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ORIGINAL RESEARCH

Are nonmeasurable sets significant for epistemology?

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Abstract
Probabilism holds that rational credence functions are probability functions defined over some probability space $(\Omega, \mathcal{F}, P)$. According to some recent philosophical arguments, in some situations, rational credence function must be *total*, i.e. $\mathcal{F} = 2^{\Omega}$, a view which I call *credence totalism*. Arguments for credence totalism are based on the premise that non-Lebesgue measurable subsets of $\mathbb{R}$ are epistemically significant, in the sense that an agent has reasons to assign probability to these sets. This paper argues that nonmeasurable sets are not epistemically significant in this sense. Consequently, the arguments for credence totalism are not successful. My argument is based on a careful consideration of the role of the Axiom of Choice in probabilistic practice. I also discuss some topics considered closely related, viz. the existence of total chance functions and the truth value of the Continuum Hypothesis. I argue that the role of nonmeasurability in epistemology does not shed light on these issues.

Keywords Probability · Credence · Measurability · Totalism · Axiom of choice

1 Introduction

A *probability space* is a triple $(\Omega, \mathcal{F}, P)$, where the *sample space* Ω is any set, $\mathcal{F}$ is a set of subsets of Ω called the set of *probability bearers*, and the *probability function* P is a function from $\mathcal{F}$ to some *probability range* R. In Bayesian epistemology, an agent's *credence* for propositions is often considered as probability functions living in such spaces. Bayesian epistemology investigates formal constraints on credence functions placed by epistemic rationality. For example, *probabilism* holds that epis-

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temic rationality requires an agent's credence function to be a probability function with $R = [0, 1]$ that satisfies the axioms of total probability and finite additivity, or to be extendible to such a function. Probabilists have developed arguments which aim to show that any agent whose credence function does not satisfy these axioms is irrational (Lin, 2024, § 2).

Does epistemic rationality require credence function to satisfy additional formal properties beyond these two axioms, especially when **Ω** is infinite? There are two extensively discussed issues in this realm. The first concerns whether finite additivity should be strengthened to countable additivity, or even full additivity (De Finetti, 2017; Bartha, 2004; Easwaran, 2013; Stewart & Nielsen, 2021). The second issue concerns the nature of the probability range *R*, for example, whether *R* should be extended beyond standard real numbers to include infinitesimals, or replaced by some partially ordered set (Williamson, 2007; Burgess, 2010). In this paper, however, we focus on an issue distinctive from these two, namely the proper choice of $\mathcal{F}$, the set of probability bearers.

$\mathcal{F}$ is typically required to satisfy some closure properties. The axiom of finite additivity naturally requires $\mathcal{F}$ to be an algebra, and countable additivity naturally requires $\mathcal{F}$ to be a σ-algebra. Further requirements on $\mathcal{F}$ are uncommon. In particular, when **Ω** is infinite, $\mathcal{F}$ is *not* commonly required to be the full powerset 2^{Ω}. We say that a probability *P* over **Ω** is *total* if its domain $\mathcal{F}$ contains *all* subsets of **Ω**, i.e. $\mathcal{F} = 2^{\Omega}$. If some $X \subseteq \Omega$ is left out of $\mathcal{F}$, that means *P* does not assign a probability to *X*. If an agent's credence is represented by such a *P*, this means she has no credence for *X*, or she leaves a *credence gap* on *X*. A question thus arises: Does epistemic rationality require an agent's credence function to be total?

Rationality places some general constraints on credence functions, which are applicable in most, if not all, epistemic situations. We may call them *global* constraints. Examples may include "avoiding Dutch Book" and "avoiding inconsistency". In contrast, *local* constraints are only derived from and applicable to specific epistemic situations. Many have argued that no global constraint implies that credence function ought to be total (Lin, 2024, §3; De Finetti, 2017, §3.3.3). However, some philosophers recently proposed that in certain epistemic situations where the sample space **Ω** has cardinality continuum, there are strong reasons to have total credence functions (Isaacs et al., 2022; Hoek, 2021). One instructive example is the following.

Box 1

Random Spinner. A spinner points to a random point on a wheel after you spin it. The spinner is modeled as $\Omega = [0, 2\pi)$, each point in **Ω** corresponds to a point on the wheel. Each subset $A \subseteq \Omega$ represents the proposition that some point $x \in A$ is picked. The spinner appears rotationally symmetric.

We make the following definition:

> CREDENCE TOTALISM is the view that for sample spaces with cardinality continuum (such as the *Random Spinner*), the credence function of an agent is rationally required to be total.

In this case, a total credence function assigns credence to propositions corresponding to sets that are not Lebesgue measurable (henceforth abbreviated "nonmeasurable propositions/sets”). According to the totalists, there are important epistemic reasons to assign credence to nonmeasurable propositions.[1] In contrast, the present paper argues that nonmeasurable propositions are not of much epistemic significance, and it is rationally permissible to ignore them. In other words, credence gaps on nonmeasurable sets are rationally permissible. Consequently, the arguments for CREDENCE TOTALISM based on nonmeasurable sets are unmotivated.

The paper is structured as follows: In Sect. 2, I describe nonmeasurable sets and explain the mathematical difficulty for giving them credence. I present several options to address this problem and give an outline of my position. In Sect. 3, I defend my position by giving two main arguments which show that it is in fact rationally permissible to ignore nonmeasurable propositions. In Sect. 4, I respond to some counterarguments in the literature and show that they are unsuccessful. Section 5 is an appendix independent from the rest of the paper, which readers can skip without losing sight of the main points of the paper. The appendix discusses two topics considered to be closely related: a) the existence of total *chance* functions; b) the consistency of certain large cardinals and the truth value of the Continuum Hypothesis (CH). I conclude that the role of nonmeasurability in epistemology does not shed light on these issues, *contra* Hoek (2021).

2 Nonmeasurable sets, totality, translation invariance

We begin with presenting the mathematical difficulty underlying assigning credence for all nonmeasurable sets. Recall that the *Random Spinner* is conceived as rotationally symmetric. This condition imposes some rational constraints on an agent’s credence P. To begin with, for example, she should think that the semicircles $[0, \pi)$ and $[\pi, 2\pi)$ are equally likely to be hit. Such intuition is generalized by the following property: we say that P is *translation invariant* if $P(A) = P(A + r)$ for any $A \subseteq [0, 2\pi)$ and $r \in \mathbb{R}$, where $A + r = \{a + r \mod 2\pi : a \in A\}$.[2] For the *Random Spinner*, the intuition thus described suggests that rationality requires P to be translation invariant. Translation invariance implies that for any interval $[a, b)$, $P([a, b)) = \frac{b-a}{2\pi}$. In turn, this implies that any countably additive P agrees with the *Borel probability measure* over the *Borel sets*, which are formed by countably many iterations of taking countable unions or intersections of open intervals. The Borel measure can be canonically extended to the *Lebesgue probability measure*, defined over the measurable sets. The σ-algebra formed by these set is denoted $\mathcal{L}$. These are sets whose symmetric difference with some Borel set is null. The Lebesgue probability measure assigns probabilities to all these measurable sets and preserves translation invariance. Furthermore, any translation invariant probability measure agrees

[1] Hoek in fact defends the closely related view that there exists a total *chance* function. However, as I will discuss later, an analysis of Hoek’s central argument reveals that its central premise is epistemological, and his defense for the premise supports CREDENCE TOTALISM.

[2] Note that according to our definition, if P is translation invariant, then for any set A in the domain of P, all its translates are also in the domain of P.

with the Lebesgue probability measure over the measurable sets. In this sense the Lebesgue probability measure is unique.

On the other hand, the totalists consider totality as another rational constraint on *P*. If we accept both totality and translation invariance, then *P* needs to be a total extension of the Lebesgue probability measure (since any translation invariant probability measure agrees with the Lebesgue measure over the measurable sets). However, as the following well-known theorem shows, such an extension does not exist, assuming the Axiom of Choice (AC):

Theorem 1 *(Vitali). Let P be a countably additive probability function over $\Omega = [0, 2\pi)$. Over Zermelo–Fraenkel set theory (*ZF*), the following are contradictory: i)* AC*; ii) P is translation invariant; iii) P is total.*

At this point, it is helpful to comment on the role of the axiom of countable additivity. Whether this axiom constitutes a rational constraint on credence function (globally or locally) is a topic of active philosophical debate, a debate that we do not consider settled. Meanwhile, it is known that there exist finitely additive, total extensions of the Lebesgue measure over $\mathbb{R}$ which preserve translation invariance. Thus weakening additivity is a possible way to accommodate both translation invariance and totality for the *Random Spinner* where $\Omega = [0, 2\pi)$. However, the following well-known theorem (which generalizes to $\mathbb{R}^n$ for $n \geq 3$) shows that this option is not available in more general cases involving spaces of higher dimensions:

Theorem 2 *[Banach–Tarski] Let P be a finitely additive (not necessarily countably additive) probability measure over the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$. Over* ZF*, the following are contradictory: i)* AC*; ii) P is isometrically invariant*[3]*; iii) P is total.*

The Banach–Tarski theorem shows that even if one is ready to weaken additivity, the tension between totality and translation invariance persists for higher-dimensional analogues of the *Random Spinner*. For this reason, we do not consider weakening additivity as an adequate response to the issue at hand. Thus, the totalist must take at least one of the following options: i) reject or weaken translation invariance as a rational constraint; ii) generalize the notion of probability to avoid Theorem 1; iii) reject or weaken AC.

