Abstract

This paper develops a theory of mathematical explanation through the lens of a separatist metaphysical grounding framework. I argue that mathematical explanations delivered by proofs are best captured by a non-causal determination relation between mathematical facts—a metaphysical grounding relation where the *explanans* determines the *explanandum*, where the relevant why questions at each step are answered; an explanatory proof is the one that establishes an explanatory chain between the *explanans* and the *explanandum*. Through a case study in algebra (e.g., the infinity of fields with characteristic zero), I argue how this relation establishes objective dependencies in terms of determination relations that answer why-questions about mathematical facts. By adopting a ground-first separatist approach—where grounding relations back explanations but are distinct from them—the theory aligns with mathematical practice, supports proof plurality, and addresses gaps in existing accounts.

Keywords:
mathematical explanation, separatism, explanatory chain, metaphysical grounding.

1 Introduction

Mathematics has a crucial role in explaining physical and social phenomena [1]. This role has been a major topic in the philosophy of science. However, until recently, the question of whether mathematics has an explanatory role in accounting for mathematical facts has not been fully addressed in the literature of analytic philosophy [2]. Nonetheless, according to Mancosu in [Man00], discussions on mathematical explanation, not necessarily in the analytic philosophy, date back to Aristotle's *Posterior Analytics*. He claims that demonstrations "of the reasoned fact" (i.e., explanations) occur in mathematics. He contrasts these demonstrations with the demonstration "of the fact." According to him, although both are logically correct, only the first type of demonstration explains the result [3] [4].

[1] This paper is a substantially revised version of the first chapter of my PhD dissertation on mathematical explanation at the University School for Advanced Studies of Pavia (IUSS). I am deeply grateful to numerous individuals for their insightful and constructive feedback on earlier drafts of this paper. I extend my sincere appreciation to Andrea Sereni, Dugald Macpherson, William D'Alessandro, Stefan Roski, Hamid Vahid, Mahmoud Morvarid, Mohammad Saleh Zarepour, Mohsen Zamani, and Davood Hosseini. In particular, I am particularly grateful for the invaluable critiques and suggestions provided by anonymous reviewers of the European Journal for Philosophy of Science. Finally, I express my heartfelt gratitude to the editors of EJPS for their unwavering support throughout the editorial process.

[2] In this literature, the issue was discussed in the works of Steiner [Ste78], Kitcher developed in [HM08] extracted from [Kit89], and Lange's [Lan14].

[3] See [MPP23], Section 5 for further details.

[4] Aristotle's idea raised a critical discussion (the *Quaestio de Certitudine Mathematicarum*) during the Renaissance. See [Man00] for further details.

On the other hand, the relation between grounding explanations and mathematical explanations has been gaining some attention, with some versions discussed in the literature. One of the early systematic contributions to the relation between the mathematical explanation and the grounding explanation is due to Bolzano [5], who argued that explanatory proofs are ultimately ground-revealing proofs. On the other hand, Lange, in a famous work [Lan19], has argued that mathematical explanation and grounding explanation diverge. Nonetheless, a couple of recent studies by Poggiolesi and Genco in [PG23] and Poggiolesi in [Pog23] suggest that mathematical explanation is a type of conceptual explanation backed by a conceptual grounding relation [6].

In a previous work [Maa25] [7], I have argued that there is a version of metaphysical grounding that is immune to Lange's criticisms. In this paper, I will take a more direct approach to mathematical explanation. I will show that the version of grounding theory that withstands Lange's arguments in [Lan19] is actually a natural suggestion for addressing problems regarding mathematical explanation [8]. The main proposal is that a mathematical explanation is a version of grounding explanation that is backed by a metaphysical grounding relation of determination (between facts that explain and the fact that is being explained), such that the relevant why questions about the fact under study are answered. In other words, a proof is explanatory when it creates an "explanatory chain" that links the items meant to explain with the item being explained.

To start, I will make a few assumptions about the grounding relation. I will consider the grounding relation as a non-causal form of determination that backs explanations and is transitive. Based on this, I will argue that grounding provides a natural way to understand mathematical explanation. It shows how mathematical facts depend on one another—specifically, how the facts that do the explaining determine the ones being explained. I will illustrate this with several mathematical examples.

A metaphysical understanding of the grounding relation, that according to Correia in [Cor14] and Smithson in [Smi20] includes the conceptual grounding [9], addresses the questions about mathematical explanation. So, the main strategy of the current research is to

[5] See [Rus22] for a survey of Bolzano's view of explanatory proofs in terms of grounding.

[6] Betti has presented a similar view in [Bet10], p.252.

[7] Marc Lange has responded to this paper in [Lan25].

[8] I thank an anonymous reviewer for pointing out that the grounding relations are generally less clear than many mathematical concepts, including those presented in the current research. However, proponents of using the grounding relation can employ it not to discuss or clarify any mathematical concepts (e.g., defining or clarifying them through metaphysical grounding) but to discuss the practice of mathematics and what appears to be a crucial aspect of it, such as explanations in mathematics. Therefore, the goal of the current research is to better understand a significant component of the practice of mathematics, clarifying which requires employing notions that are not explicitly used in the practice of mathematics.

[9] According to Correia, for example, "every case of logical grounding is a case of conceptual grounding (but not vice versa), and that every case of conceptual grounding is a case of metaphysical grounding (but not vice versa)." ([Cor14], p. 32)

extend Poggiolesi and Genco's account in [PG23] and Poggiolesi's account [Pog23] to a larger set of explanatory proof by weakening the *conceptual grounding* to a *metaphysical grounding* relation, adopting a ground-first approach, where the grounding relation is understood as a metaphysical form of determination, and the grounding explanation is backed by it. I will argue that this weakening of the conceptual grounding relation to a metaphysical relation sets the stage for studying a larger set of explanatory proofs (in an informal setting) as a variety of grounding explanations. I leave the comparison between Poggiolesi and Genco's account in [PG23] and Poggiolesi's account [Pog23], on the one hand, and the current account, on the other hand, to future work.

Here is an overview of the paper. In Subsection 1.1, the notion of mathematical explanation, along with some examples, will be introduced. Following this, Section 2.1 presents a case of mathematical explanation that motivates the current research. It is then suggested in Subsection 2.2 that, to properly address mathematical explanation, a determination relation between mathematical facts is appropriate. This approach facilitates addressing the question of mathematical explanation as a form of grounding explanation backed by a metaphysical relation of determination in Section 3. For this purpose, a separatist theory of ground will be presented in Subsection 3.1, arguing that mathematical explanation can be naturally viewed as a type of grounding explanation in Subsection 3.2.

1.1 Mathematical Explanations and Informal Mathematical Proofs

Mathematicians offer and ask for explanations; they wonder, "Why does such-and-such a mathematical fact occur?". For example, why is the number of occurrences of the numeral 7 on the list from 1 to $99,999$ exactly $50,000$? [10]. Although many accounts of mathematical explanation were focused on what an explanatory *proof* is, mathematical explanation is not restricted to mathematical proofs [11]. In addition to proofs, diagrams could, for instance, have explanatory value [12]. However, the present research only focuses on the explanations

[10] The example is discussed in Lange's [Lan19].

