.e..

We now define two functions $\delta_{1,m}$, $\delta_{2,m}$ over $(\Sigma^*)^{2m} \times (\Sigma^*)^2$ as follows. Given $((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) \in (\Sigma^*)^{2m} \times (\Sigma^*)^2$

$$\delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \rightarrow (\varphi, \wedge (\psi, \neg(\psi)))] \ .$$

$$\delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \neg(\varphi)] \ .$$

All of the two functions we have defined are computable functions from $(\Sigma^*)^{2m} \times (\Sigma^*)^2$ to Σ^*. If we define a function δ_m over $(\Sigma^*)^{2m} \times (\Sigma^*)^2$ as follows:

$$\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) = \begin{pmatrix} \delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)), \\ \delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)), \end{pmatrix}$$

then δ_m is a computable function from $(\Sigma^*)^{2m} \times (\Sigma^*)^2$ to $(\Sigma^*)^2$, therefore the set

$$D_m = \{\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) |\ ((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) \in Q_{m,2}\}$$

is a r.e. subset of $(\Sigma^*)^2$.

If we now consider the set $\bigcup_{m \geqslant 1} D_m$ then this is a r.e. subset of $(\Sigma^*)^2$ and actually this set is equal to our set $R_{10.16}$ which so is r.e. itself.

□

Then let $R_{10.16} \in \mathcal{R}$.

Lemma 10.18. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi \in S(k)$.*

Under these assumptions we have

- $\neg (\wedge(\varphi, \psi)), \rightarrow (\varphi, \neg(\psi)) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg (\wedge(\varphi, \psi))] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \neg(\psi))] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg(\wedge(\varphi, \psi))])$ *then* $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\varphi, \neg(\psi))])$.

Proof. We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg(\wedge(\varphi, \psi))])$ as

$$P_\forall(\{\#(k, \neg(\wedge(\varphi, \psi)), \sigma)|\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\neg(\#(k, \wedge(\varphi, \psi), \sigma))|\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\neg(P_\wedge(\#(k, \varphi, \sigma), \#(k, \psi, \sigma)))|\ \sigma \in \Xi(k)\})\ .$$

We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\varphi, \neg(\psi))])$ as

$$P_\forall(\{\#(k, \rightarrow(\varphi, \neg(\psi)), \sigma)|\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), \#(k, \neg(\psi), \sigma))|\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), P_\neg(\#(k, \psi, \sigma)))|\ \sigma \in \Xi(k)\})\ .$$

Thus if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg(\wedge(\varphi, \psi))])$ we have that 'for each $\sigma \in \Xi(k)$ it is false that $\#(k, \varphi, \sigma)$ and $\#(k, \psi, \sigma)$ are both true'.

In other words for each $\sigma \in \Xi(k)$ ($\#(k, \varphi, \sigma)$ is false) or ($\#(k, \psi, \sigma)$ is false).

In other words for each $\sigma \in \Xi(k)$ $P_\rightarrow(\#(k, \varphi, \sigma), P_\neg(\#(k, \psi, \sigma)))$.

The last condition clearly implies $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\varphi, \neg(\psi))])$. $\square$

We can create a set $R_{10.18}$ which is the set of all pairs

$$(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg(\wedge(\varphi, \psi))], \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\varphi, \neg(\psi))])$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Lemma 10.18 shows us that this set (which is a potential 1-ary rule) is 'sound'. In order to use $R_{10.18}$ as a rule in our system we also need to show that $R_{10.18}$ is r.e..

Lemma 10.19. $R_{10.18}$ *is r.e..*

Proof. Given a positive integer m and $(x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m$ we can notice the following:

- $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \in K$;

- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ is r.e.;
- $\{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^2$ is r.e..

Let's define

$$Q_{m,2} = \bigcup_{(x_1,\varphi_1,\ldots,x_m,\varphi_m)\in R_m} \{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^2 .$$

Clearly $Q_{m,2} \subseteq (\Sigma^*)^{2m} \times (\Sigma^*)^2$ is also r.e..

We now define two functions $\delta_{1,m}$, $\delta_{2,m}$ over $(\Sigma^*)^{2m} \times (\Sigma^*)^2$ as follows. Given $((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) \in (\Sigma^*)^{2m} \times (\Sigma^*)^2$

$$\delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \neg (\wedge(\varphi, \psi))] .$$

$$\delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \rightarrow (\varphi, \neg(\psi))] .$$

All of the two functions we have defined are computable functions from $(\Sigma^*)^{2m} \times (\Sigma^*)^2$ to Σ^*. If we define a function δ_m over $(\Sigma^*)^{2m} \times (\Sigma^*)^2$ as follows:

$$\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) = \begin{pmatrix} \delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)), \\ \delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)), \end{pmatrix}$$

then δ_m is a computable function from $(\Sigma^*)^{2m} \times (\Sigma^*)^2$ to $(\Sigma^*)^2$, therefore the set

$$D_m = \{\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) |\ ((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi)) \in Q_{m,2}\}$$

is a r.e. subset of $(\Sigma^*)^2$.

If we now consider the set $\bigcup_{m\geqslant 1} D_m$ then this is a r.e. subset of $(\Sigma^*)^2$ and actually this set is equal to our set $R_{10.18}$ which so is r.e. itself.

□

Then let $R_{10.18} \in \mathcal{R}$.

Lemma 10.20. *Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ also holds, we define $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\chi \in S(h)$, $\varphi \in S(k)$.

Under these assumptions we have

- $\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)) \in S(h)$,
- $\neg (\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))) \in S(h)$,
- $\rightarrow (\chi, \neg (\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)))) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \neg (\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))))] \in S(\epsilon)$,
- $\neg(\varphi) \in S(k)$,
- $\exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi))) \in S(h)$,
- $\rightarrow (\chi, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi)))) \in S(h)$,

- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi))))] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \neg (\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))))])$ *then*

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi))))]) \ .$$

Proof. We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \neg (\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))))])$ as

$$P_\forall(\{\#(h, \rightarrow (\chi, \neg (\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)))) , \rho)|\ \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow (\#(h, \chi, \rho), \#(h, \neg (\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))) , \rho)) \mid \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow (\#(h, \chi, \rho), P_\neg (\#(h, \forall (\{\} (x_{m+1} : \varphi_{m+1}, \varphi)) , \rho))) \mid \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow (\#(h, \chi, \rho), P_\neg (P_\forall(\{\#(k, \varphi, \sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\}))) \mid \rho \in \Xi(h)\}) \ .$$

We can furtherly express this as

'for each $\rho \in \Xi(h)$ $P_\rightarrow (\#(h, \chi, \rho), P_\neg (P_\forall(\{\#(k, \varphi, \sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})))$',
'for each $\rho \in \Xi(h)$ if $\#(h, \chi, \rho)$ then it is false that $P_\forall(\{\#(k, \varphi, \sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$',
'for each $\rho \in \Xi(h)$ if $\#(h, \chi, \rho)$ then it is false that (for each $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ $\#(k, \varphi, \sigma)$ holds)',
'for each $\rho \in \Xi(h)$ if $\#(h, \chi, \rho)$ then (there exists $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $\#(k, \varphi, \sigma)$ is false)'.

We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi))))])$ as

$$P_\forall(\{\#(h, \rightarrow (\chi, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi)))) , \rho)|\ \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow (\#(h, \chi, \rho), \#(h, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi))), \rho)) \mid \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow (\#(h, \chi, \rho), P_\exists(\{\#(k, \neg(\varphi), \sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})) \mid \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow (\#(h, \chi, \rho), P_\exists(\{P_\neg(\#(k, \varphi, \sigma))|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})) \mid \rho \in \Xi(h)\}) \ .$$

This can be furtherly rewritten as

'for each $\rho \in \Xi(h)$ $P_\rightarrow (\#(h, \chi, \rho), P_\exists(\{P_\neg(\#(k, \varphi, \sigma))|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\}))$ ',
'for each $\rho \in \Xi(h)$ if $\#(h, \chi, \rho)$ then $P_\exists(\{P_\neg(\#(k, \varphi, \sigma))|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$ ',
'for each $\rho \in \Xi(h)$ if $\#(h, \chi, \rho)$ then (there exists $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $\#(k, \varphi, \sigma)$ is false)'.

The last condition is clearly ensured by our hypothesis. □

We can create a set $R_{10.20}$ which is the set of all pairs

$$\left(\begin{array}{l} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \neg (\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))))], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi))))] \end{array} \right)$$

such that

- m is a positive integer, $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_{m+1} \in E$, $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$;
- if we define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$ and $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ then $\chi \in S(h)$, $\varphi \in S(k)$.

Lemma 10.20 shows us that this set (which is a potential 1-ary rule) is 'sound'. In order to use $R_{10.20}$ as a rule in our system we also need to show that $R_{10.20}$ is r.e..

Lemma 10.21. *$R_{10.20}$ is r.e..*

Proof. Given a positive integer m and $(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1}) \in R_{m+1}$ all of the following sets are r.e.:

- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$,
- $S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])$.

Therefore the following set is also r.e.:

$$\{(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1})\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \times S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]) \, .$$

Let's use this temporary definition

$$Q'_{m+1,2} = \bigcup_{(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1}) \in R_{m+1}} \{(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1})\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \times S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])).$$

With this $Q'_{m+1,2}$ is a r.e. subset of $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$.

We now define two functions $\delta_{1,m}$, $\delta_{2,m}$ over $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$ as follows. Given $((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) \in (\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$

$$\delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \to (\chi, \neg (\forall(\{\}(\psi_{m+1} : \varphi_{m+1}, \varphi))))] \, .$$

$$\delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \to (\chi, \exists(\{\}(\psi_{m+1} : \varphi_{m+1}, \neg(\varphi))))] \, .$$

All of the two functions we have defined are computable functions from $(\Sigma^*)^{2(m+1)} \times$

$\Sigma^* \times \Sigma^*$ to Σ^*. If we define a function δ_m over $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$ as follows:

$$\delta_m((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) = \begin{pmatrix} \delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi), \\ \delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) \end{pmatrix}$$

then δ_m is a computable function from $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$ to $(\Sigma^*)^2$, therefore the set

$$D_m = \{\delta_m((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) | ((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) \in Q'_{m+1,2}\}$$

is a r.e. subset of $(\Sigma^*)^2$.

If we now consider the set $\bigcup_{m \geqslant 1} D_m$ then this is a r.e. subset of $(\Sigma^*)^2$ and actually this set is equal to our rule $R_{10.20}$ which so is r.e. itself.

If $\xi \in R_{10.20}$ then there exist a positive integer m, $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_{m+1} \in E$ such that $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$; if we define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$ and $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ there also exist $\chi \in S(h)$, $\varphi \in S(k)$, $\xi_1, \xi_2 \in \Sigma^*$ such that

- $\xi = (\xi_1, \xi_2)$,
- $\xi_1 = \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \neg(\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))))]$,
- $\xi_2 = \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi)))))]$.

This means that $(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1}) \in R_{m+1}$, $\chi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$, $\varphi \in S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])$, so $((x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1}), \chi, \varphi) \in Q'_{m+1,2}$.

Moreover

- $\xi_1 = \delta_{1,m}((x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1}), \chi, \varphi)$,
- $\xi_2 = \delta_{2,m}((x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1}), \chi, \varphi)$.

i.e. $\xi = \delta_m((x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1}), \chi, \varphi) \in D_m$.

Conversely if there exists $p \geqslant 1$ such that $\xi \in D_p$ then there exists $((\psi_1, \varphi_1, \ldots, \psi_{p+1}, \varphi_{p+1}), \chi, \varphi) \in Q'_{p+1,2}$ such that
$\xi = \delta_p((\psi_1, \varphi_1, \ldots, \psi_{p+1}, \varphi_{p+1}), \chi, \varphi)$.

Since $((\psi_1, \varphi_1, \ldots, \psi_{p+1}, \varphi_{p+1}), \chi, \varphi) \in Q'_{p+1,2}$ we have $(\psi_1, \varphi_1, \ldots, \psi_{p+1}, \varphi_{p+1}) \in R_{p+1}$, $\chi \in S(k[\psi_1 : \varphi_1, \ldots, \psi_p : \varphi_p])$, $\varphi \in S(k[\psi_1 : \varphi_1, \ldots, \psi_{p+1} : \varphi_{p+1}])$.

It follows that $\psi_1, \ldots, \psi_{p+1} \in \mathcal{V}$, $\psi_i \neq \psi_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_{p+1} \in E$, $H[\psi_1 : \varphi_1, \ldots, \psi_{p+1} : \varphi_{p+1}]$.

Moreover

$$\xi = \delta_p((\psi_1, \varphi_1, \ldots, \psi_{p+1}, \varphi_{p+1}), \chi, \varphi) = \begin{pmatrix} \delta_{1,p}((\psi_1, \varphi_1, \ldots, \psi_{p+1}, \varphi_{p+1}), \chi, \varphi), \\ \delta_{2,p}((\psi_1, \varphi_1, \ldots, \psi_{p+1}, \varphi_{p+1}), \chi, \varphi) \end{pmatrix} =$$

$$= \begin{pmatrix} \gamma[\psi_1 : \varphi_1, \ldots, \psi_p : \varphi_p, \rightarrow (\chi, \neg(\forall(\{\}(\psi_{p+1} : \varphi_{p+1}, \varphi))))], \\ \gamma[\psi_1 : \varphi_1, \ldots, \psi_p : \varphi_p, \rightarrow (\chi, \exists(\{\}(\psi_{p+1} : \varphi_{p+1}, \neg(\varphi)))))] \end{pmatrix}$$

and so $\xi \in R_{10.20}$.

□

Then let $R_{10.20} \in \mathcal{R}$.

Lemma 10.22. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi, \chi \in S(k)$.*

Under these assumptions we have

- $\rightarrow (\wedge(\varphi, \psi), \chi), \rightarrow (\varphi, \rightarrow (\psi, \chi)) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \chi)] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \rightarrow (\psi, \chi))] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \chi)])$ then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \rightarrow (\psi, \chi))])$.

Proof. We assume $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \chi)])$ which can be rewritten

$$P_\forall(\{\#(k, \rightarrow (\wedge(\varphi, \psi), \chi), \sigma) |\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_\rightarrow(\#(k, \wedge(\varphi, \psi), \sigma), \#(k, \chi, \sigma)) |\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_\rightarrow(P_\wedge(\#(k, \varphi, \sigma), \#(k, \psi, \sigma)), \#(k, \chi, \sigma)) |\ \sigma \in \Xi(k)\}) \ .$$

Of course we now try to show $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \rightarrow (\psi, \chi))])$ which in turn can be rewritten

$$P_\forall(\{\#(k, \rightarrow (\varphi, \rightarrow (\psi, \chi)), \sigma) |\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), \#(k, \rightarrow (\psi, \chi), \sigma)) |\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), P_\rightarrow(\#(k, \psi, \sigma), \#(k, \chi, \sigma))) |\ \sigma \in \Xi(k)\}) \ .$$

Let $\sigma \in \Xi(k)$, suppose $\#(k, \varphi, \sigma)$ and $\#(k, \psi, \sigma)$, then we have $\#(k, \chi, \sigma)$ and this completes the proof. □

We can create a set $R_{10.22}$ which is the set of all pairs

$$(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \chi)], \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \rightarrow (\psi, \chi))])$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,

- $\varphi, \psi, \chi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Lemma 10.22 shows us that this set (which is a potential 1-ary rule) is 'sound'. In order to use $R_{10.22}$ as a rule in our system we also need to show that $R_{10.22}$ is r.e..