Existing literature on this topic often ignores option iii) and assumes a basic tension between translation invariance and totality, which necessarily exists under AC. For example, taking option i), Hoek (2021) argues that the chance function need not be translation invariant, while Goodsell and (2024) argues that translation invariance does not apply to all sets. Taking option ii), Isaacs et al. (2022), henceforth "IHH") attempt to accommodate both totality and translation invariance without giving up the existence of nonmeasurable sets by admitting interval-valued credence.

[3] When *P* is defined for subsets of $\mathbb{R}^n$ ($n \geq 2$), translation invariance naturally generalizes to invariance under isometry, meaning that $P(A) = P(\pi"A)$ whenever $\pi : \mathbb{R}^n \to \mathbb{R}^n$ is a transformation that preserves the Euclidean metric. Such transformations include not only translations but also rotations, for example.

Distinct from these approaches, this paper takes a novel approach by carefully considering the interaction between option iii) and Credence Totalism. The very possibility of this option is based on the following important theorem, which shows that if AC is weakened to the axiom of dependent choice (DC), then it is consistent that both translation invariance and totality can be satisfied by the Lebesgue measure, with a mild large cardinal assumption:

Theorem 3 *([Solovay, 1970]) If there is an inaccessible cardinal, then there is a model of* ZF + DC *such that every subset of the reals is Lebesgue measurable.*

The goal of this paper is to evaluate Credence Totalism in light of Theorem 3. My overall position can be stated in a disjunctive form. On the one hand of the disjunct, if nonmeasurable sets do not exist, then the standard Lebesgue probability is already total. In this case, the force of Credence Totalism is trivialized as it does not require modifying existing theories of probability and statistics.

On the other more interesting hand of the disjunct, nonmeasurable sets exist. Then the most prominent reason for Credence Totalism is the epistemic significance of assigning credence to nonmeasurable sets. As I shall argue, there is no such significance, so arguments for Credence Totalism cannot be based on it.[4] I give two arguments for my position: one based on *operability* (Sect. 3.2) and another based on *instrumentality* (Sect. 3.3). Theorem 3 plays an important role in both arguments. First, this theorem gives strong evidence that nonmeasurable sets are nonconstructive. I argue that this means that they are not empirically operable. In my second argument, I introduce an important distinction between two perspectives of mathematically conceiving real numbers: *the folklore perspective* and *the logical perspective*. I argue that in applying probability theory to situations such as the *Random Spinner* and statistical inference, we primarily take the folklore perspective. From this point of view, axiomatic specifications of set-theory are merely instrumental, and the set theory ZF + DC suffices for probabilistic applications. Consequently, by Theorem 3, this perspective requires no commitment to the existence of nonmeasurable sets. I therefore claim that nonmeasurability is merely a vestige of an instrumental choice made in probabilistic practice.

Before turning to the discussion, I make two clarifications on my position. First, my arguments are independent of whether translation invariance is a rational constraint. Hence, my position is not subjected to objections against translation invariance, such as those described by Goodsell and Nebel (2024). Specifically, these authors hold that a non-translation invariant extension of the Borel measure can be rationally permissible. The failure of translation invariance of these measures is witnessed only by nonmeasurable sets. While I consider translation invariance a natural requirement that should hold at least for the open sets, I contend that it needs not extend to nonmeasurable sets (if they exist) – in fact, in light of my position just described, the issue of whether it extends to nonmeasurable sets does not appear to be of much importance, as far as the application of probability theory is concerned.

[4] Thus my arguments do not establish the falsity of Credence Totalism but rather conclude that its most prominent motivation is invalid.

Second, my view makes no commitment on whether AC is true, or should be considered a fundamental axiom of mathematics. These issues have been extensively discussed since Zermelo's proof of the well-ordering theorem (Hamkins, 2021, §8.8). In particular, I do not suggest that full AC is false. Instead, my view is merely based on the observation that certain important applications of mathematics either do not depend on axiomatizing set theory, or merely commit to set theories weaker than full ZFC. This observation is consistent even with versions of set-theoretic universe view or Platonism that commit to the absolute truth of AC.

3 The epistemic insignificance of nonmeasurable propositions

3.1 Nonmeasurable sets and statistical inference

We begin with developing our main positive point that nonmeasurable sets can be justifiably ignored in statistical inference. In statistical inference, we typically consider a class of statistical models or hypotheses parameterized by some parameter θ, formalized as a family of *random variables* $f_\theta : \Omega \to \mathbb{R}$. After empirically observing the value of f in multiple trials, we seek to infer a parameter θ such that f_θ best fits the observed behavior of f. We focus on the following toy example closely related to the *Random Spinner*:

Box 2

Mystery Lamp. A lamp is connected to the *Random Spinner* via a wire. It is observed that the lamp sometimes blinks after a spin of the spinner. The exact relationship between the two devices is unclear.

One may suspect that the two devices are somehow correlated and reasonably ask for the probability that the lamp blinks after a spin. To answer this question, one can apply statistical inference on a class of hypotheses of the form $f_\theta : [0, 2\pi) \to \{0, 1\}$, i.e. we are picking a function that best predicts the correlation between the lamp and the spinner. Once a best parameter θ is chosen, the desired probability is then $P(f_\theta^{-1}(1))$.

At this point, the issue of choosing the algebra of probability bearers arises: if the set $f_\theta^{-1}(1)$ is not included in the algebra, then $P(f_\theta^{-1}(1))$ is undefined. To avoid this problem, statistics usually makes the assumption that all random variables are *measurable functions*.[5] Which functions are measurable is relative to the choice of the underlying algebra. When the algebra is too restrictive, there may be too few measurable functions, leading to erroneous inference results. To illustrate, consider the following example:

Box 3

[5] Recall that given measure spaces $(X, \mathcal{F})$, $(Y, \mathcal{G})$, a function $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is measurable if for any $A \in \mathcal{G}$, $f^{-1}(A) \in \mathcal{F}$.

Suppose the *Mystery Lamp* is, in fact, triggered by the event that the spinner hits $[0, \frac{\pi}{4})$. However, our probability is defined on a restricted algebra: the algebra generated by $\{[0, \frac{\pi}{2}), [\frac{\pi}{2}, \pi), [\pi, \frac{3\pi}{2}), [\frac{3\pi}{2}, 2\pi)\}$. We consider hypotheses of the form $f_\theta : [0, 2\pi) \to \{0, 1\}$, which are assumed to be measurable with respect to the restricted algebra.

In this scenario, clearly, among the functions measurable with respect to the restricted algebra, the best hypothesis is the indicator function of the interval $[0, \frac{\pi}{2})$. However, it is still a poor predictor of the behavior of the lamp (with success rate 0.5). Our result can be improved by enlarging our algebra so that more functions become measurable and hence qualify as hypotheses.

In probability and statistics, when the underlying space is $\mathbb{R}^n$, it is standard to choose $\mathcal{L}$ – the σ-algebra of the Lebesgue measurable sets – as the probability bearers, thus excluding the nonmeasurable sets and nonmeasurable functions from consideration. A natural question is, why not consider *all* hypotheses? In fact, part of the motivation for Credence Totalism is indeed the worry that these "missing hypotheses" result in suboptimal inference results, just as in the previous example (Hoek, 2021).[6] Nevertheless, we think that the credence gaps on nonmeasurable sets are not worrisome. To defend this view, our task is the following. In general, we can observe that in any context of epistemic representation or investigation, an agent can only focus on a rather limited class of propositions or features of interest. In some cases such as the previous example, ignoring some propositions which could have been engaged with is irrational or erroneous. However, sometimes such omission is reasonable, and we shall offer some positive reasons which explain why ignoring nonmeasurable sets falls into the second kind.

