Logical Dependence of Physical Determinism on Set-theoretic Metatheory

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Draft: August 15, 2025

Abstract

Baroque questions of set-theoretic foundations are widely assumed to be irrelevant to physics. In this article, I challenge this assumption. I argue that even the fundamental physical question of whether a theory is deterministic—whether it fixes a unique future given the present—can depend on one's choice of set-theoretic axiom candidates over which there is intractable disagreement. Suppose, as is customary (Earman 1986), that a deterministic theory is one whose mathematical formulation yields a unique solution to its governing equations. Then the question of whether a physical theory is deterministic becomes the question of whether there exists a unique solution to its mathematical model—typically a system of differential equations. I argue that competing axiom candidates extending standard mathematics—in particular, the Axiom of Constructibility (V = L) and large cardinal axioms strong enough to prove Projective Determinacy—can diverge on all the core dimensions of physical determinism. First, they may disagree about whether a given physical system is well-posed, and so whether a solution exists. Second, even when they agree that a solution exists, they can differ on whether that solution is unique. Finally, even when they agree that a system has a solution, and agree that this solution is unique, they may still dispute what that solution is. Whether a theory is deterministic—and even which outcome it deterministically predicts—can depend on one's choice of set-theoretic metatheory. I indicate how the conclusions extend to discrete systems and suggest directions for future research. One upshot of the discussion is that either physical theories must be relativized to set-theoretic metatheories (in which case physics itself becomes relative), or, as Quine (1951) controversially argued, the search for new axioms to settle undecidables may admit of empirical input.

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1. Introduction: Set-theoretic Foundations Meet Physical Reality

Physical theories are formulated using mathematics, but the relationship between the choice of set-theoretic metatheory and physical content is rarely examined. This paper demonstrates that

[1] Thanks to Avner Ash for catching a mistake in my treatment of PDEs, Will Cavendish for pressing me to consider the discrete case, Gabriel Goldberg for helpful discussion of complexity classes and projective uniformization in L, and Joel David Hamkins for general exchanges about the relevance of incompleteness to determinism. This article is an experiment for me. AI assistants Claude (Anthropic) and ChatGPT (OpenAI) provided important support, similar to that of research assistants who offer critical feedback, including suggestions for mathematical formulations and efficient references to known results. Needless to say, I am responsible for all errors. See Balaguer 1998; Berry 2023, Forthcoming; Colyvan 1998; Clarke-Doane 2022, 2024, 2025; Farah & Magidor 2012; Jonas 2024; and Pour-El & Richards 1989 and the references to follow for cognate discussions.

even basic physical properties—like whether a system evolves deterministically from initial conditions—can depend on which new axiom candidates for set theory we embrace.

Let **ZFC** denote Zermelo–Fraenkel set theory with the Axiom of Choice. It provides the standard foundation for most mathematics. But it must be incomplete if it is consistent, by Gödel's incompleteness theorems (Gödel 1931). Moreover, subsequent work by Cohen (1963) and others showed that the incompleteness of ZFC goes far beyond codings of apparently paradoxical sentences like "I am not provable," including Hilbert's First Problem, the Continuum Problem: Is the cardinality of the set of real numbers the next greatest after that of the set of natural numbers? This raises the question of whether we might adopt new axioms that settle all or many "natural" undecidable questions. Two starkly opposed candidates for new axioms are "large cardinal" axioms and the Axiom of Constructibility.

Large cardinal axioms are strong axioms of infinity. I will use **LC** to denote some fixed large cardinal axiom strong enough to imply the regularity principle known as **Projective Determinacy (PD)**—for concreteness, the axiom that there are ω many Woodin cardinals (cf. Kanamori 2009; Martin and Steel 1989; Woodin 1988). Projective Determinacy asserts that all projective sets are Lebesgue measurable, have the property of Baire, and satisfy other regularity conditions. I will write **V = L** to denote Gödel's Axiom of Constructibility (Gödel 1940). This postulates that every set is constructible in a precise technical sense related to Russell's ramified theory of types. It leads to a relatively minimal universe of sets and contradicts strong large cardinal axioms—in particular, LC. (In detail: let **PDef(A)** refer to the set of all subsets of **A** definable in the structure $\langle A, \in \rangle$ by first-order formulas with parameters in **A**. Then **V = L** says that every set lies in the hierarchy obtained by transfinite recursion on the ordinals: $\mathbf{L_0 = \varnothing}$, $\mathbf{L_{\alpha+1} = PDef(L_\alpha)}$, and $\mathbf{L_\gamma = \cup\{L_\alpha : \alpha < \gamma\}}$ for limit ordinals γ.)

