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RESEARCH

Deriving the Geodesic Principle

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ABSTRACT

In the recent philosophy literature, there have been several attempts to use the seminal result by Geroch and Jang (1975) to precisify Harvey Brown's claim that the geodesic principle can be recovered as a theorem in General Relativity, and then to critique it. We contend that the philosophical debate has unfolded in a curious way: even though Geroch and Jang's paper contains two distinct approaches to the problem of geodesic motion, the philosophical literature has focused on only one of them. We then argue that the neglected approach offers an alternative—and more physical—set of resources to explain geodesic motion. Motivated by this approach, we prove a new "physical Geroch-Jang theorem", which provides a scale-relative interpretation of the geodesic principle in General Relativity. We, thereby, make new resources available to re-evaluate Brown's arguments as well as those of his critics.

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1 INTRODUCTION

There has been considerable discussion of the concept of inertial motion in General Relativity (GR), beginning with Einstein himself (Einstein and Grommer 2005; Kennefick 2005; Lehmkuhl 2017): does GR have the theoretical resources to *explain* the inertial motion of "small" bodies free of external influences, as opposed to taking it for granted? Brown (2007) prominently claims that the Einstein Field Equations (EFE) of GR provide such an explanation by means of the "geodesic principle," the insight that the trajectories of "free" test bodies are well modeled by the geodesics of a spacetime manifold.[1] Brown's claim has led to a body of literature (Samaroo 2018; Sus 2014; Weatherall 2011, 2017, 2019, 2020) that attempts to conceptualize and assess this claim within the framework of a foundational paper by Geroch and Jang (1975), which aimed to derive the geodesic motion of a test body in GR.[2]

We are in agreement with the tacit assumption in the literature that a serious assessment of Brown's claim ought to engage with rigorous results about geodesic motion provable in the context of GR. Moreover, we also agree that Geroch and Jang's paper is an important reference point. However, we believe that further technical and conceptual groundwork on this point of reference is needed before it can pay philosophical dividend. Specifically, we contend in this article that the literature has unfolded in a curious way: even though the original paper (Geroch and Jang 1975) contains *two* rather distinct approaches to the problem of geodesic motion in GR—we will call them "weakest topology" and "scale-relative topology" approach, respectively[3]—the philosophical literature has only focused on the former. Since, as we will argue, the two approaches implement different interpretations of the underlying physical problem, the ensuing philosophical discussion has been one-sided and progress on the issue of geodesic motion has potentially been stifled because the full set of available resources (in particular, the "scale-relative topology" approach) has not yet been exploited. In this article, we aim to accomplish the important technical and conceptual task of revisiting the original paper, giving a careful reconstruction, and drawing important philosophical lessons from it in order to re-open the issue of geodesic motion in GR for debate within the philosophy-of-physics community. The chief philosophical contribution, as we see it, consists in the reconstruction of the *philosophy* of a *locus classicus* of the philosophical debate, which as such, merits careful attention and a thorough interpretation. We, thereby, lay important groundwork that must be conducted *before* results such as those contained in Geroch and Jang's paper can be exploited in philosophical dialectic.

In line with this motivation, we pursue three goals. First, we aim to accomplish a task of conceptual analysis, viz. distinguishing and explicating the aforementioned two approaches within the original paper: while the statement of the theorem implements the "weakest topology" approach, the proof strategy and the "the physical interpretation of the theorem" as given by Geroch and Jang themselves (Geroch and Jang 1975, 66) are closer to the "scale-relative topology" approach. Delineating these approaches carefully is novel as the literature has not recognized the two strands and thus has not taken into account the discrepancy

1 Indeed, Brown claims that GR is the historically first theory to do so (Brown 2007, 141).

2 Weatherall (2020) has recently revisited the problem in light of a result proved in collaboration with Geroch (Geroch and Weatherall 2018). Since it is not the Geroch-Weatherall theorem that has been the focal case of the literature, we still feel justified to restrict attention to the *locus classicus* of the debate, especially since it has, to our minds, not been treated sufficiently so far.

3 To avoid any misunderstanding, we would like to flag that "weakest topology" is not meant pejoratively, but as a description of the topology's "strength." Preempting a point to be made below, "weakest" can be read as "least quantitative."

between the proof strategy and the formulation of the theorem. Second, since Geroch and Jang's professed physical motivation is to model test bodies relative to scale, that is, "insofar as that body is sufficiently small compared with the curvature" (Geroch and Jang 1975, 66), we argue for adopting a "scale-relative topology" approach, which involves pursuing a theorem that explicitly incorporates such considerations of scale into its assumptions. This is a significant intervention in the literature and opposes a perspective recently advocated by Weatherall, who directs philosophical attention away from test bodies (or point particles) because for him, "the status of such objects is unclear in general relativity" (Weatherall 2020, 222).[4] Third, we show that although Geroch and Jang's argumentation is telegraphic and incomplete, their sketched proof, which follows the "scale-relative topology" approach, can be easily made rigorous by making various limits explicit that are only implicit in their argumentation. To take these limits, a new set of assumptions is needed. Choosing the simplest one possible, we obtain a new, now fully explicit and rigorous theorem. By achieving these three goals, we lay down the conceptual and technical preliminaries necessary for a fruitful employment of the "scale-relative topology" approach in the philosophical debate on inertial motion in GR and likewise for a careful re-assessment of the justification of the "weakest topology" approach.

Our article is structured as follows. Our novel reading of Geroch and Jang's paper is given in Section 2. Subsection 2.1 consists of a clean reconstruction of their treatment of geodesic motion in the case of Special Relativity (SR). Along the way, we correct a small mistake made in the philosophical literature concerning an energy condition. The reconstruction of the SR case will provide the necessary background for the GR case, which can, and in our view ought to, be understood as an extension of the SR result.[5] The GR case itself is treated in Subsection 2.2, where we show that the original paper contains the two approaches mentioned above as different strands. We also argue that only one of them is compatible with the physical interpretation motivating the paper. Section 3 then contains the precise statement of our new theorem (which we call the "physical GJ theorem") as well as a sketch of its proof—the details are given in the appendix—and an interpretation. Finally, in Section 4, we briefly comment on other approaches, besides the Geroch-Jang paper, to geodesic motion in the physics and mathematics literature.

2 AN ANALYSIS OF THE GEROCH-JANG PAPER

The paper (Geroch and Jang 1975) contains two parts: the first attempts to derive the geodesic motion of a free *extended body* in Special Relativity (SR), and the second attempts to derive the geodesic motion of a *test body* in General Relativity (GR). The SR result about extended bodies is relatively straightforward, which might explain why it has not been addressed in the philosophy literature. However, since it is essential to the overall argumentative strategy in the GR case, we include the SR result in our presentation.

Following the original paper, in Section 2.1, we lay some crucial groundwork by discussing the SR case, in order to (i) contrast the notions of "extended body" and "test body;" (ii) introduce SR concepts such as linear momentum (P_a) and angular momentum (J_{ab})

4 For transparency's sake, we note that this remark is made in the context of Geroch and Weatherall (2018), which is bracketed here. However, Weatherall's comment seems to apply equally to his interpretation of the problematic contained in the Geroch-Jang paper.

5 While this point has not been disputed, the precise connection between the SR result and the GR case has never been made explicit. On our view, this is not surprising because the connection becomes clear only upon rigorously pursuing the "scale-relative topology" approach, as we do in this article.

that will be analogously extended to the GR case; and (iii) develop the energy condition to prepare the GR case. There is one additional function played by Section 2.1: some of the philosophical literature on the Geroch-Jang paper contains a confused discussion (with a small mathematical mistake) of energy conditions, and without dispelling this confusion, it is difficult to see how one could prove a physical GJ theorem of the kind that we undertake in Section 3; we thus take the opportunity to make this minor, yet useful correction.

Having laid the groundwork in Section 2.1, Section 2.2 proceeds to discuss the two very different conceptual strands in the GR part of Geroch and Jang (1975): (i) what we call the "GJM result" (Malament's formulation of the theorem in the GR case in the Geroch-Jang paper) and (ii) the physical reasoning that Geroch and Jang use to sketch an argument for their result, which we call "the physical GJ argument." We will argue in this section that the GJM result is physically unmotivated. On the other hand, the physical GJ argument—despite being incomplete—outlines the proof strategy for a novel result, the "physical GJ theorem."[6] We proceed to sketch the statement and proof of this result in Section 3; the details are given in Appendix D.

2.1 SPECIAL RELATIVITY

The first part of Geroch and Jang (1975) concerns the extended body in SR. From a mathematical point of view, it proceeds by assuming that one has Minkowski space $(\mathcal{M}, \eta_{ab})$ and a non-vanishing spatially compactly supported energy-momentum tensor T_{ab} on $\mathcal{M}$ that represents the extended body. Energy-momentum is assumed to be covariantly conserved, that is,

$$\nabla^a T_{ab} = 0,$$

where ∇ is the Levi-Civita connection associated with the flat Minkowski metric η_{ab}.[7]

Since Minkowski space is maximally symmetric (i.e., the number of Killing fields is maximal), we can immediately derive (see Lemma B.1 in the appendix)[8]

$$P_a \xi^a + J_{ab} \nabla^a \xi^b = \int_\Sigma T^a{}_b \, \xi^b \, \mathrm{d}S_a \tag{1}$$

for any spacelike hypersurface Σ and any Killing field ξ^a. This integral identity should be interpreted as follows. For any spacelike hypersurface of Minkowski space, and any of the 10 Killing fields (i.e., any of the symmetries of the Poincaré group), the integral on the right hand side can be split into a momentum part (where P_a generalizes linear momentum) and an angular momentum part (where J_{ab} generalizes angular momentum). Note here that the one-form P_a and the two-form J_{ab} are defined on all of Σ. Accordingly, the left-hand side is a function that can be evaluated at any point on Σ, while the right-hand side (an integral) is a constant on Σ.