3.2 Nonmeasurable sets are inoperable

Recall that the existence of nonmeasurable sets were originally proved under the assumption of AC. By Solovay's Theorem 3, this assumption is difficult to eliminate in the sense that even DC – a natural weakening of AC – cannot prove that they exist.[7] This is why nonmeasurable sets are often considered *nonconstructive*.[8] AC asserts that every family of sets has a choice function: if $\{A_s : s \in S\}$ is a set of sets, then there is F with domain S such that $F(s) \in A_s$ for all $s \in S$. Meanwhile, in Solovay's model where there are no nonmeasurable sets, DC holds. DC states that for any relation R over any set X which satisfies for all x there is y such that xRy, there is a countable sequence $F : \omega \to X$ such that $F(s)RF(s+1)$ for all $s \in \omega$. In other words, DC preserves a strong fragment of choice which allows us to make countably many choices along any relation R. The upshot is, to prove the existence of nonmeasurable

[6] This objection is addressed in Section 4.1.

[7] The existence of nonmeasurable sets follows from some strict consequences of AC, for example, the existence of a well-ordering of the reals, or the existence of a nonprincipal ultrafilter. These principles are generally considered nonconstructive too, so all the points I make in this section also apply to them.

[8] The word "constructive" is used differently in constructive mathematics or constructive logic, where a constructive approach often requires weakening the base logic. Here I am discussing the constructivity of objects, instead of reasoning processes. I simply mean that nonmeasurable sets cannot result from constructive processes, or no constructive proof can prove their existence. My usage is of "nonconstructive" here is interchangable with "inexplicit" or "intangible", following Schechter (1996, § 14.77).

sets, we need a highly nonconstructive choice function not available even in a DC universe.

In scientific practice, some hypotheses are justifiably ignored on the ground that they are *inoperable*, meaning that we lack the means to empirically interact with them. I claim that because nonmeasurable sets are nonconstructive, nonmeasurable propositions are inoperable, and this constitutes a reason why nonmeasurable hypotheses can be justifiably ignored.

Some fundamental ways in which we empirically interact with theories include: making predictions based on the theory, verifying the accuracy of our prediction, and investigating other questions conditionalizing on the theory. All these tasks appear impossible for nonmeasurable propositions. For example, consider the proposition that the *Mystery Lamp* blinks iff the *Random Spinner* hits a nonmeasurable set A. We lack a method to conditionalize on this proposition, since standard theories of conditionalization only allow us to conditionalize on measurable propositions. But even the more basic tasks of prediction and verification seem impossible. If we want to predict whether a spin sparks the lamp under this hypothesis, we need to determine whether the spinner lands in the nonmeasurable set A. If we want to verify this hypothesis, we make many trials and record whether the spinner lands in A, and whether the lamp blinks. Both of these tasks involve measuring the location of the spinner and determine if it lands in A. For the sake of argument, we make the highly unrealistic assumption that we are capable of measuring the location of the spinner infinitely precisely, in the sense that there is a unique real $x \in [0, 2\pi)$ that can be empirically determined to be hit by the spinner. Even then, we are left with the task of deciding whether $x \in A$. But the only way we can specify A involves a choice function given by AC. For example, the Vitali set is the set formed by choosing an element from each element in the group $(\mathbb{R}/\mathbb{Q}, +)$. AC asserts the existence of such a choice function, but does not specify which elements are chosen. Therefore, there is no general way to determine whether the x falls into this set. Some fundamental ways in which we empirically interact with theories include: making predictions based on the theory, verifying the accuracy of our prediction, and investigating other questions conditionalizing on the theory. All these tasks appear impossible for nonmeasurable propositions. For example, consider the proposition that the *Mystery Lamp* blinks iff the *Random Spinner* hits a nonmeasurable set A. We lack a method to conditionalize on this proposition, since standard theories of conditionalization only allow us to conditionalize on measurable propositions. But even the more basic tasks of prediction and verification seem impossible. If we want to predict whether a spin sparks the lamp under this hypothesis, we need to determine whether the spinner lands in the nonmeasurable set A. If we want to verify this hypothesis, we make many trials and record whether the spinner lands in A, and whether the lamp blinks. Both of these tasks involve measuring the location of the spinner and determine if it lands in A. For the sake of argument, we make the highly unrealistic assumption that we are capable of measuring the location of the spinner infinitely precisely, in the sense that there is a unique real $x \in [0, 2\pi)$ that can be empirically determined to be hit by the spinner. Even then, we are left with the task of deciding whether $x \in A$. But the only way we can specify A involves a choice function given by AC. For example, the Vitali set is the set formed by choosing an element from each element in the group $(\mathbb{R}/\mathbb{Q}, +)$. AC asserts the existence of such a choice function, but does not specify which elements are chosen. Therefore, there is no general way to determine whether the x falls into this set.

Thus we cannot empirically engage with nonmeasurable sets because we cannot physically measure them. Another way to make this point (in the contrapositive form) is that the sets amenable to physical measurement and empirical investigations are all measurable. Here, it is helpful to briefly introduce the *projective hierarchy*.

Formally, the hierarchy is defined as following: at the lowest end of the hierarchy, $\mathbf{\Sigma}^1_1$ denotes the collection of *analytic sets*: sets of the form $\{y \in \mathbb{R} : \exists x \in \mathbb{R} Rxy\}$ where $\{(x, y) : Rxy\}$ is a countable intersection of open sets, i.e. analytic sets are formed by *projecting* the G_δ-sets. Dually, $\mathbf{\Pi}^1_1$ denotes the *coanalytic sets*, the complements of analytic sets. One climbs up the hierarchy by alternating applications of projections and complementations: $\mathbf{\Sigma}^1_{n+1}$ sets are projections of $\mathbf{\Pi}^1_n$ sets, and $\mathbf{\Pi}^1_{n+1}$ sets are complements of $\mathbf{\Sigma}^1_{n+1}$ sets. The projective sets are the collection of every set that appears in the hierarchy: $\bigcup_{n\in\omega} \mathbf{\Sigma}^1_n$.

The projective hierarchy is relevant here for the following reason: early results in descriptive set theory due to Luzin and Suslin show that the Borel sets are exactly the sets that are both analytic and coanalytic (denoted $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$), and that there are non-Borel analytic sets, i.e. $\mathbf{\Delta}^1_1 \subsetneq \mathbf{\Sigma}^1_1$ (Moschovakis, 2025). This means that even a single application of projection enables one to form more complex sets than the Borel sets, which themselves can be enormously complex (e.g. countable unions of countable intersections of countable unions of closed sets…). However, by the definition of measurability, we know that all the open sets and all the Borel sets are measurable. Moreover, Luzin went one step further along the hierarchy and showed that the analytic/coanalytic sets are measurable. Naturally, he asked whether all projective sets are measurable, a question turned out to be independent of ZFC.[9] However, Luzin's result suffices to show that even under AC, many sets of relatively low complexity are measurable.[10] It seems reasonable to think that all the sets that are physically detectable fall into these classes with low complexity. In fact, any real-world measurement device has some limited precision, so a single measurement of a quantity yields an interval that the true value likely falls in. Our measurements can be improved by measuring the same quantity repeatedly. The number of measurements can be some large finite number but never countably infinite. Considering this fact, we may reasonably think that the empirically detectable subsets are finite unions or intersections of open/closed intervals, meaning that the class of measurable sets already contains all the sets we might be practically interested in, and many more.

Some philosophers may find the perspective based on operability unsatisfactory. For example, Norton writes:

[9] For a discussion of the history of this question and some relevant results, see Woodin (2001).

[10] Here, one might object that under V = L, there is a well-ordering of the reals low in the projective hierarchy. So in this situation, there are nonmeasurable sets that are "easy to define", and so they might not be highly nonconstructive after all. See Sect. 4.4 where I address this objection.

> "Should an account of inductive inference be responsible for relations among propositions that pertain to nonmeasurable sets? To forego exploring these relations would require positive reasons for precluding nonmeasurable sets. I do not see them unless we are prepared to entertain anthropocentric perspectives on the world. This might happen if we were so committed a subjectivist that we reduce the scope of inductive inference to relations among things that we can construct. This attitude seems quite presumptuous to me". (Norton, 2021, Ch. 14)

Two comments on Norton's view in order. First, in so far as the purpose of a theory of inductive inference is to account for how knowledge is acquired and justified based on empirical evidence, propositions that are not amenable to empirical investigation appear irrelevant. It is quite natural to stay silent on hypotheses which we lack the means to investigate, and it is not clear to me why this is "presumptuous" for Norton. Second, Norton appears to assume that nonmeasurable sets admit some status of objective existence, therefore focusing on the constructible sets is limiting and "subjectivist". However, from a choiceless point of view, nonmeasurable sets only exist in idealized set-theoretic universes which are themselves constructed by human beings. Thus it is not clear which view really is the "anthropocentric" 1, unless we resolve the issue of whether AC is objectively true, an issue beyond the scope of this paper.

Norton's criticism aside, I believe there is a second positive argument for precluding nonmeasurable sets that does not involve feasibility considerations, thus directly addressing Norton's worry. This argument is to be described in the following section.

