No choice, no history

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Abstract

This paper investigates histories in Branching Space-Time (BST) structures. We start by identifying necessary and sufficient conditions for the existence of free histories, and then we turn to the intangibility problem, and we show that the existence of histories in BST structures is equivalent to the axiom of choice, yielding the punchline "history gives us choice".

1 Introduction

The question that we address in this paper is how to rigorously construct possible worlds or histories in philosophy. To answer this question one needs a rigorous modal framework: only then the construction of possible worlds/histories can be examined. Unfortunately, almost all modal frameworks in philosophy that we know (e.g., concretism of D. Lewis, abstractionism of R. Adams, A. Plantinga, and R. Stalnaker, or combinatorialism of D. Armstrong[1]) fall short of this standard of rigor, one (known to us) exception being the theory of Branching Space-Times (henceforth BST), launched by Belnap (1992). We thus investigate BST histories; we believe that our results will carry over to some other modal frameworks in philosophy, once they are appropriately formalized.

Our investigations of BST histories revolves around the question of intangibility: are BST histories intangible? To put it first informally, intangible objects can be defined, but no examples of them can be given constructively, though there is an abundance of them. More formally, typically a borderline between tangible and intangible properties is determined by the axiom of choice (AC):

[1] We took this categorization of possible world theories from Menzel (2025), where one can read more about these theories.

the definition of a tangible property does not require AC, in contrast to an intangible property, which requires AC in its construction.

Intangibility brings to the fore the question whether one can know an intangible possible world / history. Clearly, since these objects are very large, we cannot know them in every detail. Yet, intangibly arguably brings in an extra layer of unknowability: one cannot, in principle, display an example an intangible possible world / history. In this strong sense, one cannot know an an intangible possible world / history. The intangibility of possible worlds/ histories would lead to pessimism about the explanatory values of philosophical analyses carried out in terms of possible worlds/histories (there is an abundance of such analyses).

Our paper is organized as follows. The next section 2 sets the stage of our investigations. Then, in section 3 we recall the main ideas and constructions of Branching Space-Times: the later witness criterion of compossibility, the concept of histories, and the axiomatic basis of BST. In Section 4 we offer a short overview of maximal ideals and ultrafilters over sets, which will serve as a background to our investigations of maximal ideals on posets and, in particular, BST posets, to which we turn in Section 5. In Section 6 we turn to the question of intangibility of histories, where we display a particular BST structure and prove that the existence of BST histories is equivalent to the axiom of choice. A discussion and our conclusions are in Section 7.

2 Setting the stage

It is hard to imagine current logic and philosophy without the concept of possible worlds/histories: if anything has re-defined logic and philosophy over the last fifty years or so, it has been the invention and subsequent popular use of this concept in the two disciplines, although the concept has been employed somewhat differently in them. Possible worlds of logic are frugal: they are typically identified with "points" in which sentences have truth values, i.e., are true or false. As for the riches of possible worlds of philosophy, only the sky seems to be the limit. They are inhabitable by various entities (events, processes, and enduring objects), might be equipped with a temporal or a spatial structure (or just a spatiotemporal structure), its objects might satisfy, or might violate, laws of the natural sciences, as we know them. Some objects might be odd or unearthly ones, like wizards, zombies, brains-in-vat, or specters of any kind. Yet, in themselves, these unfathomable riches are not objectionable, as philosophy aims to analyze various notions of possibility, from real possibility to mere conceivability, and these notions call for resources of different richness. In what follows, however, we focus on stricter variants of modality: real possibility, natural (physical) modality, and metaphysical modality, and accordingly, leave aside the consideration of some rich possible worlds.[2] It is these stricter notions that offer a hope of specifying conditions under which something belongs to a possible world.