[11] There are generally two main views on how mathematical explanations should be understood concerning mathematical proofs. The first view considers being explanatory as a virtue of a mathematical proof. For instance, according to Lange, "Purity and explanatory power are both virtues in proofs, as are brevity, generalizability, simplicity, visualizability, theoretical fruitfulness, pedagogic value, and so forth." (Lange, [Lan19], Footnote 7). On the other hand, the second view considers being explanatory as a goal of a mathematical proof. For instance, to borrow Detlefsen's words, "... a prime goal of proof is explanation" ([Det08a], p. 17). The former views explanation as a virtue of mathematical proof and other mathematical virtues, e.g., elegance, brevity, and purity. However, the latter view assigns a more substantial role to mathematical explanation by elevating it as a prime goal of proofs. The present research subscribes to the latter view. This view is best captured by viewing explanation as a relation between a set of mathematical facts that explain and a mathematical fact that is being explained.

[12] See [Lan18] and [D17].

delivered by mathematical proofs. In what follows, I will discuss major features of this particular type of mathematical explanation [13].

Generally speaking, mathematicians develop a collection of theorems, definitions, and lemmas to address a particular question. So, the final proof appears to be just the "tip of the iceberg" in a piece of mathematical research, representing a (sometimes lengthy) process. Hence, to learn more about the nature of mathematical explanation as it appears in the practice of mathematics, knowing the details of an entire research process (and sometimes the history of the subject, the motivations, and even the failed attempts to solve the problem) seems relevant. This shift, i.e., to look for explanatoriness in mathematical research as a global and multifarious process rather than just locating it in mathematical proofs, does not diminish the role of proofs insofar as proofs are seen as the final step of a longer process. So, we will always refer to proof as a representative of a more complex piece of mathematical research, and explanation as a prime research goal.

As we are investigating explanations in mathematics, by a fact, I will always mean a mathematical fact, which is denoted by $[P]$, where P is a mathematical proposition. Here are some examples of mathematical facts: that every natural number greater than 1 has a unique prime decomposition, that the first-order theory of real closed fields eliminates the quantifiers in the language of ordered rings, or that every algebraically closed field is infinite.

A key aspect of this kind of mathematical explanation that any *bona fide* account should include is its objectivity; in a genuine case of mathematical explanation, some mathematical fact accounts for another mathematical fact. In other words, we are looking for a relation between a mathematical fact that we are about to study (the *explanandum*) and another mathematical fact that we understand better (the *explanans*), such that the latter *makes true* statements about the former. For facts $[P]$ and $[Q]$, there is an objective element to the mathematical explanation involving an objective relation that is guaranteed to hold when $[Q]$ accounts for $[P]$. Call this the *objective element* of mathematical explanation [14]:

> Mathematical explanation includes an objective element based on which the *explanans* makes some statements about the *explanandum* to be true. (1)

It is important to emphasize that, consistent with standard mathematical practice, the scope of our research includes proofs, which consist of a series of true propositions (or facts) that

[13] Mathematical explanations can be quite intricate. In their paper [HM05], Hafner and Mancosu draw a comparison between the numerous types of mathematical explanations and the diverse religious experiences described by William James. They suggest that the term "explanation" cannot be used as a catch-all for any principle or essence but rather serves as an umbrella term for the vast array of explanations that exist. As a result, it is crucial to specify the type of explanation when dealing with mathematical explanation. This research aims to offer a theory of mathematical explanation that is offered by mathematical proofs.

[14] This aspect will be addressed in terms of the grounding relation. Note that the grounding relation, as well as the distinctions relating to this topic, including partial versus full grounding, will be discussed in section 3.1.

serve as premises and another true proposition (or fact) that serves as the conclusion. This is because we are focused on the proofs that convey the reasons why some mathematical proposition is true or why some mathematical fact is the case. These reasons, usually in a mathematical proof, take the form of a series of steps (usually in an informal language including some formal element and some natural language) that eventually establish the truth of the proposition that is considered as the conclusion by answering a series of why questions regarding the fact under study.

On the other hand, when examining mathematical explanations provided by proofs, it is important to recognize that some proofs explain a given fact better than others. This does not imply that one proof is the best explanation, but rather that some offer a deeper understanding of the *explanandum*. Although comparing the relative explanatory power of different proofs is closely tied to the study of explanation, it is not the main focus of this research [15]. To the extent that this issue is relevant here, it highlights the need for a representational element—something that shows why one fact holds in virtue of another [16]. I will denote this representational structure as $\langle P$ because of $Q\rangle$ [17].

As we examine explanations delivered by proofs, it is appropriate to clarify our conception of proofs and their explanatory aspects. Following Dawson's characterization [Daw06], I understand proofs as informal arguments that establish truth while ideally explaining why that truth holds. According to him, [Daw06], p. 270, "we shall take a proof to be an informal argument whose purpose is to convince those who endeavor to follow it that a certain mathematical statement is true (and, ideally, to explain why it is true)". This aligns with mathematical practice, where proofs typically blend formal elements with natural language rather than existing as fully formalized derivations.

However, adopting this approach, i.e., considering proofs in mathematics as informal arguments, does not *ipso facto* result in saying that proofs (in an informal setting) are not rigorous [20]. The debate over mathematical rigor reveals a divergence in views regarding the relationship between formalization and rigor. According to what Hamami calls in [Ham22]

[15] For a discussion of the comparative explanatory value of proofs, see Wilhelm [Wil23].

[16] For a functional approach to mathematical explanation that centers on answering why-questions and enhancing understanding, see [IMR21].

[17] Steiner made a relevant distinction in his work [Ste78] between relative explanatory value and explanation *per se*. While the former concerns the differences in the explanatory value of various proofs, the latter focuses on the nature of mathematical explanation. Nonetheless, these questions are interrelated, and exploring one can shed light on the other. For example, in [Ste78], p.135, Steiner recalls that Feferman identifies explanation with generality or abstraction. So, according to him, if we adopt Feferman's idea about explanation, we should concede that among several possible explanations, the more abstract or general the explanation, the more explanatory it will be. The present research's main target is studying mathematical explanation *per se*. So, while I will discuss proofs with respect to their explanatory value, the main purpose will be to investigate a theory of mathematical explanation itself. However, along with Lange, I assume that the explanatory power of proofs is not "all or nothing" [18]; different proofs could have different explanatory values [19].

[20] I thank an anonymous referee of European Journal for Philosophy of Science for bringing up this issue.

the “standard view,” a mathematical proof is rigorous if it can be routinely translated into a formal proof within a formal theory (e.g., ZFC). This view regards formalization as a definitive measure of rigor, even if full formalization is rarely carried out in practice. However, this has been contested by several philosophers of mathematical practice who argue that rigor and formalization are different concerns, and that mathematical rigor can and often does operate independently of any routine translation into formal systems [21]. Detlefsen, for example, says, “Mathematical proofs are not commonly formalized, either at the time they’re presented or afterwards. Neither are they generally presented in a way that makes their formalizations either apparent or routine ... There are thus indications that rigor and formalization are independent concerns.” ([Det09], p 17). Thus, there are at least two positions in the literature regarding the relation between rigor and formalization: one that identifies rigor with formalizability, and another that sees rigor as a practice-sensitive notion that need not be tied to formal systems.

Since my focus is on how mathematical explanations are conveyed through proofs in the actual practice of mathematics, I adopt the informal presentation of proofs as my primary subject of study. The reason is, according to Hamami in [Ham22], both proponents and critics of the standard view agree that formalization is *not* routinely pursued in everyday mathematics, and that proofs as they appear in practice often combine formal and informal elements. Accordingly, I analyze mathematical proofs as they are used and understood in the everyday practice of mathematicians—proofs that contain heuristic strategies, sometimes intuitive inferences, and other informal components. Moreover, I agree with Rav that informal proofs include “topic-specific moves” that serve as “bridges between the initially given data, or between some intermediate steps, and subsequent parts of the argument” ([Rav99], p. 26).

