Range of Violations of Bell's Inequality by Entangled Photon Pairs*

Frederick M. Kronz
Department of Philosophy, The University of Texas at Austin

If the quantum states of measured pairs are entangled, then there are triplets of experimental configurations for which Bell's original inequality is violated. This paper gives a concise characterization of the entire range of possible triplets of polarization measurements on entangled photon pairs for which the inequality is violated.

PACS numbers: 03.65.Bz, 42.50.Ar
Keywords: quantum entanglement, Bell's inequality, nonlocality

Quantum entanglement is a key formal feature common to many of the well-known conceptual conundrums that arise in quantum mechanics including the famous paradoxes of Einstein, Podolsky, and Rosen (EPR) [1] and Schrödinger's cat [2]. The latter is a particularly graphic illustration of the well-known measurement problem. Quantum entanglement (EPR-correlations) typically occurs when two or more quantum systems interact, though interaction is not a necessary condition for it to occur (as in fermion systems). The state of an entangled system is represented as a nonfactorizable superposition of product states; whereas, that of an unentangled systems is represented as a product state. The EPR paradox, particularly Bohm's version of the paradox [3], is the source of continued speculations about nonlocal action-at-a-distance and the incompatibility of quantum mechanics and the special theory of relativity [4]. The measurement problem is the source of continued questioning of the adequacy of quantum mechanics to explain the emergence of classicality [5]. The original Bell inequalities [6] and reformulated versions [7] are extremely useful tools for exploring the nonlocal [8] and nonclassical [9] character of entangled systems. This essay uses one of the original Bell inequalities to characterize extensively (though not exhaustively) the range of violations of Bell's inequality by pairs of photons that have entangled polarization states. This account of the range of violations is much more extensive than any of the existing accounts.

It is useful to begin the discussion with a brief characterization of expectation values associated with polarization measurements on single photons. Recall that the expectation value for an experiment A is defined to be $\langle A \rangle = \sum_{i=1}^{n} a_i p_i$, where $\{a_1, \cdots, a_n\}$ is the set of possible outcomes of A and be $\{p_1, \cdots, p_n\}$ is the associated set of probabilities. Let P_v be an experiment that consists of measuring elements of an ensemble of photons for linear polarization in the v-direction, where v is an angle θ from the y-axis in the xy-plane. P_v has two possible outcomes, v and $v^{\perp}$ ("linearly polarized in the v-direction" and "linearly polarized in the $v^{\perp}$-direction"), which are assigned the numerical values +1 and –1, respectively. If the pre-measurement photon moves in the z-direction and is linearly polarized in the y-direction, then the probability

* Received 18 Octorber 2000; accepted 13 December 2000; published 5 February 2001 in *Physica Letters A* 279 (2001), pp. 287-290. Communicated by P.R. Holland.

associated with the measurement outcome +1 is $\cos^2\theta$, and that for –1 is $\sin^2\theta$. It follows that $\langle P_v \rangle = \cos^2\theta - \sin^2\theta$, which simplifies to $\cos 2\theta$.

The pertinent type of experiment here is P_{uv}, which consists of a compound measurement on members of an ensemble of photon pairs. The indices u, v correspond to directions in the xy-plane, the angle between them is θ_{uv}. The elements of a given pair are emitted in opposite directions along the z-axis from a common source S. One element of the photon pair is measured for P_u and the other for P_v. Each measurement has two possible outcomes, as indicated above, and as before they are assigned the numerical values +1 and –1. S emits one component of the photon pair in the z-direction towards a measuring device R, which measures photons for P_v, and the other component of the pair in the opposite direction towards another measuring apparatus L, which measures photons for P_u. The outcome of P_{uv} for a given photon pair is defined as the product of the value obtained at L with that obtained at R. Thus, P_{uv} has two possible outcomes, +1 (meaning that the numerical outcomes at L and R are the same) and –1 (meaning they are different). If S emits photon pairs in the entangled polarization state

$$|\Phi\rangle = \tfrac{1}{\sqrt{2}}\left(|yy^{\perp}\rangle + |y^{\perp}y\rangle\right), \tag{1}$$

then the probability that P_{uv} yields the outcome +1 is $\sin^2\theta_{uv}$ and that for the outcome –1 is $\cos^2\theta_{uv}$. It follows that $\langle P_{uv} \rangle = \sin^2\theta_{uv} - \cos^2\theta_{uv}$, which simplifies to $-\cos 2\theta_{uv}$.