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6 To avoid any misunderstanding, we would like to stress that we do not intend to be dismissive about the GJM result by not calling it a theorem. Rather, we merely mean to imply that the physical GJ argument is, on our view to be explained below, not a rigorous proof of the GJM result, but rather a proof sketch for the physical GJ theorem.

7 Latin indices of tensors are used as abstract indices while Greek indices designate a choice of coordinate basis. We adopt the metric convention $-+++$.

8 As noted in the appendix, we understand $\mathrm{d}S_a$ as a shorthand for $-n_a \,\mathrm{dvol}_\eta(\Sigma)$, where n^a is a future-directed unit normal to a spacelike hypersurface Σ with induced volume element $\mathrm{dvol}_\eta(\Sigma)$. While this is the more canonical choice, Geroch and Jang seem to use it as a shorthand for $+n_a \,\mathrm{dvol}_\eta(\Sigma)$, which would, however, mean that $\int_\Sigma \varphi \, n^a \, \mathrm{d}S_a$ is negative for a positive smooth function φ with compact support. This difference explains why we have $P_a \xi^a$ on the left-hand side whereas Geroch and Jang have $-P_a \xi^a$.

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Furthermore, the Riemann-flatness of Minkowski space makes it possible to derive particularly simple differential equations for P_a and J_{ab}:[9]

$$\nabla_a P_b = 0$$
$$\nabla_a J_{bc} = -g_{a[b} P_{c]}.$$

Using these ingredients, Geroch and Jang proceed to show that the extended body follows a timelike geodesic. More precisely, they prove that there is a timelike geodesic in the spatially convex hull of the support of T_{ab} by showing (1) that one can construct a center-of-motion curve whose tangent is P^a (see Proposition C.7) and (2) that P^a is timelike (see Proposition C.5).[10] To prove this, an energy condition is required. In the original paper, Geroch and Jang assume that T_{ab} satisfies the following Strict Dominant Energy Condition (Strict DEC):

Definition 2.1: *(Strict DEC) An energy-momentum tensor T_{ab} satisfies the Strict DEC if at every point, either $T_{ab} = 0$ or $T_{ab}X^aY^b > 0$ for all co-oriented timelike vectors X^a and Y^a.*[11]

However, the Strict DEC is insufficient for the argument to work in the SR setting (*a fortiori* the GR setting). A simple counterexample to the theorem is furnished by null dust. Let k^a be a constant null vector field. Define the energy-momentum tensor given by $T_{ab} = \Phi k_a k_b$ and choose a Φ with compact spatial support such that $\nabla^a T_{ab} = 0$, i.e., $k^a \nabla_a \Phi = 0$. Then T_{ab} satisfies the Strict DEC, but there can be no timelike geodesic in the convex hull of the support of T_{ab} because the spatial support of T_{ab} lies within a cylinder whose boundary is tangent to k^a, so every timelike geodesic starting in the support leaves the cylinder eventually.[12]

So which energy condition is sufficient for Geroch and Jang's SR result to go through? Analysing their argument (see Lemma C.5) reveals that in order to ensure that P^a is timelike, they require that

$$\int_\Sigma T^a{}_b \xi^b n_a \, \mathrm{dvol}_\eta(\Sigma) > 0$$

for a causal Killing field ξ^a. If we want to follow Geroch and Jang (1975) in imposing a *pointwise* energy condition, then this is satisfied if and only if

$$T^a{}_b \xi^b n_a > 0.$$

As we could have chosen another Σ and hence a different n^a, we must require more generally that

$$T_{ab} X^a Y^b > 0$$

for co-oriented non-vanishing X^a and Y^b such that X^a is timelike and Y^b is causal. By Lemma A.3, this just means that T_{ab} satisfies the Strengthened Dominant Energy Condition introduced into the discussion by Malament (2009):

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9 For more details on the following steps, see Corollary B.3 and Appendix C.

10 The spatially convex hull of the support of T_{ab} can be defined simply as the smallest set containing the support of T_{ab} such that for every spacelike hypersurface Σ, the intersection $C \cap \Sigma$ is a convex set in the Riemannian manifold $(\Sigma, \eta_{ab}|_\Sigma)$.

11 The condition is referred to as a "(strong) energy condition" in Geroch and Jang (1975, 66), although they only consider non-zero energy-momentum tensors. We adopt the standard nomenclature from Weatherall (2012, 213) and like Weatherall, also allow for energy-momentum tensors that vanish everywhere.

12 Weatherall (2012) provides a counterexample in the *non-Special Relativistic* setting by showing that the GJM result does not hold if the Strict DEC, rather than a stronger energy condition, is imposed. To show this, he constructs a cylindrical spacetime. Our counterexample, by contrast, is Special Relativistic insofar as it is formulated in Minkowski space.

Definition 2.2: *(Strengthened DEC) An energy-momentum tensor T_{ab} satisfies the Strengthened Dominant Energy Condition if given any timelike vector X^a at any point, (1) $T_{ab}X^aX^b \geq 0$ and (2) either $T_{ab} = 0$ or for all timelike X^a, $T^a{}_bX^b$ is timelike.*

To conclude our discussion of energy conditions, we note that an energy condition even stronger than the Strengthened DEC has appeared in the literature:

Definition 2.3: *(Strengthened* DEC) An energy-momentum tensor T_{ab} satisfies the Strengthened* Dominant Energy Condition if at every point, either $T_{ab} = 0$ or $T_{ab}X^aY^b > 0$ for all non-vanishing co-oriented causal vectors X^a and Y^b.*

It has been falsely claimed (Weatherall 2012, 213) that this energy condition is equivalent to the Strengthened DEC.[13] This can easily be shown to be wrong by considering $T_{ab} = -g_{ab}$, which satisfies the Strengthened DEC. To see this, one checks that for every timelike X^a, $T^a{}_bX^b = -\delta^a_bX^b = -X^a$ is timelike. Moreover, $T_{ab}X^aX^b = -X^aX_a > 0$, and thus T_{ab} satisfies the Strengthened DEC. However, this energy-momentum tensor does not satisfy the Strengthened* DEC because for every non-zero null vector N^a, $T_{ab}N^aN^b = -N^aN_a = 0$.

The Strengthened DEC is, therefore, weaker than the Strengthened* DEC. For a Geroch-Jang-style result, one ideally wants the weakest possible condition that works, so as to not rule out *a priori* certain energy-momentum tensors: Strengthened DEC fulfils this role in the SR case. The final SR result can, therefore, be stated as follows:

Theorem 2.4: *Let $(\mathbb{R}^4, \eta_{ab})$ be Minkowski space, and let T_{ab} be a conserved energy-momentum tensor with compact non-vanishing spatial support satisfying the Strengthened DEC. Then there is a timelike geodesic in the convex hull of the support of T_{ab}.*

Apart from the issue of the energy condition, the proof of Theorem 2.4 as given in Geroch and Jang (1975) is complete *mutatis mutandis*. However, their presentation does not clearly indicate which parts of the proof actually require that T_{ab} be covariantly conserved. Since this issue is crucial for the proof of the GR case, we include a more careful presentation of the proof of Theorem 2.4 in Appendix C, paying close attention to where exactly covariant conservation is needed.

Before turning to the GR result, it is worth noting that Theorem 2.4 is Special Relativistic insofar as it is genuinely about Minkowski space. For not only does the proof use that $(\mathcal{M}, \eta_{ab})$ is maximally symmetric—in order to define P_a and J_{ab} on every hypersurface Σ—but Riemann-flatness is used for the vanishing of $\nabla_a P_b$, which would otherwise take the form

$$\nabla_a P_b = J_{cd}R^{cd}{}_{ab},$$

introducing an interaction of the motion with the geometry.[14] Maximal symmetry and Riemann-flatness make the motion of an extended body lucidly tractable.

2.2 GENERAL RELATIVITY: THE TWO STRANDS

So much for the SR part of Geroch and Jang (1975). The rest of the Geroch-Jang paper purports to extend this result to GR. Crucially, the GR section operates with a different conception of "body:" while the SR result is about an extended body (of unspecified size) in Minkowski space, the second, General Relativistic part is about a "test body" in a general

13 This claim appears to have made its way into a publication by Curiel (Curiel 2017, 49), whose presentation of various "versions" of the "physical SDEC" at least seems to imply the equivalence of version 1 (= our Strengthened DEC) and version 2 (= our Strengthened* DEC).

14 See (15) in Corollary B.3.

spacetime, that is, a body "whose effect on the background spacetime structure is negligible" (Malament 2009, 8).

As noted before, the GR section of Geroch and Jang's paper itself contains two very different strands. The first is found in how Geroch and Jang *state* their General Relativistic theorem. Significantly, this strand has a global flavor, by which we mean that it requires the existence of a suitable energy-momentum tensor with non-vanishing support in *any* open neighborhood around the *whole* curve. As we will suggest shortly, this requirement has some unintuitive consequences. But there is also another strand, which can be read off from the way in which Geroch and Jang explain their proof. For instance, they write that "[t]he proof consists of noting that 'the nearer one is to Γ [the analogue of the center-of-motion curve in the GR case], the more nearly is the result of special relativity applicable.' " They also provide a "physical interpretation of the theorem which is that, for any body, 'insofar as that body is sufficiently small compared with the curvature that it may be regarded as a realization of the limit implicit in the theorem, then to that extent so may it be regarded as following some geodesic Γ' " (Geroch and Jang 1975, 66). These informal statements point to taking limits and implicitly appeal to the Equivalence Principle.[15] Both aspects will be brought out more clearly in our theorem.