As Jensen writes, "[V = L] on the one hand and large cardinals and determinacy on the other embody two radically different conceptions of the universe of sets" (Jensen 1995, pp. 400–401).[2] According to V = L, there are simply definable sets with pathological regularity behavior (e.g., they are non-measurable); under LC, such sets are regular. Differences like these can be encoded into the data of physical models. The same formulas can then describe a well-posed, uniquely solvable system in one universe, and an ill-posed or non-unique system in another. In fact, the identity of the solution can vary between models of V = L and models of LC when it is provably unique in both contexts. What changes is the set-theoretic foundations, not the interpretation.

It is instructive to contrast the foundational dependence described here with other known failures of determinism in classical physics, such as Norton's Dome (Norton 2008). The Dome describes a physical system, governed by Newtonian mechanics, that admits a spontaneous, uncaused motion. While debated (cf. Malament 2008), the source of its indeterminism is located within the physical laws as applied to a specific (albeit idealized) configuration of matter and force. The form of non-uniqueness explored in this paper is of a very different kind. It is not a feature of the physical laws alone, but emerges through the interaction between those laws and the set-theoretic foundations by which they are conceptualized. This reveals a new axis of non-uniqueness in physics that, to my knowledge, has been overlooked.

[2] For discussion of other "restrictive" axioms, besides V=L, see Fraenkel, Bar-Hillel & Levy (1973, §6.4).

It is important to clarify that changes in set-theoretic axioms can alter regularity facts only when the problem's data or the admissible solution space rely on definability properties that are independent of ZFC. Many well-posedness theorems involve only Borel coefficients or are otherwise provable within ZFC itself, without sensitivity to the V = L versus LC dichotomy. The point is that when PDEs, variational principles, or discrete dynamical systems are formulated so that the admissible sets, selection functions, or uniformization steps climb higher in the projective hierarchy, the truth of existence or uniqueness can hinge on regularity properties (measurability, Baire property, determinacy) that are independent of ZFC. In such cases, new axiom candidates like V = L (which maximizes definability but minimizes regularity) and strong large-cardinal hypotheses (which yield Projective Determinacy and robust regularity for projective sets) can enforce different outcomes for well-posedness, uniqueness, or identity of solutions. The thesis of this paper concerns these non-absolute, definability-sensitive instances. I do not claim axiom-relativity for all ordinary textbook PDEs with classical data. (See the Appendix for a basic review of set-theoretic definability and Projective Determinacy.)

The arguments to follow employ the method of conceptual analysis. Just as Gettier's (1963) cases need not be actual to reveal facts about the concept of knowledge, the physical theories that I discuss need not describe true physics to reveal facts about the concept of determinism. I argue that V = L versus LC classify different possible physical theories as deterministic or non-deterministic. This demonstrates that the concept of determinism is contested or relative in a way that has not been appreciated—regardless of whether physicists happen to encounter such theories in practice.[3] That said, my examples are not far out. The mechanisms involved—weak formulation of PDEs, symmetry breaking in variational problems, recurrence relations—are standard in mathematical physics. An important question for future research is whether the determinism of viable physical theories is also foundationally sensitive in the way that I describe.

2. The Mathematical Framework of Determinism

Physicists regiment determinism through Hadamard **well-posedness**: for admissible initial data, (i) a solution exists; (ii) it is unique; and (iii) it depends continuously on the data (Hadamard 1923; Evans 2010). This abstract template is instantiated throughout physics, from ordinary differential equation (ODE) models of point mechanics to partial differential equation (PDE) models of fluids and fields, and also in discrete models. In all cases, the verdicts rest on technical hypotheses. Coefficients are typically required to be measurable (or more regular), domains to have standard geometric properties, and function spaces to be complete (Evans 2010; Brezis 2011).

These hypotheses are not mere bookkeeping. In frameworks such as $L^2(0,T; H^1_0(\Omega))$ for parabolic equations, the weak formulation is literally undefined if basic measurability fails. The key observation of this article is that such regularity assumptions can turn on speculative extensions of standard set theory. Sets that are Δ^1_2 in the projective hierarchy are non-measurable in V = L but measurable under LC. When the same definable set is inserted into a model as a coefficient or in the data, the model's determinism profile can change with the metatheory. The rest of the paper develops this point through explicit examples.