3.3 Axiomatic set theory is instrumental

In an iconic paragraph from *Science and Hypothesis*, Poincaré writes:

> "The geometer is always seeking, more or less, to represent to himself the figures he is studying, but his representations are only instruments to him; he uses space in his geometry just as he uses chalk; and further, too much importance must not be attached to accidents which are often nothing more than the whiteness of the chalk". (Poincaré, 1952, Ch. 2)

I introduce Poincaré's remark because I think the role of models of set theory in the formal representation of belief and credence and the practice of statistical inference is analogous to the role of the chalk in mathematical investigations, in that they are merely instrumental. These applications may well mention the notion of set. However, in most cases, *sets need not be considered as members of models of axiomatic set theory*. Instead, users of sets often manipulate them on the basis of naive set theory. Moreover, these applications do not require the kind of global description that set-theoretic axioms impose on the set-theoretic universe. Even when we do need some more rigorous treatment of sets, often some axioms weaker than or different from ZFC may well serve our purpose. On this view, sets as members of models of ZFC, and full AC in particular, are merely one out of many instruments that can fulfill the demands of probability and statistics. The presence of nonmeasurable sets in some specific models of set theory is merely an accidental feature of our instrumental choice. We now motivate and explain this view.

The issue of nonmeasurability appears precisely when we decide to use $\Omega = [0, 2\pi)$ as the sample space for the *Random Spinner* and conceive various subsets of the reals as events. Importantly, different mathematical contexts or goals call for different perspectives on how the real numbers $\mathbb{R}$ are conceived. There are two relevant perspectives here: *the folklore perspective* and *the logical perspective*.,[11] From the folklore perspective, the most important properties of $\mathbb{R}$ are geometrical. A real number is construed as a dimensionless point on a line infinitely extending in both directions, deployed with a metric measuring the distance between any two point, in such a way that the line "contains no gaps", in the sense of Cauchy completeness. On the other hand, while the logical perspective acknowledges all the folklore properties of $\mathbb{R}$, it introduces some additional features. This perspective is characterized by viewing $\mathbb{R}$ as a *set*, existing in some universe of set theory V, which satisfies some set-theoretic axioms. It also becomes natural in this context to identify $\mathbb{R}$ with 2^ω, the powerset of ω, and a single real number as an infinite path through the tree $2^{<\omega}$.

Historically, the development of the logical perspective since the late 19th century was mathematically fruitful. It settled important open questions, and led to interesting new ones. In particular, set-theoretic questions concerning $\mathbb{R}$, e.g. whether nonmeasurable sets exist, whether $\mathbb{R}$ is well-orderable, whether CH holds, are decided by the existence or nonexistence of certain sets in V. However, despite these landmark discoveries in logic and set theory, it does not mean that one should fully embrace the logician's perspective in *all* mathematical contexts involving the real numbers. Instead, the folklore perspective is often all one needs for real-world application, and is what we are all familiar with when we were first introduced to mathematics as schoolchildren. From the folklore point of view, usually there is no need to consider $\mathbb{R}$ as living in V. Even though this move is perhaps necessary to answer some deeper questions which may well interest the folklore mathematicians too, the conceptual origins of the two perspectives are distinct, and many applications of the real numbers remain immune to the logical developments.

I suggest that in applications of probability – whether in statistics or philosophy – it suffices to take the folklore view, and the logical view is unnecessary. Imagine we tell Laplace that the sample spaces he considers are all part of some V, and there are different axiom candidates for V, and so on. Chances are that he would find these ideas quite irrelevant to his scientific concerns and results, intriguing as they are. In fact, even today, I think a majority of statisticians are unable to state the axioms of ZFC and have no idea what is a model of set theory. None of this prevents them from

[11] The presence of multiple perspectives on the real or the continuum has been observed by many, including Hermann Weyl (1987), §2.6), who contrasts "the intuitively given continuum" and "the concept of real number". Weyl also observes that philosophical confusions can arise if these perspectives are mixed up: "I *see* this pencil lying before me on the table throughout a certain period of time. This observation entitles me to assert that during a certain period this pencil was on the table; and even if my right to do so is not absolute, it is nonetheless reasonable and well grounded. It is obviously absurd to suppose that this right can be undermined by an 'expansion of our principles of definition' – as if new moments of time, overlooked by my intuition, could be added to this interval, moments in which the pencil was, perhaps, in the vicinity of Sirius or who knows where."

taking $\mathbb{R}$ as granted and freely use some of its basic properties, such as separability by $\mathbb{Q}$ and Cauchy completeness.

Having distinguished these two perspectives, we can now make sense of the idea that nonmeasurable sets are incidental, just as Poincaré's chalk. Measurability itself is a highly set-theoretically entangled phenomenon which only arises once we are ready to embrace the logical perspective and perform set-theoretic operations on $\mathbb{R}$, e.g. well order $\mathbb{R}$, or pick representatives from equivalence classes of equivalence relations on $\mathbb{R}$. The extent to which these operations are available depends on what kind of set-theoretic choice principles one adopts. However, once we accept that probability and statistics can be done merely from the folklore perspective, we see that the probabilist needs not decide these issues.

Furthermore, even if the probabilist is ready to embrace the logical point of view by making explicit some set-theoretic axioms and viewing her spaces, functions, and numbers as living in some universe that satisfies these axioms, she needs not assume that the universe she uses satisfies full ZFC. Recall that by Solovay's Theorem 3, in some $\mathsf{ZF} + \mathsf{DC}$ models there are no nonmeasurable sets. Fundamental theorems in probability and statistics are derived from mathematical analysis, and DC was proposed by Bernays (1942) as part of a set theory that enables analysis to be carried out. $\mathsf{ZF} + \mathsf{DC}$ is often considered "the right theory for classical analysis", on the ground that most theorems in classical analysis can be proved from these axioms (Asperó & Karagila, 2021). If this is right, then Solovay's theorem shows that there exists a robust set theory for the probabilist such that she can prove all the theorems she wants, without committing to nonmeasurable sets.[12]

In conclusion, according to our view, a probabilist is free to consider her sets as being naive, unaxiomatized without specifying a model of set theory in which they inhabit. Or, she may consider her sets as axiomatized by $\mathsf{ZF} + \mathsf{DC}$ and living in a universe where all sets are measurable. None of these set-theoretic views affect her probabilistic practice. She may also use models of full ZFC, but the existence of the alternatives suggests that nonmeasurable sets are merely instrumental features induced by particular ways of specifying her mathematical apparatus, which are neither determined nor required by probabilistic applications.

4 Objections

In this section we work under the assumption that nonmeasurable sets exist and respond to some counterarguments to our position. We begin with discussing the "missing hypothesis" worry in Sect. 3.1 proposed by Hoek (2021).

[12] The Axiom of Determinacy (AD) contradicts AC and *implies* that there are no nonmeasurable sets (as opposed to $\mathsf{ZF} + \mathsf{DC}$ being merely consistent with this). One might wonder if $\mathsf{ZF} + \mathsf{AD}$ is also a reasonable set theory for the probabilist, but the extent to which it proves theorems of analysis and probability is less clear to the author than $\mathsf{ZF} + \mathsf{DC}$. For a philosophically informed introduction to AD, see Koellner (2014).

4.1 Missing hypotheses

Hoek aims to show that for the *Random Spinner*, there exists a total *chance* function $P : \Omega \to [0, 1]$ that is a probability measure such that $P(\{r\}) = 0$ for all $r \in \Omega$. (We might call this assertion CHANCE TOTALISM – more on this in the appendix.) Chance is defined as "the objective likelihood that [an] event takes place" (p. 641). The notion of chance leads to various metaphysical issues, such as whether devices like the *Random Spinner* are in fact chancy, and what does it mean for a chance function to exist. These issues are irrelevant to the present paper, since our following analysis of Hoek's argument reveals that its key premise is epistemological in nature, nothing beyond the worry that ignoring nonmeasurable sets lead to "missing hypotheses".

Hoek begins by describing what he calls the TYPICAL INDUCTIVE INFERENCE. Suppose a scientist investigates how likely the *Mystery Lamp* is to blink (call this event *E*), i.e. the probability *P*(*E*). The class of hypotheses that the scientist considers are of the form "$P(E) = x$" where $x \in [0, 1]$. The scientist proceeds by making *n* trials and recording the number of blinks *m*. She then considers the *likelihood* of her evidence under various hypotheses: if, e.g. $P(E) = 80\%$, then the likelihood of observing $m = 794$ among $n = 1000$ is high, whereas that of observing $m = 50$ among $n = 1000$ is low. The hypothesis which maximizes the likelihood function is chosen. In this way various hypotheses about the value of *P*(*E*) can be compared and selected.