Returning to the first strand, Malament (Malament 2009, 7–8) was the first to codify it (and replace Geroch and Jang's energy condition by the Strengthened DEC), and we will thus call his statement the **GJM result**:[16]

Theorem 2.5 (GJM result)**:** *Let $(\mathcal{M}, g_{ab})$ be a spacetime and let $\gamma : I \to \mathcal{M}$ be a smooth embedded curve. Suppose that for any open neighborhood O of $\gamma(I)$, there is a smooth symmetric 2-tensor T_{ab} with the following properties:*

1. *T_{ab} satisfies the Strengthened DEC.*
2. *T_{ab} satisfies $\nabla^a T_{ab} = 0$.*
3. *T_{ab} has non-vanishing support in O.*

Then γ is timelike and can be re-parametrized as a geodesic.[17]

While the physical GJ argument, that is, the argument given by Geroch and Jang for Theorem 2.5, is incomplete and not fully rigorous, we are not aware of any counterexamples to the GJM result. Part of the difficulty in constructing a possible counterexample stems from the peculiar requirement that for *any* open neighborhood of the curve (e.g., one whose spatial volume asymptotes to zero, as depicted in Figure 1), there exists an energy-momentum tensor with the desired properties: since there is in general no *minimal radius* of a given neighborhood along the *whole* curve $\gamma(I)$, one can only easily control such neighborhoods in the case of closed curves.[18] But even if a complete proof of the GJM result is presented eventually—although we insist that the argument in Geroch and Jang (1975) is a proof sketch

15 Indeed, Geroch and Jang cite Fermi 1922, who explicitly draws on considerations of scale. This, in turn, suggests an implicit appeal to the Equivalence Principle (Linnemann, Read, and Teh 2024). There is also an interesting structural parallel to (Wallace 2017), where it is argued that the Equivalence Principle holds for isolated gravitating systems precisely because of considerations of scale.

16 We slightly streamline the presentation by collapsing Malament's two conditions 3 and 4 into our condition 3.

17 Theorem 2.5 does not hold if certain "global" properties are weakened. The philosophical discussion has of course already pointed out the indispensability of condition 1, which is Malament's correction to the formulation of the theorem given in the Geroch-Jang paper.

18 See Weatherall (2012, 212).

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for a different theorem, our physical GJ theorem—Theorem 2.5 will still be a problematic reference point for the philosophical discussion on geodesic motion in GR. We present three reasons for why we believe this to be so.

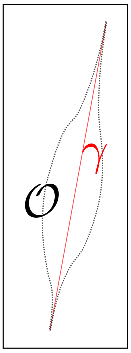

Figure 1 Timelike curve with neighborhood whose spatial volume asymptotes to zero as proper time tends to $\pm\infty$.

First, insofar as the GJM result is meant as an extension of the SR result, it gets things backwards in the following sense. In general, it seems that essentially two questions can be asked when inquiring under which conditions a curve $\gamma(I)$ corresponding to a class of energy-momentum tensors is a timelike geodesic:

(1) Given a curve $\gamma(I)$ in a spacetime $(\mathcal{M}, g_{ab})$, what are sufficient conditions on a class of energy-momentum tensors that are, in a sense to be specified, associated with $\gamma(I)$ such that γ is a timelike geodesic?
(2) Given a class of energy-momentum tensors in $(\mathcal{M}, g_{ab})$, what are sufficient conditions on that class such that a curve $\gamma(I)$ can, in a sense to be specified, associated with the class such that γ is a timelike geodesic?

Only the second question asks for a theorem analogous to the SR result. For, in that case, the class is $\{T_{ab}\}$ for an energy-momentum tensor T_{ab}; a sufficient condition for γ to be a timelike geodesic is the requirement that T_{ab} should be covariantly conserved and should satisfy the Strengthened DEC; and the associated $\gamma(I)$ is the curve of center-of-motion points.

By contrast, the GJM result clearly answers the first question by defining "associated" via the support of the energy-momentum tensors in the class. We highlight this by saying that Theorem 2.5 implements a **"weakest topology" approach**, meaning that it uses the following topological space. Let Ω be the set of all energy momentum tensors excluding the tensor identically zero everywhere. Starting from a curve $\gamma(I)$, one considers all open neighborhoods of that curve in $\mathcal{M}$. Call their set Ξ. For every $U \in \Xi$, define a subset N_U of Ω: an energy-momentum tensor T_{ab} is in N_U if and only if supp $T_{ab} \subseteq U$. Define the topology $\mathcal{T}$ as generated by the base $\{N_U \;:\; U \in \Xi\} \cup \{\emptyset\}$. Then $(\Omega, \mathcal{T})$ is a topological space. Note that this topological space cannot separate an energy-momentum tensor T_{ab} from any of its multiples λT_{ab} ($\lambda \in \mathbb{R}$), implying that it does not "control" the "size" of energy-momentum tensors and is, thus, quantitatively "weak."

The GJM result can now be construed as an answer to the following question, which is an adaptation of the first question above to the topological space $(\Omega, \mathcal{T})$: what are sufficient conditions on elements of Ω such that γ is a timelike geodesic? Figure 2 illustrates the way in which the GJM result is indeed an answer to this question. The conditions of Theorem 2.5 define a subset of Ω, here depicted by the red curve, such that γ is a timelike geodesic if any open set N_O has non-empty intersection with the red curve. When attempting to *apply* the GJM result to a curve $\gamma(I)$, one can therefore only conclude that γ is indeed a timelike geodesic if one can *prove* that the conditions of Theorem 2.5 can be satisfied by

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an energy-momentum tensor T_{ab} for *any* neighborhood, a difficult task, especially when dealing with complicated spacetimes. But as well as being technically difficult, this way of proceeding is also not conceptually straightforward. Suppose one were to consider a curve $\gamma(I)$ representing a test body of a certain type, that is to say, suppose there is a sequence of energy-momentum tensors ${}^{(1)}T_{ab}, {}^{(2)}T_{ab}, {}^{(3)}T_{ab}, \ldots$ with non-vanishing spatially compact support shrinking down to $\gamma(I)$. If there is an open set N_O that contains none of these energy-momentum tensors—a situation depicted in Figure 2—then in order to apply the GJM result, one needs to prove the existence of another energy-momentum tensor with support in O that satisfies the conditions of Theorem 2.5, even though the test body was already fully conceptualized using the sequence of energy-momentum tensors. The physical application of the GJM result and thereby the "weakest topology" approach thus come with considerable technical and conceptual difficulties.

Second, the GJM result is intended to be about test bodies, those objects whose trajectories are well modeled by the gravitational effects of the background spacetime because they are small *relative to* these effects. This means that a class of tensors can only model such test bodies that are, in a certain sense, *commensurate with* the curvature. This is the issue of "physical scale," which is crucial for the physical understanding of GR (Linnemann, Read, and Teh 2024). And yet, the GJM result does not contain any appeal to scale due to its global nature. The GJM result is therefore physically problematic; but fortunately, there is another strand in Geroch and Jang's paper that respects "physical scale." This strand is implemented in our new physical theorem, which fixes the shortcomings of the GJM result and implements the physical ideas more straightforwardly. Our resulting formulation will put more emphasis on the limit, which is merely implicit in the physical GJ argument. It thereby makes more precise the various limits that Geroch and Jang had to take in their argument, which were hidden under general phrases that some quantities can be made as small or close to each other "as we wish" or "as we please," but were acknowledged explicitly through their "physical interpretation of the theorem which is that, for any body, 'insofar as that body is sufficiently small compared with the curvature that it may be regarded as a realization of the limit implicit in the theorem, then to that extent so may it be regarded as following some geodesic Γ' " (Geroch and Jang 1975, 66). We will implement this intuition through a **"scale-relative topology" approach**, which asks the second of the two questions raised above, in the form: *which* topology should be placed on the class of energy momentum tensors based on physical *desiderata*? What is at stake is, therefore, not whether topologies play a role in the result—they clearly do—but rather, whether the topologies involved are specified physically.[19]

Third, the "weakest topology" approach excludes certain energy-momentum tensors *a priori* by not being applicable to reasonable classes of energy-momentum tensors. This can be illustrated through an example. Let $(\mathcal{M}, g_{ab})$ be Minkowski space and γ an inextendible timelike geodesic with tangent vector u^a. Consider moreover, the class C of energy-momentum tensors

$$T_{ab} = \Phi\, u_a u_b, \tag{2}$$

for a non-vanishing $\Phi \geq 0$ whose compact spatial support contains $\gamma(I)$. For T_{ab} to be conserved, $u^a \nabla_a \Phi = 0$, so Φ must have constant spatial volume for all times. However, if we consider an open neighborhood O of $\gamma(I)$ that becomes infinitesimally narrow towards the future (see Figure 1), then there is no T_{ab} from C that is in that neighborhood. Theorem 2.5

19 Our points here will, *mutatis mutandis*, also extend to the issues discussed in Fletcher (2020) and Fletcher and Weatherall (2023).

therefore does not apply.[20] Of course, one could object that there might be other energy-momentum tensors in O that do not have the form (2). Two replies can be given. First, by shifting the focus from an explicitly given example to an existence claim, the objector has the burden of proof to show that such energy-momentum tensors indeed exist. Second, the GJM result ought to be applicable in this situation as per the physical interpretation presented in the literature: that "we are representing 'point particles' as nested convergent sequences of smaller and smaller extended bodies" (Malament 2009, 8).