[3] See Earman 1986, 2007; Fletcher 2012; Malament (2008); Norton 2008; Fletcher 2012; Wilson 2009; Ismael 2016; Loewer 2001, 2020; Chen 2024;, Irndl 2009 for relevant discussion of determinism, and Butterfield 2007)for background. See Halvorson, Manchak, & Weatherall 2025 for a recent intervention.

3. The Set-Theoretic Divide: Constructibility versus Large Cardinals

The Axiom of Constructibility (V=L) and strong large-cardinal hypotheses (LC) yield different regularity facts about definable sets of reals and thereby different analytic behaviors in models that incorporate definability-sensitive data. Under V=L there is a Δ^1_2 "good" definable well-order of $\mathbb{R}$. Fix a coding of reals into [0,1] and define:

$$W := \{ (x, y) \in [0,1]^2 : x < y \},$$

where "<" is such a Δ^1_2 well-order. (For expositions see Kechris 1995 and Moschovakis 2009; for a recent discussion of definable well-orders and the "good Δ^1_2" property, see Kanovei 2022, §2.) Under this assumption one obtains definable non-Lebesgue-measurable sets by a classical Sierpiński–Fubini argument. For each y, the horizontal section:

$$W^y := \{ x : x < y \}$$

is countable (hence measure 0), while for each x, the vertical section:

$$W_x := \{ y : x < y \}$$

is co-countable (hence measure 1). If χ^W were measurable, Tonelli/Fubini would force the iterated integrals to agree. Since the inner integrals are 0 and 1, respectively, this is impossible. Hence χ^W is non-measurable. By contrast, LC, and so PD, implies that any Δ^1_2 set is measurable.

Solovay (1970) showed that (assuming an inaccessible cardinal) there is a model of ZF + the Axiom of Dependent Choice in which *every* set of reals is Lebesgue measurable. This underscores that the coherence phenomena we exploit are not idiosyncratic to PD. They reflect a general alignment between determinacy/regularity principles and the well-posedness of analytical formulations used in physics. (See Appendix B for absoluteness facts used.)

4. Three Types of Foundational Dependence

The dependence of physics on set theory appears in three progressively more subtle stages, which I call **equation coherence**, **solution uniqueness**, and **solution identity**.

(i) Equation coherence. A model's formal specification—such as the weak form of a PDE—is coherent in a given functional-analytic framework only if coefficients and data satisfy minimal regularity requirements (e.g., measurability). If a required integral cannot be defined in the standard Lebesgue σ-algebra, the model is not merely unsolved; it is undefined in that framework. Whether such regularity holds can hinge on set-theoretic assumptions.

(ii) Solution uniqueness. Given coherence and existence, determinism in the classical sense reduces to uniqueness of solution (Earman 1986, 2007). Analytic models often secure uniqueness via strict convexity or monotonicity. In our examples, the presence or absence of a symmetry-breaking term depends on the truth value of a fixed Σ^1_3 sentence, which can differ between V=L and large-cardinal universes—altering uniqueness without changing the formal functional.

(iii) Solution identity. Even if two backgrounds agree on coherence, existence, and uniqueness, they may yield different "unique" solutions when the initial data encode non-absolute information (e.g., Σ^1_3 membership). The models remain deterministic in each universe's sense, yet the predicted evolutions differ.

5. Continuous Systems: When PDEs Depend on Axioms

The following examples explore how the determinism of physical systems can become sensitive to set-theoretic axioms when physical quantities or boundary conditions are allowed to be specified by their definability properties. Physical laws are often expressed in terms of functions (e.g., density, potential, temperature), and an important question is what constitutes an admissible function. While physical intuition often favors smooth or piecewise continuous functions, more complex functions are possible and may arise in idealized models (e.g., fractals in dynamical systems). I consider the case where a physical parameter is specified by a formula in the projective hierarchy by taking a fixed Δ^1_2 set C from Section 3 and studying physical models whose coefficients or data depend on its characteristic function, χ_C. In a universe satisfying LC, χ_C is a measurable function, and standard analytic techniques apply. In the constructible universe (i.e., under V=L), χ_C is non-measurable, leading to a breakdown in the standard formulation of the physical problem – at the level of coherence, uniqueness, and solution identity.