Lemma 10.23. *$R_{10.22}$ is r.e..*

Proof. Given a positive integer m and $(x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m$ we can notice the following:

- $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \in K$;
- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ is r.e.;
- $\{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^3$ is r.e..

Let's define

$$Q_{m,3} = \bigcup_{(x_1,\varphi_1,\ldots,x_m,\varphi_m) \in R_m} \{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^3 .$$

Clearly $Q_{m,3} \subseteq (\Sigma^*)^{2m} \times (\Sigma^*)^3$ is also r.e..

We now define two functions $\delta_{1,m}$, $\delta_{2,m}$ over $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ as follows. Given $((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) \in (\Sigma^*)^{2m} \times (\Sigma^*)^3$

$$\delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \to (\wedge(\varphi, \psi), \chi)] .$$

$$\delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \to (\varphi, \to (\psi, \chi))] .$$

All of the two functions we have defined are computable functions from $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ to Σ^*. If we define a function δ_m over $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ as follows:

$$\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \begin{pmatrix} \delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)), \\ \delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)), \end{pmatrix}$$

then δ_m is a computable function from $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ to $(\Sigma^*)^2$, therefore the set

$$D_m = \{\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) | \; ((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) \in Q_{m,3}\}$$

is a r.e. subset of $(\Sigma^*)^2$.

If we now consider the set $\bigcup_{m \geqslant 1} D_m$ then this is a r.e. subset of $(\Sigma^*)^2$ and actually this set is equal to our set $R_{10.22}$ which so is r.e. itself. □

Then let $R_{10.22} \in \mathcal{R}$.

Lemma 10.24. *Let $\varphi, \psi, \chi \in S(\epsilon)$. We have*

- $\to (\varphi, \to (\psi, \chi)) \in S(\epsilon)$,

- $\to(\wedge(\varphi,\psi),\chi) \in S(\epsilon)$.

Moreover if $\#(\to(\varphi, \to(\psi,\chi)))$ *then* $\#(\to(\wedge(\varphi,\psi),\chi))$.

Proof. Suppose $\#(\to(\varphi, \to(\psi,\chi)))$ holds. It can be rewritten

$$P_{\to}(\#(\varphi), \#(\to(\psi,\chi))) \ ,$$

$$P_{\to}(\#(\varphi), P_{\to}(\#(\psi), \#(\chi))) \ .$$

In turn, $\#(\to(\wedge(\varphi,\psi),\chi))$ can be rewritten

$$P_{\to}(\#(\wedge(\varphi,\psi)), \#(\chi)) \ ,$$

$$P_{\to}(P_{\wedge}(\#(\varphi), \#(\psi)), \#(\chi)) \ .$$

Suppose $\#(\varphi)$ and $\#(\psi)$ both hold, we need to show that $\#(\chi)$ holds. This is granted by

$$P_{\to}(\#(\varphi), P_{\to}(\#(\psi), \#(\chi))) \ .$$

□

We can create a set $R_{10.24}$ which is the set of all pairs

$$\begin{pmatrix} \to(\varphi, \to(\psi,\chi)), \\ \to(\wedge(\varphi,\psi),\chi) \end{pmatrix}$$

such that $\varphi, \psi, \chi \in S(\epsilon)$.

Lemma 10.24 shows us that this set (which is a potential 1-ary rule) is 'sound'. In order to use $R_{10.24}$ as a rule in our system we also need to show that $R_{10.24}$ is r.e..

Lemma 10.25. $R_{10.24}$ *is r.e.*

Proof. Clearly $S(\epsilon)$ is .r.e. and so is $S(\epsilon)^3$.

Let's define two functions $\delta_{1,1}$, $\delta_{2,1}$ over $(\Sigma^*)^3$ as follows:

$$\delta_{1,1}(\varphi,\psi,\chi) = \to(\varphi, \to(\psi,\chi)) \ ,$$

$$\delta_{2,1}(\varphi,\psi,\chi) = \to(\wedge(\varphi,\psi),\chi) \ .$$

The two functions we have defined are both computable functions from $(\Sigma^*)^3$ to Σ^*. If we define a function δ_1 over $(\Sigma^*)^3$ as follows

$$\delta_1(\varphi, \psi, \chi) = \begin{pmatrix} \delta_{1,1}(\varphi, \psi, \chi), \\ \delta_{2,1}(\varphi, \psi, \chi) \end{pmatrix},$$

then δ_1 is a computable function from $(\Sigma^*)^3$ to $(\Sigma^*)^2$, therefore the set

$$D_1 = \{\delta_1(\varphi, \psi, \chi) | (\varphi, \psi, \chi) \in S(\epsilon)^3\}$$

is a r.e. subset of $(\Sigma^*)^2$, and D_1 is equal to our set $R_{10.24}$ which so is r.e. itself. □

Then let $R_{10.24} \in \mathcal{R}$.

11. Example of a proof

As an example of proof, we want to prove a form of the Bocardo syllogism. In Ferreirós' referenced paper ([4]), on paragraph 3.1, the syllogism is expressed as follows:

Some A are not B. All C are B. Therefore, some A are not C.

Suppose A, B and C represent sets, the statement we actually want to prove is the following:

If ((there exists $x \in A$ such that $x \notin B$) and (for each $y \in C$ $y \in B$)) then (there exists $z \in A$ such that $z \notin C$).

In order to formalize this, we will use a language $(\mathcal{V}, \mathcal{F}, \mathcal{C}, \#, \{D_1, \ldots, D_p\}, q_{max})$ which must be as follows

$$\mathcal{V} = \{x, y, z\},$$

$$\mathcal{F} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in, =\},$$

$$\mathcal{C} = \{A, B, C\},$$

where A, B, C are constants each representing a set.

Moreover, we do not need the additional sets $\{D_1, \ldots, D_p\}$ so we can set $p = 0$ and we also set a conventional value of 1 for q_{max}.

At this point we suppose we can formalize the statement as

$$\rightarrow \left(\wedge \begin{pmatrix} \exists \left(\{\} \left(x : A, \neg \left(\in (x, B) \right) \right) \right), \\ \forall \left(\{\} \left(y : C, \in (y, B) \right) \right) \end{pmatrix}, \exists \left(\{\} \left(z : A, \neg \left(\in (z, C) \right) \right) \right) \right). \qquad (Th_1)$$

We'll soon see a proof of this statement and of course if we can show a proof of a statement then we have also proved the statement is a sentence in our language.

First of all we need the following lemma, that can be applied to any language which includes all the symbols $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in$ in the set $\mathcal{F}$, and therefore it can also be applied to our current language.

Lemma 11.1. *Let m be a positive integer, $x_1, \dots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $A_1, \dots, A_m \in \mathcal{C}$ such that for each $i = 1 \dots m$ $\#(A_i)$ is a set. Let $D \in \mathcal{C}$ such that $\#(D)$ is a set. We have $H[x_1 : A_1, \dots, x_m : A_m]$. If we define $k = k[x_1 : A_1, \dots, x_m : A_m]$ then for each $i = 1 \dots m$*

- $\in (x_i, D) \ \in S(k)$,
- *for each $\sigma \in \Xi(k)$ $\#(k, \in (x_i, D), \sigma) = P_{\in}(\#(k, x_i, \sigma), \#(D))$.*

Proof. We first consider that $A_1 \in E(\epsilon)$ and $\#(A_1)$ is a set, so $A_1 \in E_s(\epsilon)$ and $H[x_1 : A_1]$. Let $k_1 = k[x_1 : A_1]$.

If $m > 1$ then for each $i = 1 \dots m-1$ we suppose $H[x_1 : A_1, \dots, x_i : A_i]$ holds and we define $k_i = k[x_1 : A_1, \dots, x_i : A_i]$.
Clearly by lemma 9.4 $A_{i+1} \in E(k_i)$ and for each $\rho \in \Xi(k_i)$ $\#(k_i, A_{i+1}, \rho) = \#(A_{i+1})$ is a set.
So $A_{i+1} \in E_s(k_i)$, which implies $H[x_1 : A_1, \dots, x_{i+1} : A_{i+1}]$ (and we can define $k_{i+1} = k[x_1 : A_1, \dots, x_{i+1} : A_{i+1}]$).

This proves that $H[x_1 : A_1, \dots, x_m : A_m]$ holds.

Let $i = 1 \dots m$. Using lemma 9.1 we obtain that $x_i \in E(k)$.

Moreover $D \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, D, \sigma) = \#(D)$ is a set. By lemma 9.2 we have

- $\in (x_i, D) \ \in S(k)$,
- for each $\sigma \in \Xi(k)$ $\#(k, \in (x_i, D), \sigma) = P_{\in}(\#(k, x_i, \sigma), \#(D))$.

$\square$

In order to provide a proof of statement Th_1 we'll make use of a deductive system which includes all the axioms and rules listed in section 10.

Using the former lemma we can derive $H[x : A]$ and we can define $h = k[x : A]$. Moreover $\in (x, B) \in S(h)$, so $\neg(\in (x, B)) \in S(h)$.

We also have $H[x : A, y : C]$ and we define $k_y = k[x : A, y : C]$.
We have $\in (y, B) \in S(k_y)$ and by lemma 8.21 $\forall(\{\}(y : C, \in (y, B))) \in S(h)$.

Thus $\wedge(\neg(\in (x, B)), \forall(\{\}(y : C, \in (y, B))))$ also belongs to $S(h)$.

Moreover $H[x : A, z : A]$ and we define $k_z = k[x : A, z : A]$.
We have $\in (z, C) \in S(k_z)$ and by lemma 8.21 $\forall(\{\}(z : A, \in (z, C))) \in S(h)$.

The first sentence in our proof is an instance of axiom $A_{10.2}$.

$$\gamma \left[x : A, \rightarrow \left(\wedge \left(\begin{array}{c} \wedge \left(\begin{array}{c} \neg(\in (x, B)), \\ \forall(\{\}(y : C, \in (y, B))) \end{array} \right), \\ \forall(\{\}(z : A, \in (z, C))) \end{array} \right), \wedge \left(\begin{array}{c} \neg(\in (x, B)), \\ \forall(\{\}(y : C, \in (y, B))) \end{array} \right) \right) \right]. \tag{11.1}$$

By $A_{10.2}$ we also obtain

$$\gamma\left[x : A, \rightarrow \left(\wedge \begin{pmatrix} \neg(\in(x, B)), \\ \forall(\{\}(y : C, \in(y, B))) \end{pmatrix}, \neg(\in(x, B))\right)\right]. \tag{11.2}$$

By 11.1, 11.2 and rule $R_{10.4}$

$$\gamma\left[x : A, \rightarrow \left(\wedge \begin{pmatrix} \wedge \begin{pmatrix} \neg(\in(x, B)), \\ \forall(\{\}(y : C, \in(y, B))) \end{pmatrix}, \\ \forall(\{\}(z : A, \in(z, C))) \end{pmatrix}, \neg(\in(x, B))\right)\right]. \tag{11.3}$$

Another instance of $A_{10.2}$ is the following

$$\gamma\left[x : A, \rightarrow \left(\wedge \begin{pmatrix} \wedge \begin{pmatrix} \neg(\in(x, B)), \\ \forall(\{\}(y : C, \in(y, B))) \end{pmatrix}, \\ \forall(\{\}(z : A, \in(z, C))) \end{pmatrix}, \forall(\{\}(z : A, \in(z, C)))\right)\right]. \tag{11.4}$$

By axiom $A_{10.6}$ we obtain

$$\gamma[x : A, \in(x, A)]. \tag{11.5}$$

By 11.5 and rule $R_{10.8}$ we also get

$$\gamma\left[x : A, \rightarrow \left(\wedge \begin{pmatrix} \wedge \begin{pmatrix} \neg(\in(x, B)), \\ \forall(\{\}(y : C, \in(y, B))) \end{pmatrix}, \\ \forall(\{\}(z : A, \in(z, C))) \end{pmatrix}, \in(x, A)\right)\right]. \tag{11.6}$$

Since $x \in E(h)$, $C \in E_s(h)$ etc. we can apply rule $R_{10.10}$ to 11.4 and 11.6 and obtain

$$\gamma\left[x : A, \rightarrow \left(\wedge \begin{pmatrix} \wedge \begin{pmatrix} \neg(\in(x, B)), \\ \forall(\{\}(y : C, \in(y, B))) \end{pmatrix}, \\ \forall(\{\}(z : A, \in(z, C))) \end{pmatrix}, \in(x, C)\right)\right]. \tag{11.7}$$

By axiom $A_{10.2}$

$$\gamma\left[x : A, \rightarrow \left(\wedge \begin{pmatrix} \neg(\in(x, B)), \\ \forall(\{\}(y : C, \in(y, B))) \end{pmatrix}, \forall(\{\}(y : C, \in(y, B)))\right)\right]. \tag{11.8}$$

By 11.1, 11.8 and rule $R_{10.4}$

$$\gamma\left[x : A, \rightarrow \left(\wedge \begin{pmatrix} \wedge \begin{pmatrix} \neg(\in(x, B)), \\ \forall(\{\}(y : C, \in(y, B))) \end{pmatrix}, \\ \forall(\{\}(z : A, \in(z, C))) \end{pmatrix}, \forall(\{\}(y : C, \in(y, B)))\right)\right]. \tag{11.9}$$

Since $x \in E(h)$, $B \in E_s(h)$ etc. we can apply rule $R_{10.10}$ to 11.7 and 11.9 and obtain

$$\gamma\left[x : A, \rightarrow \left(\wedge \begin{pmatrix} \wedge \begin{pmatrix} \neg(\in(x, B)), \\ \forall(\{\}(y : C, \in(y, B))) \end{pmatrix}, \\ \forall(\{\}(z : A, \in(z, C))) \end{pmatrix}, \in(x, B)\right)\right]. \tag{11.10}$$

By 11.10, 11.3 and $R_{10.14}$

$$\gamma\left[x:A,\rightarrow\left(\wedge\left(\begin{array}{c}\wedge\left(\begin{array}{c}\neg(\in(x,B)),\\ \forall(\{\}(y:C,\in(y,B)))\end{array}\right),\\ \forall(\{\}(z:A,\in(z,C)))\end{array}\right),\wedge\left(\begin{array}{c}\in(x,B),\\ \neg(\in(x,B))\end{array}\right)\right)\right]. \tag{11.11}$$

By $R_{10.16}$

$$\gamma\left[x:A,\neg\left(\wedge\left(\begin{array}{c}\wedge\left(\begin{array}{c}\neg(\in(x,B)),\\ \forall(\{\}(y:C,\in(y,B)))\end{array}\right),\\ \forall(\{\}(z:A,\in(z,C)))\end{array}\right)\right)\right]. \tag{11.12}$$

By $R_{10.18}$

$$\gamma\left[x:A,\rightarrow\left(\wedge\left(\begin{array}{c}\neg(\in(x,B)),\\ \forall(\{\}(y:C,\in(y,B)))\end{array}\right),\neg\left(\forall(\{\}(z:A,\in(z,C)))\right)\right)\right]. \tag{11.13}$$

By $R_{10.20}$

$$\gamma\left[x:A,\rightarrow\left(\wedge\left(\begin{array}{c}\neg(\in(x,B)),\\ \forall(\{\}(y:C,\in(y,B)))\end{array}\right),\exists(\{\}(z:A,\neg(\in(z,C))))\right)\right]. \tag{11.14}$$

Since $\exists(\{\}(z:A,\neg(\in(z,C)))) \in S(h)$ we can apply $R_{10.22}$ and obtain

$$\gamma\left[x:A,\rightarrow\left(\neg(\in(x,B)),\rightarrow\left(\begin{array}{c}\forall(\{\}(y:C,\in(y,B))),\\ \exists(\{\}(z:A,\neg(\in(z,C))))\end{array}\right)\right)\right]. \tag{11.15}$$

Using lemma 11.1 we obtain that $\in(y,B) \in S(k[y:C])$ and $\in(z,C) \in S(k[z:A])$.