One might further object to our example that the timelike dust of (2) is not a physically reasonable example *tout court*. However, it was supposed to be "a considerable advance to prove theorems that dispense with special modeling assumptions in favor of generic ones" and the GJM result was taken to be an example of this (Malament 2009, 3). Nevertheless, the GJM result has maneuvered us into the paradoxical situation that being maximally permissive about which neighborhoods of $\gamma(I)$ ought to be admitted also meant being again restrictive about classes of energy-momentum tensors one can consider. So rather than "dispens[ing] with special modeling assumptions," we have imported implicit ones. Therefore, although the GJM result without doubt achieves a degree of genericity—insofar as it does not presuppose a specific matter model (for example, that of a perfect fluid) as did some of the works prior to Geroch and Jang (1975)—it is not applicable to reasonable classes of energy-momentum tensors.

The GJM result, which implements a "weakest topology" approach, is thus seen to be an at least not unproblematic reference point for philosophical discussions about geodesic motion in GR. So new results that avoid the "weakest topology" approach are desirable. In the next section, we show how such a result can be obtained by using other resources from the original paper by Geroch and Jang.

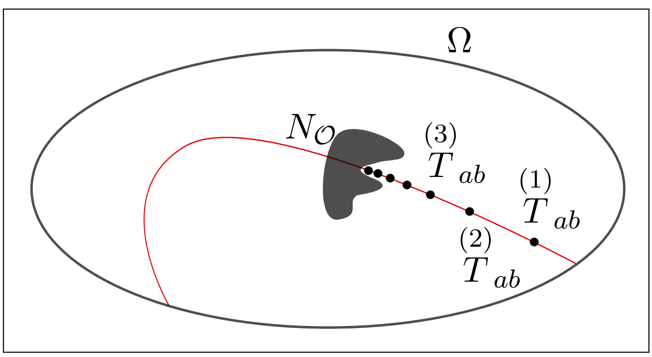

Figure 2 This is a schematic representation of the topological space $(\Omega, \mathcal{T})$. The red curve represents the subset of energy momentum tensors satisfying the conditions of Theorem 2.5. The sequence $^{(1)}T_{ab}, ^{(2)}T_{ab}, \ldots$ exemplifies a class of energy-momentum tensors whose support is uniformly shrinking down to the curve. The shaded region represents an open set N_O (induced by a neighborhood O) that does not contain any of the energy-momentum tensors of the given class.

3 THE PHYSICAL GJ THEOREM AND ITS RELATION TO THE FIELD EQUATIONS

The physical motivation provided by Geroch and Jang provides a proof strategy for our physical GJ theorem. We now proceed to sketch the statement and proof of this result, referring the reader to Appendix D for further details.

20 One might wonder whether we have not been too unimaginative in our application of the GJM result. For example, could one not divide $\gamma(I)$ into countably many open segments γ_i with finite length such that $\gamma(I) = \bigcup_i \gamma_i$ and then apply the GJM result to each γ_i? But the problem is that here again, a neighborhood (of any γ_i) can be chosen whose spatial volume tends to zero as we approach the ends of the segment, and no non-zero dust energy-momentum tensor exists whose support lies in such a neighborhood.

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First, we will introduce the notion of a *Geroch-Jang particle*,[21] which is a pair $\left(\gamma, \left({}^{(n)}T_{ab}\right)_{n\in\mathbb{N}}\right)$ such that $\gamma : I \to \mathcal{M}$ is a smooth curve and $\left({}^{(n)}T_{ab}\right)_{n\in\mathbb{N}}$ is a sequence of energy-momentum tensors that is *Geroch-Jang admissible*. We call $\left({}^{(n)}T_{ab}\right)_{n\in\mathbb{N}}$ Geroch-Jang admissible if it *shrinks down* to $\gamma(I)$, that is to say, every ${}^{(n)}T_{ab}$ has compact support "supp ${}^{(n)}T_{ab}$" and

$$\bigcap_{n\in\mathbb{N}} \operatorname{supp} \overset{(n)}{T}_{ab} = \gamma(I),$$

and additionally satisfies what we call the *Uniform Strengthened DEC* and the *Integrated Size* and *Integrated Conservation Conditions*. We introduce these conditions in turn.

Definition 3.1 (Uniform Strengthened DEC): *A sequence of energy-momentum tensors* $\left({}^{(n)}T_{ab}\right)_{n\in\mathbb{N}}$ *shrinking down to* $\gamma(I)$ *satisfies the Uniform Strengthened DEC if for every* $q \in \gamma(I)$ *and every co-oriented non-zero* V^a *and* W^a *at* q *with* $V_aV^a < 0$ *and* $W_aW^a \leq 0$, *there exists a constant* $c > 0$ *such that for every* $m \in \mathbb{N}$,

$$\overset{(m)}{T}_{ab} V^a W^b \geq c > 0.$$

This definition is uniform in m. Uniformity is a crucial ingredient of the proof in the following way. For any point p on the curve $\gamma(I)$, there is an open neighborhood U about p such that there exist normal coordinates, that is, coordinates in which $g_{\mu\nu} = \eta_{\mu\nu}$ at p and $\Gamma^{\mu}_{\nu\rho} = 0$ at p. We can consider (U, η_{ab}) as a manifold on its own, which constitutes something like a "pre-frame" about p.[22] Now, the uniformity in the Uniform Strengthened DEC guarantees that if U is sufficiently small, then every ${}^{(n)}T_{ab}$ satisfies the Strengthened DEC in the pre-frame (U, η_{ab}) with a bound from below (in an appropriate sense).[23]

The remaining conditions for a Geroch-Jang admissible sequence of energy-momentum tensors are contained in the following definitions.

Definition 3.2 (Integrated Size Condition): *A sequence* $\left({}^{(n)}T_{ab}\right)_{n\in\mathbb{N}}$ *satisfies the Integrated Size Condition with respect to a smooth curve* $\gamma : I \to \mathcal{M}$ *if for every spacelike hypersurface* Σ *and for all vector fields* X^a *and* Y^a, *there exists a constant* $C > 0$ *such that*

$$\int_{\Sigma} \left| \overset{(n)}{T}_{ab} X^a Y^b \right| \operatorname{dvol}_g(\Sigma) < C \tag{3}$$

for all $n \in \mathbb{N}$.

The Integrated Size Condition enforces that $\left|{}^{(n)}T_{ab}X^aY^b\right|$ does not grow "too much." Importantly, if the sequence $\left({}^{(n)}T_{ab}\right)_{n\in\mathbb{N}}$ satisfies the Integrated Size Condition *and* shrinks down to $\gamma(I)$, then for every spacelike hypersurface Σ and for all vector fields X^a and Y^a,

$$\int_{\Sigma} \operatorname{dist}(x) \left| \overset{(n)}{T}_{ab} X^a Y^b \right| \operatorname{dvol}_g(\Sigma) \leq \sup_{x \in \operatorname{supp}({}^{(n)}T_{ab})} \operatorname{dist}(\mathrm{x}) \cdot \int_{\Sigma} \left| \overset{(n)}{T}_{ab} X^a Y^b \right| \operatorname{dvol}_g(\Sigma) \to 0 \tag{4}$$

as $n \to \infty$. Here, "dist" denotes the geodesic distance from a point $x \in \Sigma$ to $\gamma(I) \cap \Sigma$.

Definition 3.3 (Integrated Conservation Condition): *A sequence* $\left({}^{(n)}T_{ab}\right)_{n\in\mathbb{N}}$ *satisfies the Integrated Conservation Condition with respect to a smooth curve* $\gamma : I \to \mathcal{M}$ *if for every spacelike hypersurface* Σ *and for every vector field* X^a,

$$\int_{\Sigma} \left| \nabla^a \overset{(n)}{T}_{ab} X^b \right| \operatorname{dvol}_g(\Sigma) \to 0. \tag{5}$$

as $n \to \infty$.

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21 The name was introduced by Tamir, though with a different meaning (Tamir 2012, 147).

22 We call (U, η_{ab}) a "pre-frame" because the physical GJ theorem shows that under the conditions of the theorem, it can, in fact, be considered an inertial frame for the test body as $n \to \infty$.

23 For more details, see the proof in Appendix D.

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Clearly, the Integrated Conservation condition, though entailed by

$$\nabla^a \overset{(n)}{T}_{ab} = 0,$$

is significantly weaker: by (5), conservation need only hold "in the limit" as the volume of the energy-momentum tensors' support becomes smaller and smaller.