5.1. Equation coherence: The heat equation with a non-measurable potential

Let $\Omega = \mathbb{T}^2$ (a two-torus, i.e., $[0,1]^2$ with periodic boundary). Consider

$$\partial u/\partial t - \Delta u + V(x)u = f \quad \text{on } (0,T)\times\Omega, \quad u(0,\cdot) = u_0,$$

in the standard weak framework $X ≔ L^2(0,T; H^1(\Omega)) \cap C([0,T]; L^2(\Omega))$, with test space $H^1(\Omega)$. Assume $u_0 \in L^2(\Omega)$ and $f \in L^2(0,T; H^{-1}(\Omega))$ (or $f \in L^2((0,T)\times\Omega)$ if preferred). The weak form contains

$$\int_\Omega V(x)\, u(t,x)\, v(x)\, dx \quad \text{for all } v \in H^1(\Omega).$$

Fix a lightface Δ^1_2 set $C \subseteq \Omega$ as in §3 and set $V ≔ \chi_C$. Under LC,: χ_C is Lebesgue measurable and essentially bounded. Then $u \mapsto \int_\Omega Vuv$ is a bounded bilinear form on L^2, and standard parabolic theory yields existence and uniqueness in X for the above data (equivalently: $A = -\Delta + V$ is a bounded perturbation of $-\Delta$ on $L^2(\Omega)$, so $u' + Au = f$ is well-posed). However, under V=L, χ_C can be non-measurable. Taking $u(t,\cdot) \equiv 1$ and $v \equiv 1$, the term reduces to $\int_\Omega \chi_C$, which is undefined in the Lebesgue sense. The universal quantification "for all v" fails already at $v \equiv 1$, so the weak formulation is not a well-formed statement. Coherence thus depends on the set-theoretic metatheory (LC vs. V=L), with the same displayed PDE and data classes.

5.2. Solution uniqueness: Symmetry breaking in a variational problem

Let $\Omega = \mathbb{T}^d$. Fix $0 < \varepsilon < 1$ and define

$$W_0(u) ≔ (u^2 - 1)^2 + \varepsilon u^2, \quad E(\gamma)[u] ≔ \int_\Omega [\tfrac{1}{2}\|\nabla u\|^2 + W_0(u) + \gamma u^2]\, dx.$$

Fix γ by a background-sensitive rule using a single Σ^1_3 sentence $\varphi(n_0)$:

$\gamma^M \coloneqq \{ \gamma_1$ if $M \vDash \varphi(n_0), \quad \gamma_0$ otherwise $\}$,

with $0 \le \gamma_0 < 2 - \varepsilon < \gamma_1$, and LC-models satisfying $\varphi(n_0)$ while V = L does not. Write $W\_\gamma(u) \coloneqq u^4 + (\varepsilon + \gamma - 2)u^2 + 1$. Then

$W\_\gamma''(u) = 12u^2 + 2(\varepsilon + \gamma - 2)$.

Hence strict convexity holds globally when $\varepsilon + \gamma > 2$, and the functional $u \mapsto E(\gamma)[u]$ is strictly convex and coercive on $H^1(\Omega)$, so it admits a unique minimizer by the direct method. When $\varepsilon + \gamma < 2$, W_γ has two equal minima at $u = \pm a_0$ with $a_0 \coloneqq \sqrt{(1 - (\varepsilon + \gamma)/2)} > 0$; since the gradient term vanishes on constants, the global minimizers of $E(\gamma)$ are precisely $u \equiv \pm a_0$. Thus the formal functional is fixed, but uniqueness toggles with γ—and hence with V=L and LC.

5.3. Solution identity: Heat equation with Σ^1_3-coded initial data

Fix a Σ^1_3 formula $\varphi(n)$ and n_0 with LC $\vDash \varphi(n_0)$ but V = L $\nvDash \varphi(n_0)$. For any background M, define $C^M \coloneqq \{ n \in \mathbb{N} : M \vDash \varphi(n) \}$. Choose a smooth bump $\psi \equiv 1$ on $B(0,½)$, $\mathrm{supp}(\psi) \subseteq B(0,1)$, and disjoint centers $x_□$ so that $B(x_□,2)$ are pairwise disjoint. Set $\psi_□(x) \coloneqq \psi(x - x_□)$ and