By lemma 8.21 we obtain that $\forall(\{\}(y:C,\in(y,B))) \in S(\epsilon)$ and similarly $\exists(\{\}(z:A,\neg(\in(z,C)))) \in S(\epsilon)$.

We can apply rule $R_{10.12}$ to 11.15 and obtain

$$\rightarrow\left(\exists\left(\{\}(x:A,\neg(\in(x,B)))\right),\rightarrow\left(\begin{array}{c}\forall(\{\}(y:C,\in(y,B))),\\ \exists(\{\}(z:A,\neg(\in(z,C))))\end{array}\right)\right) \tag{11.16}$$

Finally, by $R_{10.24}$, we obtain

$$\rightarrow\left(\wedge\left(\begin{array}{c}\exists\left(\{\}(x:A,\neg(\in(x,B)))\right),\\ \forall(\{\}(y:C,\in(y,B)))\end{array}\right),\exists(\{\}(z:A,\neg(\in(z,C))))\right) \tag{11.17}$$

□

We have proved statement Th_1, this also means that Th_1 is a sentence in our language.

12. Extending our deductive system

In this section we are going to extend our deductive systems, in other words we are going to add axioms and rules to some of the deductive sytems $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ which we have built in section 10. We are going to do this in order to be able to show another example of proof in the next section. Our new deductive systems can refer to any language $\mathcal{L} = (\mathcal{V}, \mathcal{F}, \mathcal{C}, \#, \{D_1, \ldots, D_p\}, q_{max})$ such that all of these symbols: $N, *$ are in our set $\mathcal{C}$, all of these symbols x, y, z, u, v, w are in our set $\mathcal{V}$, all of these symbols: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in, =$ are in our set $\mathcal{F}$. For each of these operators f $A_f(x_1, \ldots, x_n)$ and $P_f(x_1, \ldots, x_n)$ are defined as specified at the beginning of section 3. Moreover we require $p \geqslant 1$ and $D_1 = \mathbb{N}$.

The constant symbol N represents the set of natural numbers $\mathbb{N}$, so that we have $\#(N) = \mathbb{N}$.

The symbol $*$ that stands for the product (or multiplication) operation in the domain $\mathbb{N}$ of natural numbers. Therefore $\#(*)$ is a function defined on $\mathbb{N} \times \mathbb{N}$ and for each $\alpha, \beta \in \mathbb{N}$ $\#(*)(\alpha, \beta)$ is the product of α and β, in other words $\#(*)(\alpha, \beta) = \alpha \cdot \beta$.

Given a language $\mathcal{L} = (\mathcal{V}, \mathcal{F}, \mathcal{C}, \#, \{D_1, \ldots, D_p\}, q_{max})$ as above, in section 10 we have defined a deductive system for this language, and we assume that all the axioms and rules we have defined for that deductive system apply to our new deductive system. We are now going to add new axioms and rules to our new deductive system.

Lemma 12.1. *$H[x : N, y : N, z : N, u : N, v : N]$ holds.*

Proof. Follows from lemma 11.1. □

Lemma 12.2. *Let $k \in K$ and $\varphi, \psi \in E(k)$. Then*

- $= (\varphi, \psi) \in S(k)$
- *for each* $\sigma \in \Xi(k)$ $\#(k, = (\varphi, \psi), \sigma) = (\#(k, \varphi, \sigma) = \#(k, \psi, \sigma))$.

Proof. It's a simply a case of lemma 8.15. □

Lemma 12.3. *Let $k \in K$ and let $\varphi, \psi \in E(k)$. Assume for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma) \in \mathbb{N}$ and $\#(k, \psi, \sigma) \in \mathbb{N}$, then*

- $(*)(\varphi, \psi) \in E(k)$
- *for each* $\sigma \in \Xi(k)$ $\#(k, (*)(\varphi, \psi), \sigma) = (\#(k, \varphi, \sigma) \cdot \#(k, \psi, \sigma))$.

Proof. It's simply a case of lemma 8.17. □

Lemma 12.4. *Let $k \in K$ and let $\varphi \in E(k)$, then*

- $\in (\varphi, N) \in S(k)$,
- *for each* $\sigma \in \Xi(k)$ $\#(k, \in (\varphi, N), \sigma) = P_\in(\#(k, \varphi, \sigma), \mathbb{N})$.

Proof. It's a simply a case of lemma 8.15. □

Lemma 12.5. *Let $k = k[x : N, y : N, z : N, u : N, v : N]$, $\varphi \in E(k)$ such that for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma) \in \mathbb{N}$ then*

- $= (y, \varphi) \in S(k)$
- $= (z, *(y, v)) \in S(k)$
- $= (z, *(\varphi, v)) \in S(k)$

Moreover

$$\#\left(\gamma[x : N, y : N, z : N, u : N, v : N, \to (\wedge(= (y, \varphi), = (z, yv)), = (z, \varphi v))]\right)$$

is true.

Proof. By lemma 9.1 $y \in E(k)$ and by 12.2 $= (y, \varphi) \in S(k)$.

Moreover by 9.1 $v \in E(k)$. If we define $k_v = k[x : N, y : N, z : N, u : N]$ then for each $\sigma \in \Xi(k)$ $\sigma_{/dom(k_v)} \in \Xi(k_v)$ and $\#(k, v, \sigma) \in \#(k_v, N, \sigma_{/dom(k_v)}) = \#(N) = \mathbb{N}$.

Similarly by lemma 9.1 if we define $k_y = k[x : N]$ then for each $\sigma \in \Xi(k)$ $\sigma_{/dom(k_y)} \in \Xi(k_y)$ and $\#(k, y, \sigma) \in \#(k_y, N, \sigma_{/dom(k_y)}) = \#(N) = \mathbb{N}$.

Similarly by the same lemma $z \in E(k)$ and if we define $k_z = k[x : N, y : N]$ then for each $\sigma \in \Xi(k)$ $\sigma_{/dom(k_z)} \in \Xi(k_z)$ and $\#(k, z, \sigma) \in \#(k_z, N, \sigma_{/dom(k_z)}) = \#(N) = \mathbb{N}$.

By lemma 12.3 it follows that $(*)(y, v) \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, (*)(y, v), \sigma) = (\#(k, y, \sigma) \cdot \#(k, v, \sigma)) \in \mathbb{N}$.

Similarly $(*)(\varphi, v) \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, (*)(\varphi, v), \sigma) = (\#(k, \varphi, \sigma) \cdot \#(k, v, \sigma)) \in \mathbb{N}$.

By lemma 12.2 $= (z, *(y, v)) \in S(k)$ and $= (z, *(\varphi, v)) \in S(k)$.

Clearly it follows that $\to (\wedge(= (y, \varphi), = (z, yv)), = (z, \varphi v)) \in S(k)$ and we can rewrite

$$\#\left(\gamma[x : N, y : N, z : N, u : N, v : N, \to (\wedge(= (y, \varphi), = (z, yv)), = (z, \varphi v))]\right)$$

as follows

$$P_\forall(\{\#(k, \to (\wedge(= (y, \varphi), = (z, yv)), = (z, \varphi v)), \sigma)|\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\to(\#(k, \wedge(= (y, \varphi), = (z, yv)), \sigma), \#(k, = (z, \varphi v), \sigma))|\ \sigma \in \Xi(k)\})\ ,$$

for each $\sigma \in \Xi(k)$
$\#(k, \wedge(= (y, \varphi), = (z, yv)), \sigma)$ is false or $\#(k, = (z, \varphi v), \sigma)$.

Given $\sigma \in \Xi(k)$ we assume $\#(k, \wedge(= (y, \varphi), = (z, yv)), \sigma)$ holds and want to show that $\#(k, = (z, \varphi v), \sigma)$ then holds.

We have

$$P_\wedge(\#(k, = (y, \varphi), \sigma), \#(k, = (z, yv), \sigma)) ,$$

$$P_\wedge(\#(k, y, \sigma) = \#(k, \varphi, \sigma), \#(k, z, \sigma) = \#(k, yv, \sigma)) ,$$

$$P_\wedge(\#(k, y, \sigma) = \#(k, \varphi, \sigma), \#(k, z, \sigma) = \#(k, y, \sigma) \cdot \#(k, v, \sigma)) .$$

From there it follows that $\#(k, z, \sigma) = \#(k, \varphi, \sigma) \cdot \#(k, v, \sigma)$.

We have shown that $\#(k, = (z, \varphi v), \sigma)$ holds, in fact it can be rewritten

$$\#(k, z, \sigma) = \#(k, \varphi v, \sigma),$$

$$\#(k, z, \sigma) = \#(k, \varphi, \sigma) \cdot \#(k, v, \sigma) .$$

□

We can create a set $A_{12.5}$ which is the set of all sentences

$$\gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\wedge(= (y, \varphi), = (z, yv)), = (z, \varphi v))]$$

such that

- $\varphi \in E(k[x : N, y : N, z : N, u : N, v : N])$,
- for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma) \in \mathbb{N}$.

Lemma 12.5 shows us that this set of sentences (which is a potential axiom) is 'sound'. In order to use $A_{12.5}$ as an axiom in our system we also need to show that $A_{12.5}$ is r.e..

Lemma 12.6. *$A_{12.5}$ is r.e..*

Proof. $A_{12.5}$ is the set of all sentences

$$\gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\wedge(= (y, \varphi), = (z, yv)), = (z, \varphi v))]$$

such that $\varphi \in E_{\mathbb{N}}(k[x : N, y : N, z : N, u : N, v : N])$.

Let's define a function η over Σ^* with $\eta(\varphi) = \gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\wedge(= (y, \varphi), = (z, yv)), = (z, \varphi v))]$.

Then $A_{12.5}$ is simply the set $\{\eta(\varphi) | \ \varphi \in E_{\mathbb{N}}(k[x : N, y : N, z : N, u : N, v : N])\}$.

Since $E_{\mathbb{N}}(k[x : N, y : N, z : N, u : N, v : N])$ is r.e. then $A_{12.5}$ is also r.e..

□

Then let $A_{12.5} \in \mathcal{A}$.

Lemma 12.7. *Let* $k = k[x : N, y : N, z : N, u : N, v : N]$, $\chi \in S(k)$ *then*

- $= ((xu)v, x(uv)) \in S(k)$

Moreover

$$\#\left(\gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\chi, = ((xu)v, x(uv)))]\right)$$

is true.

Proof. By lemma 9.1 $u \in E(k)$. If we define $k_u = k[x : N, y : N, z : N]$ then for each $\sigma \in \Xi(k)$ $\sigma_{/dom(k_u)} \in \Xi(k_u)$ and $\#(k, u, \sigma) \in \#(k_u, N, \sigma_{/dom(k_u)}) = \#(N) = \mathbb{N}$.

Similarly by 9.1 $x \in E(k)$. If we define $k_x = \epsilon$ then for each $\sigma \in \Xi(k)$ $\sigma_{/dom(k_x)} \in \Xi(k_x)$ and $\#(k, x, \sigma) \in \#(k_x, N, \sigma_{/dom(k_x)}) = \#(N) = \mathbb{N}$.

Similarly by 9.1 $v \in E(k)$. If we define $k_v = k[x : N, y : N, z : N, u : N]$ then for each $\sigma \in \Xi(k)$ $\sigma_{/dom(k_v)} \in \Xi(k_v)$ and $\#(k, v, \sigma) \in \#(k_v, N, \sigma_{/dom(k_v)}) = \#(N) = \mathbb{N}$.

By lemma 12.3 $(*)(x, u) \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, (*)(x, u), \sigma) = (\#(k, x, \sigma) \cdot \#(k, u, \sigma)) \in \mathbb{N}$.

Also by lemma 12.3 $(*)(xu, v) \in E(k)$ and for each $\sigma \in \Xi(k)$

$$\#(k, (*)(xu, v), \sigma) = \#(k, xu, \sigma) \cdot \#(k, v, \sigma) = (\#(k, x, \sigma) \cdot \#(k, u, \sigma)) \cdot \#(k, v, \sigma) \ .$$

By lemma 12.3 $(*)(u, v) \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, (*)(u, v), \sigma) = (\#(k, u, \sigma) \cdot \#(k, v, \sigma)) \in \mathbb{N}$.

Also by lemma 12.3 $(*)(x, uv) \in E(k)$ and for each $\sigma \in \Xi(k)$

$$\#(k, (*)(x, uv), \sigma) = \#(k, x, \sigma) \cdot \#(k, uv, \sigma) = \#(k, x, \sigma) \cdot (\#(k, u, \sigma) \cdot \#(k, v, \sigma)) \ .$$

Clearly it follows that for each $\sigma \in \Xi(k)$

$$\#(k, (*)(x, uv), \sigma) = \#(k, (*)(xu, v), \sigma) \ .$$

By lemma 12.2 it also follows that $= ((xu)v, x(uv)) \in S(k)$ and that for each $\sigma \in \Xi(k)$ $\#(k, = ((xu)v, x(uv)), \sigma)$ is true.

Finally we observe that

$$\#\left(\gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\chi, = ((xu)v, x(uv)))]\right)$$

can be rewritten as

$$P_\forall(\{\#(k, \rightarrow (\chi, = ((xu)v, x(uv)))\,, \sigma)|\ \sigma \in \Xi(k)\}) \ ,$$

$$P_\forall(\{P_\rightarrow(\#(k,\chi,\sigma),\#(k,=((xu)v,x(uv)),\sigma))|\ \sigma\in\Xi(k)\})\ ,$$

for each $\sigma \in \Xi(k)$ $\#(k,\chi,\sigma)$ is false or $\#(k,=((xu)v,x(uv)),\sigma)$.

So we have proved it is true. □

We can create a set $A_{12.7}$ which is the set of all sentences

$$\gamma[x:N,y:N,z:N,u:N,v:N,\rightarrow(\chi,=((xu)v,x(uv)))]$$

such that

- $\chi \in S(k[x:N,y:N,z:N,u:N,v:N])$.