The original version of Bell's inequality [6] involves three experiments. Each of the three is similar to P_{uv} in that it involves a pair of measurements. The experiments are related in that each has exactly one measurement component in common with each of the other two, as in the triplet $\{P_{uv}, P_{vw}, P_{uw}\}$, where u, v, w are three distinct directions in the xy-plane. In what follows, triplets of experiments $\{E_{ab}, E_{bc}, E_{ac}\}$, which involve compound measurements that are related in the manner indicated, are referred to as "Bell triplets." The Bell inequality establishes a relationship that must hold between the expectation values associated with the elements of a Bell triplet, if certain locality conditions are satisfied:

$$1 + \langle E_{bc} \rangle - \left| \langle E_{ab} \rangle - \langle E_{ac} \rangle \right| \geq 0. \tag{2}$$

It turns out that Bell's inequality is violated by numerous quantum-mechanical Bell-triplets.

The Bell triplets of interest here is the class of triplets of experiments $\{P_{uv}, P_{vw}, P_{uw}\}$ on photon pairs that are prepared in the entangled polarization state (1). Such triplets violate Bell's inequality to varying degrees. The purpose of this essay is to specify triplet types and then to show the extent to which the various triplet types violate (2). The triplet $\{P_{uv}, P_{vw}, P_{uw}\}$ may be classified with respect to a single parameter q that relates the three angles θ_{uv}, θ_{vw}, θ_{uw}. It may be assumed without loss of generality that $\theta_{uw} = \theta_{uv} + \theta_{vw}$. Thus, there is real number q, $0 < q < 1$, such that $\theta_{uv} = q\theta_{uw}$ and $\theta_{vw} = (1-q)\theta_{uw}$. Violations of Bell's inequality for triplets of type q may be explored for the range of angles $0 < \theta_{uw} < 2\pi$. It is useful to examine a particular case. To do so, let $q = \frac{1}{2}$ and let $2\theta_{uv} = 2\theta_{vw} = \theta_{uw} = \phi$. Now consider the function

$$f(\phi) = 1 - \cos\phi - \left| \cos 2\phi - \cos\phi \right|. \tag{3}$$

It follows that Bell's inequality is violated for Bell triplets of type-$\frac{1}{2}$ whenever $f(\phi)<0$. The diagram of $f(\phi)$ below shows the range of angles for which Bell's inequality is violated.

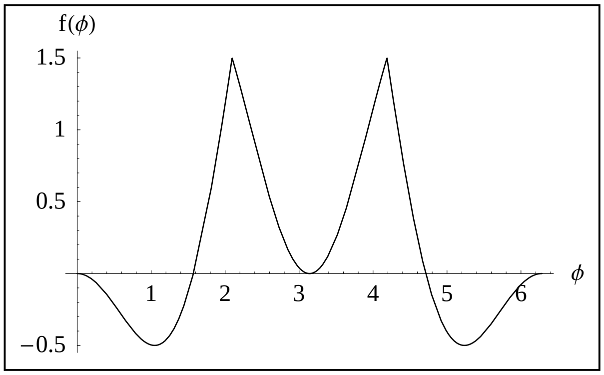

Figure 1: $q=\frac{1}{2}$

Figure 1 shows that that Bell's inequality is violated when $0<\phi<\frac{\pi}{2}$ and when $\frac{3\pi}{2}<\phi<2\pi$. Equation (3) is easily generalized to Bell triplets of type-q. The relevant function is

$$f_q(\phi)=1-\cos\frac{2\phi}{q}-\left|\cos 2\phi-\cos\frac{2\phi\,q}{q-1}\right|. \tag{4}$$

Four additional cases follow (Figures 2-5) that correspond to the set of q-values $\left\{\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{32}\right\}$.