Once this method is defined, Hoek proposes the following argument:

Box 4

1. If some event has no chance at all, then TYPICAL INDUCTIVE INFERENCE is unreliable.

2. TYPICAL INDUCTIVE INFERENCE is, in fact, reliable.

1. If some event has no chance at all, then TYPICAL INDUCTIVE INFERENCE is unreliable.

3. Therefore, every event has a chance.

How is the first premise justified? Hoek reasons: if some event has no chance, then we have some additional hypotheses to consider. Here, one such hypothesis is that *E* has no chance (NO CHANCE), in the sense that the chance function *P* gives no value with the input *E*. However, TYPICAL INDUCTIVE INFERENCE is unable to examine this hypothesis, since it only applies to evaluating hypotheses for which we know how likely certain data is observed, given the hypothesis. But we have no idea how likely we are going to observe *m* blinks among *n* trials, given the hypothesis NO CHANCE. This is a problem because "if [NO CHANCE] fits the data equally well as the hypothesis that $[\mathrm{Ch}(E)]$ is around 80%, then that substantially undermines the strong justification we thought we had for our inductive predictions about the future behavior of the lamp". (p. 646) Consequently, the inference fails to justify its conclusion and so it is unreliable.

Clearly, Hoek's consideration is closely related to the statistical inference setting we introduced earlier in Sect. 3.1. The scientist described above considers the class of hypotheses of the form $E : \Omega \to \{0, 1\}$ where $\Omega = [0, 2\pi)$. The problematic statement NO CHANCE describes situations where *E* is not a measurable function, i.e. the set $E^{-1}(1)$ is non-measurable. As we discussed earlier, these nonmeasurable functions are typically excluded from consideration in statistical inference. Essentially Hoek worries that a nonmeasurable function "fits the data equally well" with a measurable one chosen by statistical inference and thinks in this case the justification for choosing the measurable function is undermined. To avoid this problem, Hoek concludes that the scientist must set up her algebra $\mathcal{F}$ to include all the subsets, so that all functions are measurable and her credence function is total.

> Key Claim If an agent's algebra $\mathcal{F}$ is not equal to the full powerset 2^{Ω}, then her Typical Inductive Inference is unreliable.

I claim that Hoek's argument does not suffice to establish Key Claim. The argument clearly relies on the worry that nonmeasurable hypotheses might "fit the data equally well" with measurable hypotheses. However, upon closer scrutiny, we can see that this worry is misplaced.

To begin with, "fitting the data" might mean being consistent with the data, but in the probabilistic setting at issue, mere consistency is irrelevant. "This coin is fair" and "this coin is unfair" are both consistent with any finite number of coin toss results. This fact does not prevent us from employing Typical Inductive Inference to decide between these hypotheses.

Statistics works by identifying some notions of fitness with data more useful then mere consistency. In this case, the most straightforward (although not necessarily the most appropriate) notion is maximum likelihood estimation. However, if we are to compare the likelihood of the relevant hypotheses, we immediately see that nonmeasurable hypotheses *do not* "fit the data equally well" with measurable hypotheses. This is simply because the likelihood of a nonmeasurable hypothesis is undefined, so it cannot appear in the comparison of likelihood.

Moreover, even if we somehow define a degree of fitness that works for nonmeasurable hypotheses and they end up "fitting the data equally well" with some measurable hypothesis according to the notion thus defined, this does not mean that our chosen hypothesis is unjustified. For in general, any method for hypothesis choice typically cannot isolate a single best hypothesis. In the philosophy of science literature, this is well-known as the underdetermination of hypothesis by evidence. Sometimes, a best hypotheses can be isolated once a criterion is chosen. For example, given some data points, there might be a unique set of parameters generating a curve which maximizes a certain function that measures the degree of fitness, e.g. the likelihood function. However, this would not resolve the underdetermination phenomenon. Still, there exist other plausible candidate criteria for evaluating hypotheses which may yield different results. Let alone there may exist other plausible but different hypotheses (which may involve alternative kinds parameters) to choose from. The typical lesson from underdetermination is not that some method is unjustified unless it can always isolate one single best hypothesis. Instead, the lesson is that we can make theory choice based on criteria beyond fitness with data, e.g. theoretical virtues, pragmatic considerations, and more. These criteria often arise in scientific practice and results in effective theory choices.

In conclusion, the mere *possibility* of some ignored hypothesis fitting the data equally well with a chosen hypothesis does not undermine that choice. Someone advocating for considering the ignored hypothesis has the burden to prove that it is unjustifiably ignored. In particular, if one thinks nonmeasurable hypotheses undermine theory choice, one must offer some specific positive reasons, yet they are lacking in the current discussion. Such reasons may include, for example, some theory predicting the plausibility of nonmeasurable hypotheses. But as of today, no scientific theory describes or predicts any empirical bearing of nonmeasurable sets. On this issue, the only worry Hoek mentions concerns losing justifications for "future predictions" of the chosen theory, but in a lot of underdetermination cases, the competing hypotheses also predict the future differently, which do not prevent us from making theory choices. In fact, if our data underdetermine two theories that make the same predictions, then a choice seems even less pressing. As these instances of underdetermination do not appear to undermine scientific practice, there seems to be no reason to think that nonmeasurable hypotheses yielding different predictions threatens our conclusions about the behavior of the *Mystery Lamp*. Meanwhile, as described in the previous section, there are what we consider as strong positive reasons *against* the empirical bearing of nonmeasurable sets. Therefore we reject the "missing hypotheses" worry.

4.2 Undue restriction

Now we turn to a different set of arguments proposed by IHH (2022) for *imprecise credence*: the idea of representing credence with a generalized probability function that assigns intervals (as opposed to mere numbers) as probabilities. For the *Random Spinner*, the credence assigned to $A \subseteq \Omega$ would be the interval whose endpoints are A's inner measure and outer measure, respectively. In this manner, translation invariance is preserved while nonmeasurable sets receive probabilities. IHH did not explicitly endorse Credence Totalism, and it is not clear whether they think non-total credence functions are rationally impermissible.[13] However, in order to motivate the advantage of imprecise credence over non-total credence functions, they suggest that it is important to give nonmeasurable sets credence. This is the main claim we shall discuss. We think that IHH did not give a successful argument for this claim.

The first argument proposed by IHH is based on the idea that any subset should be permitted to enter an agent's algebra if she wants.[14] More precisely, it relies on the following premise.

> Extendability states that P is permissible only if for any set A, there exists P' extending P such that P' is defined on A.

[13] One may understand IHH as merely arguing that imprecise credence is rationally permissible, as opposed to rationally required (Dorr, 2024). This view is consistent with the idea that non-total credence functions are also rationally permissible. In this paper we do not intend to argue against the permissibility of imprecise credence. However, from our point of view, imprecise credence is not *more* permissible or desirable than non-total credence.

[14] "…if [a proposition] doesn't appear in [an agent's] algebra, she simply has no doxastic attitude to it. To be sure, it's fine if the algebra excludes some propositions - one need not think about everything. But if an agent wants to assign a credence to the proposition that the spinner will land on some non-measurable set of points, *she should not be doomed to failure*… An epistemology that abjures such agents is too restrictive." (2022, p. 900)

To begin our objections, we first observe that Extendability does not imply any problem with the Lebesgue measure. Indeed, Vitali's construction shows that there are *some* nonmeasurable sets – we shall call these nonmeasurable sets *Vitali-style* – for which the Lebesgue measure cannot be extended to, *if the extension preserves translation invariance*.[15] From the existence of Vitali-style nonmeasurable sets, the argument appears to conclude that the Lebesgue measure is not rationally permissible, because it violates Extendability. But in fact, it does not: a translation invariant extension to the Vitali-style sets does not exist, but it is consistent with ZFC that a total extension of the Lebesgue measure *can* exist (which by Theorem 3, cannot be translation invariant), under suitable large cardinal hypotheses. (See Section 4.3 and the appendix.) Thus, Extendability needs to be modified for the argument to go through:

> Extendability* states that P is permissible only if for any set A, there exists P' extending P such that P' is defined on A and preserves certain properties of P, e.g. translation invariance.

One might think that it is *prima facie* permissible to assign credence to any set A, and this is how Extendability is justified. This can be granted for the sake of argument, even if A is of Vitali-style. However, this justification for Extendability does not straightforwardly apply to Extendability*, for the following reason.