Lemma 12.7 shows us that this set of sentences (which is a potential axiom) is 'sound'. In order to use $A_{12.7}$ as an axiom in our system we also need to show that $A_{12.7}$ is r.e..

Lemma 12.8. *$A_{12.7}$ is r.e..*

Proof. $A_{12.7}$ is the set of all sentences

$$\gamma[x:N,y:N,z:N,u:N,v:N,\rightarrow(\chi,=((xu)v,x(uv)))]$$

such that

- $\chi \in S(k[x:N,y:N,z:N,u:N,v:N])$.

Let's define a function η over Σ^* with

$$\eta(\chi)=\gamma[x:N,y:N,z:N,u:N,v:N,\rightarrow(\chi,=((xu)v,x(uv)))]\ .$$

Then $A_{12.7}$ is simply the set $\{\eta(\chi)|\ \chi\in S(k[x:N,y:N,z:N,u:N,v:N])\}$.

Since $S(k[x:N,y:N,z:N,u:N,v:N])$ is r.e. then $A_{12.7}$ is also r.e.. □

Then let $A_{12.7} \in \mathcal{A}$.

Lemma 12.9. *Let m be a positive integer. Let $x_1,\ldots,x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1,\ldots,\varphi_m \in E$ and assume $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$. Define $k = k[x_1:\varphi_1,\ldots,x_m:\varphi_m]$ and let $\chi \in S(k)$, $\varphi,\psi,\theta \in E(k)$.*

Then
$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\chi,\rightarrow(\wedge(=(\varphi,\psi),=(\psi,\theta)),=(\varphi,\theta)))])$.

Proof. We can rewrite

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\chi,\rightarrow(\wedge(=(\varphi,\psi),=(\psi,\theta)),=(\varphi,\theta)))])$$

as

$$P_{\forall}(\{\#(k, \rightarrow(\chi, \rightarrow(\wedge(=(\varphi,\psi), =(\psi,\theta)), =(\varphi,\theta))), \sigma)|\ \sigma \in \Xi(k)\})\ ,$$

$$P_{\forall}(\{P_{\rightarrow}(\#(k,\chi,\sigma), \#(k, \rightarrow(\wedge(=(\varphi,\psi), =(\psi,\theta)), =(\varphi,\theta)), \sigma))|\ \sigma \in \Xi(k)\})\ ,$$

for each $\sigma \in \Xi(k)$ $\#(k,\chi,\sigma)$ is false or $\#(k, \rightarrow(\wedge(=(\varphi,\psi), =(\psi,\theta)), =(\varphi,\theta)), \sigma)$.

We can rewrite $\#(k, \rightarrow(\wedge(=(\varphi,\psi), =(\psi,\theta)), =(\varphi,\theta)), \sigma)$ as

$$P_{\rightarrow}(\#(k, \wedge(=(\varphi,\psi), =(\psi,\theta)), \sigma), \#(k, =(\varphi,\theta), \sigma))\ ,$$

$$P_{\rightarrow}(P_{\wedge}(\#(k, =(\varphi,\psi), \sigma), \#(k, =(\psi,\theta), \sigma)), \#(k, =(\varphi,\theta), \sigma))\ ,$$

$$P_{\rightarrow}(P_{\wedge}(\#(k,\varphi,\sigma) = \#(k,\psi,\sigma), \#(k,\psi,\sigma) = \#(k,\theta,\sigma)), \#(k,\varphi,\sigma) = \#(k,\theta,\sigma))\ ,$$

$(\#(k,\varphi,\sigma) = \#(k,\psi,\sigma)$ and $\#(k,\psi,\sigma) = \#(k,\theta,\sigma))$ is false or $\#(k,\varphi,\sigma) = \#(k,\theta,\sigma)$.

If $(\#(k,\varphi,\sigma) = \#(k,\psi,\sigma)$ and $\#(k,\psi,\sigma) = \#(k,\theta,\sigma))$ is false then $\#(k, \rightarrow(\wedge(=(\varphi,\psi), =(\psi,\theta)), =(\varphi,\theta)), \sigma)$ is true.

Otherwise clearly $\#(k,\varphi,\sigma) = \#(k,\theta,\sigma)$ holds and so $\#(k, \rightarrow(\wedge(=(\varphi,\psi), =(\psi,\theta)), =(\varphi,\theta)), \sigma)$ is true all the same.

□

We can create a set $A_{12.9}$ which is the set of all sentences

$$\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m, \rightarrow(\chi, \rightarrow(\wedge(=(\varphi,\psi), =(\psi,\theta)), =(\varphi,\theta)))]$$

such that

- m is a positive integer, $x_1,\ldots,x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1,\ldots,\varphi_m \in E$, $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$,
- $\varphi,\psi,\theta \in E(k[x_1:\varphi_1,\ldots,x_m:\varphi_m])$,
- $\chi \in S(k[x_1:\varphi_1,\ldots,x_m:\varphi_m])$.

Lemma 12.9 shows us that this set of sentences (which is a potential axiom) is 'sound'. In order to use $A_{12.9}$ as an axiom in our system we also need to show that $A_{12.9}$ is r.e..

Lemma 12.10. *$A_{12.9}$ is r.e..*

Proof. Given a positive integer m and $(x_1,\varphi_1,\ldots,x_m,\varphi_m) \in R_m$ we can notice the following:

- $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \in K$;
- $E(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ is r.e.;
- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ is r.e.;
- $\{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times E(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^3 \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ is r.e..

So we can define the following

$$Q_{m,4} = \bigcup_{(x_1,\varphi_1,\ldots,x_m,\varphi_m)\in R_m} \{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times E(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^3 \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \,.$$

Clearly $Q_{m,4} \subseteq (\Sigma^*)^{2m} \times (\Sigma^*)^3 \times \Sigma^*$ is r.e..

We can define a function η over $(\Sigma^*)^{2m} \times (\Sigma^*)^3 \times \Sigma^*$ such that for each $((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \theta), \chi) \in (\Sigma^*)^{2m} \times (\Sigma^*)^3 \times \Sigma^*$

$$\eta(((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \theta), \chi)) = \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \to (\wedge\, (= (\varphi, \psi), = (\psi, \theta))\,, = (\varphi, \theta)))] \,.$$

Now η clearly is a computable function and so the set $\{\eta(((x_1, \varphi_1, \ldots, x_m, \varphi_m), (\varphi, \psi, \theta), \chi))|\, ((x_1, \varphi_1, \ldots, x_m, \varphi_m), (\varphi, \psi, \theta), \chi) \in Q_{m,4}\}$ is a r.e. subset of Σ^*. And finally the set

$$\bigcup_{m\geqslant 1} \{\eta(((x_1, \varphi_1, \ldots, x_m, \varphi_m), (\varphi, \psi, \theta), \chi))|\, ((x_1, \varphi_1, \ldots, x_m, \varphi_m), (\varphi, \psi, \theta), \chi) \in Q_{m,4}\}$$

is itself a r.e. set. It should be clear at this point that this set is actually our set $A_{12.9}$, and so that $A_{12.9}$ is r.e.. □

Then let $A_{12.9} \in \mathcal{A}$.

Lemma 12.11. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi, \chi \in S(k)$. Under these assumptions, if*

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \varphi)]),$$

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \to (\varphi, \psi))])$$

then

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \psi)]).$$

Proof. We can rewrite

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \varphi)])$$

as

$$P_\forall(\{\#(k, \to (\chi, \varphi)\,, \sigma)|\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\to(\#(k,\chi,\sigma),\#(k,\varphi,\sigma))|\ \sigma\in\Xi(k)\})\ .$$

We can rewrite

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\chi,\to(\varphi,\psi))])$$

as

$$P_\forall(\{\#(k,\to(\chi,\to(\varphi,\psi)),\sigma)|\ \sigma\in\Xi(k)\})\ ,$$

$$P_\forall(\{P_\to(\#(k,\chi,\sigma),\#(k,\to(\varphi,\psi),\sigma))\ |\ \sigma\in\Xi(k)\})\ ,$$

$$P_\forall(\{P_\to(\#(k,\chi,\sigma),P_\to(\#(k,\varphi,\sigma),\#(k,\psi,\sigma)))\ |\ \sigma\in\Xi(k)\})\ .$$

Finally we can rewrite

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\chi,\psi)])$$

as

$$P_\forall(\{\#(k,\to(\chi,\psi),\sigma)|\ \sigma\in\Xi(k)\})\ ,$$

$$P_\forall(\{P_\to(\#(k,\chi,\sigma),\#(k,\psi,\sigma))|\ \sigma\in\Xi(k)\})\ .$$

If we assume both

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\chi,\varphi)])\ ,$$

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\chi,\to(\varphi,\psi))])$$

then for each $\sigma\in\Xi(k)$

- $\#(k,\chi,\sigma)$ is false or $\#(k,\varphi,\sigma)$ is true,
- $\#(k,\chi,\sigma)$ is false or $\#(k,\varphi,\sigma)$ is false or $\#(k,\psi,\sigma)$ is true.

Clearly this implies $\#(k,\chi,\sigma)$ is false or $\#(k,\psi,\sigma)$ is true.
Therefore in our assumptions $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\chi,\psi)])$ holds. □

We can create a set $R_{12.11}$ which is the set of all 3-tuples

$$\left(\begin{array}{l}\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\chi,\varphi)],\\ \gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\chi,\to(\varphi,\psi))],\\ \gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\chi,\psi)]\end{array}\right)$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi, \chi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Lemma 12.11 shows us that this set (which is a potential 2-ary rule) is 'sound'. In order to use $R_{12.11}$ as a rule in our system we also need to show that $R_{12.11}$ is r.e..

Lemma 12.12. $R_{12.11}$ *is r.e.*

Proof. Given a positive integer m and $(x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m$ we can notice the following:

- $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \in K$;
- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ is r.e.;
- $\{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^3$ is r.e..

Let's define

$$Q_{m,3} = \bigcup_{(x_1,\varphi_1,\ldots,x_m,\varphi_m)\in R_m} \{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^3 \ .$$

Clearly $Q_{m,3} \subseteq (\Sigma^*)^{2m} \times (\Sigma^*)^3$ is also r.e..

We now define three functions $\delta_{1,m},\ \delta_{2,m},\ \delta_{3,m}$ over $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ as follows. Given $((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) \in (\Sigma^*)^{2m} \times (\Sigma^*)^3$

$$\delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \rightarrow (\chi, \varphi)] \ .$$

$$\delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \rightarrow (\chi, \rightarrow (\varphi, \psi))] \ .$$

$$\delta_{3,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \rightarrow (\chi, \psi)] \ .$$

All of the three functions we have defined are computable functions from $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ to Σ^*. If we define a function δ_m over $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ as follows:

$$\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \begin{pmatrix} \delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)), \\ \delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)), \\ \delta_{3,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)), \end{pmatrix}$$

then δ_m is a computable function from $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ to $(\Sigma^*)^3$, therefore the set

$$D_m = \{\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) |\ ((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) \in Q_{m,3}\}$$

is a r.e. subset of $(\Sigma^*)^3$.

If we now consider the set $\bigcup_{m \geqslant 1} D_m$ then this is a r.e. subset of $(\Sigma^*)^3$ and actually this set is equal to our set $R_{12.11}$ which so is r.e. itself. □

Then let $R_{12.11} \in \mathcal{R}$.

Lemma 12.13. *Let* $k = k[x : N, y : N, z : N, u : N, v : N]$, $\chi \in S(k)$, $\varphi \in E(k)$ *such that for each* $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma) \in \mathbb{N}$ *then*

- $= (z, x\varphi) \in S(k)$,
- $\exists(\{\}(w : N, = (z, xw))) \in S(k)$.

Under these assumptions if

$$\#\,(\gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\chi, = (z, x\varphi))])$$

then

$$\#\,(\gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\chi, \exists(\{\}(w : N, = (z, xw))))])$$

is true.

Proof. By 9.1 $x \in E(k)$. If we define $k_x = \epsilon$ then for each $\sigma \in \Xi(k)$ $\sigma_{/dom(k_x)} \in \Xi(k_x)$ and $\#(k, x, \sigma) \in \#(k_x, N, \sigma_{/dom(k_x)}) = \#(N) = \mathbb{N}$.

By lemma 12.3 it follows that $(*)(x, \varphi) \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, (*)(x, \varphi), \sigma) = (\#(k, x, \sigma) \cdot \#(k, \varphi, \sigma)) \in \mathbb{N}$.

By 9.1 $z \in E(k)$, so we can apply lemma 12.2 and obtain that $= (z, x\varphi)$ belongs to $S(k)$.

Let $h = k+ < w, N >$, we have $N \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, N, \sigma) = \#(N) = \mathbb{N}$. So $N \in E_s(k)$. Moreover $w \in (\mathcal{V} - var(k))$ so by lemma 8.21 $h \in K$.

We now want to show that $= (z, xw)$ belongs to $S(h)$. Since $N \in E_s(k)$ we have $H[x : N, y : N, z : N, u : N, v : N, w : N]$. We have

$$k[x : N, y : N, z : N, u : N, v : N, w : N] = k+ < w, N >= h\ .$$

Using lemma 9.1 we obtain that $z, x, w \in E(h)$. If we define $h_x = \epsilon$ then for each $\rho \in \Xi(h)$ $\rho_{/dom(h_x)} \in \Xi(h_x)$ and $\#(h, x, \rho) \in \#(h_x, N, \rho_{/dom(h_x)}) = \#(N) = \mathbb{N}$.

Moreover for each $\rho \in \Xi(h)$ $\rho_{/dom(k)} \in \Xi(k)$ $\#(h, w, \rho) \in \#(k, N, \rho_{/dom(k)}) = \#(N) = \mathbb{N}$.

By lemma 12.3 it follows that $(*)(x, w) \in E(h)$ and for each $\rho \in \Xi(h)$ $\#(h, (*)(x, w), \rho) = (\#(h, x, \rho) \cdot \#(h, w, \rho)) \in \mathbb{N}$.

By lemma 12.2 $= (z, xw)$ belongs to $S(h)$. We can now apply lemma 8.21 and obtain that $\exists(\{\}(w : N, = (z, xw))) \in S(k)$ and for each $\sigma \in \Xi(k)$

$$\#(k, \exists(\{\}(w : N, = (z, xw))), \sigma) = P_{\exists}(\{\#(h, = (z, xw), \rho)|\ \rho \in \Xi(h),\ \sigma \sqsubseteq \rho\})\ .$$

We can rewrite

$$\#\,(\gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\chi, = (z, x\varphi))])$$

as:

$$P_\forall(\{\#(k, \rightarrow(\chi, =(z, x\varphi)), \sigma)|\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\rightarrow(\#(k, \chi, \sigma), \#(k, =(z, x\varphi), \sigma))|\ \sigma \in \Xi(k)\})\ ,$$

for each $\sigma \in \Xi(k)$
$\#(k, \chi, \sigma)$ is false or $\#(k, =(z, x\varphi), \sigma)$.