$u_0^M(x) \coloneqq \sum_{□\ge1} 3^{-n}\, \chi\_\{C^M\}(n)\, \psi_□(x)$,

which converges in C^∞ by the M-test. For the Dirichlet heat equation

$\partial u/\partial t - \Delta u = 0$ in $(0,T)\times\Omega, \quad u(0,\cdot) = u_0^M, \quad u|\_\{\partial\Omega\} = 0$,

standard parabolic theory yields a unique classical solution u^M. Because C^{PD} and C^L differ at n_0, we have $u^{PD} \neq u^L$ for all $t > 0$. Determinism (existence + uniqueness) holds according to both V=L and LC. But the deterministic evolution predicted differs.[4]

6. Discrete Systems: Foundational Sensitivity Without the Continuum

The same three types of foundational dependence identified in Section 4 appear in discrete settings. As in the continuous case, "coherence" refers to the well-posedness of the intended formulation in the standard analytic framework for the model; "uniqueness" means uniqueness of the admissible solution within that framework; and "identity" means agreement on determinism but disagreement on the deterministic evolution. These discrete analogues are constructed to parallel their continuous counterparts in Sections 5.1, 5.2, and 5.3, with the underlying definability features playing the same roles.

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[4] We might also consider the metatheory dependence of continuous dependence. Recall Hadamard (iii): stability of the solution map with respect to data in the prescribed norms. In Section 5.1, under LC the solution map $(u_0,f) \mapsto u$ is continuous in X because the standard parabolic estimates hold with $V \in L^\infty$. Under V=L, the weak formulation is ill-formed for certain Δ^1_2 potentials, so (iii) fails vacuously (there is no map). To see a non-vacuous contrast, fix the *same* measurable coefficient V and instead encode Σ^1_3-sensitive data as in §5.3. Then both backgrounds satisfy (i)–(ii) and (iii), but they yield different continuous solution maps $S^{PD} \neq S^L$ because $u_0^{PD} \neq u_0^L$. Roughly: there are models M and N and admissible data classes D such that the solution operators $S^M, S^N : D \to X$ are continuous and single-valued, yet $S^M \neq S^N$ (indeed, $S^M(d) \neq S^N(d)$ for a comeagre set of $d \in D$ constructed by the Σ^1_3 bump encoding).

6.1 Coherence difference: Discrete evolution with continuum-defined parameter

Let **C** $\subseteq$ $[0,1]^2$ be the fixed lightface Δ^1_2 set from Section 3, non-measurable in **V = L** and Lebesgue measurable under LC. Define:

$\alpha = \iint_{(0,1)^2} \boldsymbol{\chi_c(x,y)}\, dx\, dy, \quad \beta = e^{-\alpha^2}.$

(*Note:* This example concerns discrete dynamics but references a continuum-defined parameter. While the system evolves on $\ell^2(\mathbb{N})$, we define parameters using α above. Such continuum–discrete mixing is common in physics—for example, lattice gauge theory (Wilson 1974). The gate $\beta = e^{-\alpha^2}$ then determines the discrete evolution operator.)

Under **LC**, α is a well-defined real in [0,1], so β > 0 is defined. Under **V = L**, $\boldsymbol{\chi_c}$ is non-measurable, so α and β are undefined in the Lebesgue sense.

Now consider the linear difference equation on $\ell^2(\mathbb{N})$:

$f_{\square+1} = A f_{\square} + g_{\square}, \quad A = \beta \cdot I.$

Under LC, A is bounded and the recursion is well-posed. Under V=L, β is undefined, so the discrete equation is incoherent.

6.2. Discrete solution uniqueness

Let sites i = 1,…,N with periodic boundary condition $u_{(N+1)} = u_1$. Fix $0 < \varepsilon < 1$ and define, for $\gamma \geq 0$:

$\mathbf{E_\gamma[u]} = \Sigma_{i=1}^{N} [\tfrac{1}{2}(u_{i+1} - u_i)^2 + (u_i^2 - 1)^2 + (\varepsilon + \gamma)u_i^2].$

As in Section 5.2, set γ^M by the same Σ^1_3 predicate $\varphi(n_0)$, with $0 \leq \gamma_0 < 2 - \varepsilon < \gamma_1$.