To comment on a difference of terminology, (i.e., worlds or histories), it is important in some quarters of modal metaphysics, but we will ignore it in the present paper. Roughly, the controversy is whether possible worlds are *modally thin*, meaning that modalities are not represented in a single possible world, but by relations between them, or are *modally thick*, that is, there are some structures within a world that represent modalities. In our usage in this paper, both histories and worlds are assumed to be modally thin.[3]

After these preliminaries, we come to an issue to which Nuel Belnap contributed: how are possible worlds (histories) to be introduced, or constructed? A dominant way of introducing these objects that philosophers follow appears to go roughly like this. One begins with a set of possible entities of a sort, call it a base set, after choosing which notion of possibility is to be modeled and what category of entities are handy and adequate for the purpose. One next assumes a condition, call it a compossibility condition, which specifies when two objects can occur together, that is, are *compossible*. As a final move, one needs to prove that there are maximal subsets of compossible objects in the base set of possible objects. If there are such maximal subsets, one identifies them with possible worlds/histories.

A succinct overview of compossibility criteria might help to illustrate this approach. Leibniz, who pioneered the compossibility approach, is usually interpreted as identifying composibility with logical consistency: two objects are compossible iff the supposition of their joint existence is logically consistent.[4] Lewis' (1986, p. 208) criterion, as read from his concept of the demarcation of

[2]For an illuminating classification of modalities, see e.g., Rumberg (2020).

[3]To give two examples, a Lewisian possible world is modally thin, whereas Our World of Belnap's Branching Space-Times is modally thick, and comprises histories which are modally thin structures.

[4]The consistency interpretation was assumed by Mates, Hintikka, Rescher and others. Whether consistency is to be understood as non-contradictoriness is discussed by Messina and Rutherford (2009). For other interpretations that have some currency today see Brown and Chiek (2016).

possible worlds, says that two objects are compossible (share a possible world) if they "stand in suitable external relations, preferably spatiotemporal." Proponents of modal combinatorialism assume, after Armstrong (1989) that any pattern of instantiation of any fundamental properties and relations is metaphysically possible. More demanding is compatibilism, which claims that two objects are compossible iff their joint occurrence is not prohibited by the laws of nature. A variant of this idea says that a pair of objects is compossible iff the laws of nature assign to it a non-zero probability. In the non-relativistic approach of Prior (1967), two events are compossible iff they form a chain, that is, one event is above the other with respect to some temporal-modal partial ordering. To finish this snap overview with Belnap's (1992) idea, working in a relativity-friendly framework of Branching Space-Times (BST), he based his criterion on the intuition of later witness: if after two events a third event might occur, then the former two events are compossible. The bulk of this paper is a discussion of what concept of history emerges from the criterion based on the later witness intuition.

Turning to the topic of our paper, it builds upon a mathematical observation that Belnap's criterion delivers histories that are maximal ideals in a partially ordered set (poset). For, in BST the collection of possible objects is a poset, the elements of which are partially ordered by a relation interpreted as a spatio-temporal-modal ordering. We look at BST histories as maximal ideals on a poset, investigating in particular how BST axioms further delineate them. Our focus is free ideals on a poset.[5] In the context of ideals over *sets* (rather than posets) maximal free ideals are sometimes classified as intangible objects: they can be defined, but no example of them can be constructed, though there is an abundance of them. This predicament reflects a discrepancy between criteria of definability and criteria of constructibility. Clearly, there is a considerable debate in the philosophy of mathematics regarding which procedures are constructive, and which are not. In the tangibility / intangibility controversy, a typical way of drawing the line points to the two maximalisation principles, axiom of choice (AC) and principle of dependent choice (DC): constructive procedures are taken to include DC but not AC. Accordingly, a property is tangible if it can be defined in Zermelo–Fraenkel set theory with DC (ZF+DC). Otherwise, it is intangible. The question that we address in this paper is whether being a BST history is an intangible property. One might be worried by the

[5] An ideal is free if it is not the downset of any element of the poset.

prospect of possible histories turning out to be intangible, while raising no objection to the use of AC in mathematics. Possible histories are intended to systematize and clarify our intuitions about possibilities and necessities. By their global character, they are not fully epistemically accessible. Intangibility seems to raise the inaccessibility bar to the extreme. If our modal intuitions are encoded in possible histories that are intangible, we cannot, in principle, learn what is possible and what is not.