We can rewrite

$$\#\left(\gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow(\chi, \exists(\{\}(w : N, =(z, xw))))]\right)$$

as:

$$P_\forall(\{\#(k, \rightarrow(\chi, \exists(\{\}(w : N, =(z, xw)))), \sigma)|\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\rightarrow(\#(k, \chi, \sigma), \#(k, \exists(\{\}(w : N, =(z, xw))), \sigma))|\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\rightarrow(\#(k, \chi, \sigma), P_\exists(\{\#(h, =(z, xw), \rho)|\ \rho \in \Xi(h),\ \sigma \sqsubseteq \rho\}))|\ \sigma \in \Xi(k)\})\ ,$$

for each $\sigma \in \Xi(k)$
$\#(k, \chi, \sigma)$ is false or $P_\exists(\{\#(h, =(z, xw), \rho)|\ \rho \in \Xi(h),\ \sigma \sqsubseteq \rho\})$.

We now assume

$$\#\left(\gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow(\chi, =(z, x\varphi))]\right)$$

and try to prove

$$\#\left(\gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow(\chi, \exists(\{\}(w : N, =(z, xw))))]\right)\ .$$

Let $\sigma \in \Xi(k)$, if $\#(k, \chi, \sigma)$ is false then our proof is already finished. So we assume $\#(k, \chi, \sigma)$ is true. In this case $\#(k, =(z, x\varphi), \sigma)$ holds.

It follows that

$$\#(k, z, \sigma) = \#(k, x\varphi, \sigma) = \#(k, x, \sigma) \cdot \#(k, \varphi, \sigma)\ .$$

We have to show there exists $\rho \in \Xi(h)$ such that $\sigma \sqsubseteq \rho$ and $\#(h, =(z, xw), \rho)$. We can rewrite $\#(h, =(z, xw), \rho)$ as

$$\#(h, z, \rho) = \#(h, xw, \rho) = \#(h, x, \rho) \cdot \#(h, w, \rho)\ .$$

Let's define $\rho = \sigma + (w, \#(k, \varphi, \sigma))$. There exists a positive integer n such that $h \in K(n)$. Since $h \neq \epsilon$ we have $n \geqslant 2$ and by lemma 8.1 there exists $q < n$ such that $h \in K(q)^+$. Then there exist $g \in K(q)$, $\phi \in E_s(q,g)$, $\alpha \in (\mathcal{V} - var(g))$ such that $h = g + < \alpha, \phi >$ and

$$\Xi(h) = \{\delta + (\alpha, s) | \, \delta \in \Xi(g), s \in \#(g, \phi, \delta)\} \, .$$

Now we have also $h = k + < w, N >$ therefore

$$\Xi(h) = \{\delta + (w, s) | \, \delta \in \Xi(k), s \in \mathbb{N}\} \, .$$

It follows that $\rho \in \Xi(h)$ and moreover

$$\#(h, w, \rho) = \#(h, w, \rho)_{(q+1,h,a)} = \#(k, \varphi, \sigma) \; .$$

We have also $\#(h, z, \rho) = \#(k, z, \sigma)$. In fact $z \in E(h) \cap E(k)$, $k \sqsubseteq h$, $\sigma \sqsubseteq \rho$ and we can use lemma 8.14. Similarly we obtain $\#(h, x, \rho) = \#(k, x, \sigma)$. Since

$$\#(k, z, \sigma) = \#(k, x, \sigma) \cdot \#(k, \varphi, \sigma) \; .$$

we have

$$\#(h, z, \rho) = \#(h, x, \rho) \cdot \#(h, w, \rho) \; .$$

and then of course $\#(h, = (z, xw), \rho)$. □

We can create a set $R_{12.13}$ which is the set of all pairs

$$\left(\begin{array}{l} \gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\chi, = (z, x\varphi))], \\ \gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\chi, \exists(\{\}(w : N, = (z, xw))))] \end{array} \right)$$

such that $\chi \in S(k)$, $\varphi \in E(k)$ such that for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma) \in \mathbb{N}$.

Lemma 12.13 shows us that this set (which is a potential 1-ary rule) is 'sound'. In order to use $R_{12.13}$ as a rule in our system we also need to show that $R_{12.13}$ is r.e..

Lemma 12.14. $R_{12.13}$ *is r.e..*

Proof. Our set $R_{12.13}$ is the set of all pairs

$$\left(\begin{array}{l} \gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\chi, = (z, x\varphi))], \\ \gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\chi, \exists(\{\}(w : N, = (z, xw))))] \end{array} \right)$$

such that $\chi \in S(k[x : N, y : N, z : N, u : N, v : N])$, $\varphi \in E_{\mathbb{N}}(k[x : N, y : N, z : N, u : N, v : N])$.

We now define two functions δ_1, δ_2 over $(\Sigma^*)^2$ as follows. Given $\chi, \varphi \in \Sigma^*$

$$\delta_1(\chi, \varphi) = \gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\chi, = (z, x\varphi))] \; ,$$

$$\delta_2(\chi, \varphi) = \gamma[x : N, y : N, z : N, u : N, v : N, \to (\chi, \exists(\{\}(w : N, = (z, xw))))] \ .$$

All of the two functions we have defined are computable functions from $(\Sigma^*)^2$ to Σ^*. If we define a function δ from $(\Sigma^*)^2$ to $(\Sigma^*)^2$ as follows:

$$\delta(\chi, \varphi) = \begin{pmatrix} \delta_1(\chi, \varphi), \\ \delta_2(\chi, \varphi) \end{pmatrix}$$

then δ is a computable function from $(\Sigma^*)^2$ to $(\Sigma^*)^2$. We can actually rewrite $R_{12.13}$ as

$$\{\delta(\chi, \varphi) | \, (\chi, \varphi) \in S(k[x : N, y : N, z : N, u : N, v : N]) \times E_{\mathbb{N}}(k[x : N, y : N, z : N, u : N, v : N]\} \ .$$

Since $S(k[x : N, y : N, z : N, u : N, v : N])$ and $E_{\mathbb{N}}(k[x : N, y : N, z : N, u : N, v : N]$ are r.e. then $R_{12.13}$ is r.e. itself. □

Then let $R_{12.13} \in \mathcal{R}$.

Lemma 12.15. *Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ also holds, we define $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\chi \in S(h) \cap S(k)$, $\varphi \in S(k)$.

Under these assumptions we have

- $\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)) \in S(h)$,
- $\to (\chi, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)))] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}, \to (\chi, \varphi)] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}, \to (\chi, \varphi)])$ then

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)))]) \ .$$

Proof. By lemma 8.21 $\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)) \in S(h)$, and clearly all the other 'preliminary' results hold.

We can rewrite

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}, \to (\chi, \varphi)])$$

as

$$P_\forall(\{\# (k, \to (\chi, \varphi), \sigma) \mid \sigma \in \Xi(k)\}) \ ,$$

$$P_\forall(\{P_\to (\# (k, \chi, \sigma), \# (k, \varphi, \sigma)) \mid \sigma \in \Xi(k)\}) \ .$$

We can furtherly express this as

'for each $\sigma \in \Xi(k)$ $P_{\to}(\#(k,\chi,\sigma), \#(k,\varphi,\sigma))$',

'for each $\sigma \in \Xi(k)$ $\#(k,\chi,\sigma)$ is false or $\#(k,\varphi,\sigma)$'.

We can rewrite

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)))])$$

as

$$P_{\forall}(\{\#(h, \to (\chi, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))), \rho) \mid \rho \in \Xi(h)\}) \ ,$$

$$P_{\forall}(\{P_{\to}(\#(h,\chi,\rho), \#(h, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)), \rho)) \mid \rho \in \Xi(h)\}) \ ,$$

$$P_{\forall}(\{P_{\to}(\#(h,\chi,\rho), P_{\forall}(\{\#(k,\varphi,\sigma) \mid \sigma \in \Xi(k), \ \rho \sqsubseteq \sigma\})) \mid \rho \in \Xi(h)\}) \ .$$

We can furtherly express this as
'for each $\rho \in \Xi(h)$
$P_{\to}(\#(h,\chi,\rho), P_{\forall}(\{\#(k,\varphi,\sigma) \mid \sigma \in \Xi(k), \ \rho \sqsubseteq \sigma\}))$',

'for each $\rho \in \Xi(h)$
$\#(h,\chi,\rho)$ is false or $P_{\forall}(\{\#(k,\varphi,\sigma) \mid \sigma \in \Xi(k), \ \rho \sqsubseteq \sigma\})$',

'for each $\rho \in \Xi(h)$ $\#(h,\chi,\rho)$ is false or
for each $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ $\#(k,\varphi,\sigma)$'.

Let $\rho \in \Xi(h)$ and $\#(h,\chi,\rho)$, let $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$, we want to show that $\#(k,\varphi,\sigma)$ holds. To show this it is clearly enough to show that $\#(k,\chi,\sigma)$ holds. To do this we can use lemma 8.14. In fact there exists a positive integer n such that $h \in K(n)$, $\chi \in E(n,h)$, $k \in K(n)$, $\chi \in E(n,k)$. Given that $\rho \in \Xi(h)$, $\sigma \in \Xi(k)$, $\rho \sqsubseteq \sigma$ we can apply that lemma and get $\#(h,\chi,\rho) = \#(k,\chi,\sigma)$, so $\#(k,\chi,\sigma)$ is proved. □

We can create a set $R_{12.15}$ which is the set of all pairs

$$\left(\begin{array}{l} \gamma[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}, \to (\chi, \varphi)], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)))] \end{array} \right)$$

such that

- m is a positive integer, $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_{m+1} \in E$, $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$;
- if we define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$ and $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ then $\chi \in S(h) \cap S(k)$, $\varphi \in S(k)$.

Lemma 12.15 shows us that this set (which is a potential 1-ary rule) is 'sound'. In order to use $R_{12.15}$ as a rule in our system we also need to show that $R_{12.15}$ is r.e..

Lemma 12.16. $R_{12.15}$ *is r.e..*

Proof. Given a positive integer m and $(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1}) \in R_{m+1}$ all of the following sets are r.e.:

- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$,
- $S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])$,
- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \cap S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])$.

Therefore the following set is also r.e.:

$$\{(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1})\} \times (S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \cap S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])) \\ \times S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]) \ .$$

Let's use this temporary definition

$$Q'_{m+1,2} = \bigcup_{(x_1,\varphi_1,\ldots,x_{m+1},\varphi_{m+1}) \in R_{m+1}} \{(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1})\} \\ \times (S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \cap S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])) \\ \times S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])).$$

With this $Q'_{m+1,2}$ is a r.e. subset of $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$.

We now define two functions $\delta_{1,m}$, $\delta_{2,m}$ over $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$ as follows. Given $((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) \in (\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$

$$\delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_{m+1} : \varphi_{m+1}, \rightarrow (\chi, \varphi)] \ .$$

$$\delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) = \\ \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \rightarrow (\chi, \forall(\{\}(\psi_{m+1} : \varphi_{m+1}, \varphi)))] \ .$$

All of the two functions we have defined are computable functions from $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$ to Σ^*. If we define a function δ_m over $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$ as follows:

$$\delta_m((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) = \begin{pmatrix} \delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi), \\ \delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) \end{pmatrix}$$

then δ_m is a computable function from $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$ to $(\Sigma^*)^2$, therefore the set

$$D_m = \{\delta_m((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) | ((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi) \in Q'_{m+1,2}\}$$

is a r.e. subset of $(\Sigma^*)^2$.

If we now consider the set $\bigcup_{m\geqslant 1} D_m$ then this is a r.e. subset of $(\Sigma^*)^2$ and actually this set is equal to our set $R_{12.15}$ which so is r.e. itself. □

Then let $R_{12.15} \in \mathcal{R}$.

Lemma 12.17. *Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ also holds, we define $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\chi \in S(h)$, $\varphi \in S(k)$, $\psi \in S(h) \cap S(k)$.

Under these assumptions we have

- $\forall(\{\}(x_{m+1} : \varphi_{m+1}, \rightarrow (\varphi, \psi))) \in S(h)$,
- $\rightarrow (\chi, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \rightarrow (\varphi, \psi)))) \in S(h)$,
- $\exists(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)) \in S(h)$,
- $\rightarrow (\chi, \rightarrow (\exists(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)), \psi)) \in S(h)$
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \rightarrow (\varphi, \psi))))] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \rightarrow (\exists(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)), \psi))] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \rightarrow (\varphi, \psi))))])$ then

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \rightarrow (\exists(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)), \psi))]) \ .$$

Proof. Clearly $\rightarrow (\varphi, \psi) \in S(k)$ and by lemma 8.21

$$\forall(\{\}(x_{m+1} : \varphi_{m+1}, \rightarrow (\varphi, \psi))) \in S(h).$$

Similarly $\exists(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)) \in S(h)$ and all the other 'preliminary' results hold.

We can rewrite

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \rightarrow (\varphi, \psi))))])$$

as

$$P_\forall(\{\#\left(h, \rightarrow (\chi, \forall\left(\{\}\left(x_{m+1} : \varphi_{m+1}, \rightarrow (\varphi, \psi)\right)\right)\right), \rho) \mid \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow\left(\#\left(h, \chi, \rho\right), \#\left(h, \forall\left(\{\}\left(x_{m+1} : \varphi_{m+1}, \rightarrow (\varphi, \psi)\right)\right), \rho\right)\right) \mid \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow\left(\#\left(h, \chi, \rho\right), P_\forall\left(\{\#\left(k, \rightarrow (\varphi, \psi), \sigma\right) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\}\right)\right) \mid \rho \in \Xi(h)\}) \ ,$$

$$P_\forall(\{P_\rightarrow\left(\#\left(h, \chi, \rho\right), P_\forall\left(\{P_\rightarrow\left(\#(k, \varphi, \sigma), \#(k, \psi, \sigma)\right) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\}\right)\right) \mid \rho \in \Xi(h)\}) \ .$$

We can furtherly express this as
'for each $\rho \in \Xi(h)$
$P_\rightarrow\left(\#\left(h, \chi, \rho\right), P_\forall\left(\{P_\rightarrow\left(\#(k, \varphi, \sigma), \#(k, \psi, \sigma)\right) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\}\right)\right)$',

‘for each $\rho \in \Xi(h)$ $\#(h,\chi,\rho)$ is false or
$P_\forall(\{P_\to(\#(k,\varphi,\sigma),\#(k,\psi,\sigma)) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$’,

‘for each $\rho \in \Xi(h)$ $\#(h,\chi,\rho)$ is false or
for each $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ $P_\to(\#(k,\varphi,\sigma),\#(k,\psi,\sigma))$’,

‘for each $\rho \in \Xi(h)$ $\#(h,\chi,\rho)$ is false or
for each $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ $\#(k,\varphi,\sigma)$ is false or $\#(k,\psi,\sigma)$’.