In this setting, the objective element appears as a sequence of steps, including propositions that begin with those about the facts that explain and lead to propositions about the fact being explained. This corresponds to consecutive major steps of the proof in informal language and will be referred to as the *explanatory chain* of an explanatory proof (further discussed and illustrated in Section 3.2). However, adopting this does not exclude the possibility that, in a formal approach, the steps of the proof include purely inferential and grounding steps [22]. In the current approach, while not ruling out the formal approach to the steps of the argument in a specific language, the main focus remains on the informal version of the proof and its major steps.

To illustrate the objective element through a series of informal steps, let us consider the example presented in [IMR21]. This example includes a proof for the sum of the first n

[21] See Larvor’s paper [Lar12], Tanswell’s work [Tan15], and Detlefsen’s [Det09] for arguments against the standard view.

[22] For example, Genco in [Gen21] provides a calculus in a language that enables combining logical derivations and formal explanations to distinguish the explanatory parts of derivations from their non-explanatory parts.

natural numbers. It's a well-known fact that this sum is equal to $\frac{n(n+1)}{2}$. This is the sum of the first n numbers:

$$1 + 2 + \ldots + n. \tag{2}$$

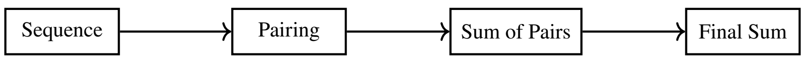
Now, let us write the sum from n to 1:

$$n + n - 1 + \ldots + 1. \tag{3}$$

The sum of both sequences is the same; the only difference is the order of the addition. Next, add the first element of the first sequence to the first element of the second sequence, which is $(1 + n)$. Do the same for each of the two elements of both sequences. The result is the following:

$$\begin{pmatrix} 1 \\ + \\ n \end{pmatrix} + \begin{pmatrix} 2 \\ + \\ n-1 \end{pmatrix} + \cdots + \begin{pmatrix} n \\ + \\ 1 \end{pmatrix} \tag{4}$$

Each of the elements in the above summation amounts to $n + 1$. Hence, we have n-many summations of $n + 1$, i.e., $n(n + 1)$. However, we added the sequence to itself first. So, the sum itself will be $\frac{n(n+1)}{2}$. The sketch presented here shows why the sum of the first n natural numbers is equal to $\frac{n(n+1)}{2}$. It also produces a chain consisting of the following steps:



In this proof, each major step is not merely a procedural move but is determined by the facts established in the preceding steps. To begin with, the step of sequencing—the move from the original sum $S = 1 + 2 + \ldots + n$ to the symmetrical arrangement with its reverse, $n + (n - 1) + \ldots + 1$—is underwritten by the mathematical fact that the addition in the ring of integers is commutative [23]. This ensures that reordering the terms of the sum does not change the outcome. Without this structural fact about integers, the move to a reversed sequence would be unjustified. Once this symmetry is established, it determines the facts for the next step: pairing each element in the original sequence with its counterpart in the reversed sequence to form n many pairs of $(n + 1)$. The possibility of such uniform pairing is a direct consequence of the sequencing fact. Thus, the structure of the integers and the way the two sequences mirror one another determine the pairing step.

Following the pairing step, the sum of pairs step is similarly determined. Since each pair now sums to $(n + 1)$ and there are n such pairs, the total becomes $n(n + 1)$. But because this total emerged from pairing two sequences of S, this total is exactly $2S$. The fact that each pairing corresponds one-to-one with terms from the original and reversed sequences—combined with the previously justified symmetry—determines that $2S = n(n +$

[23] Note that addition is not always commutative, for example, $1 + \omega \neq \omega + 1$.

1). This, in turn, directly determines the final step: solving for S yields $S = \frac{n(n+1)}{2}$. Crucially, at each stage, it is not that the next step merely follows in a purely formal sense, but rather that the mathematical facts revealed at each point make the next fact to be the case by answering the why questions fixed by the context.

This research aims to explore this aspect of explanatory proofs through a separate metaphysical form of the grounding relation. However, before moving on to the main proposal, I provide an example from the theory of fields that further motivates the present research in the next section.

2 Mathematical Explanation and Determination Relation

2.1 A Case of Mathematical Explanation

While the account advocated by the current research agrees with Poggiolesi and Genco's account in [PG23] and Poggiolesi's account in [Pog23] that mathematical explanations are a variety of grounding explanations, I propose extending their framework (by weakening the conceptual grounding relation to a metaphysical version of grounding) to encompass a broader range of explanatory proofs. I suggest that grounding explanations in mathematics should be backed by a *metaphysical* grounding relation that the facts appearing as *explanans* determine the fact that appears as *explanandum*. To motivate this, we present a case from field theory where an explanatory proof relies on concepts not explicitly present in the theorem's statement. This example illustrates how explanatory proofs often involve dependency relations between mathematical facts, thereby motivating the need for a metaphysical determination relation to fully capture mathematical explanation.

Let us consider the case from the field theory. By a field, I mean a structure of the form $(F, +, \times, 0, 1)$, including two group structures, i.e., $(F, +, 0)$, and $(F - \{0\}, \times, 1)$, which are the additive and the multiplicative groups. Some familiar examples are the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$, and the field of rational numbers $\mathbb{Q}$. These examples are infinite fields. Some fields, however, contain only finitely many elements. An example of a field with finitely many elements is the finite field $\mathbb{F}_p$, where p is a prime number. This field consists of the integers $\{0, 1, 2, \dots, p-1\}$, and addition and multiplication are defined modulo p. For example, the field $\mathbb{F}_5$ has the elements $\{0, 1, 2, 3, 4\}$, with operations performed modulo 5. In this field:

$$3 + 4 = 2 \mod 5,$$

$$3 \times 4 = 12 = 2 \mod 5.$$

A field F has characteristic p, if for all $a \in F$:

$$\underbrace{a + a + ... + a}_{\text{p-times}} = 0.$$

A field F has characteristic 0 if no such p exists. This is a fact that any field with characteristic 0 is infinite. Here, I discuss a proof for this mathematical fact. The proof highlights this point by employing the concept of field embedding, which is not explicitly part of the theorem's initial formulation. This proof draws on mathematical tools and structures from outside the theorem's immediate scope, thus providing a richer, more constructive understanding of the theorem. This example motivates the proposal that mathematical explanation is a type of grounding explanation that is backed by a metaphysical determination relation. Here is the statement that fields with characteristic 0 are infinite:

Fact 2.1.1. *Let F be a field with characteristic* 0*. Then, F is infinite.*

Proof. Since F has characteristic 0, the field of rational numbers $\mathbb{Q}$ can be embedded in F using the following map:

$$\phi : \mathbb{Q} \to F$$

defined by:

$$\phi\left(\frac{a}{b}\right) = \left(a \cdot 1_F\right) \cdot \left(b \cdot 1_F\right)^{-1}$$

for any rational number $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. Here, 1_F denotes the multiplicative identity in F. This map is a field homomorphism since it preserves both addition and multiplication:

$$\phi\left(\frac{a}{b} + \frac{c}{d}\right) = \phi\left(\frac{ad + bc}{bd}\right) = \phi(ad + bc) \cdot (bd)^{-1} = \phi\left(\frac{a}{b}\right) + \phi\left(\frac{c}{d}\right),$$

and similarly for multiplication. Since F contains a copy of the field of rationals, F must contain infinitely many elements, as $\mathbb{Q}$ is infinite. Therefore, F must be at least as large as the field of rational numbers, which is infinite. □

Here is a breakdown of the major steps of the proof. There is a copy of the rational numbers, i.e., $\mathbb{Q}$, in every field with characteristic 0. Let us denote this copy of the rational numbers by $\overline{\mathbb{Q}}$. As a crucial remark, $\overline{\mathbb{Q}}$ denotes a series of formal objects that are unique up to isomorphism. Although elements of $\overline{\mathbb{Q}}$ in different fields could have different names, as a field, they are the same. However, the infinite structure of rational numbers $\mathbb{Q}$ is preserved in every isomorphic copy. Hence, the infinity of F is proved via a structural understanding of the infinite subfield that exists in every field with characteristic 0.