3 Branching Space-Times

3.1 The later witness criterion of compossibility

BST is a formal theory of modal and local indeterminism, with the underlying kind of possibilities, called real possibilities, playing out in a rudimentarily relativistic spacetime. It extends Prior's (1967) theory of Branching Time (BT) by accounting for spacial and relativistic features of histories.[6] BST has been applied to analyze Bell-type arguments in quantum mechanics as well as some issues in general philosophy, like causation, flow of time, or causal probabilities – see Belnap et al. (2022). Belnap (2011) applied BST to analyze agency in spatio-temporal settings as well.

We rehearse now the initial steps of the construction of a BST structure. For the set of objects, Belnap takes the set of events, motivated mostly by reasons of simplicity. Examples of events include a given photon impinging upon a smooth screen, or Leszek's winking a moment ago (each idealized to be point-like). To produce the required set of events, however, Belnap starts with an indexically-given event, like for instance, Zalan's blink now (he is blinking now). The construction then needs a distinction between what is *really* possible and what is possible, but not really. To illustrate, it is possible for the two us, Zalan and Tomasz, to go for a beer tonight in Kraków as well as in Budapest, but only the Kraków option looks like a real possibility, as it is already late afternoon, and we are both in Kraków (and thirsty); yet the Kraków-Budapest journey takes at least four hours; we will not really able to make it to Budapest before late evening. (We recommend that the reader formulate their own examples of real possibilities). Having the notion of real possibility at hand, starting

[6] BT structures were first discussed in an exchange between Saul Kripke and Arthur Prior in the late 1950s (see Ploug and Øhrstrøm (2012)), and then published in Prior (1967). For more on BT consult Belnap et al. (2022, p.99).

with a given indexically-specified present event, we consider all events that are really possible at this present event, the events that will be really possible in its future and events that were once really possible in its past. The totality of all such events is non-empty, and Belnap calls it *Our World*, denoting it by $\mathcal{W}$. The notion of real possibility gives rise to an ordering of $\mathcal{W}$ by a modally-spatiotemporal relation $<$, with $x < y$ interpreted as saying that y can really happen in the future of x. Arguably, this relation is irreflexive and transitive (consequently, it is anti-symmetric). With the remaining BST axioms added, it turns out that $\mathcal{W}$ has some cohesion: provably, any two events of $\mathcal{W}$ can be linked by an M-zigzagging chain founded on the $<$ ordering (see Belnap (1992, Fact 14)).

Yet, even without appealing to these axioms we have all the resources we need to introduce histories, just by appealing to $\mathcal{W}$ being pre-ordered, and hence partially ordered (the partial ordering $\leqslant$ is standardly defined as $x \leqslant y$ iff $[x < y$ or $x = y]$. In line with the general method alluded to above, we need a compatibility criterion that would provide us with sufficient and necessary conditions for two events being jointly possible. Belnap's ingenious idea was the intuition of a later witness: if there is a later witness to the two events x and y, then x and y are compossible. More precisely, elaborating on the meaning of $<$, if there is an event z that can really happen after x and can really happen after y, then x and y can occur together. A later witness z offers a vantage point from which the two events, x and y, are seen as occurring jointly in z's past. If z is possible (and it is since it belongs to Our World), then x and y are jointly possible. We thus take it as uncontroversial that the later witness criterion provides a sufficient condition for the two events being compossible, that is, if there is a later witness to the two events, then they are compossible. However, the opposite direction is not adequate in some contexts, see e.g., Müller (2014). A simple counterexample is a half-plane equipped with the Minkowskian ordering,[7] which is a BST structure, intuitively understood as modeling exactly one history. Now, two events close enough to the verge of this half-plane do not have a later witness, so they are non-compossible by the criterion, contrary to the intuition that there is just one history, the half-plane, to which they both belong. The later witness criterion does not provide the necessary condition for compossibility, nor does it on some BST structures based on general relativistic models. The reaction to these findings is to restrict the applicability of BST

[7]For Minkowskian ordering $<_M$, $x <_M y$ iff y is in the future light-cone of x and distinct from x.

to the special theory of relativity, and generalize the criterion before venturing into more complex settings.[8] In this spirit, we thus formulate the criterion as providing necessary and sufficient conditions for two events being compossible, to be valid in BST, the application of which is appropriately curtailed:

Definition 1 (later witness criterion)**.** Let $\mathcal{W} = \langle W, < \rangle$ be Our World, where $<$ is a pre-order and W is nonempty. Then $x, y \in W$ are compossible iff there is $z \in W$ such that $x \leqslant z$ and $y \leqslant z$, where $\leqslant$ is the partial order on W derived standardly from $<$.