We can rewrite

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\chi,\to(\exists(\{\}(x_{m+1}:\varphi_{m+1},\varphi)),\psi))])$$

as

$$P_\forall(\{\#(h,\to(\chi,\to(\exists(\{\}(x_{m+1}:\varphi_{m+1},\varphi)),\psi)),\rho) \mid \rho \in \Xi(h)\}) ,$$

$$P_\forall(\{P_\to(\#(h,\chi,\rho),\#(h,\to(\exists(\{\}(x_{m+1}:\varphi_{m+1},\varphi)),\psi),\rho)) \mid \rho \in \Xi(h)\}) ,$$

$$P_\forall(\{P_\to(\#(h,\chi,\rho),P_\to(\#(h,\exists(\{\}(x_{m+1}:\varphi_{m+1},\varphi)),\rho),\#(h,\psi,\rho))) \mid \rho \in \Xi(h)\}) ,$$

$$P_\forall(\{P_\to(\#(h,\chi,\rho),P_\to(P_\exists(\{\#(k,\varphi,\sigma) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\}),\#(h,\psi,\rho))) \mid \rho \in \Xi(h)\}) .$$

We can furtherly express this as

‘for each $\rho \in \Xi(h)$
$P_\to(\#(h,\chi,\rho),P_\to(P_\exists(\{\#(k,\varphi,\sigma) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\}),\#(h,\psi,\rho)))$’,

‘for each $\rho \in \Xi(h)$ $\#(h,\chi,\rho)$ is false or
$P_\to(P_\exists(\{\#(k,\varphi,\sigma) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\}),\#(h,\psi,\rho))$’

‘for each $\rho \in \Xi(h)$ $\#(h,\chi,\rho)$ is false or
$(P_\exists(\{\#(k,\varphi,\sigma) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$ is false or $\#(h,\psi,\rho))$’.

‘for each $\rho \in \Xi(h)$ $\#(h,\chi,\rho)$ is false or
((there exists $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $\#(k,\varphi,\sigma)$) is false or $\#(h,\psi,\rho)$)’.

We now assume

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\chi,\forall(\{\}(x_{m+1}:\varphi_{m+1},\to(\varphi,\psi))))])$$

and try to prove

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\to(\chi,\to(\exists(\{\}(x_{m+1}:\varphi_{m+1},\varphi)),\psi))]) .$$

Let $\rho \in \Xi(h)$ and $\#(h,\chi,\rho)$, suppose there exists $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $\#(k,\varphi,\sigma)$. Clearly under our assumptions $\#(k,\psi,\sigma)$ holds. We need to prove $\#(h,\psi,\rho)$, and to do this we can use lemma 8.14. In fact there exists a positive integer n such that $h \in K(n)$, $\psi \in E(n,h)$, $k \in K(n)$, $\psi \in E(n,k)$. Given that $\rho \in \Xi(h)$, $\sigma \in \Xi(k)$, $\rho \sqsubseteq \sigma$ we can apply that lemma and get $\#(h,\psi,\rho) = \#(k,\psi,\sigma)$, so $\#(h,\psi,\rho)$ is proved. $\square$

We can create a set $R_{12.17}$ which is the set of all pairs

$$\left(\begin{array}{l} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \to (\varphi, \psi)))))], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \to (\exists(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)), \psi))] \end{array} \right)$$

such that

- m is a positive integer, $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_{m+1} \in E$, $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$;
- if we define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$ and $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ then $\chi \in S(h)$, $\varphi \in S(k)$, $\psi \in S(h) \cap S(k)$.

Lemma 12.17 shows us that this set (which is a potential 1-ary rule) is 'sound'. In order to use $R_{12.17}$ as a rule in our system we also need to show that $R_{12.17}$ is r.e..

Lemma 12.18. $R_{12.17}$ *is r.e..*

Proof. Given a positive integer m and $(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1}) \in R_{m+1}$ all of the following sets are r.e.:

- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$,
- $S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])$,
- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \cap S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])$.

Therefore the following set is also r.e.:

$$\begin{aligned} \{(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1})\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \\ \times S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]) \\ \times (S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \cap S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])) \; . \end{aligned}$$

Let's use this temporary definition

$$\begin{aligned} Q'_{m+1,3} = \bigcup_{(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1}) \in R_{m+1}} \{(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1})\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \\ \times S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]) \\ \times (S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \cap S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])) \; . \end{aligned}$$

With this $Q'_{m+1,3}$ is a r.e. subset of $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^* \times \Sigma^*$.

We now define two functions $\delta_{1,m}$, $\delta_{2,m}$ over $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^* \times \Sigma^*$ as follows. Given $((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi, \psi) \in (\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^* \times \Sigma^*$

$$\begin{aligned} \delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \chi, \varphi, \psi) = \\ \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \to (\chi, \forall(\{\}(\psi_{m+1} : \varphi_{m+1}, \to (\varphi, \psi)))))] \; . \end{aligned}$$

$$\delta_{2,m}((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\chi,\varphi,\psi) = \\ \gamma[\psi_1:\varphi_1,\ldots,\psi_m:\varphi_m,\rightarrow(\chi,\rightarrow(\exists(\{\}(\psi_{m+1}:\varphi_{m+1},\varphi)),\psi))]\ .$$

All of the two functions we have defined are computable functions from $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^* \times \Sigma^*$ to Σ^*. If we define a function δ_m over $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^* \times \Sigma^*$ as follows:

$$\delta_m((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\chi,\varphi,\psi) = \begin{pmatrix} \delta_{1,m}((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\chi,\varphi,\psi), \\ \delta_{2,m}((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\chi,\varphi,\psi) \end{pmatrix}$$

then δ_m is a computable function from $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^* \times \Sigma^*$ to $(\Sigma^*)^2$, therefore the set

$$D_m = \{\delta_m((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\chi,\varphi,\psi) | ((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\chi,\varphi,\psi) \in Q'_{m+1,3}\}$$

is a r.e. subset of $(\Sigma^*)^2$.

If we now consider the set $\bigcup_{m\geqslant 1} D_m$ then this is a r.e. subset of $(\Sigma^*)^2$ and actually this set is equal to our set $R_{12.17}$ which so is r.e. itself. □

Then let $R_{12.17} \in \mathcal{R}$.

Lemma 12.19. *Let m be a positive integer. Let $x_1,\ldots,x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1,\ldots,\varphi_{m+1} \in E$ and assume $H[x_1:\varphi_1,\ldots,x_{m+1}:\varphi_{m+1}]$.*

Define $k = k[x_1:\varphi_1,\ldots,x_{m+1}:\varphi_{m+1}]$. Of course $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$ also holds, we define $h = k[x_1:\varphi_1,\ldots,x_m:\varphi_m]$. Let $\varphi \in S(k)$, $\psi \in S(h)\cap S(k)$.

Under these assumptions we have

- $\forall(\{\}(x_{m+1}:\varphi_{m+1},\rightarrow(\varphi,\psi))) \in S(h)$,
- $\exists(\{\}(x_{m+1}:\varphi_{m+1},\varphi)) \in S(h)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\forall(\{\}(x_{m+1}:\varphi_{m+1},\rightarrow(\varphi,\psi)))] \in S(\epsilon)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\exists(\{\}(x_{m+1}:\varphi_{m+1},\varphi)),\psi)] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\forall(\{\}(x_{m+1}:\varphi_{m+1},\rightarrow(\varphi,\psi)))])$ then

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\exists(\{\}(x_{m+1}:\varphi_{m+1},\varphi)),\psi)])\ .$$

Proof. Clearly $\rightarrow(\varphi,\psi) \in S(k)$ and by lemma 8.21

$$\forall(\{\}(x_{m+1}:\varphi_{m+1},\rightarrow(\varphi,\psi))) \in S(h).$$

Similarly $\exists(\{\}(x_{m+1}:\varphi_{m+1},\varphi)) \in S(h)$ and all the other 'preliminary' results hold.

We can rewrite

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\forall(\{\}(x_{m+1}:\varphi_{m+1},\rightarrow(\varphi,\psi)))])$$

as

$$P_\forall(\{\#\,(h,\forall\,(\{\}\,(x_{m+1}:\varphi_{m+1},\rightarrow(\varphi,\psi)))\,,\rho)\mid\rho\in\Xi(h)\})\ ,$$

$$P_\forall(P_\forall\,(\{\#\,(k,\rightarrow(\varphi,\psi),\sigma)\mid\sigma\in\Xi(k),\ \rho\sqsubseteq\sigma\})\mid\rho\in\Xi(h)\})\ ,$$

$$P_\forall(\{P_\forall\,(\{P_\rightarrow(\#(k,\varphi,\sigma),\#(k,\psi,\sigma))\mid\sigma\in\Xi(k),\ \rho\sqsubseteq\sigma\})\mid\rho\in\Xi(h)\})\ .$$

We can furtherly express this as
'for each $\rho\in\Xi(h)$ $P_\forall\,(\{P_\rightarrow(\#(k,\varphi,\sigma),\#(k,\psi,\sigma))\mid\sigma\in\Xi(k),\ \rho\sqsubseteq\sigma\})$',

'for each $\rho\in\Xi(h)$
for each $\sigma\in\Xi(k)$ such that $\rho\sqsubseteq\sigma$ $P_\rightarrow(\#(k,\varphi,\sigma),\#(k,\psi,\sigma))$',

'for each $\rho\in\Xi(h)$
for each $\sigma\in\Xi(k)$ such that $\rho\sqsubseteq\sigma$ $\#(k,\varphi,\sigma)$ is false or $\#(k,\psi,\sigma)$'.

We can rewrite

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\exists(\{\}(x_{m+1}:\varphi_{m+1},\varphi)),\psi)])$$

as

$$P_\forall(\{\#\,(h,\rightarrow(\exists(\{\}(x_{m+1}:\varphi_{m+1},\varphi)),\psi),\rho)\mid\rho\in\Xi(h)\})\ ,$$

$$P_\forall(\{P_\rightarrow(\#(h,\exists(\{\}(x_{m+1}:\varphi_{m+1},\varphi)),\rho),\#(h,\psi,\rho))\mid\rho\in\Xi(h)\})\ ,$$
$$P_\forall(\{P_\rightarrow(P_\exists\,(\{\#\,(k,\varphi,\sigma)\mid\sigma\in\Xi(k),\ \rho\sqsubseteq\sigma\})\,,\#(h,\psi,\rho))\mid\rho\in\Xi(h)\})\ .$$

We can furtherly express this as
'for each $\rho\in\Xi(h)$
$P_\rightarrow(P_\exists\,(\{\#\,(k,\varphi,\sigma)\mid\sigma\in\Xi(k),\ \rho\sqsubseteq\sigma\})\,,\#(h,\psi,\rho))$',

'for each $\rho\in\Xi(h)$
($P_\exists\,(\{\#\,(k,\varphi,\sigma)\mid\sigma\in\Xi(k),\ \rho\sqsubseteq\sigma\})$ is false or $\#(h,\psi,\rho)$)'.

'for each $\rho\in\Xi(h)$
((there exists $\sigma\in\Xi(k)$ such that $\rho\sqsubseteq\sigma$ and $\#\,(k,\varphi,\sigma)$) is false or $\#(h,\psi,\rho)$)'.

We now assume

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\forall(\{\}(x_{m+1}:\varphi_{m+1},\rightarrow(\varphi,\psi)))])$$

and try to prove

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\exists(\{\}(x_{m+1}:\varphi_{m+1},\varphi)),\psi)])\ .$$

Let $\rho\in\Xi(h)$, suppose there exists $\sigma\in\Xi(k)$ such that $\rho\sqsubseteq\sigma$ and $\#\,(k,\varphi,\sigma)$. Clearly under our assumptions $\#(k,\psi,\sigma)$ holds. We need to prove $\#(h,\psi,\rho)$, and to do this we can use lemma 8.14. In fact there exists a positive integer n such that $h \in K(n)$, $\psi \in E(n,h)$, $k \in K(n)$, $\psi \in E(n,k)$. Given that $\rho \in \Xi(h)$, $\sigma \in \Xi(k)$, $\rho \sqsubseteq \sigma$ we can apply that lemma and get $\#(h,\psi,\rho) = \#(k,\psi,\sigma)$, so $\#(h,\psi,\rho)$ is proved. $\square$

We can create a set $R_{12.19}$ which is the set of all pairs

$$\left(\begin{array}{l} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \to (\varphi, \psi)))], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\exists(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)), \psi)] \end{array} \right)$$

such that

- m is a positive integer, $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_{m+1} \in E$, $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$;
- if we define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$ and $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ then $\varphi \in S(k)$, $\psi \in S(h) \cap S(k)$.

Lemma 12.19 shows us that this set (which is a potential 1-ary rule) is 'sound'. In order to use $R_{12.19}$ as a rule in our system we also need to show that $R_{12.19}$ is r.e..

Lemma 12.20. $R_{12.19}$ *is r.e..*

Proof. Given a positive integer m and $(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1}) \in R_{m+1}$ all of the following sets are r.e.:

- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$,
- $S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])$,
- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \cap S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])$.

Therefore the following set is also r.e.:

$$\{(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1})\} \times S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]) \\ \times (S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \cap S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])) \ .$$

Let's use this temporary definition

$$Q'_{m+1,2} = \bigcup_{(x_1,\varphi_1,\ldots,x_{m+1},\varphi_{m+1}) \in R_{m+1}} \{(x_1, \varphi_1, \ldots, x_{m+1}, \varphi_{m+1})\} \times S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]) \\ \times (S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \cap S(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}])) \ .$$

With this $Q'_{m+1,2}$ is a r.e. subset of $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$.

We now define two functions $\delta_{1,m}$, $\delta_{2,m}$ over $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$ as follows. Given $((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \varphi, \psi) \in (\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$

$$\delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_{m+1}, \varphi_{m+1}), \varphi, \psi) = \\ \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \forall(\{\}(\psi_{m+1} : \varphi_{m+1}, \to (\varphi, \psi)))] \ .$$

$$\delta_{2,m}((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\varphi,\psi) = \gamma[\psi_1:\varphi_1,\ldots,\psi_m:\varphi_m, \rightarrow(\exists(\{\}(\psi_{m+1}:\varphi_{m+1},\varphi)),\psi)] \ .$$

All of the two functions we have defined are computable functions from $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$ to Σ^*. If we define a function δ_m over $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$ as follows:

$$\delta_m((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\varphi,\psi) = \begin{pmatrix} \delta_{1,m}((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\varphi,\psi), \\ \delta_{2,m}((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\varphi,\psi) \end{pmatrix}$$

then δ_m is a computable function from $(\Sigma^*)^{2(m+1)} \times \Sigma^* \times \Sigma^*$ to $(\Sigma^*)^2$, therefore the set

$$D_m = \{\delta_m((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\varphi,\psi) | ((\psi_1,\varphi_1,\ldots,\psi_{m+1},\varphi_{m+1}),\varphi,\psi) \in Q'_{m+1,2}\}$$

is a r.e. subset of $(\Sigma^*)^2$.

If we now consider the set $\bigcup_{m\geqslant 1} D_m$ then this is a r.e. subset of $(\Sigma^*)^2$ and actually this set is equal to our set $R_{12.19}$ which so is r.e. itself. □

Then let $R_{12.19} \in \mathcal{R}$.