Let us delve into the reasons why the proof is indeed explanatory. First, for any field F of characteristic zero, the proof proceeds by showing that there exists a *canonical* embedding of the rational numbers $\mathbb{Q}$ into F. This embedding arises naturally from the inclusion of the integers $\mathbb{Z}$ into F via the map $n \mapsto n \cdot 1_F$, where 1_F denotes the multiplicative identity of F. Since F has characteristic zero, this assignment is injective and thus extends *uniquely* to $\mathbb{Q}$ by defining $\iota(a/b) = (a \cdot 1_F)(b \cdot 1_F)^{-1}$ for any integers a, b with $b \neq 0$. The map ι

preserves all field operations, rendering the copy of $\mathbb{Q}$ in F as a distinguished subfield of F. Hence, The embedding is canonical in the sense that given any homomorphism $\varphi : F \to F'$ between fields of characteristic zero, the restriction of φ to the $\mathbb{Q}$-subfields commutes with ι in the sense that $\iota \circ \varphi = \iota'$. This is illustrated in the following diagram:

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\iota} & F \\ & \underset{\iota'}{\searrow} & \downarrow{\scriptstyle \varphi} \\ & & F' \end{array}$$

Hence, in every field with characteristic 0, a particular structure, i.e., $\mathbb{Q}$-like structure, is preserved and is isomorphically unique, which, among many other features, is infinite. This is because the canonical embedding denoted by $\iota : \mathbb{Q} \hookrightarrow F$ preserves not only the algebraic structure but also reflects the infinite nature of F; since $\mathbb{Q}$ is infinite and ι is injective, the image $\iota(\mathbb{Q})$ forms an infinite subset of F.

Second, the rational numbers in this example are what mathematicians call an example of a *prime field*. The prime field of F is the intersection of all subfields of F:

$$P = \bigcap_{\substack{K \subseteq F \\ K \text{ is a subfield}}} K.$$

This intersection is itself a field and is contained in every other subfield of F. Prime subfields are one of the central notions in the study of fields and field extensions [24]. In this case, the prime field of F is exactly the rational numbers $\mathbb{Q}$. So, not only does $\mathbb{Q}$ embed in every field with characteristic 0, it embeds in every subfield of F. This showcases how the embedding of the prime subfield provides a structural understanding of fields and field extensions. Hence, the infinite structure not only appears in every field with characteristic 0, but it also appears in every subfield of F. Hence, if F is an extension of $\mathbb{Q}$ (e.g., $\mathbb{R}$, or $\mathbb{C}$), this embedding ensures that F inherits a densely ordered structure if F is ordered (e.g., $\mathbb{R}$).

Furthermore, studying prime fields helps characterize fields based on their characteristics. This is because the prime field is the smallest subfield generated by the multiplicative identity 1_F, i.e., it includes all finite sums and differences of 1_F, along with their multiplicative inverses (when nonzero). Its structure depends on the characteristic of F. So, we have:

- If $\mathrm{Char}(F) = 0$, the prime field is isomorphic to the field of rational numbers $\mathbb{Q}$.
- If $\mathrm{Char}(F) = p$ (where p is prime), the prime field is isomorphic to the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

[24] See [Lan05], Chapter VII for more details.

Consequently, $\mathbb{Q}$ serves as the prime field in characteristic zero, playing a role analogous to $\mathbb{F}_p$ in characteristic $p > 0$. This underscores $\mathbb{Q}$ as the minimal algebraic structure common to all fields characteristic 0. Therefore, the embedding in question provides (to use a term used by Steiner [Ste78]) a “characterizing property” of fields with characteristic 0:

$$Char(F) \text{ is } 0 \text{ if and only if } \mathbb{Q} \text{ is canonically embedded in } F. \tag{5}$$

The main premise, i.e., that $\mathbb{Q}$ is canonically embedded in every field with characteristic 0, is generalizable in two ways: first, it generalizes to every field with characteristic 0. This ensures that the embedding condition is crucial because $\mathbb{Q}$ as a set is not a subset of every field with characteristic 0, but it embeds in every field with characteristic 0. Hence, the property necessary for any reconstruction of the proof is the “embedding of $\mathbb{Q}$ in every field with characteristic 0”, which amounts to a structural understanding of every such field (including the infinite structure).

The canonical embedding also generalizes to the case of algebraically closed fields. The same method, i.e., embedding the smallest field of a particular property in a family of fields, applies to the case of algebraically closed fields [25]. The inclusion of the prime field of algebraically closed fields is a significant fact in studies regarding algebraically closed fields, which explains many facts about the structure of algebraically closed fields. In addition, this proof explains the infinity of F by showing structural similarity between all of the algebraically closed fields with characteristic 0. This generalization is further illustrated in the following diagram:

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\iota} & F \\ {\scriptstyle \text{acl}}\downarrow & & \downarrow{\scriptstyle \text{acl}} \\ \widetilde{\mathbb{Q}} & \xrightarrow{\widetilde{\iota}} & \widetilde{F} \end{array}$$

I conclude that using the canonical embedding to prove the infinity of F is an explanatory proof, as it offers a structural understanding of why F is infinite, uniquely characterizes fields with characteristic 0, and provides a generalizable method that can be extended to other cases.

The situation described above is quite common in the practice of mathematics. To prove a theorem, especially in the case of solving an “open” question, i.e., a known conjecture or unsolved problem that garners some mathematician’s attention, they employ various additional lemmas, theorems, or facts that extend those appearing in the problem’s statement. Moreover, having proofs for a given theorem is more than providing additional support for a

[25] See more details in [Lan05], Chapter VII, Section 2.

known theorem. This aspect is relevant to discussions regarding mathematical explanation because each proof offers a unique understanding of mathematical facts. There are often multiple proofs for a single mathematical theorem. According to Rota, fully understanding a new theorem often manifests as a series of proofs, with each subsequent proof being more straightforward than its predecessor ([Rot97], p.192). The significance of the plurality of proofs and its role in metaphysical and epistemological questions regarding mathematical knowledge is understudied. Proofs not only prove the truth of a theorem but also offer diverse perspectives on it, highlight different aspects of a mathematical fact, and establish connections with various areas of mathematics. For instance, a mathematical fact is sometimes best studied using specific fields, such as topology or algebra, via the connections that mathematicians naturally establish between various theorems. Therefore, a *bona fide* theory of mathematical explanation that is faithful to the practice of mathematics should address this issue.