Compossibility naturally associates to the notion of histories, by means of equivalence: two events are compossible iff they share a history. To define histories, however, we need an auxiliary notion of upward directed subsets of a poset: we say that $E \subseteq W$ is an upward directed subset of W, where $\mathcal{W} = \langle W, \leqslant \rangle$ is a poset iff for all $e_1, e_2 \in E$, there is some $e_3 \in E$ such that $e_1 \leqslant e_3$ and $e_2 \leqslant e_3$.

Finally, in line with a longstanding tradition, we require a history to contain *all* compossible events, that is, to be maximal in a sense. As in the BST context, maximality is based on inclusion, the following definition ensues:

Definition 2 (BST histories)**.** Let $\mathcal{W} = \langle W, < \rangle$ be non-empty strict partial order that satisfies the BST axioms.[9] Then $h \subseteq W$ is a history in $\mathcal{W}$ iff h is a maximal upward directed subset of W, i.e., if h' is an upward directed subset of W and $h \subseteq h'$, then $h = h'$. The set of histories in $\mathcal{W}$ is denoted by Hist.

By the Zorn-Kuratowski lemma (ZK), there is at least one maximal upward directed subset in $\mathcal{W}$ – see Belnap (1992). Depending on what $\mathcal{W}$ is like, it might comprise multiple histories, or not, in which case it is identical to a single history. Intuitively, in the former case, Our World is indeterministic, whereas it is deterministic in the latter case.

Observe that since every two elements of an upward-directed subset E have an upper bound, every finite subset of E has an upper bound as well. However, an infinite subset of E might fail to have an upper bound. We thus say that the compossibility criterion is finitely-generalizable but not infinitely-generalizable. This is in stark contrast to Prior's criterion in Branching Time, which defines

[8] The criterion might be expressed as the existence of $\wedge$-like shape connecting two bottom nodes. This leads to one generalization, proposed in Placek (2011), which requires connectability by a W-like shape with finitely many zigzags, instead of the $\wedge$-like connectability.

[9] The BST axioms are recalled in Subsection 3.3.

compossibility in terms of chains (with respect to a partial temporally-modal order). Clearly, if any two elements in E form a chain, then any subset of E, finite or not, forms a chain. However, many compossibility criteria do not generalize from pairwise cases to infinite, or even finite cases. Besides, the fact that the later witness criterion is not infinitely generalizable is a welcome feature, given the important/intended BST structures are structures with histories isomorphic to Minkowski space-time, with BST ordering generalizing the Minkowskian ordering to modal context. An infinitely-generalizable later witness condition will require that a maximal element of Minowski space-time exists, but since it does not, there will be no BST histories isomorphic to Minkowski space-time.

We finish this section by mentioning a few properties of BST histories. First, every element of the base set W of $\mathcal{W}$ can be extended to a history, so, by maximality, histories are non-empty, and, since $W \neq \emptyset$, $\mathcal{W}$ has at least one history. Then, importantly, a history is closed downward: if $e_1 < e_2$ and $e_2 \in h$, then $e_1 \in h$. Consequently, the complement of a history is closed upward (for the proofs, see Belnap et al. (2022, 29–30).

3.2 BST histories are the maximal ideals on a BST poset

We come to our main mathematical observations, announced in this section's title. Let us rehearse the relevant terminology:

1. A non-empty subset $F \subseteq X$ is called an *ideal* on the poset $\langle X, \leqslant \rangle$ iff F is upward directed (i.e., $\forall x,y \in F\ \exists z \in F\ [x \leqslant z \wedge y \leqslant z]$) and F is downward closed (if $y \leqslant x \wedge x \in F \wedge y \in X$, then $y \in F$).