High-γ regime ($\varepsilon + \gamma_1 > 2$):

For any u, the discrete Hessian is

$\nabla^2 E_\gamma(u) \geq \mathbf{L} + (-4 + 2(\varepsilon + \gamma))\mathbf{I},$

where **L** is the ring Laplacian (positive semidefinite, kernel = constants). If $-4 + 2(\varepsilon + \gamma_1) > 0$, this sum is positive definite. Hence $\mathbf{E}_\gamma$ with $\gamma = \gamma_1$ is strictly convex and admits a unique minimizer.

Low-γ regime ($\varepsilon + \gamma_0 < 2$):

The on-site potential has minima at $\pm a_0$, where $a_0 = \sqrt{(1 - (\varepsilon + \gamma_0)/2)}$. The gradient term $\Sigma_i \tfrac{1}{2}(u_{i+1} - u_i)^2 \geq 0$ vanishes only for constants, so the global minimizers are exactly $u \equiv \pm a_0$. Thus:

- LC: $\gamma^{PD} = \gamma_1 \Rightarrow$ strict convexity $\Rightarrow$ unique minimizer.
- V=L: $\gamma^{L} = \gamma_0 \Rightarrow$ double-well, flat only at constants $\Rightarrow$ two minimizers.

Coherence holds in both; uniqueness changes solely via the γ value set by a Σ^1_3 predicate (cf. Brezis 2011; Dacorogna 2008).

6.3. Discrete solution identity

Let $\varphi(n,r)$ be a Σ^1_3 formula with $n \in \mathbb{N}$ and real parameter r such that for some n_0:

- Under LC: $\varphi(n_0,r)$ is true
- Under V=L: $\varphi(n_0,r)$ is false

For each model **M**, define:

$\mathbf{C}^{\mathbf{M}} = \{n \in \mathbb{N} : \mathbf{M} \vDash \varphi(n,r)\}$.

Then $\mathbf{C}^{\mathbf{PD}}$ and $\mathbf{C}^{\mathbf{L}}$ differ at least at n_0. Define the recurrence:

$a_0 = 0, \quad a_m = a_{m-1} + 3^{-m} \cdot \chi_c^{\mathbf{M}}(m), \quad m \geq 1.$

The sequence converges to:

$a\_\infty^{M} = \Sigma_{m\geq 1} 3^{-m} \cdot \chi_c^{\mathbf{M}}(m).$

Both backgrounds prove uniqueness of the limit, but since $\chi_c^{\mathbf{L}}(n_0) = 0$ and $\chi_c^{\mathbf{PD}}(n_0) = 1$, we have $|a\_\infty^{PD} - a\_\infty^{L}| \geq (½)\cdot 3^{-n0} > 0$.

7. Directions for Future Research

Although I have focused on specific constructions (especially the measurability of simply definable sets), many possible sources of foundational sensitivity exist. Several lines of investigation appear promising:

1. **Other regularity properties.** PD ensures that all projective sets have the Baire property and the perfect set property, in addition to measurability (Kechris 1995; Moschovakis 2009). In V = L there are definable counterexamples to each. PDEs or discrete systems whose analysis depends essentially on one of these other properties could yield new examples of foundational dependence.
2. **Higher definability complexity.** Moving from Δ^1_2 to Σ^1_3 or Π^1_3 sets allows one to exploit genuine disagreement on the membership of specific natural numbers between models. At higher complexity, it may be possible to construct other systems whose existence or

uniqueness properties themselves change between universes.

3. **Physically natural models.** The examples here are designed to be standard in analytic structure but to carry definability-coded coefficients or data. It would be valuable to identify equations arising directly in mainstream physics—for example in general relativity (Wald 1984), quantum field theory (Haag 1996), or statistical mechanics (Ruelle 1969)—where such definability enters from modeling assumptions rather than being inserted for illustrative purposes.

4. **Hybrid systems.** Many real-world models couple continuous and discrete components. If definable sets appear in one part of such a system, foundational sensitivity may propagate to the other, possibly producing effects not seen in purely continuous or purely discrete settings.

5. **Computational complexity analogues.** In discrete settings, definability considerations connect with questions about the computational difficulty of solving the system. It would be interesting to explore whether there are complexity-theoretic counterparts to the examples here, where foundational differences change the computational feasibility of prediction (see Pour-El and Richards 1989; Blum et al. 1998).[5]

6. **Robustness analysis.** A systematic study of the extent to which real physical systems are robust to foundational differences would help determine the significance of these results. It would also be illustrative to study more radical alternatives to ZFC + LC, such as ZF + AD (Axiom of Determinacy), NF (New Foundations), Kripke–Platek set theory, and perhaps systems of intuitionistic analysis (Bishop and Bridges 1985).