The combination of (4) and (5) implies that in a pre-frame (U, η_{ab}), integrals of the divergence of the energy-momentum tensors vanish in the limit as $n \to \infty$:

$$\begin{aligned}\int_\Sigma \partial^\mu \overset{(n)}{T}_{\mu\nu} X^\nu \mathrm{dvol}_\eta(\Sigma) &= \int_\Sigma \eta^{\mu\rho}\partial_\mu \overset{(n)}{T}_{\rho\nu} X^\nu \,\mathrm{dvol}_\eta(\Sigma)\\ &= \int_\Sigma \eta^{\mu\rho}\partial_\mu \overset{(n)}{T}_{\rho\nu} X^\nu \sqrt{|\mathrm{det}\eta|_\Sigma|}\,\mathrm{d}x^1 \dots \mathrm{d}x^3\\ &= \int_\Sigma \eta^{\mu\rho}\partial_\mu \overset{(n)}{T}_{\rho\nu} X^\nu \frac{\sqrt{|\mathrm{det}\eta|_\Sigma|}}{\sqrt{|\mathrm{det g}|_\Sigma|}}\sqrt{|\mathrm{det g}|_\Sigma|}\,\mathrm{d}x^1 \dots \mathrm{d}x^3\\ &= \int_\Sigma \eta^{\mu\rho}\partial_\mu \overset{(n)}{T}_{\rho\nu} X^\nu \frac{\sqrt{|\mathrm{det}\eta|_\Sigma|}}{\sqrt{|\mathrm{det g}|_\Sigma|}}\,\mathrm{dvol}_g(\Sigma)\\ &= \int_\Sigma \left[\eta^{\mu\rho}\nabla_\mu \overset{(n)}{T}_{\rho\nu} + \eta^{\mu\rho}\left(\Gamma^\sigma_{\mu\rho}\overset{(n)}{T}_{\sigma\nu} + \Gamma^\sigma_{\mu\nu}\overset{(n)}{T}_{\rho\sigma}\right)\right]\times\\ &\qquad \times X^\nu \frac{\sqrt{|\mathrm{det}\eta|_\Sigma|}}{\sqrt{|\mathrm{det g}|_\Sigma|}}\,\mathrm{dvol}_g(\Sigma).\end{aligned} \tag{6}$$

For the calculation, we first express $\mathrm{dvol}_\eta(\Sigma)$ in coordinates on Σ and then do the same for $\mathrm{dvol}_g(\Sigma)$. In the last step, ∇ is expressed using Christoffel symbols. The last integral of (6) can be interpreted as an integral in $(\mathcal{M}, g_{ab})$, so the Integrated Size and Conservation Conditions can be applied. In particular, since $\Gamma^\sigma_{\mu\rho} = O(\mathrm{dist}(x))$, we can use (4) and (5) to see that the last integral tends to zero as $n \to \infty$. This calculation also provides a further insight. The reader might have wondered why a weight occurs in (4). The last integral of (6) shows why: there, (4) enters to control the first non-trivial term of $\Gamma^\sigma_{\mu\rho}$.[24]

In Appendix D, we give the full proof yielding the following theorem:

Theorem 3.4 (Physical GJ theorem)**:** *Let* $(\mathcal{M}, g_{ab})$ *be a spacetime and let* $\left(\gamma, \left({}^{(n)}T_{ab}\right)_{n\in\mathbb{N}}\right)$ *be a Geroch-Jang particle. Then* γ *is a timelike geodesic (upon possible re-parametrization).*

The benefit of this theorem and the "scale-relative topology" approach is that it provides a quantitative understanding of the limits and approximations needed to state and prove a theorem about geodesic motion. The full extent of the quantitative character of the physical GJ theorem is further elucidated by the fact that the "scale-relative topology" approach makes possible a heuristic connection between the theorem and the dynamics of GR. This connection also further motivates the two conditions (4) and (5). Consider first (4). As the support of ${}^{(n)}T_{ab}$ shrinks as $n \to \infty$, this condition prevents a blow-up stronger than $1/\mathrm{dist}(x)$. This is an assumption on the ${}^{(n)}T_{ab}$'s being "well behaved." What is interesting is that (5) can be heuristically derived from (4) *using the additional ingredient* that every ${}^{(n)}T_{ab}$ is conserved with respect to the Levi-Civita connection of the metric "sourced" by it and the energy-momentum tensor of the background spacetime, *as well as particular modeling assumptions* (8) to (10), as we will now explain.

24 This can be construed as the order controlled by the ratio of probe scale and curvature strength (Linnemann, Read, and Teh 2024).

Let T_{ab} be the energy momentum tensor associated with the background spacetime $(\mathcal{M}, g)$, that is to say,

$$G_{ab} =: 8\pi T_{ab}.$$

Moreover, let ${}^{(n)}g_{ab}$ be a metric on $\mathcal{M}$ associated with the Einstein tensor

$$\overset{(n)}{G}_{ab} := 8\pi\left(T_{ab} + \overset{(n)}{T}_{ab}\right), \tag{7}$$

and let ${}^{(n)}\nabla_a$ be the Levi-Civita connection associated with ${}^{(n)}g$.[25] We take the test-particle concept to entail that in any coordinates,

$$\overset{(n)}{g}{}^{\mu\nu} \to g^{\mu\nu} \tag{8}$$

$$\overset{(n)}{\Gamma}{}^{\mu}_{\nu\sigma} \to \Gamma^{\mu}_{\nu\sigma} \tag{9}$$

$$\overset{(n)}{\nabla}_{\mu}\overset{(n)}{G}_{\nu\sigma} \to \nabla_{\mu}G_{\nu\sigma} \tag{10}$$

pointwise as $n \to \infty$. The underlying intuition is that a test particle does not significantly perturb the background metric.

In Appendix E, we present the calculation, which uses the EFE, of how

$$\int_{\Sigma} \nabla^a \overset{(n)}{T}_{ab} X^b \, \mathrm{dvol}_g(\Sigma)$$

can be expressed as a sum of five integrals I_1 to I_5. Since the energy-momentum tensors shrink down to $\gamma(I)$, the spacelike hypersurface can be assumed to have finite volume. Then, by the modeling assumptions, one sees that I_1 to I_3 tend to zero as $n \to \infty$. Finally, to see that I_4 and I_5 tend to zero, we use these limits as well as (4).

4 CONTEXTUALIZING THE PHYSICAL GJ THEOREM

In Section 3, we have introduced our new physical GJ theorem, which implements Geroch and Jang's "scale-relative topology" approach fully. We have thus provided a *realization of the physical GJ argument* by making explicit the scale relativity of the concept of a test body. In addition, we have given a *heuristic justification* of the Integrated Conservation Condition by means of the EFE.

In this section, we add a brief discussion of the physical GJ theorem in the context of multiple other results from the literature, to wit, the results due to Geroch and Weatherall (2018), Ehlers and Geroch (2004),[26] Gralla and Wald (2011), and Yang (2014). A general feature of our result is that just as is the case with Geroch and Jang, we are concerned with the question of which assumptions are sufficient to guarantee the timelikeness of the resulting geodesic. By contrast, the first three results simply assume the timelike character. We will say a little bit more about each of the four approaches in turn.

While we do not wish to say much about the recent study by Geroch and Weatherall, leaving a full evaluation for future work, it should be remarked that their approach can be seen as an implementation of the "weakest topology" approach because of their qualitative

25 The reader should note that in (7), n is used to label the perturbation on the right-hand side while it denotes the metric corresponding to the "base" energy-momentum tensor plus the perturbation on the left-hand side.

26 A rigorous proof of a generalized version of their theorem is given by Bezares et al. (2015).

assumptions on the class of energy-momentum tensors and the absence of quantitative bounds. Moreover, Geroch and Weatherall's theorem represents the energy-momentum tensors by distributions, which leads us away from the intuitive concept of a test body as employed in the original Geroch-Jang paper. While a distributional approach does, of course, not constitute a problem as such, we deem it to be sufficiently different to justify postponing a discussion for now.

The result by Ehlers and Geroch, by contrast, shares an important similarity with our heuristic justification of the assumptions of the physical GJ theorem. The Ehlers-Geroch theorem is itself essentially perturbative, in the way in which the energy-momentum tensors associated with the test body are being absorbed (via the EFE) into the Einstein tensor modeling the world tube of the small body, thereby using an approximation similar to our heuristic derivation in Section 3. However, there are also important differences between the two theorems. First, the Ehlers-Geroch theorem does not implement the "scale-relative topology" approach fully: since no explicit limits are taken and no quantitative bounds specified, the nature of the approximation and limiting procedures remains less explicated than in the statement and proof of the physical GJ theorem. Second, while the result by Ehlers and Geroch operates at a higher level of abstraction (so that it can be interpreted as encompassing both the physical GJ theorem and our heuristic justification of the assumptions from the EFE), this gain in generality comes at the price of losing the intuitive concept of a test body.

Gralla and Wald's result is explicitly perturbative. They model the test body by introducing a formal power series expansion (in the perturbation parameter) around an ambient spacetime solution of the EFE, and consider the linearized equations of motion around that solution. Thus, one can think of the Gralla-Wald approach as being more explicitly perturbative than that by Ehlers and Geroch as it allows for quantitative control over the perturbative fields and their dynamics.

Lastly, consider Yang's mathematical proof of the geodesic motion of test bodies modeled by a more specific choice of energy-momentum tensor. He adopts the matter model of complex scalar fields given by non-linear Klein-Gordon equations, which are then coupled to the Einstein equations. He then shows that there is a sequence of initial data with shrinking spatial support that is more and more centered on a timelike geodesic. Note that Yang gives a genuinely dynamical argument, using the EFE as a system of partial differential equations whose solutions are obtained from an initial-value problem. Here, the timelike nature of the geodesic is obtained "for free," or more accurately, from a careful analysis of the partial differential equations, without having to be imposed "by hand." The price to pay is that he has to assume a fairly specific matter model.