2. An ideal F is proper iff $F \neq X$.

3. If F is a maximal ideal, it cannot be properly extended into an ideal.

Filters are duals of ideals: they are non-empty, downward directed, and upward closed subsets of X.
The important distinction is that between *free* and *fixed* ideals. An ideal is called free if it does not have a greatest element, i.e., there is no $m \in F$ such that $F = \{x \in X \mid x \leqslant m\}$. An ideal that is not free has a greatest element, so it can be written in the form above, and is called fixed.
Putting together Def. 2 and the terminology rehearsed above, we say:

> A BST history is a maximal ideal on a strict partial order that satisfies BST axioms.

For the sake of self-containment, we recall below the relevant axioms and basic definitions.[10]

3.3 The BST axioms

We give here the "official definition" of a common BST structure, as presented in Belnap et al. (2022):

Definition 3 (Common BST structure)**.** A *common BST structure* is a pair $\langle W, < \rangle$ that fulfills the following conditions:

1. (Nonempty) W is a non-empty set of possible point events.
2. (Order) $<$ is a strict partial ordering.
3. (Density) The ordering $<$ is dense;
4. (Infima) The ordering contains infima for all lower bounded chains;
5. (H-Suprema) The ordering contains history-relative suprema for all upper bounded chains; For a history h and a chain $C \subseteq h$ the history-relative supremum $\sup_h(C)$ is an element such that

   $$\sup_h(C) \in h, \qquad C \leq \sup_h(C), \qquad \text{and}$$

   for any $e \in h$ such that $C \leq e$, we have that $\sup_h(C) \leq e$.
6. (Weiner's postulate) Let $C, C' \subseteq h_1 \cap h_2$ be upper bounded chains in histories h_1 and h_2. Then the order of the suprema in these histories is the same:

   $$sup_{h_1} C \leqslant sup_{h_1} C' \quad \text{iff} \quad sup_{h_2} C \leqslant sup_{h_2} C'.$$
7. (Historical connection) Any two histories have a non-empty intersection, i.e., for $h_1, h_2 \in \text{Hist}$, $h_1 \cap h_2 \neq \emptyset$.

There are various ways, topologically speaking, of how historical connection can be implemented. The axiom of the prior choice principle (below) encodes the pattern of histories' branching that defines the theory known as BST_{1992}. For an alternative axiom, defining an alternative theory, BST_{NF}, see Belnap et al. (2021). Each of the two axioms entails historical connection.

[10] Readers seeking a broader overview with further motivations, illustrative examples, and background material are encouraged to consult Belnap et al. (2022).

Definition 4 (BST$_{1992}$ prior choice principle, PCP)**.** A common BST structure $\mathcal{W}$ satisfies the BST$_{1992}$ *prior choice principle* (PCP) iff it fulfills the following condition:

> Let h_1, h_2 be two histories, and let $l \subseteq (h_1 \setminus h_2)$ be a lower-bounded chain. Then there is an event c maximal in $h_1 \cap h_2$ such that $c < l$ and lies properly below l.

A point that is maximal in $h_1 \cap h_2$ is called a *choice point* for h_1 and h_2.

We end this section with mentioning three BST notions that we need below. First, two histories h_1 and h_2 in $\mathcal{W}$ are said to be undivided at point event e, $h_1 \equiv_e h_2$, iff either e is not maximal in $h_1 \cap h_2$, or e is maximal in $\mathcal{W}$. Provably, $\equiv_e$ is an equivalence relation on the set of histories containing e. For e and $h \ni e$, the equivalence class of h with respect to $\equiv_e$ is defined as an open possibility at e and denoted by $[h]_e$. Second, point events e and e' are called *space-like related* (SLR) iff they are incomparable by $<$, but there is a history to which they belong. Finally, Modal Funny Business (MFB, see Belnap et al. (2022)) intends to capture the modal aspect of (idealized) EPR-like scenarios, in which some combinatorially possible histories are not possible.[11] Formally, *a BST structure exhibits MFB* if and only if it contains SLR point events e and e' and histories $h \ni e$ and $h' \ni e'$ such that $[h]_e \cap [h']_{e'} = \emptyset$.