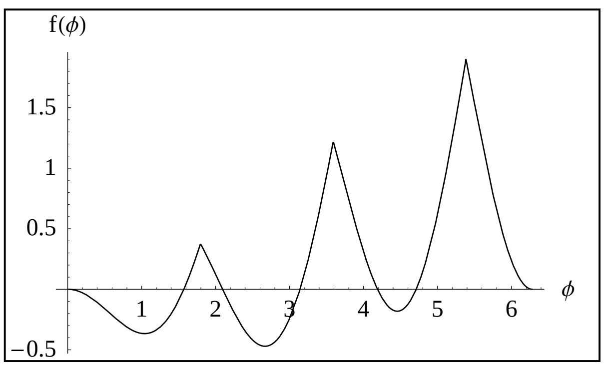

Figure 2: $q=\frac{1}{4}$

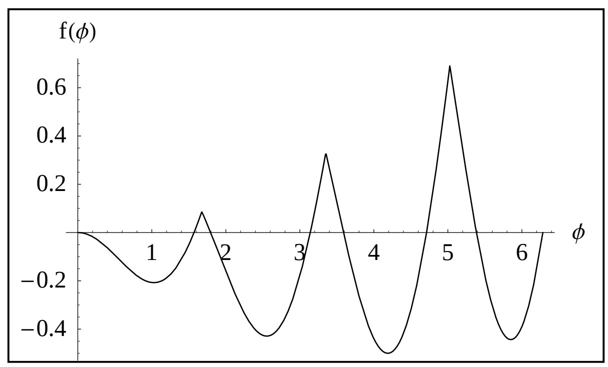

Figure 3: $q=\frac{1}{8}$

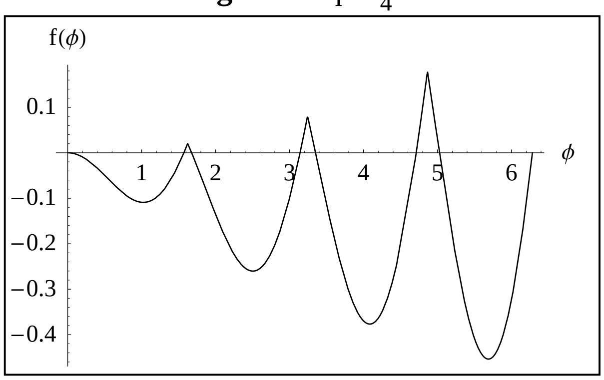

Figure 4: $q=\frac{1}{16}$

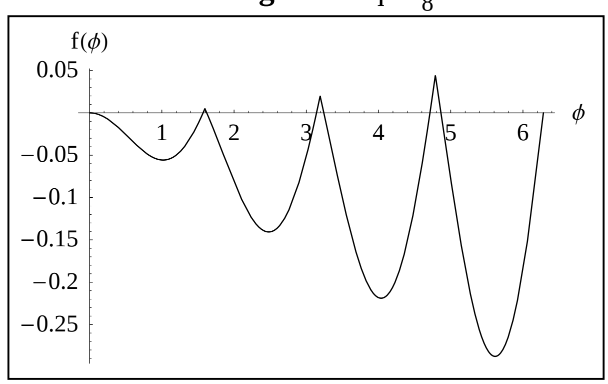

Figure 5: $q=\frac{1}{32}$

The entire set of q-types may be represented using either a three-dimensional graph or a contour graph by regarding (4) as a two parameter function of ϕ and q (see Figures 6-7 below).