5 CONCLUSION

Brown's claim about the special status of GR vis-à-vis the explanation of geodesic motion of free bodies has attracted considerable scholarly attention. One strain of interpreting Brown's claim—and criticizing it—goes back to Malament, who, accepting that "the geodesic principle can be recovered as a theorem in general relativity," points out that "it is not a consequence of Einstein's equation (or the conservation principle) *alone*" (Malament 2009, 2). To arrive at his conclusion, Malament needed to *choose* a particular way of conceptualizing this recovery of the geodesic principle and thus choose an appropriate theorem. His preferred choice was what we call the GJM result. Apart from Malament, this line of engaging with geodesic motion and Brown's claim was adopted by Weatherall, who largely followed Malament in the explication of "explanation" as a "consequence of Einstein's equation"

(Weatherall 2019, 144)[27] and, at least initially, in probing the issue through the lens of the GJM result. On that basis, he also stressed that to prove geodesic motion, one needs an "assumption that the energy-momentum fields associated with test matter are divergence free just in case the fields are non-interacting"—the second condition in Theorem 2.5—but one does not "get that assumption directly from Einstein's equation," implying that the condition "is a bare assumption about test matter" and not a consequence of the EFE (Weatherall 2011, 280). The strategy of using the GJM result for the philosophical study of inertial motion was also taken up by Sus who called "the result proposed by Geroch and Jang in 1975" one of the "most promising attempts" to "model a body associated with the energy-momentum tensor of the theory" (Sus 2014, 300). Samaroo likewise stated that "[t]here are various geodesic theorems, but Geroch and Jang's (1975) has a claim to being the most perspicuous" and "if any geodesic theorem can be said to figure in a deductive-nomological explanation of inertial motion, the Geroch-Jang theorem can be said to do so" (Samaroo 2018, 972). All these commentators have followed Malament's lead and have made the GJM result and thus the "weakest topology" approach, rather than the "scale-relative topology" approach, the focus of their analysis of geodesic motion in GR.

Our results show that the GJM result is but one strand of the original Geroch-Jang paper, and the less physically motivated strand at that because it does not capture many physicists'—including Geroch and Jang's—intuitive understanding of the test-body concept. Furthermore, we have demonstrated that Geroch and Jang's physical motivation can indeed be clarified and turned into a rigorous argument and theorem in favor of geodesic motion. As argued, the Integrated Conservation Condition also affords a connection (in the sense of an approximation and limit) to the dynamics of the EFE via the heuristic derivation given in Section 3 and Appendix E. While Malament's assessment—that geodesic motion "is not a consequence of Einstein's equation (or the conservation principle) *alone*"—is valid also here because of the requirement of the Uniform Strengthened DEC, Weatherall's worry about covariant conservation might therefore be somewhat assuaged. Consequently, the possibility of a heuristic derivation of the Integrated Conservation Condition could play into the hands of a defender of Brown's claim, who needs a tangible way to cash out the logical relation between the EFE and the dynamics of "small" bodies. We submit that based on our results, the time is ripe for a re-evaluation of Brown's arguments as well as those of his critics.

A ENERGY CONDITIONS

Lemma A.1: *The energy-momentum tensor T_{ab} satisfies the Strengthened DEC if and only if (1) it satisfies the Strict DEC, and (2) if $T_{ab} \neq 0$ at a point, then ${T^a}_b X^b$ is timelike for every timelike X^a.*

Proof. It suffices to show that the Strengthened DEC implies the Strict DEC. Let $T_{ab} \neq 0$ and X^a and Y^a be co-oriented timelike vectors. Then $T_{ab}X^aY^b \neq 0$ because ${T^a}_b X^b$ is timelike. Moreover, since $T_{ab}X^aX^b \geq 0$, X^a and $-{T^a}_b X^b$ are co-oriented. Therefore, $T_{ab}X^aY^b > 0$. □

Remark A.2: *The proof shows in particular that if T_{ab} satisfies the Strengthened DEC and $T_{ab} \neq 0$, then X^a and $-{T^a}_b X^b$ are co-oriented timelike for every timelike X^a.*

Lemma A.3: *An energy-momentum tensor T_{ab} satisfies the Strengthened DEC if and only if at every point, $T_{ab} = 0$ or $T_{ab}X^aY^b > 0$ for all co-oriented X^a and Y^a such that X^a is timelike and Y^a is causal.*

27 It should be noted that in the article, Weatherall also offers an alternative account of cashing out what it means for GR to "explain" geodesic motion.

Proof. It follows from Remark A.2 that if T_{ab} satisfies the Strengthened DEC and $T_{ab} \neq 0$, then for every co-oriented timelike X^a and causal Y^a, $-T^a{}_b X^b$ and Y^a are co-oriented, hence $T_{ab} X^a Y^b > 0$. For the other direction, assume that $T_{ab} \neq 0$ and $T_{ab} X^a Y^b > 0$ for all co-oriented X^a and Y^a such that X^a is timelike and Y^a is causal. If $T^a{}_b X^b$ were null or spacelike, there would be a causal Y^a such that $T^a{}_b X^b Y_a = 0$, a contradiction. Therefore, $T^a{}_b X^b$ is timelike, and T_{ab} satisfies the Strengthened DEC. □

B ENERGY FLUX

The equality (1) is well-known. As such, it appears in Geroch and Jang (1975) without derivation. In this appendix, we derive (1) in a slightly more general setting.

Let $(\mathcal{M}, g_{ab})$ be a maximally symmetric Lorentzian manifold[28] of dimension m, and let Σ be a spacelike hypersurface with future-directed unit normal n^a. The induced volume $(m-1)$-form $\mathrm{dvol}(\Sigma)$ (or $\mathrm{dvol}_g(\Sigma)$ if we need to indicate the corresponding metric) is given by $\iota_n \epsilon$, where ϵ is the volume m-form of $\mathcal{M}$ and ι is the interior product operator. This means that[29]

$$\epsilon = -n^\flat \wedge \mathrm{dvol}(\Sigma).$$

In Geroch and Jang (1975), the one-form $\mathrm{d}S_a$ is used, which satisfies

$$\int_\Sigma X^a \, \mathrm{d}S_a = \int_\Sigma \iota_X \epsilon = -\int_\Sigma \iota_X \left(n^\flat \wedge \mathrm{dvol}(\Sigma)\right) = -\int_\Sigma X^a n_a \, \mathrm{dvol}(\Sigma). \tag{11}$$

If $(x^1, \ldots, x^{m-1})$ are coordinates on Σ and $g_{ab}|_\Sigma$ denotes the induced metric tensor on Σ and $\mathrm{detg}|_\Sigma$ its determinant, then

$$\mathrm{dvol}(\Sigma) = \sqrt{\mathrm{detg}|_\Sigma}\, \mathrm{d}x^1 \wedge \ldots \wedge x^{m-1}.$$

Let a tensor T_{ab} be *admissible* with respect to Σ if and only if the map from the Lie algebra of Killing fields into the reals

$$\xi^a \mapsto \int_\Sigma T^b{}_c \xi^c \mathrm{d}S_b$$

is well-defined for all Killing fields ξ^a.

Lemma B.1: *Let $(\mathcal{M}, g_{ab})$ and Σ be as above. Moreover, let T_{ab} be admissible. Then there exists a unique P_a and a unique anti-symmetric J_{ab} on Σ such that for every Killing field ξ^a,*

$$P_a \xi^a + J_{ab} \nabla^a \xi^b = \int_\Sigma T^a{}_b \xi^b \, \mathrm{d}S_a \tag{12}$$

holds at any point $p \in \Sigma$.

Proof. Set $N := m(m+1)/2$. Let $\mathcal{K}$ be the N-dimensional Lie algebra of Killing fields on $\mathcal{M}$. For every $p \in \Sigma$, a Killing field is uniquely determined by the values of ξ^a and $\nabla^b \xi^c$ $(b < c)$ at p via the ordinary differential equation

$$\nabla_a \nabla_b \xi^c = R^c{}_{bad} \xi^d. \tag{13}$$

We define P_a and J_{ab} as follows. Fix $p \in \Sigma$. From (13), we know that there exist a linear bijection

$$\Phi_p : \mathbb{R}^N \to \mathcal{K}$$

28 A (pseudo-)Riemannian manifold of dimension m is called maximally symmetric if it has $m(m+1)/2$ independent Killing fields.

29 With $\flat$ denoting the musical isomorphism, indeed $\iota_n \left(n^\flat \wedge \mathrm{dvol}(\Sigma)\right) = \iota_n n^\flat \wedge \mathrm{dvol}(\Sigma) - n^\flat \wedge \iota_n \mathrm{dvol}(\Sigma) = -\mathrm{dvol}(\Sigma)$.

mapping the values of ξ^a and $\nabla^b \xi^c$ ($b < c$) at p onto the vector space of Killing fields. We also define the linear map

$$Q : \mathcal{K} \to \mathbb{R},\ \xi^a \mapsto \int_\Sigma T^a{}_b \xi^b \, \mathrm{d}S_a.$$

Since the map $Q \circ \Phi_p$ is linear, there are N coefficients P_a and $J_{[ab]}$ at p such that at p,

$$P_a \xi^a \Big|_p + J_{ab} \nabla^a \xi^b \Big|_p = \int_\Sigma T^a{}_b \xi^b \, \mathrm{d}S_a. \tag{14}$$

Since p is arbitrary, and since the right hand side is independent of p, (14) defines a one-form P_a and a two-form $J_{[ab]}$ on Σ. Moreover, (14) determines P_a and $J_{[ab]}$ uniquely at every p, so P_a and $J_{[ab]}$ are unique. □

Remark B.2: *Both P_a and J_{ab} depend crucially on T_{ab}* and Σ. *Note also that the boundary $\partial(\Sigma)$ of Σ need not be empty.*

Corollary B.3: *Suppose that the intersection of $\partial\Sigma$ with the support of T_{ab} on Σ is empty. Then*

$$\left[\nabla_a P_b - J_{cd} R^{cd}{}_{ab}\right] \xi^b + \left[\nabla_a J_{bc} + g_{a[b} P_{c]}\right] \nabla^b \xi^c = K_a. \tag{15}$$

such that the values of the one-form K_a are determined as follows. If X^a is parallel to Σ, then $\iota_X K = 0$. The remaining component is determined by

$$\iota_n K = \int_\Sigma \nabla^a T_{ab} \xi^b \, \mathrm{dvol}(\Sigma). \tag{16}$$