Before we turn to ideals on partially ordered sets, we begin with some preliminary reflections on ideals over sets.

Unless explicitly stated otherwise, in what follows by a BST structure we mean a common BST structure that satisfies the PCP.

4 Ideals and ultrafilters over sets

Ideals (filters) on the powerset $\mathcal{P}(X)$ of a set X, ordered by inclusion (equivalently, a complete atomic Boolean algebra) is a well-investigated topic. Such objects are called ideals (filters) over the set X. As for maximal free ideals and free ultrafilters over a set X, the important result is that they abound if X is infinite. A necessary and sufficient condition for an ultrafilter over set X to be free is that it contains the cofinite filter (aka the Fréchet filter). Similarly, a

[11] MFB can be analyzed in a few provably equivalent ways, see Belnap et al. (2022); here our focus is space-like related MFB of the simplest kind, explained in Def. 5.2, *ibid.*

maximal proper ideal over X is free iff it contains the ideal of all finite subsets of X (aka the Fréchet ideal). According to the results of Pospíšil (1937), for an infinite set X, the cardinality of the set of its ultrafilters is $2^{2^{|X|}}$, whereas the cardinality of the set of its fixed ultrafilters is $|X|$, where $|X|$ is the cardinality of X. This translates into a result concerning Boolean algebras: for an infinite, complete atomic Boolean algebra $\mathcal{B}$, if the cardinality of the set of its atoms is κ, the cardinality of the set of maximal free ideals on $\mathcal{B}$ is $2^{2^{\kappa}}$. In short, maximal free ideals on an infinite complete atomic Boolean algebra abound.[12]

This fact is philosophically loaded since a proof of the existence of fixed ideals over an infinite set is constructive, whereas the existence of maximal free ideals (free ultrafilters) over an infinite set requires, in its proof, the Ultrafilter Theorem, which is weaker than the axiom of choice.[13] Being a maximal free ideal (free ultrafilter) over an infinite set is thus what is sometimes called *intangible property* (see Schechter (1997, p. 105)): they exist and moreover, exist in abundance (given the Ultrafilter Theorem), but no example of them can be given in the sense of being constructible in ZF set theory with DC. To stress again, these results concern maximal free ideals (ultrafilters) over *sets*, and not the more general case of ideals (ultrafilters) on posets. It is thus interesting to learn if being a maximal free ideal on an infinite poset is an intangible property. If it is the case, BST histories will be even less knowable than one might initially think. Clearly, a history is too large to be grasped, and even an idealized agent might learn at most an initial part of it (i.e., the agent's causal past). But intangibility will make a history even less accessible. We tend to think of histories as being in principle inaccessible, if they turn out to be intangible. Thus, the issues of free vs. fixed ideals on posets and intangibility figure somehow high on this paper's agenda.

Now, being a maximal free ideal over a set is an intangible property, as in this case each maximal free ideal is a maximal extension, via the Zorn lemma, of the Fréchet ideal. One might expect, however, that these notions might fail to coincide in the context of posets, as there might be maximal free ideals on a poset that are tangible, and also, ones that are intangible. Accordingly, our investigation will be divided into two parts. In the first one, we investigate sufficient and necessary conditions for a poset to have a maximal free ideal,

[12]We remark that the completeness of the Boolean algebra is important for this result. For example, consider the Boolean algebra of finite and cofinite sets of an infinite set X. This Boolean algebra has exactly one free ultrafilter, consisting of the cofinite elements.

[13]In Zermelo-Frankel (ZF) set theory the ultrafilter theorem and the dependent choice principle are independent, and together they are strictly weaker than the axiom of choice.

and analogously, the conditions for a BST structure (i.e., a poset satisfying BST axioms) to have a maximal ideal (aka history). In Section 6 we turn to the intangibility issue and prove the Hauptsatz of this paper is that the claim "every BST structure has a history" implies the axiom of choice.

5 On maximal ideals on a poset

We begin with a simple fact concerning the existence of maximal ideals and maximal proper ideals on a poset.