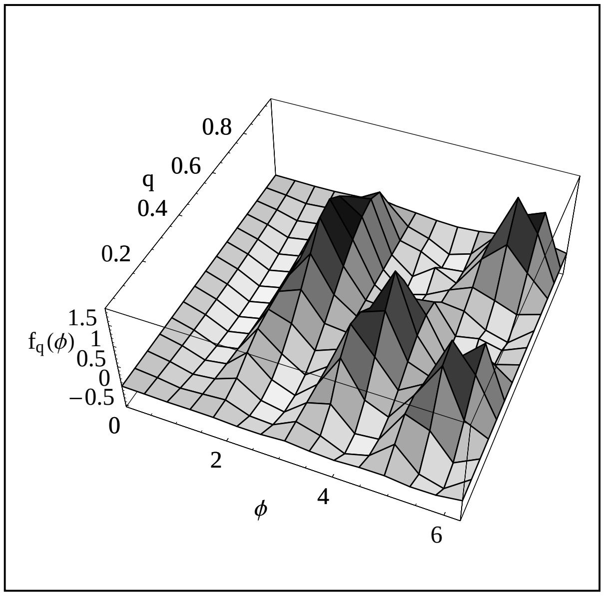

Figure 6

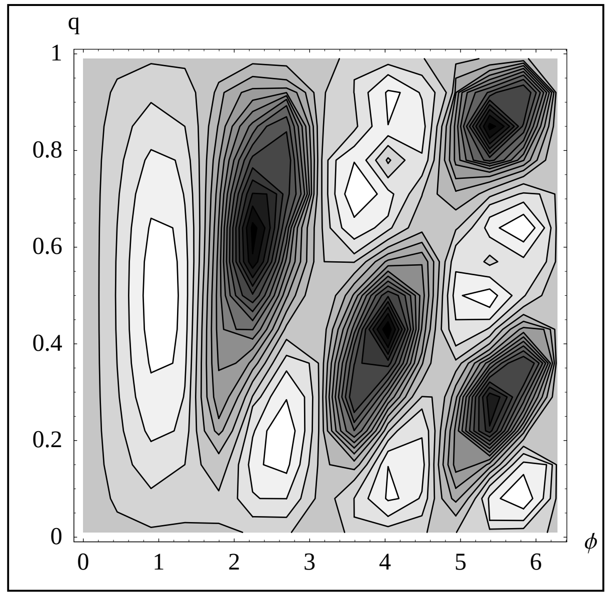

Figure 7

The valleys of quantum nonlocality are in white and the peaks of local realism are in black (Figures 6-7). The peaks are three times as high as the valleys are deep, though the valleys outnumber the peaks two-to-one. The largest quantum valley is the one that occurs when the angle ϕ (that is, θ_{uw}) equals one radian, in which case the corresponding valley of quantum nonlocality occurs for nearly the entire range of q-values, $0 < q < 1$. It is not as obvious which q-value has the most robust nonlocal character for the range of ϕ-values, $0 \le \phi \le 2\pi$. One way to make this determination is to calculate the area enclosed by $f_q(\phi)$ underneath the y-axis for that range. The function from q-values to areas is plotted below in Figure 8.

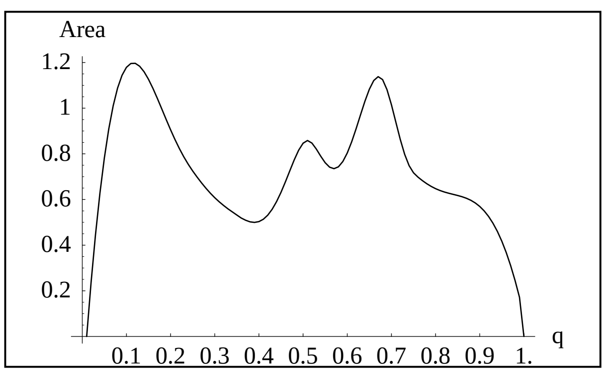

Figure 8

There are two distinct peaks, one at $q = 0.11$ and the other at $q = .67$.

The significance of the specific values where quantum nonlocality is maximized and where local realism is maximized are subjects for additional research. It is likely a new focus for speculations regarding quantum nonlocality and the emergence of classicality.

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