Lemma 12.21. *Let m be a positive integer. Let $x_1,\ldots,x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1,\ldots,\varphi_m \in E$ and assume $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$. Define $k = k[x_1 : \varphi_1,\ldots,x_m:\varphi_m]$ and let $\varphi,\psi,\chi \in S(k)$.*

Under these assumptions we have

- $\rightarrow(\wedge(\varphi,\psi),\chi), \rightarrow(\varphi,\rightarrow(\psi,\chi)) \in S(k)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\varphi,\rightarrow(\psi,\chi))] \in S(\epsilon)$,
- $\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\wedge(\varphi,\psi),\chi)] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\varphi,\rightarrow(\psi,\chi))])$ then $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\wedge(\varphi,\psi),\chi)])$

Proof. We assume $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\varphi,\rightarrow(\psi,\chi))])$ which can be rewritten

$$P_\forall(\{\#(k,\rightarrow(\varphi,\rightarrow(\psi,\chi)),\sigma)|\ \sigma\in\Xi(k)\})$$

$$P_\forall(\{P_\rightarrow(\#(k,\varphi,\sigma),\#(k,\rightarrow(\psi,\chi),\sigma))|\ \sigma\in\Xi(k)\})$$

$$P_\forall(\{P_\rightarrow(\#(k,\varphi,\sigma),P_\rightarrow(\#(k,\psi,\sigma),\#(k,\chi,\sigma)))|\ \sigma\in\Xi(k)\})\ ,$$

'for each $\sigma \in \Xi(k)$ $\#(k,\varphi,\sigma)$ is false or ($\#(k,\psi,\sigma)$ is false or $\#(k,\chi,\sigma)$)'.

We now try to show $\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\wedge(\varphi,\psi),\chi)])$ which in turn can be rewritten

$$P_\forall(\{\#(k,\rightarrow(\wedge(\varphi,\psi),\chi),\sigma)|\ \sigma\in\Xi(k)\})$$

$$P_\forall(\{P_\to(\#(k, \wedge(\varphi,\psi),\sigma), \#(k,\chi,\sigma))|\ \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_\to(P_\wedge(\#(k,\varphi,\sigma), \#(k,\psi,\sigma)), \#(k,\chi,\sigma))|\ \sigma \in \Xi(k)\}) \ ,$$

'for each $\sigma \in \Xi(k)$ it is false that $(\#(k,\varphi,\sigma)$ and $\#(k,\psi,\sigma))$ or $\#(k,\chi,\sigma)$'.

Let $\sigma \in \Xi(k)$, let's also keep in mind that $\#(k,\varphi,\sigma)$ is false or $\#(k,\psi,\sigma)$ is false or $\#(k,\chi,\sigma)$. If $\#(k,\varphi,\sigma)$ is false then it is false that $(\#(k,\varphi,\sigma)$ and $\#(k,\psi,\sigma))$. Similarly if $\#(k,\psi,\sigma)$ is false then it is false that $(\#(k,\varphi,\sigma)$ and $\#(k,\psi,\sigma))$. Finally if $\#(k,\chi,\sigma)$ holds then it holds itself and what we wanted to show is true. □

We can create a set $R_{12.21}$ which is the set of all pairs

$$(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\varphi, \to (\psi,\chi))], \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\wedge(\varphi,\psi),\chi)])$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi, \chi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Lemma 12.21 shows us that this set (which is a potential 1-ary rule) is 'sound'. In order to use $R_{12.21}$ as a rule in our system we also need to show that $R_{12.21}$ is r.e..

Lemma 12.22. *$R_{12.21}$ is r.e..*

Proof. Given a positive integer m and $(x_1, \varphi_1, \ldots, x_m, \varphi_m) \in R_m$ we can notice the following:

- $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \in K$;
- $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$ is r.e.;
- $\{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^3$ is r.e..

Let's define

$$Q_{m,3} = \bigcup_{(x_1,\varphi_1,\ldots,x_m,\varphi_m)\in R_m} \{(x_1, \varphi_1, \ldots, x_m, \varphi_m)\} \times S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])^3 \ .$$

Clearly $Q_{m,3} \subseteq (\Sigma^*)^{2m} \times (\Sigma^*)^3$ is also r.e..

We now define two functions $\delta_{1,m}$, $\delta_{2,m}$ over $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ as follows. Given $((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) \in (\Sigma^*)^{2m} \times (\Sigma^*)^3$

$$\delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \to (\varphi, \to (\psi, \chi))] \ .$$

$$\delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \gamma[\psi_1 : \varphi_1, \ldots, \psi_m : \varphi_m, \to (\wedge(\varphi,\psi), \chi)] \ .$$

All of the two functions we have defined are computable functions from $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ to Σ^*. If we define a function δ_m over $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ as follows:

$$\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) = \begin{pmatrix} \delta_{1,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)), \\ \delta_{2,m}((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)), \end{pmatrix}$$

then δ_m is a computable function from $(\Sigma^*)^{2m} \times (\Sigma^*)^3$ to $(\Sigma^*)^2$, therefore the set

$$D_m = \{\delta_m((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) | \; ((\psi_1, \varphi_1, \ldots, \psi_m, \varphi_m), (\varphi, \psi, \chi)) \in Q_{m,3}\}$$

is a r.e. subset of $(\Sigma^*)^2$.

If we now consider the set $\bigcup_{m \geqslant 1} D_m$ then this is a r.e. subset of $(\Sigma^*)^2$ and actually this set is equal to our set $R_{12.21}$ which so is r.e. itself. □

Then let $R_{12.21} \in \mathcal{R}$.

13. Another proof

For each x, y natural numbers we say that x divides y if there exists a natural number α such that $y = x\alpha$.

In our example we want to show that for each x, y, z natural numbers if x divides y and y divides z then x divides z.

Of course, we first need to build an expression in our language to express this. To build that expression we must add to our language a constant symbol N to represent the set of natural numbers $\mathbb{N}$, so that we have $\#(N) = \mathbb{N}$.

And we need to add another constant symbol in our language. This is the symbol $*$ that stands for the product (or multiplication) operation in the domain $\mathbb{N}$ of natural numbers. Therefore $\#(*)$ is a function defined on $\mathbb{N} \times \mathbb{N}$ and for each $\alpha, \beta \in \mathbb{N}$ $\#(*)(\alpha, \beta)$ is the product of α and β, in other words $\#(*)(\alpha, \beta) = \alpha \cdot \beta$.

The set $\mathcal{F}$ of operators is the same we have assumed in our former example, so it must contain all of these symbols: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in, =$.

So, in order to formalize our statement and a proof of it, we will use a language $(\mathcal{V}, \mathcal{F}, \mathcal{C}, \#, \{D_1, \ldots, D_p\}, q_{max})$ which must be as follows

$$\mathcal{V} = \{x, y, z, u, v, w\}.$$

$$\mathcal{F} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in, =\},$$

$$\mathcal{C} = \{N, *\};$$

Moreover, we need to include the set $\mathbb{N}$ of natural numbers in our additional sets, so let $p = 1$ and $D_1 = \mathbb{N}$, and we also set a conventional value of 1 for q_{max}.

At this point, the statement we wish to prove is the following:

$$\gamma\left[x : N, y : N, z : N, \rightarrow \left(\wedge\left(\begin{array}{c}\exists(\{\}(u : N, = (y, *(x, u)))),\\ \exists(\{\}(v : N, = (z, *(y, v))))\end{array}\right), \exists(\{\}(w : N, = (z, *(x, w))))\right)\right]. \qquad (Th_1)$$

Let $k = k[x : N, y : N, z : N, u : N, v : N]$. By lemma 9.1 $u \in E(k)$. If we define $k_u = k[x : N, y : N, z : N]$ then for each $\sigma \in \Xi(k)$ $\sigma_{/dom(k_u)} \in \Xi(k_u)$ and $\#(k, u, \sigma) \in \#(k_u, N, \sigma_{/dom(k_u)}) = \#(N) = \mathbb{N}$.

Similarly by 9.1 $x \in E(k)$. If we define $k_x = \epsilon$ then for each $\sigma \in \Xi(k)$ $\sigma_{/dom(k_x)} \in \Xi(k_x)$ and $\#(k, x, \sigma) \in \#(k_x, N, \sigma_{/dom(k_x)}) = \#(N) = \mathbb{N}$.

By lemma 12.3 it follows that $(*)(x, u) \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, (*)(x, u), \sigma) = (\#(k, x, \sigma) \cdot \#(k, u, \sigma)) \in \mathbb{N}$.

The first sentence in our proof is an instance of axiom $A_{12.5}$.

$$\gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\wedge(= (y, xu), = (z, yv)), = (z, (xu)v))] \qquad (13.1)$$

The following also hold:

- $\wedge(= (y, xu), = (z, yv)) \in S(k)$.

By $A_{12.7}$ we obtain

$$\gamma[x : N, y : N, z : N, u : N, v : N, \rightarrow (\wedge(= (y, xu), = (z, yv)), = ((xu)v, x(uv)))] \qquad (13.2)$$

The following also hold:

- $= (z, (xu)v) \in S(k)$,
- $= ((xu)v, x(uv)) \in S(k)$.

By 13.1, 13.2 and rule $R_{10.14}$

$$\gamma\left[x : N, y : N, z : N, u : N, v : N, \rightarrow \left(\wedge\left(\begin{array}{c}= (y, xu),\\ = (z, yv)\end{array}\right), \wedge\left(\begin{array}{c}= (z, (xu)v),\\ = ((xu)v, x(uv))\end{array}\right)\right)\right]. \qquad (13.3)$$

The following also hold:

- $z \in E(k)$,
- $(xu)v \in E(k)$,
- $x(uv) \in E(k)$.

By axiom $A_{12.9}$

$$\gamma\left[x : N, \ldots, v : N, \rightarrow \left(\wedge\left(\begin{array}{c}= (y, xu),\\ = (z, yv)\end{array}\right), \rightarrow \left(\wedge\left(\begin{array}{c}= (z, (xu)v),\\ = ((xu)v, x(uv))\end{array}\right), = (z, x(uv))\right)\right)\right]. \qquad (13.4)$$

The following also hold:

- $=(z,x(uv)) \in S(k)$.

By 13.3, 13.4 and rule $R_{12.11}$

$$\gamma\left[x:N,y:N,z:N,u:N,v:N,\to\left(\wedge\left(\begin{array}{c}=(y,xu),\\=(z,yv)\end{array}\right),=(z,x(uv))\right)\right]. \tag{13.5}$$

The following also hold: $(*)(u,v) \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k,(*)(u,v),\sigma) = (\#(k,u,\sigma)\cdot\#(k,v,\sigma)) \in \mathbb{N}$ (cfr lemma 12.7).

By 13.5 and rule $R_{12.13}$

$$\gamma\left[x:N,y:N,z:N,u:N,v:N,\to\left(\wedge\left(\begin{array}{c}=(y,xu),\\=(z,yv)\end{array}\right),\exists(\{\}(w:N,=(z,xw)))\right)\right]. \tag{13.6}$$

The following also holds: $\exists(\{\}(w:N,=(z,xw))) \in S(k)$ (cfr lemma 12.13).

By 13.6 and rule $R_{10.22}$

$$\gamma\left[x:N,y:N,z:N,u:N,v:N,\to\left(=(y,xu),\to\left(\begin{array}{c}=(z,yv),\\\exists(\{\}(w:N,=(z,xw)))\end{array}\right)\right)\right]. \tag{13.7}$$

Let $h = k[x:N,y:N,z:N,u:N]$. By lemma 9.1 $u \in E(h)$. If we define $h_u = k[x:N,y:N,z:N]$ then for each $\rho \in \Xi(h)$ $\rho_{/dom(h_u)} \in \Xi(h_u)$ and $\#(h,u,\rho) \in \#(h_u,N,\rho_{/dom(h_u)}) = \#(N) = \mathbb{N}$.

Similarly by 9.1 $x \in E(h)$. If we define $h_x = \epsilon$ then for each $\rho \in \Xi(h)$ $\rho_{/dom(h_x)} \in \Xi(h_x)$ and $\#(h,x,\rho) \in \#(h_x,N,\rho_{/dom(h_x)}) = \#(N) = \mathbb{N}$.

By lemma 12.3 it follows that $(*)(x,u) \in E(h)$ and for each $\rho \in \Xi(h)$ $\#(h,(*)(x,u),\rho) = (\#(h,x,\rho)\cdot\#(h,u,\rho)) \in \mathbb{N}$.

Still by 9.1 $y \in E(h)$ and by lemma 12.2 $=(y,xu) \in S(h)$.

By 13.7 and rule $R_{12.15}$

$$\gamma\left[x:N,y:N,z:N,u:N,\to\left(=(y,xu),\forall\left(\{\}\left(v:N,\to\left(\begin{array}{c}=(z,yv),\\\exists(\{\}(w:N,=(z,xw)))\end{array}\right)\right)\right)\right)\right]. \tag{13.8}$$

We now want to prove that $\exists(\{\}(w:N,=(z,xw))) \in S(h)$. We start by defining $g = h+ < w,N >$.

We have $N \in E(h)$ and for each $\rho \in \Xi(h)$ $\#(h,N,\rho) = \#(N) = \mathbb{N}$. So $N \in E_s(h)$. Moreover $w \in (\mathcal{V} - var(h))$ so by lemma 8.21 $g \in K$.

We now want to show that $=(z,xw)$ belongs to $S(g)$. Since $N \in E_s(h)$ we have $H[x:N,y:N,z:N,u:N,w:N]$. We have

$$k[x:N,y:N,z:N,u:N,w:N] = h+(w,N) = g\ .$$

Using lemma 9.1 we obtain that $z, x, w \in E(g)$. If we define $g_x = \epsilon$ then for each $\sigma \in \Xi(g)$ $\sigma_{/dom(g_x)} \in \Xi(g_x)$ and $\#(g, x, \sigma) \in \#(g_x, N, \sigma_{/dom(g_x)}) = \#(N) = \mathbb{N}$.

Moreover for each $\sigma \in \Xi(g)$ $\sigma_{/dom(h)} \in \Xi(h)$ $\#(g, w, \sigma) \in \#(h, N, \sigma_{/dom(h)}) = \#(N) = \mathbb{N}$.

By lemma 12.3 it follows that $(*)(x, w) \in E(g)$ and for each $\sigma \in \Xi(g)$ $\#(g, (*)(x, w), \sigma) = (\#(g, x, \sigma) \cdot \#(g, w, \sigma)) \in \mathbb{N}$.

By lemma 12.2 $= (z, xw)$ belongs to $S(g)$. We can now apply lemma 8.21 and obtain that $\exists(\{\}(w : N, = (z, xw))) \in S(h)$.

To sum up we have $= (y, xu) \in S(h)$, $= (z, yv) \in S(k)$,
$\exists(\{\}(w : N, = (z, xw))) \in S(h) \cap S(k)$.

By 13.8 and rule $R_{12.17}$

$$\gamma \left[x : N, y : N, z : N, u : N, \to \left(= (y, xu), \to \left(\begin{array}{c} \exists(\{\}(v : N, = (z, yv))), \\ \exists(\{\}(w : N, = (z, xw))) \end{array} \right) \right) \right] . \tag{13.9}$$

Using lemma 7.6, we can rewrite 13.9 as

$$\gamma \left[x : N, y : N, z : N, \forall \left(\{\} \left(u : N, \to \left(= (y, xu), \to \left(\begin{array}{c} \exists(\{\}(v : N, = (z, yv))), \\ \exists(\{\}(w : N, = (z, xw))) \end{array} \right) \right) \right) \right) \right] . \tag{13.10}$$

Let $\kappa = k[x : N, y : N, z : N]$. We have proved that
$\exists(\{\}(w : N, = (z, xw))) \in S(h)$ and $\exists(\{\}(v : N, = (z, yv))) \in S(h)$.