2.2 Dependencies in (Explanatory) Mathematical Proofs

Following a causal understanding of scientific explanation, assuming a dependency relation between a scientific explanation's *relata* is commonplace. For example, according to Koslicki, a successful explanation encapsulates or depicts an inherent real-world dependency between the phenomena mentioned in the given explanation ([Kos12], p.212). Similarly, Salmon identifies three primary types of scientific explanation—epistemic, modal, and ontic—with the ontic explanations showcasing how the explained phenomena integrate into natural patterns or regularities, typically understood as causal ([Sal84], p.293). D'Alessandro in [D20] proposes a more nuanced, perhaps stronger, thesis of dependence, suggesting that explanations mirror ontic dependence relations between the components of the *explanandum* and *explanans*. As the motivating idea of the separatist theory of grounding (to be discussed shortly in section 3.1) suggests, we seek a similar idea in a non-causal domain.

In mathematical explanations delivered by proofs, there is a dependency between the *explanandum* and the *explanans*. To illustrate, the truth of a conclusion depends on the truth of its premises; if any premise is false, the conclusion cannot be guaranteed by these premises. Thus, a true mathematical proposition that is considered the conclusion depends on other true propositions, as is common in proofs. Additionally, there is a more robust sense in which there is a dependency between the premises and the conclusion: the premises *make* the conclusion true. This type of dependency is best understood as a determination relation within a non-causal domain—the realm of mathematics: a determination relation between *relata* of the explanation. Once we assume the truth of the premises, the truth of the conclusion is determined. Therefore, the truth of the premises determines the truth of the conclusion, establishing a direct metaphysical link between them. On the other hand, the conclusion holds in virtue of the premises. To sum up, in a genuine case of mathematical explanation, a fact is the case in virtue of other facts mentioned in the premises.

Let us see an example to highlight the significance of the dependency relation between mathematical facts in a mathematical proof [26]. Consider the case of the unsolvability of the quintic polynomial equation discussed at length in Pincock's [Pin15] [27]. We know that the polynomial equation $ax^2 + bx + c = 0$, in general, is solvable by the following quadratic formula:

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a}, x = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

A similar but slightly more complicated formula exists for the polynomial equations of degree 3. These formulas only use plus, minus, multiplication, and radicals. When an equation is solvable only by these operations, we say that the polynomial equation is *solvable by radicals*. It was an open question whether polynomial equations of degree 5 or higher are also solvable by radicals. It turns out that the answer to the general case is negative. Galois's theory establishes the connections between the "solvability of a polynomial equation" and the "automorphisms of algebraic field extensions." Using these connections, roughly speaking, a polynomial equation is solvable if the extension contains the roots of the equation [28]. Galois theory showed that, in Pincock's terms, "What makes a given polynomial equation solvable, we should say, is that the Galois group is solvable" ([Pin15], p.11). Therefore, it illustrates how the solvability of a polynomial equation with radicals *depends* on the solvability of the Galois group. The dependency in question is also illustrated by the fact that the solvability of the Galois group determines the solvability of a given polynomial equation. Hence, the mathematical explanation is the result of revealing the relation between the mathematical fact of the solvability of a group and the solvability of a polynomial of a particular degree, rather than a mere conceptual relation between the propositions. To be more specific, the mathematical fact that explains another mathematical fact by determining it to be the case. So, the facts about the automorphism group of algebraic field extensions explain why a polynomial equation is unsolvable by determining it.

The discussion thus far can be encapsulated using a generalized framework for a dependence relation as outlined by Schnieder [Sch20a] [29] ("DPSC" for future reference):

$$((x\text{'s being true) depends on (}y\text{'s being true)) iff ((}x\text{ is true)} \otimes (y\text{ is true)).} \quad (6)$$

Within the context of this research, x and y represent mathematical facts, with x serving as the *explanandum* (the fact to be explained) and y as the *explanans* (the fact that provides the

[26] This example is a recognized case of an explanatory proof that demonstrates the metaphysical link of determination. One might argue that this case could also be approached through the conceptual theory. Although this discussion serves as an interesting case study, it lies beyond the scope of the current research.

[27] Steiner [Ste78] mentions this as a possible counterexample to his theory.

[28] See [Pin15], p.6 for more details.

[29] Here, we only use a simplified version of the dependency scheme. The original scheme is as follows: ((x's being F) depends on (y's being G)) if and only if ((x is F) $\otimes$ (y is G)). We replace F and G with "true", following the discussion on the determination relation between the *relata* of mathematical explanation.

explanation). The left-hand side of the DPSC posits that the *explanandum* is true in virtue of the *explanans*' being true. The main idea is that "$\otimes$" stands for a determination relation in a non-causal domain, such that fully understanding that such a determination relation is established between these mathematical facts provides one with answers to why one is true in virtue of the other.

Building on the previous discussion, we now set the stage for the rest of this paper. To develop a comprehensive account of mathematical explanation, we need to propose a specific relation to replace the placeholder "$\otimes$" in the DPSC scheme. This proposed relation should capture the essential idea that, in a genuine mathematical explanation, one mathematical fact determines another. Moreover, it should demonstrate how this determination relationship enhances our understanding of mathematical facts.

In the sections that follow, I will expand on this framework by introducing a version of the metaphysical grounding relation as a natural replacement for "$\otimes$".

3 Separatist Metaphysical Grounding and Mathematical Explanation

3.1 A Metaphysical Separatist Theory of Ground

In simple terms, grounding relations seek to address questions of the form "In virtue of what is it the case that X?" As a broad-brush picture, there are various concepts of grounding in the literature, with semantic, epistemic, and metaphysical notions being the most commonly recognized. According to semantic theories, grounding denotes a semantic relationship between sentences [30]. On the other hand, metaphysical grounding theories encompass a range of theories that primarily interpret grounding as a metaphysical relation between facts. The present research deals with the grounding relation as a metaphysical relation. As a reminder, we do not distinguish between a true proposition and a fact. So the *relata* of the grounding relation are considered to be either true propositions or facts. Moreover, the grounding relation in the current research is assumed to be factive. However, metaphysical theories of grounding relations do not form homogeneous views. While some ground theorists have formulated ground claims using the sentential operators [31], some other ground theorists consider the ground as a relation between true propositions or facts [32]. Following Schnieder [Sch20b], I adopt the following constraint (which he refers to as the "factual constraint") on the version of the grounding that I employ:

> Grounding is a relation the *relata* of which consists of either true propositions, or facts. (7)

[30] See [Smi20] for more details on conceptual versus metaphysical grounding.
[31] See [Fin12] as an example.
[32] See [Lit23], Section 2.1 for some of the variations of such views.

Let $[C]$ be a fact. Say that $[A]$ is a *full ground* for $[C]$ if $[A]$ alone is what it is in virtue of which $[C]$ obtains. Say $[B]$ a *partial ground* for $[C]$ if $[B]$ with some other fact $[D]$ will be a full ground for $[C]$. Another distinction is between *mediate* and *immediate* ground. Consider the fact $[A \wedge (B \wedge C)]$. As the names suggest, the collection $[A], [B \wedge C]$ is the immediate ground for $[A \wedge (B \wedge C)]$, while the collection $[A], [B], [C]$ is the mediate ground for $[A \wedge (B \wedge C)]$. For the rest of the paper, until explicitly mentioned, by ground, I will mean an immediate full ground.