In general, it is false that any permissible probability function must be able to be further extended to accommodate yet another feature which is permissible by itself. In other words, two features of a credence function can be *individually permissible but jointly impermissible*. For example, regarding the spinner (not assumed to be symmetric), it is clearly permissible for someone to have $P([0, \pi)) = 0.9$ (e.g. when she thinks that the spinner is very likely to land in the left half). It is also permissible to have $P([\pi, 2\pi)) = 0.9$, that is she thinks that the spinner is very likely to land in the right half. But there is no P which has both properties, since such P contradicts the axiom of total probability. Precisely because of this contradiction, we conclude that a credence function which satisfies $P([0, \pi)) = P([\pi, 2\pi)) = 0.9$ is not rationally permissible. In the present case, the relevant properties are i) P is translation invariant, ii) P is defined on some Vitali-style set A. Both features are permissible, but jointly they contradict our chosen axioms. In fact, the Vitali-style sets are precisely the sets for which we *cannot* assign a translation invariant credence under the chosen axioms! In so far as we are ready to consider features that contradict these axioms impermissible (unless there are independent reasons against these axioms, which are absent in IHH's discussion), we should conclude that the desired extension is impermissible.

[15] Note that for certain nonmeasurable A, the Lebesgue measure *can* be extended to $\sigma(\mathcal{L} \cup \{A\})$, while preserving translation invariance. In other words, there are nonmeasurable sets which are not of Vitali-style.

4.3 Contagious credence gap

Another argument proposed by IHH is based on the observation that credence gaps for nonmeasurable propositions can propagate to other propositions. uppose an agent is certain that the following proposition holds, i.e. assigns credence 1 to it:

> Nonmeasurable Blink: the lamp blinks iff the spinner hits A, where A is Vitali-style.

IHH claimed that in this case, she cannot assign credence to the proposition that "the lamp blinks after a spin". For whatever credence this proposition receives, the proposition that the spinner will hit A receives the same credence, which is not possible if translation invariance is to be preserved, since A is of Vitali-style. Hence the credence gap propagates to mundane propositions which supposedly should be able to receive credence. Even worse, if the agent merely has a "low but non-zero credence" to "a causal connection between a non-measurable set of points on a spinner and the light", she still cannot assign credence to the proposition that "the lamp blinks after a spin" (IHH, 2022, p. 899)

Before we discuss this argument, we note that both the proposition that the lamp blinks and Nonmeasurable Blink are not represented in the probability space with $\Omega = [0, 2\pi)$. All subsets of this space represent propositions of the form "the spinner will hit A" where $A \subseteq \Omega$, and these are the only credence bearers according to the formalism. To address this, we let the new sample space $\Omega = \{0, 1\} \times [0, 2\pi)$, where each trial has its outcome recorded as $(a, b) \in \Omega$, a is either 1 or 0 (meaning that the lamp blinks or not), and $b \in [0, 2\pi)$ records the point hit by the spinner. We can rigorously formulate IHH's argument in this setting.

Definition 1 For $X \subseteq [0, 2\pi)$, we denote $X^+ = \{1\} \times X$, $X^- = \{0\} \times X$, $X^* = X^+ \cup X^-$, and $X^c = [0, 2\pi) - X$. Translation invariance for the new sample space means for any r, $P(A^*) = P((A + r)^*)$. Let μ be the Lebesgue measure on $[0, 2\pi)$.

Proposition 1 *Let $A \subseteq [0, 2\pi)$ be Vitali-style, let $B = (A^c)^- \cup A^+$ (representing Nonmeasurable Blink), and $C = [0, 2\pi)^+$ (representing "the lamp blinks"). Assume P is translation invariant. Then no translation invariant extension of P takes value on both B and C.*

Proof Clearly $P(\cdot^*)$ induces a probability measure over $[0, 2\pi)$ and it is translation invariant. So $P(X^*) = \mu(X)$ for all measurable $X \subseteq [0, 2\pi)$. For a contradiction, assume that P' extends P, is translation invariant, and takes value on B and C. We have $A^+ = C \cap B$, $A^- = \Omega - B - C$, both in the domain of P', hence A^* is in the domain of P'. But then $P(\cdot^*)$ is a translation invariant extension of $P(\cdot^*)$ over $[0, 2\pi)$ with A in its domain, contradicting A being Vitali-style.

Let $\mathcal{F}$ be the algebra of credence bearers. IHHs argument goes as: *suppose* $B \in \mathcal{F}$, by the above Proposition, $C \notin \mathcal{F}$. But C is such a natural proposition to consider that it should always be in $\mathcal{F}$. Since the standard probability formalism does not allow this, we need to modify it.

We agree that it is natural to have $C \in \mathcal{F}$. But why should we accept the supposition that $B \in \mathcal{F}$? In fact, the very same analysis in Sect. 4.2 applies to the set B in this case. There, we discussed two reasons against giving Vitali-style sets credence: a) even if they are *prima facie* permissible to receive credence without any background assumptions, there is no reason to think that this is still permissible if we are to preserve translation invariance; and b) they should not receive credence since putting credence on them contradicts our chosen axioms and accepted local constraints on credence functions.[16] Proposition 4 shows that B (i.e. NONMEASURABLE BLINK) plays the exact same role as the Vitali-style set in our previous analysis, namely extending a translation invariant probability (whose domain naturally includes C) to be defined on B while preserving translation invariance results in contradiction. Therefore, the natural conclusion to draw from Proposition 4 is that B should not enter $\mathcal{F}$ in the first place. An argument which concludes that we should have $A \in \mathcal{F}$ based on the supposition that $B \in \mathcal{F}$ is circular, since it already supposes that something analogous to a Vitali-style set should enter the algebra.

4.4 Discussion

We end this section by treating some secondary issues and objections arising from our discussion so far.

4.4.1 Physical possibility of nonmeasurable propositions

Footnote 4 of IHH (2022) discusses (and criticizes) the idea that because of their "infinite precision", nonmeasurable propositions are physically impossible to be true. Importantly, my reasons for not engaging with these propositions are entirely different and should be distinguished from this idea.

To begin with, "infinite precision" is a poor characterization of both nonmeasurable sets and physical possibility. My issue with nonmeasurable propositions is not that they are "infinitely precise" (whatever that means), but rather they are nonconstructive. Specifically, their construction relies on unspecified choice functions with uncountable cardinality.[17] In particular, my view can be maintained while acknowledging that there are some set constructed from some *countably* infinitary process of epistemic interest. For example, the question of how to conditionalize on certain measure 0 sets has been extensively discussed (Meehan & Zhang 2020). My view does not imply that this question is pointless. In fact, these measure 0 sets are often highly constructive, which is precisely what renders the conditionalization issue relevant.

[16] A statement contradicting some chosen axioms consitutes a strong reason to reject the statement, but sometimes it is a defeasible reason. In some cases, the contradiction may give motivations to modify or improve the axioms. Our arguments in Sect. 3 suggest that credence assignment to Vitali-style sets does not constitute such a case and so dispel this concern.

[17] Since DC implies countable choice (CC), it follows from Theorem 3 that we can have all the countable choice functions but no nonmeasurable sets.

As for whether nonmeasurable propositions are physically possible to be true, I stay neutral on this issue. While some nonmeasurable propositions may be trivially true (e.g. a specific nonmeasurable set being hit), many nonmeasurable propositions assert correlations between non-measurable sets and some empirical phenomenon (e.g. NONMEASURABLE BLINK), or even causal connections. It is not clear to me whether nonmeasurable propositions of the second kind can be true. I am happy to grant that they may possibly be true, but it is not clear to me how this possibility can be empirically verified.

4.4.2 Credence gaps on possible propositions

In light of the previous point, one may worry: if someone believes that a proposition may be true, then how can it be reasonable to ignore it in scientific investigations? How can it be reasonable not to give it credence? Does my position force me to acknowledge that all nonmeasurable propositions are false? To dispel this apparent puzzle, we make two clarifications.

First, the *mere possibility* of a proposition does not require a rational agent to give it a credence. Leaving credence gaps on nonmeasurable propositions does not mean that we rule them out (i.e. consider them as false or impossible), it merely means that they are not under consideration in a specific epistemic situation. A proposition which an agent considers false or impossible receives credence 0 instead. An agent can consider a proposition as possibly true and yet reasonably ignore it by not assigning it a credence.

Second, we observe that there are many examples where an epistemic agent ignores propositions which she considers possible. One example most salient to the *Mystery Lamp* is the practice of machine learning. Indeed, the *Mystery Lamp* can be suitably viewed as a learning problem: we are given many data points of the form $(r, x) \in [0, 2\pi) \times \{0, 1\}$, and the machine attempts to learn a classifier $X \subset [0, 2\pi)$ that is a best predictor of getting 1. There are many machine learning algorithms implementing this task, their details do not matter here. It suffices to say that the class of candidate classifiers that can theoretically be learned by a given algorithm is always contained in a class of well-behaved functions and excludes the pathological functions. In particular, these functions would certainly admit only finitely many parameters and so place very low in the Borel hierarchy, let alone involving anything into the projective hierarchy. Certainly no machine learning algorithm returns nonmeasurable functions! Consequently, many functions are ignored, and which functions are ignored partly depends on our choice of algorithm. However, in machine learning practice, making these inevitable choices does not mean rejecting the possibility that other classifiers may work better, or turn out to be the "true" classifier. What matters is explaining why it is reasonable to ignore certain functions, which is precisely our task for nonmeasurable sets.