Proof. The identity (15) can be obtained by taking the Lie derivative of both sides of (12) with respect to an arbitrary vector field X^a. The left-hand side is obtained by a straightforward calculation. The right-hand side vanishes if X^a is parallel to Σ. If it is orthogonal to the hypersurface, we need to calculate $\mathcal{L}_X \left(T^a{}_b \xi^b \, \mathrm{d}S_a\right)$ in the integral. By (11), setting $Y^a = T^a{}_b \xi^b$, we can use Cartan's magic formula:[30]

$$\begin{aligned} \mathcal{L}_X (\iota_Y \epsilon) &= \iota_X (\mathrm{d}(\iota_Y \epsilon)) + \mathrm{d}(\iota_X \iota_Y \epsilon) \\ &= \iota_X ((\mathrm{div} Y)\epsilon) - \mathrm{d}(\iota_Y \iota_X \epsilon) \\ &= \mathrm{div} Y \, \iota_X \epsilon - \mathrm{d}(\iota_Y \iota_X \epsilon). \end{aligned}$$

Now set $X^a = n^a$. Since the integral of the second term of the last line over Σ vanishes by Stokes' Theorem (because the intersection of $\partial\Sigma$ and the support of T_{ab} was assumed to be empty), we obtain (16). □

C THE SR CASE

In this appendix, we present a systematic review of Geroch and Jang's argument for geodesic motion of extended bodies in Minkowski space $(\mathbb{R}^4, \eta_{ab})$. Our presentation is more careful than that of Geroch and Jang (1975), in particular it pays closer attention to the precise step in the argument in which one needs to assume that the energy-momentum tensor is divergence-free.

First, for a non-vanishing admissible energy-momentum tensor T_{ab} with compact support, define a one-form P_a and two-form J_{ab} relative to a spacelike hypersurface $\Sigma = \{t = t_0\}$ via (12). From this one derives the following

30 We also use that $\iota_A \iota_B = -\iota_A \iota_B$ as well as $\mathrm{d}(\iota_A \epsilon) = \mathrm{div} A \, \epsilon$ for all vector fields A^a and B^a, where $\mathrm{div} A$ denotes $\nabla_a A^a$.

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Proposition C.1: *Let* $(\mathbb{R}^4, \eta_{ab})$ *be Minkowski space with global coordinates* $(x^\mu) = (t, x^1, \ldots, x^3)$ *and let* Σ *be a spacelike hypersurface with constant* $t = t_0$. *Then*

$$\partial_i P_\mu = 0$$
$$\partial_i J_{\mu\nu} = -\eta_{i[\mu} P_{\nu]}.$$

Moreover, there exist a one-form A_a *and a two-form* B_{ab} *such that*

$$\partial_t P_\mu = A_\mu$$
$$\partial_t J_{\mu\nu} = -\eta_{i[\mu} P_{\nu]} + B_{[\mu\nu]}.$$

The forms A_a *and* B_{ab} *are defined via integrals of linear combinations of* $\partial^\mu T_{\mu\nu}$ *with linear weights over* Σ.

Proof. The proposition is a direct application of Corollary B.3 with $n^a = \partial_t$, noting that Minkowsi space has vanishing Riemann tensor. □

Remark C.2: *Evidently, if* T_{ab} *is covariantly conserved, i.e.,* $\partial^\mu T_{\mu\nu} = 0$, *then* A_μ *and* $B_{\mu\nu}$ *vanish.*

We now construct a center-of-motion point p (of the extended body) on Σ under the assumption that P^a is timelike.

Lemma C.3: *Under the same assumptions as above, assume furthermore that* $P_a P^a < 0$ *on* Σ. *Then there is a unique point* $p \in \Sigma$ *such that*

$$J_{i\mu} P^\mu = 0.$$

Proof. One calculates

$$\partial_i \left(J_{\mu\nu} J^{\mu\nu}\right) = -2\eta_{i\mu} P_\nu J^{\mu\nu}$$

and

$$H_{ij} := \partial_j \partial_i \left(J_{\mu\nu} J^{\mu\nu}\right) = 2\eta_i{}^\mu P^\nu \eta_{j[\mu} P_{\nu]} = |P|^2 \delta_{ij} - P_i P_j, \tag{17}$$

where $|P|^2 = P_\mu P^\mu$.

$$\Pi_{\mu\nu} := \eta_{\mu\nu} - \frac{1}{|P|^2} P_\mu P_\nu \tag{18}$$

is the projection operator orthogonal to P, which is positive definite when restricted to spacelike vectors. The matrix H_{ij} is therefore negative definite (because $|P|^2 < 0$). Moreover, H_{ij} is constant on Σ, and hence $J_{\mu\nu} J^{\mu\nu}$ achieves a unique maximum $p \in \Sigma$ where its gradient vanishes. □

Remark C.4: *Since* P^a *is timelike and* J_{ab} *anti-symmetric, one has, in fact,*

$$J_{\mu\nu} P^\nu = 0 \tag{19}$$

at the point p *defined by Lemma C.3.*

Next, we use the Strengthened DEC to show that the conditions of the previous lemma are indeed satisfied, that is to say, that P^a is timelike.

Lemma C.5: *If* T_{ab} *satisfies the Strengthened DEC and has non-vanishing support, then* P^a *is future-directed timelike.*

Proof. The one-form P_a is constant on Σ. Hence, we can write $P^a = \lambda (\partial_t)^a + \alpha^a$ on Σ, where $\lambda \in \mathbb{R}$ and α^a spacelike and constant (in Minkowski space). We show that P^a is timelike by distinguishing two cases. If $\alpha^a = 0$, then trivially, P^a is timelike. Otherwise, the

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constant (on Σ) non-vanishing vector field $\xi^a = \text{sign}(\lambda)|\alpha|(\partial_t)^a + \alpha^a$ is Killing, null, and future-directed, where $|\alpha| = \sqrt{\alpha_a\alpha^a}$. We have

$$P_a\xi^a = -\text{sign}(\lambda)\lambda|\alpha| + |\alpha|^2 = |\alpha|(|\alpha| - |\lambda|).$$

Therefore, from (12), we obtain using (11) that

$$|\alpha|(|\alpha| - |\lambda|) = -\int_\Sigma T^a{}_b\xi^b n_a \,\text{dvol}(\Sigma).$$

By Lemma A.3, $T^a{}_b\xi^b n_a > 0$, so $|\alpha| \in (0, |\lambda|)$, whence

$$P_aP^a = (|\alpha| + |\lambda|)(|\alpha| + |\lambda|) < 0.$$

Therefore, P^a is timelike. Now, choosing $\xi^a = (\partial_t)^a$ in (12) shows that $P_a(\partial_t)^a < 0$, hence $\lambda > 0$ and P^a is future-directed. ☐

We now use a particular Killing field (a boost around P^a that leaves p invariant) to show that the center-of-motion point p lies in the convex hull of the support of T^{ab}, that is to say, "in the extended body."

Lemma C.6: *Let T_{ab} have non-vanishing compact support on Σ satisfying the Strengthened DEC. Let $p \in \Sigma$ such that $J_{i\mu}P^\mu = 0$ at p. Then p lies in the convex hull C of the support of T_{ab} in Σ.*

Proof. Suppose p to lie outside of the convex hull C. Then there exists a unique geodesic in Σ from p to the boundary of C with minimal length. Let V^a be its constant tangent vector. By Lemma C.5, P^a is future-directed timelike, so there is a constant vector field X^a such that P^a, V^a, X^a are linearly dependent as well as $V_aX^a > 0$, $P_aX^a = 0$, and $X_aX^a = -P_aP^a$. Now let ξ^a be the the unique Killing field such that at p,

$$\xi^a = 0, \quad X^a\nabla_a\xi^b = P^b, \quad P^a\nabla_a\xi^b = X^b, \quad Y^a\nabla_a\xi^b = 0$$

for any Y^a orthogonal to P^a and X^a.[31] Then ξ^a is future-directed timelike on C. Then by construction, the left-hand side of (12) vanishes at p where $J_{\mu\nu}P^\nu = 0$ by (19). Yet, the right-hand side is strictly negative, a contradiction. ☐

It is worth noting that we have not yet made use of T_{ab}'s being covariantly conserved up to this point. As the following proposition shows, the conservation condition is used to "follow" the points where $J_{i\mu}P^\mu = 0$. It proves that the resulting center of motion moves on a geodesic.

Proposition C.7: *Let T_{ab} be a conserved energy-momentum tensor with compact spatial support satisfying the strengthened DEC. There then exists a well-defined smooth curve γ in the convex hull of T_{ab} such that*

$$J_{i\mu}P^\mu = 0$$

on the curve. Then P^a is parallel to the tangent vector $\dot{\gamma}^a$.

Proof. The claim can be easily seen by computing the derivatives of $J_{j\mu}P^\mu$. Since T is conserved, we obtain

$$\partial_i\left(J_{j\mu}P^\mu\right) = -\frac{1}{2}|P|^2\Pi_{ij}$$
$$\partial_t\left(J_{j\mu}P^\mu\right) = -\frac{1}{2}|P|^2\Pi_{tj},$$

31 One can easily check that this specification indeed gives rise to a well-defined Killing field by checking that at p, $\nabla_a\xi_b$ is anti-symmetric.

where Π as in (18). The $\nabla_\nu\left(J_{j\mu}P^\mu\right)$ are therefore just the components of the projection operator orthogonal to P^a. Since $J_{i\mu}P^\mu = 0$ on γ, $\nabla^a\left(J_{j\mu}P^\mu\right)$ is orthogonal to $\dot{\gamma}^a$. This immediately yields the claim. □

The above results can now be summarized in the following theorem:

Theorem C.8: *Let $(\mathbb{R}^4, \eta_{ab})$ be Minkowski space, and let T_{ab} be a conserved energy-momentum tensor with non-vanishing compact spatial support satisfying the Strengthened DEC. Then there is a timelike geodesic in the convex hull of the support of T_{ab}.*

D PROOF OF THE GR CASE

In Section 3, we introduced the definitions necessary to formulate Theorem 3.4. Moreover, we presented the conceptual ingredients of the proof. Here, we give the full details.