Fact 1. *(i) A poset has at least one maximal ideal. (ii) A poset that is not upward directed has at least one proper maximal ideal.*

Proof. Part (i) follows from the ZK lemma. Part (ii) follows by ideals being upward directed. □

Next we give the necessary and sufficient conditions for a poset to have proper maximal ideals:

Theorem 1. *Let $\mathcal{P} = (P, <)$ be a poset that is not upward directed. Then $\mathcal{P}$ has a free proper maximal ideal if and only if $\mathcal{P}$ has a maximal chain with no upper bound.*

Proof. Let $a, b \in P$ witness $\mathcal{P}$ not being upward directed, i.e., there is no $z \in P$ with $a \leqslant z$ and $b \leqslant z$.
Observe first that P has an ideal, and it can be extended, by Zorn's lemma, to a maximal ideal. Then any maximal ideal $\mathcal{I}$ of P must be proper, because $\mathcal{I}$ cannot contain both a and b, by upward directedness of $\mathcal{I}$.
($\Leftarrow$) Let C be a chain with no upper bound. C is contained in a maximal ideal $\mathcal{I}$. This $\mathcal{I}$ is proper by the above, and free, as otherwise the greatest element of $\mathcal{I}$ would be an upper bound of C.
($\Rightarrow$) Let $\mathcal{I}$ be a free proper maximal ideal, and consider $C \subseteq \mathcal{I}$, a maximal chain in $\mathcal{I}$. First, C is maximal in $\mathcal{P}$, too. For if not, and D were a chain properly extending C, then D would be contained in a proper ideal that properly extends $\mathcal{I}$. Second, C cannot have an upper bound. For if $C \leqslant x$, then $x \notin C$. Otherwise, given upward directedness of $\mathcal{I}$ and maximality of C, x would be the greatest element of $\mathcal{I}$, contradicting I's being free. Thus, $C \subsetneq C \cup \{x\}$, so C is not maximal in P, contrary to what was established above. □

To link this result to BST, any BST structure with more than one history is not upward directed. The continuity-enforcing axioms, Infima and History-relative-Superema together with the Density axiom, entail that, unless Our World $\mathcal{W}$ has one element only, every maximal chain in $\mathcal{W}$ has continuously many elements. It is enough that one of these chains be unbounded that the BST history that extends it, is a free proper maximal ideal. Also, in a BST structure with histories isomorphic to Minkowski space-time, every maximal chain has no upper bound (and no lower bound either). If this structure is not upward directed, it has multiple histories and each history is a free proper maximal ideal. Finally, in some axiomatic formulations of BST, structures with maximal/minimal elements are explicitly forbidden, as bringing in some unnecessary complications.[14] In these structures, histories are maximal ideals (proper or not). Thus, apart from some exotic BST structures, histories come out as maximal *free* ideals.

We continue with a simple fact linking unbounded free ideals to proper maximal free ideals:

Fact 2. *Let $\mathcal{P}$ be a poset that is not upward directed and has a free ideal $\mathcal{I}$ that is not upper bounded in $\mathcal{P}$. Then either $\mathcal{I}$ is a free proper maximal ideal on $\mathcal{P}$, or it can be extended to a free proper maximal ideal on $\mathcal{P}$.*

Proof. By Maximal Ideal Theorem $\mathcal{I}$ can be extended to a maximal ideal $\mathcal{J}$ on $\mathcal{P}$. Since $\mathcal{P}$ is not upward directed, $\mathcal{J}$ is a proper ideal. And if $\mathcal{J}$ were not free, it would have an upper bound b, and this b would upper bound $\mathcal{I}$, contrary to the assumption. Therefore $\mathcal{J}$ is a free proper maximal ideal on $\mathcal{P}$. □

To comment on this result, note that the extension mentioned in this proof might be trivial (i.e., if $\mathcal{I}$ is already a maximal free ideal), or not require a maximality principle to construct it. Obviously the Maximal Ideal Theorem might be replaced in these cases in the proof by some weaker principle. Clearly, there is no conflict between Theorem 1 and Fact 2: if $\mathcal{P}$ has a free ideal that is not upper bounded in $\mathcal{P}$, then $\mathcal{P}$ has a maximal chain with no upper bound.