On the other hand, it is common to assume that grounding relation and explanation are intertwined [33]. Call the explanation that is conveyed by a genuine case of grounding, *grounding explanation*. Two main frameworks address the connection between grounding explanation and grounding. According to *unionism*, a ground is identical to its explanation. According to the other view, called *separatism*, a ground is different from the grounding explanation [34]. The grounding relation and the grounding explanation are connected with another relation called *backing*. However, the nature of the backing relation is a matter of some debate. Some theorists define backing in terms of representation [Tro18], while others use the explanation itself to clarify the backing relation [Kov20]. Finally, according to Poggiolesi and Genco [PG23], a grounding relation backs an explanation when one adds to the grounding relation the generalization, which allows one to link the grounds to the conclusion. For my research, I stick to the intuitive meaning of it. So, if $[A]$ grounds $[B]$, the grounding explanation it backs is $\langle B$ in virtue of $A\rangle$. Here is an example of metaphysical grounding by Bliss and Trodgon in [BT21]:

$$\text{[The truckers are picketing]} \prec \text{[The truckers are striking]}. \tag{8}$$

The fact that truckers are picketing determines the fact that the truckers are striking. On the other hand, the truckers are striking in virtue of the fact that they are picketing. This is the proposition that is backed by the grounding relation presented above:

$$\langle \text{The truckers are striking in virtue of truckers' picketing} \rangle. \tag{9}$$

Here, the property of striking is realized and determined by picketing on this occasion. What is backed is the fine-grained proposition 9 that is backed by the determination relation expressed by 8. Note that the instance of the grounding relation expressed by 8 is not conceptual because the mere conceptual analysis of the proposition expressed by "Truckers are picketing" does not yield the proposition "Truckers are striking". However, the property expressed by the latter is realized and hence determined by the property expressed by the former.

The separatist theory adopted in the present research is minimal and governed by only a handful of principles, as stated below. First is the definition of ground, which is a metaphysical form of determination as presented in Lange [Lan19]. This view serves as the official

[33] See [Gla20] for more details.

[34] See [Sch16] for a detailed separatist view of grounding.

definition of ground and as a criterion using which we evaluate what should be considered as a ground:

"A fact's grounds are whatever it is in virtue of which that fact obtains, and a truth-bearer (such as a proposition) is grounded in its truth-makers." (10)

Moreover, most ground theorists consider the grounding relation to be transitive. So, we assume it here as well (a ground is denoted by "$\prec$"):

$$\text{If } [P] \prec [Q] \text{ and } [Q] \prec [R], \text{ then } [P] \prec [R]. \tag{11}$$

Finally, we have the following assumption, based on the discussion above about the backing relation:

$$\text{If } [P] \prec [Q], \text{ then } [P] \prec [Q] \text{ backs } \langle Q \text{ in virtue of } P \rangle. \tag{12}$$

As a remark, by assuming 10, 11, and 12, and assuming that $[P] \prec [Q]$ and $Q \to R$, one can not immediately conclude that $[P] \prec [R]$ [35]. In other words, a ground should be established and properly understood to provide an explanation. Therefore, it is not to be identified with logical entailment; neither is it defined in terms of it.

The version of the grounding relation presented here is a weakened version of the conceptual relation suggested by Poggiolesi and Genco in [PG23] and by Poggiolesi in [Pog23]. Because every case of conceptual grounding is a case of metaphysical grounding, but the example above, while an example of metaphysical grounding, is not a case of conceptual grounding. Litland in [Lit23] has leveled objections to Poggiolesi and Genco's version of grounding, and I will not rehearse these objections here. However, the present, more robust version of the grounding relation is immune to many of these objections, especially to a major one. Litland's "commonality objection" [36] says that Poggiolesi and Genco's version of grounding does not say what is common to all instances of grounding. Just as one cannot define "color" as a bundle including red, yellow, blue, etc., we cannot define grounding by enumerating the instances. We should identify what is common to all instances of grounding. The present view addresses this issue: what is common in all instances of grounding is the determination of the relation between the grounds and the grounded.

With this understanding of grounding, I argue in the next section that the version of the grounding relation mentioned before is a natural suggestion for a mathematical explanation.

3.2 Mathematical Explanation and Metaphysical Grounding Relation

To show that the grounding explanation is a good candidate for the mathematical explanation, one should show that the grounding relation satisfies the objective aspect of the

[35] See [Lit23] for a detailed critical review of defining grounding in terms of entailment.

[36] See [Lit23], Section 4.2.

mathematical explanation discussed earlier. To be more specific, one should argue that the version of the grounding relation previously discussed accounts for the dependency between the *relata* of the explanation. Several philosophers have analyzed factual dependence—that is, when one fact owes its being the case to other facts, or when a true proposition owes its truth to other true propositions— through the lens of grounding, interpreting it as a relation where one truth depends on another. According to Schierder in [Sch20b], p. 99, "in the recent debate, factual dependence has usually been discussed by the name of 'grounding', see, e.g., Rosen (2010), Correia (2010), and Fine (2012a), and other papers in Correia and Schnieder (2012a)". This grounding relation is typically seen as asymmetric and transitive, providing a robust framework for capturing metaphysical dependence between true propositions or facts, which aligns with the assumptions about the grounding relation in Section 3.

To begin with, the grounding relation is primarily understood as a determination relation in a non-causal domain. So, as a candidate for what provides a mathematical explanation, it should address the fact that the items appearing in the *explanans* determine the facts appearing in the *explanandum*. Hence, the grounding relation, if replaced with "$\otimes$" in DPSC, should provide, first, a sort of dependency. Recall that DPSC says ((x's being true) depends on (y's being true)) if and only if ((x is true) $\otimes$ (y is true)). Let us replace "$\otimes$" with the grounding relation denoted by "$\prec$", with $P :=$ "x is true" and $Q :=$ "y is true". We have:

$$[Q] \text{ depends on } [P] \text{ if and only if } [P] \prec [Q]. \tag{13}$$

So, the left-hand side of the 13 is a version of the "in-virtue-of" relation. What the left-hand side of 13 says is that the truth of the proposition $[Q]$ depends on the truth of the proposition $[P]$. So, $[P]$ is whatever it is in virtue of which $[Q]$ obtains. By the definition of ground 10, $[P]$ is a ground for $[Q]$. In addition, if the grounding relation on the right-hand side holds, then $[Q]$'s being the case depends on $[P]$'s being the case (or, if considered as true propositions, the truth of the former depends on the truth of the latter). Therefore, we can replace the left-hand side with the more familiar "in-virtue-of" relation. I conclude that grounding dependence is a *bona fide* dependency relation.

$$\langle Q \text{ in virtue of } P \rangle \text{ if and only if } [P] \text{ determines } [Q]. \tag{14}$$

The grounding relation, as understood in this research, satisfies two notions of dependency discussed in Section 2.2. On the one hand, there is a weaker notion, which I call *negative dependency*: in general, if one of the premises of a proof is false, then the conclusion does not follow. This captures the idea that the truth of the conclusion is conditional on the truth of the premises.

On the other hand, there is a stronger notion, which I call *positive dependency*. According to this notion, the facts stated in the premises *make* the other fact to be the case.

That is, the conclusion owes its truth to the truth of the premises. This more robust kind of dependency, referred to by Schnieder [Sch20b] as factual dependence, is central to the view that metaphysical grounding offers a suitable framework for understanding mathematical explanation.