4.4.3 Extent of idealization

At the end of Sect. 3.3, the logic-infused probabilist who considers her $\mathbb{R}$ as living in Solovay's model is a *partially* idealized agent who is i) capable of engaging with infinite sample spaces; ii) willing to consider the *Random Spinner* as rotationally symmetric; iii) capable of utilizing some nonconstructive objects given by $\mathsf{ZF} + \mathsf{DC}$, or at least countable choice functions; and yet iv) incapable of utilizing some uncountable choice functions. Following the literature, we may call the combination of i)–iv) a *semiconstructive* or *quasiconstructive* point of view (Massas, 2023; Schechter, 1996, §14).

One might ask whether this standpoint is coherent. On the one hand, from a fully realistic point of view, one may think that finite sample spaces are all that an agent can engage with. One may also think that the *Random Spinner* should not be considered rotationally symmetric, due to friction and finite initial angular velocity. On the other hand, if we consider a highly idealized, God-like agent capable of utilizing arbitrary choice functions, then our discussion in Sect. 3.2 is irrelevant, and nonmeasurability seems more salient. Nevertheless, we think both extremes are unsatisfactory.

Although the fully realistic point of view is reasonable, we note that there are many fruitful mathematical and scientific practices that drift away from it. Infinitary concepts (e.g. limit, the continuum, integrals, etc.) are applied from physics to statistics, without much concern on how actual human brains grapple with them. As for ii), a tradition known as *the method of arbitrary functions* stemmed from Poincaré seeks to derive uniform probability for the *Random Spinner*, even when its dynamics is taken into account (Myrvold, 2021). On iii), we observe that as evident in classical analysis textbooks, mathematicians typically consider making a countable sequence of choices unproblematic.[18] As for the fully ideal perspective, we are dissatisfied with it because it treats highly nonconstructive phenomena such as nonmeasurability on an equal footing with idealizations naturally considered in practice. In contrast, the semiconstructive point of view helps to isolate the nonconstructive phenomena, without necessitating radical reformulations of existing practices. As Schechter (1996, § 14.76) observes, it is "a compromise between constructivist mathematics and mainstream mathematics, which should be easily understood by most analysts and other 'ordinary' mathematician". We therefore think that the semiconstructive standpoint is a robust one from which formal epistemology can benefit.

4.4.4 Nonmeasurable phenomena are in fact empirically detectable

In Sect. 3.2 we argued that nonmeasurable phenomena are not empirically detectable. However, Dorr (2024) suggests otherwise. He imagines God asking us to distinguish between a lamp triggered by a nonmeasurable subset of the *Random Spinner* ("the interesting spinner") and another lamp triggered by one-third of the spinner ("the boring spinner"), and claims:

> "After thirty spins, your tally … has ten 1s and twenty 0s: just the proportion that would be most likely if you had the boring spinner. You wonder how you

[18] Here, our treatment of countably infinite processes may draw comparisons with a philosophy of set theory known as *countablism*, as outlined by Builes and Wilson (2022). However, they are quite different. Countablism there is understood as a metaphysical thesis highlighting the fact that for any infinite set there is a forcing extension of the universe such that it becomes countable.

should react. I say you should react by becoming more confident that you have the boring spinner than you were before your experiments. I hope readers will share my sense that this is *obviously the right reaction*. For those who demand an argument, I offer the following: *your observations should leave you pretty confident that you have the boring spinner*. Beforehand, you should not have been pretty confident. But necessarily, if you are pretty confident afterwards and not before, you are more confident afterwards than before…" (Emphasis added)

Dorr claims that we *can* empirically distinguish nonmeasurable phenomena from measurable ones, at least in the restricted setting of choosing between just these two spinners. However, from his description, it is entirely unclear on what basis can we be "pretty confident that we have the boring spinner". Perhaps the thought is, our confidence in having the boring spinner should increase because it has high likelihood of generating the data. But what we really need is to *compare* the likelihood of the two hypotheses. How can we compute the likelihood of the nonmeasurable hypothesis? This brings us back to the discussion in Sect. 4.1 of Hoek, who observes that there is in general no way to do this.[19]

On this point, I think Hoek is right in that if nonmeasurable hypotheses are taken into consideration, then we cannot empirically rule them out. However, Dorr's intuition that an outcome of ten 1s and twenty 0s strongly supports the boring hypothesis is also important. This very natural intuition, I think, is based on the law of large numbers: if our n trials are considered as n i.i.d. random variables, this law asserts that their average $\overline{X_n}$ approaches μ as $n \to \infty$, where μ is the mean of X_n. However, as we formulate this law as a mathematical theorem, we make the assumption that X is a measurable function. For the "interesting spinner", X is not measurable, for which the theorem says nothing. So one cannot apply intuitions based on this law to refute nonmeasurable hypotheses. The fact that Dorr overlooks this and considers it obvious to apply the law seems to me to suggest that we very naturally tends to implicitly make measurability assumptions. And from our point of view, these assumptions are very much justified.

4.4.5 Nonmeasurable sets of low complexity

It is well known that under $\mathsf{V} = \mathsf{L}$, there exist nonmeasurable sets relatively low in the projective hierarchy, e.g. at the (lightface) Π^1_2/Σ^1_2 level. A recent article by Hanson (2025) gives a comprehensive exposition of this phenomenon. The upshot of such results is that nonmeasurable sets can be more explicit than one may think – an example given by Hanson (2025, Prop. 3.5) is: there is an open set $U \subseteq \mathbb{R}^3 \times \mathbb{N}$ which yields a Π^1_2 nonmeasurable set after taking its projection and complement successively for 3 times. We think such results have no bearing on our position, for

[19] In some specific cases this might not be true, e.g. we are told that "the interesting spinner" is correlated with a nonmeasurable set whose outer measure is quite small. But the general case appears intangible. What if it is correlated with the union of one third of the spinner with a nonmeasurable set with small outer measure?

several reasons. First, our final discussion in Sect. 3.2 suggests that the empirically relevant phenomena fall within the range of the analytic/coanalytic sets, or even the Borel sets. If this is right, then the presence of nonmeasurable sets at the next level is irrelevant. Second, $\mathsf{V} = \mathsf{L}$ in particular implies a strong form of AC, that every set is uniformly well-ordered by $<_L$, an order that exists at the Δ^1_2 level. This means that nonconstrutive objects can be made definable (in the logical sense that provably there exists a formal formula describing the object) via metamathematical techniques. But a constructivist or semiconstructivist would likely doubt whether an object being definable in this sense is any indication of its explicitness, if she is concerned with an empirically motivated sense of explicitness. Finally, from the point of view discussed in Sect. 3.3, for the probabilist, there is no difference between various axiomatic set theories as long as they allow her to do the basic work, and there is no reason to pay special attention to byproducts of exotic set theories. Strengthening AC to $\mathsf{V} = \mathsf{L}$ is analogous to replacing an ordinary chalk with a particularly fancy chalk with properties of independent interest.

5 Appendix: chance totalism and set-theoretic implications

This appendix discusses some views of Hoek (2021) closely related to the theme of the paper. The issues involved here chiefly concern metaphysics and the philosophy of set theory. Readers primarily interested in the role of nonmeasurable sets in epistemology may skip the appendix without loosing sight of the main contributions of the paper.

Recall that Hoek's argument introduced in Sect. 4.1 intends to prove Chance Totalism, which appears as an intermediate step towards the ultimate goal of demonstrating the falsity of CH. Furthermore, Hoek claims that the argument against CH has an empirical character, since on his view, Chance Totalism best explains the success of inductive inference. The purpose of this appendix is to show that Chance Totalism does not imply ¬CH, unless one assumes certain forms of AC. Since the relevant form of AC is not amenable to empirical justification, Hoek's argument does not have the empirical character he claims. We need to introduce some definitions for the discussion.

Definition 2 Let κ be an infinite set, a *measure* over κ is a function $\mu : 2^\kappa \to [0, 1]$ such that i) $\mu(\kappa) = 1$; ii) $\mu(\{s\}) = 0$ for all $s \in \kappa$; iii) if $X_n \subseteq \kappa$ are pairwise disjoint, then $\mu(\bigcup_{n\in\omega} X_n) = \sum_{n\in\omega} \mu(X_n)$.

Definition 3 Hoek (2021) M denotes the assertion that there exists a measure over $\mathbb{R}$.