These directions suggest that the dependence of determinism on set-theoretic background is a rich phenomenon for further investigation, not confined to the constructions presented here.

8. Conclusion: Implications and Interpretations

Physicists and philosophers of physics commonly assume that disagreements in the foundations of mathematics—such as whether to accept large cardinals or V = L—have no bearing on physical inquiry. I have argued otherwise.

The examples surveyed demonstrate that the Axiom of Constructibility (V = L) and large cardinal axioms (LC) strong enough to prove Projective Determinacy can diverge on all the key aspects of determinism. What should we make of this phenomenon?

One reaction is that, despite widespread opinion to the contrary, Quine (1951, 1990) was right, and the search for new axioms to settle undecidables is not different in kind from the search for fundamental physical laws. Quine writes: “sentences such as the continuum hypothesis...which are independent of [standard] axioms, can...be submitted to the considerations of simplicity, economy, and naturalness that contribute to the molding of scientific theories generally. Such

[5] Thanks to Gabriel Goldberg for suggesting this line of inquiry.

considerations support... [the] Axiom of Constructibility, 'V = L'" (Quine 1990, p. 95). This position raises familiar puzzles. Set theorists are not responsive to experiment even to the extent that the most theoretical physicists are (Maddy 1997, p. 155). But maybe they should be.

My own view (following Clarke-Doane 2022, 2024, 2025) is that there is no fact of the matter as to which of V = L or LC is really right. Each is a legitimate arena in which to carry out mathematical reasoning—broadly like Euclidean and non-Euclidean geometries. The heady difference is that all the geometries live in a single set theory. But a given metatheory takes itself to be maximal. This raises unresolved questions about how to formulate the "monism–pluralism" debate (Clarke-Doane 2025, Sec. 1), and whether it is even factual (cf. Carnap's Principle of Tolerance). However, if one is a set-theoretic pluralist, then, by the arguments above, one must be a pluralist about core physical concepts too. ("Concepts" rather than "concept" because if determinism is relative to set-theoretic foundations, then so too presumably is physical necessity, prediction, explanation, causation and more.) The same physical theory can be deterministic in one universe and non-deterministic in another. More precisely, all the defining aspects of the determinism of a theory may vary with speculative extensions of standard mathematics. The question of whether a theory is deterministic along a given dimension is, in this case, either misconceived or practical—whether to assume V = L or LC for a physical purpose at hand.

(Note that the results of this article also suggest a new perspective on mathematical explanation in science (Batterman 2010; Baker 2005). If set-theoretic metatheory can affect physical predictions, the boundary between mathematical and physical explanation is blurred in ways that, to my knowledge, have not been investigated.)

It might be objected that the above constructions are too artificial to matter for "real" physics. But many physical systems do involve coefficients or boundary conditions defined through limiting processes or optimization procedures that could produce sets high in the projective hierarchy. For instance, solutions to inverse problems in geophysics (Tarantola 2005) or optimal control boundaries in plasma physics (Freidberg 2014) often involve constructions that climb the definability hierarchy. Whether such systems actually reach the complexity needed for foundational sensitivity is open. More importantly, the objection misses the point. I have argued that the *concept of determinism* is entangled with speculative set theory. The method of conceptual analysis does not require naturally occurring examples - just as Gettier cases need not be realistic to reveal facts about knowledge, or Davidson's "swampman" (Davidson 1987, 443–4) cases need not be even physically possible to reveal facts about our memories. A great deal of excellent work has examined the concept of determinism using toy models. But this work has so far ignored the role played by the ambient metatheory, and this turns out to matter.

It must be emphasized that the point is not just that we need to supplement the standard axioms in order to decide some questions of determinism (as with, say, ZFC + Con(ZFC)). That is an easy application of Gödel's theorems, assuming that those axioms are consistent (and recursively axiomatizable). V = L and LCs are seriously entertained extensions of standard mathematics. We have to take a stand on them like the stand that pioneers of set theory took with respect to the Axiom of Foundation or the Axiom of Choice. The question of which of V = L or LCs is true—or whether it even makes sense to say that one of them is true—is not a question that admits of proof or refutation (short, perhaps, of a proof that LCs are inconsistent). The fact that determinism varies depending on serious axiom candidates (as opposed to varying with the likes of ZFC + ¬Con(ZFC)) reveals that our understanding of core physical concepts is contested—and maybe indeterminate—in the way that our concept of set has long been argued to be (Skolem 1922; Putnam 1980; Field 2008).