The metaphysical grounding relation satisfies both negative and positive dependencies in proofs. First, according to the factual constraint 7 introduced earlier, the grounding relation (as understood here) holds *only* between true propositions or facts. That is, if a proposition is false, it cannot serve as one of the *relata* in a grounding relation. This ensures that the grounding relation respects negative dependency: the falsity of a premise rules out the grounding relation. Second, metaphysical grounding is understood as a relation of determination between mathematical facts. In such a relation, the grounds not only support but determine the grounded fact—they make it the case. Thus, the grounding relation also satisfies positive dependency, since the truth of the grounded fact depends on, or is made to hold by, the truth of what appears as grounds. In sum, metaphysical grounding—as a factive determination relation—captures both the negative dependency and the positive dependency as a variety of factual dependence.

In addition, this view aligns with what Roski refers to as "explanatory realism" in [Ros21]—a view rooted in the works of philosophers like Jaegwon Kim and David Lewis [37] —which posits that information is genuinely explanatory only if the *explanans* involves entities in a specific determination relation to those addressed by the *explanandum*. Importantly, this determination relation is metaphysical and holds between facts; it involves a substantive link where the truth of one proposition, or a mathematical fact, depends on another. This occurs through relations that establish a metaphysical link between the *explanans* and the *explanandum*, reinforcing the idea that mathematical facts are interconnected in a genuine case of mathematical explanation.

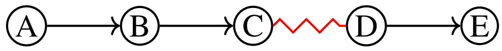
The proposal states that a proof is explanatory if it traces the truth of the proposition being proved back to the truth of the premises, ensuring that the relevant why questions are answered at each step. In terms of grounding, a proof is explanatory if it reveals the grounds for the fact being explained, so there is no explanatory gap in the chain from the explaining facts to what is being explained. This illustrates what I call an explanatory chain. An explanatory chain is made up of a series of grounding relations that, at each step, answer the relevant why questions and determine the next step. Here is the scheme for an explanatory chain, where every node shows a fact and the edges represent the grounding relation between them.

$$\textcircled{A} \longrightarrow \textcircled{B} \longrightarrow \textcircled{C} \longrightarrow \textcircled{D} \longrightarrow \textcircled{E}$$

It contrasts with cases in which a proof fails to account for a relevant why question or includes an explanatory gap. In these cases, while the grounds may be noted, they are not

[37] See [Kim88], and [Lew86]. See Roski's [Ros21], Footnote 3, for a list of previous accounts of explanatory realism.

revealed [38]:



The account presented here addresses the situations discussed in Subsection 2.1. Consider the example discussed in subsection 2.1. The proof via embedding for the Fact 2.1.1 uses notably different sets of concepts from those appearing in the problem's statement. the proof proves the infinity of F using the fact that each field with characteristic 0 contains a copy of rational numbers denoted by $\overline{\mathbb{Q}}$. However, as discussed earlier, the notion of "canonical embedding," which plays a crucial role in the proof of the infinity of F, does not appear in the conclusion.

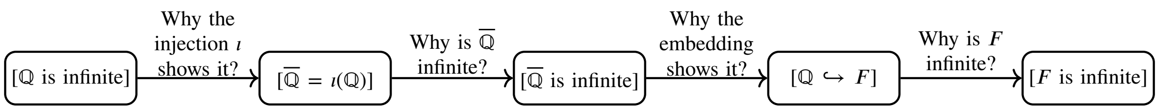
As a reminder, $[\mathbb{Q} \hookrightarrow F]$ denotes the fact that the set of rational numbers is embedded in the field F. We have:

$$[\mathbb{Q} \hookrightarrow F] \text{ determines } [F \text{ is infinite}]. \tag{15}$$

By the definition of ground, the above cases represent genuine grounding relations. So we have:

$$[\mathbb{Q} \hookrightarrow F] \prec [F \text{ is infinite}]. \tag{16}$$

Moreover, the determination relation expressed by 15 is established in a way that provides answers to the why-questions regarding the infinity of F. For example, the main question is "Why is F infinite?", and the answer is "Due to the canonical embedding of the rational numbers in F". Again, consider the relevant question, "Why does the canonical embedding of rational numbers show that F is infinite?", and the answer is "Because $[\mathbb{Q} \hookrightarrow F]$ shows that the image of $\mathbb{Q}$ in F, i.e., $\overline{\mathbb{Q}}$ is infinite". Finally, "Why is $\overline{\mathbb{Q}}$ infinite?", and the answer is "Because it is an isomorphic copy of $\mathbb{Q}$, and $\mathbb{Q}$ is infinite." This exemplifies the explanatory chain in the explanatory proof of the fact that F is infinite if $Char(F) = 0$ using the canonical embedding of the rational numbers in F. The explanatory chain of the explanatory proof discussed is depicted in the following diagram:



The statement 16 also showcases the dependency relation between the fact $[F \text{ is infinite}]$ and $[\mathbb{Q} \hookrightarrow F]$. In the sense that they show that the truth of the fact under study (i.e., $[F$ is infinite$]$) is dependent on the truth of the grounds. By doing this, as per Statement 14, these grounds provide the following representational elements:

$$\langle F \text{ is infinite in virtue of the fact that } \mathbb{Q} \text{ is canonically embedded in } F\rangle. \tag{17}$$

[38] See the example of the proof of the Fundamental Theorem of Algebra in this section for an example that the chain includes explanatory gaps.

One of the merits of the current account of the mathematical explanation in terms of the grounding explanations is the focus on the existence of a determination relation regardless of the concepts used in the immediate scope of the theorem's statement; this approach to mathematical explanation is "cross-categorical," to use a term used by Raven in [Rav12]. In other words, the present theory supports the view that mathematical proofs that offer genuine explanations are not always "pure proofs" [39], a term referring to the proofs that only employ the concepts that are occurring in the theorem's statements. However, as discussed in [Lan19], many pure proofs fall short of offering explanations as much as the impure proofs do. The view advocated here suggests that while purity may be a goal in the explanations that mathematicians seek to find (it could be a measure of the merit of an explanatory proof), it is not a prerequisite for a proof to be explanatory. In some cases, the items appearing in the theorem and the items in the proof explaining it belong to different mathematical categories, i.e., these elements are not of the same nature. In some cases, a topological proof explains an algebraic fact or a semantic fact explained via purely syntactic methods.

Let us illustrate this aspect via an example. Steiner has addressed the example. He says, "Chang and Keisler, to cite two more logicians, propose to 'explain' preservation phenomena ... "just by the syntactical form of the axioms." ([Ste78], p.135). Chang and Keisler in [CK90], p.147, talk about "preservation theorems," the general scheme of which is T is preserved under X if and only if the set of models of T is closed under Y. To be more exact, let T be a first-order $\mathcal{L}$-theory. The goal is to find necessary and sufficient criteria that explain why the T set of $\mathcal{L}$-models of T is closed under a condition X. As an example of the preservation phenomena, an $\mathcal{L}$-theory T is said to be *preserved under submodels* if any $\mathcal{L}$-submodel $\mathcal{N}$ of an $\mathcal{L}$-model $\mathcal{M}$ of T, is a model of T. As another example, an $\mathcal{L}$-theory T is *preserved under homomorphisms* if the for any $\mathcal{L}$-model $\mathcal{M}$, and an $\mathcal{L}$-homomorphism h, $h(\mathcal{M})$ i.e., the homomorphic image of $\mathcal{M}$ under h is an $\mathcal{L}$-model of T. The following is Theorem 3.2.2 of [CK90]:

Theorem 3.2.1. *Let T be a first-order $\mathcal{L}$-theory, then T is preserved under submodels if and only if T has a set of universal axioms.*

Proof. See [CK90], proof of the Theorem 3.2.2. □

Let us examine Theorem 3.2.1 more closely. If T has a set of universal axioms, it is easier to see that T is maintained under submodels. The main point is to demonstrate that if T is preserved under submodels, then T has a universal set of axioms. Therefore, we have two mathematical facts: the preservation phenomenon under submodels and the existence of a universal axiomatization. Note that these two facts are different in that the first is purely semantic, while the second is purely syntactic.