Chance Totalism is the statement that there is a chance function for the *Random Spinner* witnessing M. The principle M turns out to have the following series of set-theoretic implications.[20]

[20] See Kanamori (2008, Ch. 2) for some references to these results.

Definition 4 Let κ be a cardinal, we say κ is *real-valued measurable* if there is a κ-additive measure on κ. κ-additive means additivity holds for any mutually disjoint family of size $< \kappa$. We say κ is *measurable* if there is a κ-additive measure on κ whose range is $\{0, 1\}$.

Proposition 2 Let κ be the least cardinal such that there exists a measure on κ. Then any measure on κ is in fact κ-additive, i.e. κ is also the least real-valued measurable cardinal.

Proof Fix $\lambda < \kappa$. Let $\{X_\alpha : \alpha < \lambda\}$ be mutually disjoint and $m(X_\alpha) = 0$, it suffices to show that $m(\bigcup X_\alpha) = 0$. Suppose $m(\bigcup X_\alpha) = r > 0$, define the following measure over λ: for $A \subseteq \lambda, m(A) = \frac{\sum_{\alpha \in A} m(X_\alpha)}{r}$, contradicting the minimality of κ.

Definition 5 An *atom* for a measure *m* is some set $A \subseteq \kappa$ such that $m(A) > 0$ and for all $B \subseteq A$, either $m(B) = m(A)$ or $m(B) = 0$. If *m* has an atom we say *m* is *atomic*.

Proposition 3 *κ is measurable iff κ is real-valued measurable with an atomic, κ-additive measure.*

Proof On the one hand, let κ be measurable with measure μ, then in particular μ is an atomic real-valued measure with κ as an atom. On the other hand, if $A \subseteq \kappa$ is an atom for a measure *m*, then $\mu(X) = \frac{m(X \cap A)}{m(A)}$ defines a 2-valued measure.

Proposition 4 *If κ is measurable then $|\mathbb{R}| < \kappa$.*

Remark 1 In hindsight, it is known that a measurable cardinal is inaccessible and in fact has many inaccessibles below it, which implies Proposition 7.

Theorem 4 *[Ulam] Let κ be real-valued measurable with a κ-additive measure m. If m is atomless then $\kappa \leq |\mathbb{R}|$, moreover, there is a total extension of the Lebesgue measure over $\mathbb{R}$ which is κ-additive.*

Proof This is proved as Theorem 2.5 in Kanamori (2008).

We now have a dichotomy where κ is the least real-valued measurable cardinal.

$$\begin{cases} |\mathbb{R}| < \kappa \Leftrightarrow \kappa \text{is measurable} \Leftrightarrow \text{There is an atomic } \kappa\text{-additive measure over } \kappa \\ \kappa \leq |\mathbb{R}| \Leftrightarrow \kappa \text{is not measurable} \Leftrightarrow \text{All } \kappa\text{-additive measures over } \kappa \text{ are atomless} \end{cases}$$

We can also see that the principle M is equivalent to the second disjunct: if M holds, then there is a measure over $\mathbb{R}$, by Proposition 5, there is a least real-valued measurable cardinal $\kappa \leq |\mathbb{R}|$. On the other hand, If there is a least real-valued measurable cardinal $\kappa \leq |\mathbb{R}|$, by the extension part of Ulam's Theorem 8, M is witnessed by a κ-additive total extension of the Lebesgue measure.

In particular, M implies that there is a real-valued measurable cardinal $\kappa \leq |\mathbb{R}|$, which has some striking consequences.

Proposition 5 *[Ulam,* AC*] Let $\kappa \leq |\mathbb{R}|$ be real-valued measurable, then κ is weakly inaccessible.*

κ being weakly inaccessible means that κ is regular, and for any cardinal $\lambda < \kappa$, $\lambda^+ < \kappa$. Hence if M holds, we have $\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \kappa < |\mathbb{R}|$, meaning that CH fails considerably. Because Hoek claims that his argument earlier described in Sect. 4.1 establishes Chance Totalism, which implies that M is witnessed by a chance function for the *Random Spinner*, Hoek thinks his argument refutes CH. In fact, it has numerous other consequences: since CH holds under V = L, Hoek also refuted Gödel's V = L. Similarly, since $|\mathbb{R}| = \aleph_2$ holds under forcing axioms such as PFA, he also refuted these axioms… Moreover, we know:

Theorem 5 *[Solovay]* ZFC+ *"there is a real-valued measurable cardinal" is equiconsistent with* ZFC+ *"there is a measurable cardinal".*[21]

So if a real-valued measurable cardinal is consistent, then we have a model of ZFC +"there is a measurable cardinal", and in particular ZFC itself. Assuming that Hoek thinks M is not only true but consistent, then he also showed that ZFC as well as many large cardinals are consistent!

All these conclusions are supposed to follow from Chance Totalism. And since chance function is considered objective, it is argued that these conclusions hold for the "real" set theoretic universe.

So the argument described in Sect. 4.1, if sound, indeed has striking consequences. There, we have already discussed some of its flaws. If my criticisms are right, then the argument does not prove Chance Totalism. In this section we make a separate criticism: even if Chance Totalism is true, the alleged set-theoretic conclusions do not follow. The reason is as follows.

As in Sect. 3.3, we need to pay attention to the distinction between mere sets and sets as members of set-theoretic universes. Suppose indeed there is a chance function that is a measure defined on all the subsets of the infinitely thin rim of the *Random Spinner*, whose existence can be confirmed via some empirical or abductive considerations, as Hoek wishes. I claim that even so, the existence of this chance function does not fix the set-theoretic universe in which it lives (or in which there exists a function isomorphic to the chance function in some sense), if the notion of *the* set-theoretic universe makes sense at all. We may consider the set formed by the points on the rim of the spinner – it is a set of physical objects. *There is no reason to think that this set inhabits any specific model of set theory (e.g. a model of* ZFC*), or indeed to assume the existence of any coordination between this set and the/a set-theoretic universe at all*. With whatever local information we are given about the set of points on the spinner's rim and the functions defined on them, we have no reason to make conclusions on global questions such as whether the axiom of replacement

[21] Note that $\kappa < / = / > |\mathbb{R}|$ are all equiconsistent with a measurable.

holds, whether AC holds, or whether there is a well-ordering of the reals. Therefore, the total chance function may come from a ZFC universe, or it may well come from a choiceless universe.

This constitutes a problem for the argument against CH, because when AC is absent, the existence of a total chance function on $\mathbb{R}$ (i.e. M) does not imply $\neg$CH. M $\Rightarrow \neg$CH follows from Ulam's Proposition 9, whose proof depends on constructing an Ulam matrix using AC.[22] In fact, it turns out that in some choiceless situations CH is consistent with M.

To briefly describe such situations, we first note that without AC, $\mathbb{R}$ may not be well-orderable, so CH should not be stated as $2^{\aleph_0} = \aleph_1$. (In fact, the existence of a well-ordering of $\mathbb{R}$ implies the existence of nonmeasurable sets.) Instead, it is standard to consider the formulation known as the *weak continuum hypothesis* (WCH): any uncountable set of reals is bijective with $\mathbb{R}$.

In Solovay's model, every subset of $\mathbb{R}$ has the perfect set property, meaning that it is either countable or contains a subset bijective with $\mathbb{R}$. So in particular, any uncountable set of reals is bijective with $\mathbb{R}$, by the Schröder–Bernstein theorem which holds in ZF. So WCH holds. Since the Solovay model contains a total measure on $\mathbb{R}$ (i.e. the Lebesgue measure) and satisfies WCH, the implication from Credence Totalism to $\neg$CH does not hold.

The same phenomenon appears in many models other than Solovay's model. In descriptive set theory, measurability, the Baire property, and the perfect set property are considered the three most important regularity properties of subsets of $\mathbb{R}$. In many choiceless contexts they simultaneously hold for all subsets of $\mathbb{R}$. For example, AD implies these properties. Therefore, in AD models, we have a total measure coexisting with WCH as well.

To summarize, Hoek's alleged empirical argument against CH involves two consequences of AC: i) nonmeasurable sets exist; ii) M $\Rightarrow \neg$CH. But inductive inference goes on as usual without assuming that nonmeasurable sets admit credence or chance, because the empirical point of view does not force us to adjudicate whether i) holds (Sects. 3.3). If our empirical investigations are unaffected if we see our mathematical objects as living in Solovay's model, for example, then our above analysis shows that from the empirical point of view, we need not accept M $\Rightarrow \neg$CH either.[23] Therefore we should not consider Hoek's argument successful.

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[22] It is not known to the author whether the existence of Ulam matrices is the optimal principle for proving M $\Rightarrow \neg$CH.

[23] In Solovay's model, both i) and ii) fail. One reasonable question to ask here is whether i) and ii) are independent but the answer is not immediately clear to the author.

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Conflicts of interest Not applicable.

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