Jensen writes:

> Most proponents of V = L and similar axioms support their belief with a mild version of Ockham's razor. L is adequate for all of mathematics; it gives clear answers to deep questions; it leads to interesting mathematics. Why should one assume more? … I do not understand … why a belief in the objective existence of sets obligates one to seek ever stronger existence postulates. Why isn't Platonism compatible with the mild form of Ockham's razor? … I doubt that one could, with the sort of evidence I have, convert the mathematical world to one or the other point of view. Deeply rooted differences in mathematical taste are too strong and would persist (Jensen 1995, pp. 400–401).[6]

Quantum-gravity theorists sometimes remark that we may need "new math" to formulate a final theory. I have argued that this "new math" may go much deeper than anticipated—not just new tools within a familiar framework, but new foundational frameworks. Future progress in physics may therefore depend on novel interactions between physics and the foundations of mathematics.

Appendix A: The Projective Hierarchy and Definability

A.1. Basic definitions

For sets of natural numbers, the **arithmetical hierarchy** is defined by alternating quantifiers over a decidable relation. Σ^0_1 sets are those definable by an existential quantifier over a decidable predicate; Π^0_1 sets are their complements; Δ^0_1 sets are both Σ^0_1 and Π^0_1. Higher levels are defined inductively: Σ^0_{n+1} formulas start with an existential quantifier over a Π^0_n formula, and Π^0_{n+1} are their complements.

For sets of reals (identified with functions from $\mathbb{N}$ to $\mathbb{N}$), the **analytical hierarchy** or **projective hierarchy** is defined similarly, but with quantifiers over sets of natural numbers. Σ^1_1 sets (analytic) are projections of Borel sets. Π^1_1 sets are their complements. Δ^1_1 sets are both Σ^1_1 and Π^1_1 (the Borel sets). Higher levels are obtained by alternating existential and universal second-order quantifiers.

The **lightface** hierarchy allows no real parameters in the definition; the **boldface** hierarchy allows parameters.

A.2. Key properties

- Δ^1_1 = Borel sets.
- Σ^1_n sets are projections of Π^1_{n-1} sets.
- Each inclusion is proper: Σ^1_n is strictly contained in Σ^1_{n+1}.

[6] Jensen adds: "The author confesses to being emotionally a[n advocate of V=L]" (1995, p. 401, n. 17). See Woodin 2010 for a different perspective on V=L, and Koellner 2008 for a critical discussion of'pluralism in set theory.

- Δ^1_2 sets are those both Σ^1_2 and Π^1_2.

A.3. Projective Determinacy

Projective Determinacy (PD) is the statement that for every projective set A of reals, the associated perfect-information infinite game is determined (one player has a winning strategy).

Appendix B: Absoluteness Facts Used

Let us fix a real parameter r (if any) shared by the backgrounds under comparison.

(i) **Shoenfield absoluteness (lightface/boldface Δ^1_2):** For any $\Delta^1_2(r)$ predicate $\theta(x,r)$ with $x, r \in \mathbb{R}$ (or $x \in [0,1]^k$), membership $\boldsymbol{\theta(x,r)}$ is absolute between suitable transitive ZFC models containing r. Consequently, if $\mathbf{C} \coloneqq \{x : \theta(x,r)\}$ is $\Delta^1_2(r)$, then **C** is the *same set* across those backgrounds, although its regularity (measurability, Baire) may differ. This is the hinge in §§5–6.

(ii) **Regularity divide:** In L there is a lightface Δ^1_2 well-order $<$ of $\mathbb{R}$; the associated set $\mathbf{W} \coloneqq \{(x,y) \in [0,1]^2 : x < y\}$ witnesses Δ^1_2 non-measurability via the Sierpiński–Fubini argument (horizontal sections countable; vertical co-countable). Under LC, all projective sets (hence Δ^1_2) are Lebesgue measurable and have the Baire property.

(iii) **Higher complexity toggling.** At Σ^1_3 one can arrange background-sensitive truth of a fixed arithmetical index n_0. In Section 5.3 we encode this into smooth initial data to obtain background-sensitive solution identity with uniqueness preserved.

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