We also need to prove that $\exists(\{\}(w : N, = (z, xw))) \in S(\kappa)$
and $\exists(\{\}(v : N, = (z, yv))) \in S(\kappa)$.

In order to prove $\exists(\{\}(w : N, = (z, xw))) \in S(\kappa)$ we redefine g as $\kappa + < w, N >$.

We have $N \in E(\kappa)$ and for each $\rho \in \Xi(\kappa)$ $\#(\kappa, N, \rho) = \#(N) = \mathbb{N}$. So $N \in E_s(\kappa)$. Moreover $w \in (\mathcal{V} - var(\kappa))$ so by lemma 8.21 $g \in K$.

We now want to show that $= (z, xw)$ belongs to $S(g)$. It follows from lemma 11.1 that $H[x : N, y : N, z : N, w : N]$. We have

$$k[x : N, y : N, z : N, w : N] = \kappa + < w, N > = g \ .$$

Using lemma 9.1 we obtain that $z, x, w \in E(g)$. If we define $g_x = \epsilon$ then for each $\sigma \in \Xi(g)$ $\sigma_{/dom(g_x)} \in \Xi(g_x)$ and $\#(g, x, \sigma) \in \#(g_x, N, \sigma_{/dom(g_x)}) = \#(N) = \mathbb{N}$.

Moreover for each $\sigma \in \Xi(g)$ $\sigma_{/dom(\kappa)} \in \Xi(\kappa)$ $\#(g, w, \sigma) \in \#(\kappa, N, \sigma_{/dom(\kappa)}) = \#(N) = \mathbb{N}$.

By lemma 12.3 it follows that $(*)(x, w) \in E(g)$ and for each $\sigma \in \Xi(g)$ $\#(g, (*)(x, w), \sigma) = (\#(g, x, \sigma) \cdot \#(g, w, \sigma)) \in \mathbb{N}$.

By lemma 12.2 $= (z, xw)$ belongs to $S(g)$. We can now apply lemma 8.21 and obtain that $\exists(\{\}(w : N, = (z, xw))) \in S(\kappa)$.

In order to prove $\exists(\{\}(v : N, = (z, yv))) \in S(\kappa)$ we redefine g as $\kappa + < v, N >$.

We have $N \in E(\kappa)$ and for each $\rho \in \Xi(\kappa)$ $\#(\kappa, N, \rho) = \#(N) = \mathbb{N}$. So $N \in E_s(\kappa)$. Moreover $v \in (\mathcal{V} - var(\kappa))$ so by lemma 8.21 $g \in K$.

We now want to show that $=(z, yv)$ belongs to $S(g)$. It follows from lemma 11.1 that $H[x : N, y : N, z : N, v : N]$. We have

$$k[x : N, y : N, z : N, v : N] = \kappa + < v, N >= g \ .$$

Using lemma 9.1 we obtain that $z, y, v \in E(g)$. If we define $g_y = k[x : N]$ then for each $\sigma \in \Xi(g)$ $\sigma_{/dom(g_y)} \in \Xi(g_y)$ and $\#(g, y, \sigma) \in \#(g_y, N, \sigma_{/dom(g_y)}) = \#(N) = \mathbb{N}$.

Moreover for each $\sigma \in \Xi(g)$ $\sigma_{/dom(\kappa)} \in \Xi(\kappa)$ $\#(g, v, \sigma) \in \#(\kappa, N, \sigma_{/dom(\kappa)}) = \#(N) = \mathbb{N}$.

By lemma 12.3 it follows that $(*)(y, v) \in E(g)$ and for each $\sigma \in \Xi(g)$ $\#(g, (*)(y, v), \sigma) = (\#(g, y, \sigma) \cdot \#(g, v, \sigma)) \in \mathbb{N}$.

By lemma 12.2 $=(z, yv)$ belongs to $S(g)$. We can now apply lemma 8.21 and obtain that $\exists(\{\}(v : N, =(z, yv))) \in S(\kappa)$.

Then if we apply rule $R_{12.19}$ to 13.10 we obtain

$$\gamma \left[x : N, y : N, z : N, \rightarrow \left(\exists(\{\}(u : N, =(y, xu))), \rightarrow \left(\begin{array}{c} \exists(\{\}(v : N, =(z, yv))), \\ \exists(\{\}(w : N, =(z, xw))) \end{array} \right) \right) \right] . \tag{13.11}$$

We have also $\exists(\{\}(u : N, =(y, xu))) \in S(\kappa)$, so if we apply rule $R_{12.21}$ we finally obtain

$$\gamma \left[x : N, y : N, z : N, \rightarrow \left(\wedge \left(\begin{array}{c} \exists(\{\}(u : N, =(y, xu))) \\ \exists(\{\}(v : N, =(z, yv))) \end{array} \right), \exists(\{\}(w : N, =(z, xw))) \right) \right] . \tag{13.12}$$

14. Expression with mixed orders

We mentioned in the introduction that in our system we can express statements in which both quantifiers over individuals and quantifiers over sets of individuals occur. We made the simple example of the following statement:

for each subset X of $\mathbb{N}$ and for each x $\in \mathbb{N}$ we have x $\in$ X or x $\notin$ X .

Let's see how we can map the statement within our system. In our language we need two constants: N whose meaning is the set of natural numbers, Π which has a predefined meaning of a function that produces the power set of the provided argument.

The set $\mathcal{F}$ of operators is the same we have assumed in our other examples, so it must contain all of these symbols: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in, =$.

So, in order to formalize our statement and a proof of it, we will use a language $(\mathcal{V}, \mathcal{F}, \mathcal{C}, \#, \{D_1, \ldots, D_p\}, q_{max})$ which must be as follows

$$\mathcal{V} = \{x, X\}.$$

$$\mathcal{F} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in, =\},$$

$$\mathcal{C} = \{N, \Pi\};$$

Moreover, we need to include the set $\mathbb{N}$ of natural numbers in our additional sets, so let $p = 1$ and $D_1 = \mathbb{N}$, and we also set a conventional value of 1 for q_{max}.

Since $\mathbb{N} \in \mathcal{P}(\mathbb{N})$, $\mathbb{N}$ belongs to the domain of Π.

With this setup, we can express the statement as follows

$$\gamma[x : N, X : \Pi(N), \vee(\in (x, X), \neg(\in (x, X)))] \ .$$

Let's now verify this is an expression of our language.

First of all we want to verify that $H[x : N, X : \Pi(N)]$ holds.

Clearly $N \in E_s(\epsilon)$ so $H[x : N]$ holds.

In order to show that $H[x : N, X : \Pi(N)]$ is true we have to show that $\Pi(N) \in E_s(k[x : N])$.

We have $N \in E(k[x : N])$ and for each $\sigma \in \Xi(k[x : N])$ $\#(k[x : N], N, \sigma) = \#(N) = \mathbb{N} \in \mathcal{P}(\mathbb{N})$.

So we can apply lemma 8.18 and obtain that $\Pi(N) \in E(k[x : N])$ and for each $\sigma \in \Xi(k[x : N])$ $\#(k[x : N], \Pi(N), \sigma) = \#(\Pi)(\#(k[x : N], N, \sigma)) = \mathcal{P}(\mathbb{N})$.

Therefore $\Pi(N) \in E_s(k[x : N])$ holds and $H[x : N, X : \Pi(N)]$ holds.

Let now $k = k[x : N, X : \Pi(N)]$, we try to show that $\in (x, X) \in S(k)$.

Using lemma 9.1 we obtain that $x, X \in E(k)$.

Moreover, let $h = k[x : N]$, then for each $\sigma \in \Xi(k)$, $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k, X, \sigma) \in \#(h, \Pi(N), \sigma_{/dom(h)}) = \mathcal{P}(\mathbb{N})$. Therefore $\#(k, X, \sigma)$ is a set, we can apply lemma 9.2 and obtain that $\in (x, X) \in S(k)$.

As a consequence of this $\vee(\in (x, X), \neg(\in (x, X))) \in S(k)$ and finally

$$\gamma[x : N, X : \Pi(N), \vee(\in (x, X), \neg(\in (x, X)))] \in S(\epsilon) \ .$$

15. Further study

Of course, further investigations about our approach to logic can be performed. We have mentioned in section 8.2 the topic on the completeness or incompleteness of our deductive systems. Then we have introduced some example of a deductive system. Some questions that I have not investigated in depth are the following:

- can we describe a deductive system within our logic system as a recursively axiomatised formal system?

- given a language that does not include arithmetic, under which conditions, if any, a deductive system within our logic system is complete?

Another interesting (and not extremely easy) topic is about comparing the expressive power of our system with the one of standard logic systems.

Another topic to consider is substitution. First-order logic features the notion of 'substitution' (see e.g. Enderton's book [2]). Under appropriate assumptions, we can apply substitution to a formula φ and obtain a new formula φ^x_t, by replacing the free occurrences of the variable x by the term t. In our approach we could be able to define a similar notion, with the difference that for us t could be a generic expression. I have somehow studied how the topic of substitution could be applied to this type of system, but with respect to a former version of my system. I am rather confident that general substitution mechanisms can be introduced for this type of logic, but I'm not sure how much work this would require. After all I suppose the introduction of general substitution mechanisms could be considered as not being properly a core topic about this approach, since for instance we can use simplified substitution mechanisms.

Finally, let's also briefly talk about paradoxes. A paradox is usually a situation in which a contradiction or inconsistency occurs, in other words a paradox arises when we can build a sentence φ such that both φ and $\neg(\varphi)$ can be derived. Since our system is consistent it shouldn't be possible to have true paradoxes in it. If we have proved the consistency of our system, what can we do more than this to exclude that the system is vulnerable to paradoxes?

It could anyway not be wrong to discuss some of the most known paradoxical arguments to ask ourselves if our system could be vulnerable to one of them.

We begin with Russell's paradox. Assume we can build the set A of all those sets X such that X is not a member of X. Clearly, if $A \in A$ then $A \notin A$ and conversely if $A \notin A$ then $A \in A$. We have proved both $A \in A$ and its negation, and this is the Russell's paradox.
It seems in our system we cannot generate this paradox since building a set is permitted only if you rely on already defined sets. When trying to build set A in our language we could obtain something like this:

$$\{\}(\neg(\in (X, X)), X) \ .$$

However it is clear this isn't a legal expression in our language, since in our language if you want to build a context-independent expression using a variable X, then you have to assign a domain to X.

Finally we want to examine the liar paradox. Let's consider how the paradox is stated in Mendelson's book.

A man says, 'I am lying'. If he is lying, then what he says is true, so he is not lying. If he is not lying, then what he says is false, so he is lying. In any case, he is lying and he is not lying.

Mendelson classifies this paradox as a 'semantic paradox' because it makes use of concepts which need not occur within our standard mathematical language. I agree that, in his formulation, the paradox has some step which seems not mathematically rigorous.

We'll try to provide a more rigorous wording of the paradox.

Let A be a set, and let δ be the condition 'for each x in A x is false'. Suppose δ is the only member of A. In this case if δ is true then it is false; if on the contrary δ is false then it is true.

The explanation of the paradox is the following: simply δ cannot be the only item in set A. In fact, suppose A has only one element, and let's call it φ. This implies δ is equivalent to 'φ is false' so it seems acceptable that δ is not φ.

Another approach to the explanation is the following.

If δ is true then for each x in A x is false, so δ is not in A. By contraposition if δ is in A then δ is false.

Moreover if δ is false and the uniqueness condition 'for each x in A $x = \delta$' is true then δ is true, thus if δ is false then 'for each x in A $x = \delta$' is false too. By contraposition if 'for each x in A $x = \delta$' then δ is true.

Therefore if δ is the only element in A then δ is true and false at the same time. This implies δ cannot be the only item in A.

On the basis of this argument I consider the liar paradox as an apparent paradox that actually has an explanation. What is the relation between our approach to logic and the liar paradox?

Standard logic isn't very suitable to express this paradox. In fact first-order logic is not designed to construct a condition like our condition δ (= 'for each x in A x is false'), and moreover, it is clearly not designed to say 'δ belongs to set A'. These conditions aren't plainly leading to inconsistency, so it is desirable they can be expressed in a general approach to logic. And our system permits to express them. The paradox isn't ought to simply using these conditions, it is due to an assumption that is clearly false, and the so-called paradox is simply the proof of its falseness.

Related to the liar paradox is the Cretan 'paradox', which is actually not a proper paradox, but is perhaps even more 'unsettling' and we quote again Mendelson in this regard: ([5]).

> The Cretan "paradox", known in antiquity, is similar to the Liar Paradox. The Cretan philosopher Epimenides said, "All Cretans are liars". If what he said is true, then, since Epimenides is a Cretan, it must be false. Hence, what he said is false. Thus, there must be some Cretan who is not a liar. This is not logically impossible, so we do not have a genuine paradox. However, the fact that the utterance by Epimenides of that false sentence could imply the existence of some Cretan who is not a liar is rather unsettling.

If we try to put this argument in a more formal statement, it still refers to a sentence δ of the type 'for each x in A x is false', where this time A is the set of all the statements made by a Cretan and δ is a member of A. Here if δ is true then it is false, so we have to conclude that δ is false, hence there exists $x \in A$ such that x is true. As noticed by Mendelson, it can be unsettling to accept this just because δ is a member of A.

We can still use an argument we have shown above with respect to the liar paradox: If δ is true then for each x in A x is false, so δ is not in A. By contraposition if δ is in A then δ is false. And another formulation is the following: δ is false or δ is not in A.

Let A be a set of true/false statements (think to an actual list of statements) and δ be the statement 'for each x in A x is false'. We know from the discussion on the liar paradox that if A has just one element then δ cannot belong to A.

In the case of the Cretan paradox we have that δ could belong to A and there is not a constraint that A has just one element. Is it possible in this case that δ belongs to A? The basic problem is that δ, if it belongs to A, makes a reference to itself and this can lead us to suspect that δ in this case is not something well defined.

We could therefore conclude that also in this case it cannot be accepted that δ belongs to A. In this case we could 'resolve' the problem by using axioms like

$$\neg(\in (\forall(\{\}(x : \psi, \neg(x))), \psi)) \ ,$$

for each expression ψ that represents a set.

If instead we accept the possibility that δ belongs to A it is evident that we must also accept that if δ belongs to A then it is false, in fact if it were true then it would not belong to A.

As a conclusion, with respect to paradoxes, we cannot state that our system is designed to prevent for sure every possible form of paradox, for instance it doesn't prevent anyone to conceive something which is unsettling or contradictory. Anyway although I have made some assessments on the matter, I currently have no reason to suppose that the system is subject to some paradox.

References

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[5] E. Mendelson, *Introduction to Mathematical Logic - Fourth Edition*, Chapman & Hall, 1997 (first edition 1964).