The proof proceeds in two steps. In the first step, we show that γ is timelike. We first define a Minkowskian manifold $(\mathcal{N}, \eta_{ab})$, the "pre-frame," about an arbitrary point on the curve, say $p \in \gamma(I)$. This is done as follows. There exist a neighborhood U of p and normal coordinates $(t, x^1, \dots, x^3)$ in U such that in these coordinates, $p = (0, 0, 0, 0)$ and $g_{\mu\nu} = \eta_{\mu\nu}$ at p. Moreover, all Christoffel symbols vanish at p.

Now view (U, η_{ab}) as a manifold (which is isometrically isomorphic to a subset of Minkowski space). By the Uniform Strengthened DEC (Definition 3.1), one can choose U sufficiently small such that for all n, ${}^{(n)}T_{ab}$ satisfies the Strengthened DEC in (U, η_{ab}). Furthermore, there is an $\epsilon > 0$ such that for all $s \in (-\epsilon, \epsilon)$, $\Sigma_s \cap \gamma(I) \neq \emptyset$, where $\Sigma_s := \{t = s\}$. Set $\mathcal{N}_p := U \cap \{-\epsilon < t < \epsilon\}$. The manifold $(\mathcal{N}, \eta_{ab})$ is also, evidently, isometrically isomorphic to a subset of Minkowski space.

In $(\mathcal{N}, \eta_{ab})$, we can now simply apply results from Minkowski space for every ${}^{(n)}T_{\mu\nu}$ and obtain tensors ${}^{(n)}P_\mu$ and ${}^{(n)}J_{\mu\nu}$ via

$$\overset{(n)}{P}{}_\mu \xi^\mu + \overset{(n)}{J}{}_{\mu\nu} \nabla^\mu \xi^\nu = -\int_{\Sigma_0} \overset{(n)}{T}{}_{t\nu} \xi^\nu \,\mathrm{dvol}_\eta(\Sigma_0) \tag{20}$$

as in (12). The momentum ${}^{(n)}P_\mu$ is future-directed timelike because every energy-momentum satisfies the Strengthened DEC in $(\mathcal{N}, \eta_{ab})$. This follows in the same way as in the SR case (Lemma C.5). Moreover, we can likewise define a center-of-motion curve ${}^{(n)}\gamma$ defined by

$$\overset{(n)}{J}{}_{i\mu} \overset{(n)}{P}{}^\mu = 0,$$

analogous to Lemmas C.3 and C.6. The center-of-motion curve then lies in the convex hull of the energy-momentum tensor's support.

Since the support of the energy-momentum tensors shrinks to $\gamma(I)$, we have a convergence of the curves ${}^{(n)}\gamma$ to γ. Moreover, by (3), the sequences of both ${}^{(n)}P_a$ and ${}^{(n)}J_{ab}$ are bounded, which means that there is a subsequence such that they converge to a P_a and a J_{ab}, respectively. We pass to this subsequence. Now, since every ${}^{(n)}P_a$ is constant on Σ_0 and timelike, P_a is constant on Σ_0 and timelike or null. To show that P_a is in fact timelike, consider the Killing field in $(\mathcal{N}, \eta_{ab})$ that is uniquely determined by

$$\xi_\mu = P_\mu, \quad \partial_\mu \xi_\nu = 0$$

at $(0, 0, 0, 0)$. Moreover, by the Uniform Strengthened DEC, there exists a $c > 0$ such that at $(0, 0, 0, 0)$,

$$\overset{(n)}{T}{}_{t\mu} P^\mu \geq c$$

for all n. For this ξ^a, the right-hand side of (20) is therefore negative for all large n and hence P_a timelike.

We now show that at p, P_a is parallel to the tangent vector of γ. For this, we prove that

$$\overset{(n)}{P}{}^{\mu}\partial_\mu\left(\overset{(n)}{J}{}_{j\nu}\overset{(n)}{P}{}^{\nu}\right)\to 0$$

for all $j = 1, 2, 3$ as $n \to \infty$. First, one calculates the derivatives of ${}^{(n)}P_\mu$ and ${}^{(n)}J_{\mu\nu}$ as in Proposition C.1. One obtains

$$\partial_i\left({}^{(n)}J_{j\mu}\,{}^{(n)}P^\mu\right) = -\frac{1}{2}\left|{}^{(n)}P\right|^2\,{}^{(n)}\Pi_{ij}$$
$$\partial_t\left({}^{(n)}J_{j\mu}\,{}^{(n)}P^\mu\right) = -\frac{1}{2}\left|{}^{(n)}P\right|^2\,{}^{(n)}\Pi_{tj} +{}^{(n)} B_{j\mu}\,{}^{(n)}P^\mu +{}^{(n)} J_{j\mu}\,{}^{(n)}A^\mu,$$

where ${}^{(n)}\Pi$ is the projection operator defined as in (18). It is easy to see that the product of ${}^{(n)}P_a$ with these components tends to zero at p once one realizes that the tensors ${}^{(n)}A_a$ and ${}^{(n)}B_{ab}$ go to zero: recall that the components of these tensors are given by integrals of the form

$$\int_\Sigma \eta^{\mu\rho}\partial_\mu\,{}^{(n)}T_{\rho\nu}\eta^{\nu\sigma}X^\nu\,\mathrm{dvol}_\eta(\Sigma); \tag{21}$$

as explained in Section 3 (see (6)), these integrals tend to zero as $n \to \infty$. This proves that γ is timelike with a tangent vector proportional to P^a at p. Since p is arbitrary, the result holds for the entire curve.

In the second step of this proof, we show that γ can be re-parametrized as a geodesic. For this, we use the fact that γ is timelike to construct Fermi normal coordinates along γ via standard methods.[32] We can now repeat the argument of the first step along the whole curve to obtain a timelike P_a defined along γ and tangent to γ. It remains to show that

$$\overset{(n)}{P}{}^{\mu}\,\nabla_\mu\,\overset{(n)}{P}{}_{\nu}\to 0.$$

as $n \to \infty$. Since we have chosen Fermi normal coordinates, we know that the Christoffel symbols vanish along γ. Therefore, on $\gamma(I)$,

$$\overset{(n)}{P}{}^{\mu}\nabla_\mu\overset{(n)}{P}{}_{\nu} = \overset{(n)}{P}{}^{\mu}\partial_\mu\overset{(n)}{P}{}_{\nu} = \overset{(n)}{P}{}^{t}\overset{(n)}{A}{}_{\nu}.$$

Calculating (21) as before, we see that the quantity tends to zero as $n \to \infty$. This finishes the proof.

E CALCULATING THE DIVERGENCE OF THE *n*TH ENERGY-MOMENTUM TENSOR

We have

$$\begin{aligned}\nabla^a\overset{(n)}{T}{}_{ab} &= g^{ac}\nabla_c\overset{(n)}{T}{}_{ab}\\ &= g^{ac}\nabla_c\left(T_{ab}+\overset{(n)}{T}{}_{ab}\right)\\ &= \overset{(n)}{g}{}^{ac}\overset{(n)}{\nabla}{}_c\tau_{ab}+\left(g^{ac}-\overset{(n)}{g}{}^{ac}\right)\overset{(n)}{\nabla}{}_c\tau_{ab}-g^{ac}\left(\Gamma^d_{ac}-\overset{(n)}{\Gamma}{}^d_{ac}\right)\tau_{db}-g^{ac}\left(\Gamma^d_{bc}-\overset{(n)}{\Gamma}{}^d_{bc}\right)\tau_{ad},\end{aligned} \tag{22}$$

where we use the shorthand

$$\tau_{ab} = T_{ad}+\overset{(n)}{T}{}_{ad}.$$

32 See Poisson (2004, 37–38).

The first term in the last expression in (22) is zero by (7) and the contracted Bianchi identity. We, thus, obtain:

$$\begin{aligned}\int_\Sigma \nabla^a \overset{(n)}{T}_{ab} X^b \,\mathrm{dvol}_g(\Sigma) = &\int_\Sigma \left(g^{ac} - \overset{(n)}{g}{}^{ac}\right) \overset{(n)}{\nabla}_c \overset{(n)}{G}_{ab} X^b \,\mathrm{dvol}_g(\Sigma) \\ &- \int_\Sigma g^{ac}\left(\Gamma^d_{ac} - \overset{(n)}{\Gamma}{}^d_{ac}\right) T_{db} X^b \,\mathrm{dvol}_g(\Sigma) \\ &- \int_\Sigma g^{ac}\left(\Gamma^d_{bc} - \overset{(n)}{\Gamma}{}^d_{bc}\right) T_{ad} X^b \,\mathrm{dvol}_g(\Sigma) \\ &- \int_\Sigma g^{ac}\left(\Gamma^d_{ac} - \overset{(n)}{\Gamma}{}^d_{ac}\right) \overset{(n)}{T}_{db} X^b \,\mathrm{dvol}_g(\Sigma) \\ &- \int_\Sigma g^{ac}\left(\Gamma^d_{bc} - \overset{(n)}{\Gamma}{}^d_{bc}\right) \overset{(n)}{T}_{ad} X^b \,\mathrm{dvol}_g(\Sigma)\end{aligned}$$

Denote the first integral on the right-hand side by I_1, the second one by I_2, and so forth.

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