[39] See Detlefsen and Arana's paper [DA11], and Detlefsen's work [Det08b] for an overview of the concept of purity of proofs.

The theorem above shows that these two facts are connected. If T has a universal axiomatization, then T is preserved under submodels, and if T does not have a universal axiomatization, then T is not preserved under submodels. So, the syntactic fact determines whether or not the semantic fact occurs. Hence, it is in virtue of the universal axiomatization that the preservation phenomenon occurs, so we have:

$$[T \text{ has universal axiomatisation}] \prec [T \text{ is preserved under submodels}]. \tag{18}$$

Theorem 3.2.1 shows that there is a determination relation between these mathematical facts. The explanation that is backed here is that ⟨T is preserved under submodels in virtue of the fact that T has a universal axiomatization⟩. Moreover, this is a genuine case of mathematical explanation when the following lemma is fully incorporated into the proof:

Lemma 3.2.2. *Let T be a first-order $\mathcal{L}$-theory, and Γ, a set of $\mathcal{L}$-sentences such that Γ is closed under finite disjunctions. Then, T has a set of axioms $T_0 \subset \Gamma$ if and only if for any $\mathcal{L}$-model $\mathcal{M} \models T$, and every $\mathcal{L}$-sentence $\phi \in \Gamma$ such that $\mathcal{M} \models \phi$, then $\mathcal{N} \models \phi$, where $\mathcal{N}$ is any $\mathcal{L}$-model, then $\mathcal{N} \models T$.*

Proof. See [CK90], proof of Lemma 3.2.1. □

Finally, the present account of mathematical explanation provides ways to understand mathematical research better by adopting the ground plurality. So, along the theses 10, 11, and 12, I adopt the following thesis, which states that a single mathematical fact can be multiply grounded. I call this assumption the "Ground Plurality Thesis":

> A single mathematical fact can have multiple grounds, each offering a unique explanation. (19)

The Ground Plurality Thesis emphasizes the complexities involved in mathematical research and aligns more closely with how mathematicians actually carry out their work. Further exploration of this thesis and its role in enhancing our understanding of research practices is beyond the scope of this study.

However, merely showcasing the ground connections is not sufficient for a proof to be considered explanatory. The proof must illustrate how the facts within the *explanandum* depend on those within the *explanans*. In other words, a ground-revealing proof should trace the truth from the propositions about the *explanandum* all the way back to the truths concerning the *explanans*. This transition must enable one to answer the relevant why questions regarding what is proved.

Let us illustrate this with an example in which a mathematician knows a theorem has been proved but is unsure *why* the theorem holds (i.e., the explanation is not fully known to the mathematician). Indeed, in many cases, while conducting mathematical research, mathematicians "quote" some theorem with no further explanation. In some cases, the "quoted theorem" plays a significant role in the practice of mathematical research, yet some mathematicians are unsure about the proof of the quoted theorem itself. Consider the case of the Fundamental Theorem of Algebra. For simplicity, we only state the case where the coefficients are in the complex numbers $\mathbb{C}$:

Theorem 3.2.3. *(Fundamental Theorem of Algebra) Every polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

The proof of the Fundamental Theorem of Algebra 3.2.3 can be approached from various angles, including topological methods, analysis, or algebra. One of the straightforward proofs for the Theorem 3.2.3 comes from Liouville's Theorem [40]. Liouville's Theorem states that:

Theorem 3.2.4. *(Liouville's Theorem) Let f be an entire function (analytic everywhere in the complex plane) and bounded. Then, f is constant.*

According to Ahlfors in [Ahl79], p.122, Liouville's Theorem 3.2.4 provides an almost trivial proof for the Fundamental Theorem of Algebra 3.2.3. Let us review the proof sketch:

Proof. Let $p(x)$ be a polynomial equation over $\mathbb{C}$ such that for all $x \in \mathbb{C}$, we have $p(x) \neq 0$. Then, consider $p(z) = 1/p(x)$, which is well-defined and analytic as $p(x)$ is never 0. Again, as $x \to \infty$, then $|p(x)| \to \infty$ and hence $p(z) \to 0$, showcasing that $p(z)$ is bounded. Using Liouville's Theorem 3.2.4, $p(z)$ is a constant equation, which is a contradiction. So, $p(x)$ should have a solution for some $x \in \mathbb{C}$, proving the Fundamental Theorem of Algebra 3.2.3. □

The proof sketch presented above does not fully explain why the Theorem 3.2.3 holds, as it leaves the whole proof relying on a "black box" (i.e., Liouville's Theorem 3.2.4), a theorem that plays a significant role in the flow of the argument — a significant step in the informal proof, but it is not explained. Therefore, we have a case in which we know that a theorem holds, but we cannot address the why questions regarding why it holds. The situation above is compatible with the theory of mathematical explanation advocated here. Knowing that a ground exists does not imply knowledge of the ground itself; showing mere grounding connections does not guarantee a ground-revealing proof. One needs to fully understand the ground in order to grasp the explanation that it backs. However, proofs using quoted theorems only ensure the existence of the ground. Nevertheless, to understand the ground, i.e., to understand the representational elements that the ground backs, one should identify the elements that play a significant role in the proof. In our case, we know a metaphysical relation exists between the items in the proof and the theorem proved. However, we do not fully understand the ground because a significant part is not understood. In such a

[40] See Ahlfors' [Ahl79], p.122.

case, once we have a proper understanding of Liouville's Theorem 3.2.4, we can claim a full understanding of the ground and, consequently, the proof using Liouville's Theorem 3.2.4. To sum up, mere knowledge of the existence of (even a proper) ground does not guarantee a full understanding of the ground itself. As per separatism thesis (i.e., Statement 12), while for the former, one only needs to trust the peers or have a general view of the theorem, for the latter, one needs to understand the significant elements of the ground and how these elements contribute to determining the result.

This example demonstrates that simply revealing grounding connections between the grounds and the grounded is not sufficient to make a proof explanatory. An explanatory proof should properly reveal the grounds, ensuring that the transitions between them are seamless and without unexplained gaps. While highlighting these foundational connections might suggest to mathematicians that an explanation could be constructed using a particular method, it does not mean that the explanation has actually been provided. Only when the method is fully developed—with the consecutive steps shaping an explanatory chain—can we move from the grounds to what is grounded, thereby delivering the explanation; an explanatory proof is one that reveals the ground in a continuous way, i.e., the explanatory chain starting from the facts that are about to explain the fact that is supposed to be explained should answer the why questions. So, while the above proof sketch can ensure a mathematician that there exists an explanation for the Fundamental Theorem of Algebra using Liouville's Theorem, it does not mean that we have the explanation via this proof.

4 Conclusion

In this study, I aimed to formulate a theory of mathematical explanation through the lens of a particular metaphysical grounding theory. I argued that we need a determination relation captured by a separatist theory of metaphysical grounding. A separatist approach to metaphysical grounding emerges as an intuitive candidate for what is typically recognized as a genuine mathematical explanation. This theory acknowledges that various proofs contribute different kinds of explanations to the mathematics being examined, each with its own explanatory merits (without implying one is the best explanation). This diversity reflects the inherent diversity found within the practice of mathematical research.

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