We next attempt to estimate the number of maximal free ideals on a poset. There might be no uniform answer to this query, as posets might be too multifarious, but the question has a clear-cut answer in certain Boolean algebras, which form a subclass of posets. Interestingly, infinitely large complete atomic

[14] A case in point is the axiomatization given in Belnap et al. (2021). The existence of maximal (or minimal) elements in a BST structure has a consequence that the proofs of translatability of BST_{1992} and BST_{NF} are more complex.

Boolean algebras have a large amount of maximal free ideals: if κ is the cardinality of the set of atoms of a complete atomic Boolean algebra, then the number of maximal free ideals is 2^{2^κ}. This result does not have an interesting application in the context of BST, since a Boolean algebra is upward directed, so a Boolean BST structure has just one history. We nevertheless want to mimic this result by considering posets that embed infinitely large Boolean algebras in a particular way. To explain this, we begin with a definition.

Definition 5 (cofinally embeddable)**.** Let $\mathcal{B} = \langle B, \leqslant_B\rangle$ and $\mathcal{P} = \langle P, \leqslant\rangle$ be posets. We say that $\mathcal{B}$ is cofinally embeddable in $\mathcal{P}$ iff there is an injective function: $h : B \to P$ such that (i) $x \leqslant_B y$ if and only if $h(x) \leqslant h(y)$ and (ii) if $h(x) \leqslant y$, then there is $z \in B$ such that $y \leqslant h(z)$. If h satisfies (i) and (ii) we also say that h cofinally embeds $\mathcal{B}$ into $\mathcal{P}$.

Lemma 1. *Let h cofinally embed $\mathcal{B}$ into $\mathcal{P}$. Then:*
(i) For every free proper maximal ideal U on $\mathcal{B}$, $h(U)$ extends to a free proper maximal ideal on $\mathcal{P}$.
(ii) For free proper maximal ideals U_1, U_2 on $\mathcal{B}$, if $U_1 \neq U_2$, then every free proper maximal ideal obtained by the extension of $h(U_1)$ is different from every free proper maximal ideal obtained by the extension of $h(U_2)$.

Proof. (A) First observe that no upward directed subset of P can contain $h(U_1)$ and $h(U_2)$ for different free proper maximal ideals U_1, U_2 on $\mathcal{B}$. Since U_1, U_2 are different proper maximal ideals, there are $x \in U_1, y \in U_2$ that have no upper bound in $\mathcal{B}$. But then $h(x), h(y)$ have no upper bound in $\mathcal{P}$ either. If they had an upper bound, say p, then by condition (ii) of cofinal embedding, since $p \geqslant h(x), p \geqslant h(y)$, there is $z \in B$ such that $h(z) \geqslant p$. Hence $h(z) \geqslant h(x)$ and $h(z) \geqslant h(y)$, so by (i) of Def. 5 x and y would have an upper bound in $\mathcal{B}$, contrary to the above.
(B) To extend $h(U)$, for U a free proper maximal ideal on $\mathcal{B}$, to a free proper maximal ideal in $\mathcal{P}$, consider the set

$$\mathcal{Q} := \{Q \subseteq P \mid Q \text{ is a proper free ideal on } \mathcal{P} \text{ and } h(U) \subseteq Q.\}$$

Take C – an arbitrary chain in poset $\langle \mathcal{Q}, \subseteq\rangle$. $\bigcup C$ is an upper bound of C and it is easy to see it is an ideal as well. Since by (A) it cannot contain $h(U')$, for U' a different free proper maximal ideal on $\mathcal{B}$, $\bigcup C$ is a proper filter. Hence, by the ZK lemma, there is a maximal proper ideal on $\mathcal{P}$ extending $h(U)$. Moreover, this proper maximal ideal must be free; otherwise, there would be its greatest

element, $p \in P$. This p would be an upper bound of $h(U)$, and hence, by properties of top-embedding, there would be $z \in B$ such that $h(z) = p$, with z a greatest element of U, contradicting U's being free.
(C) We get (ii) by (A): every maximal element in $\mathcal{Q}$ and every maximal element in $\mathcal{Q}' := \{Q \subseteq P \mid Q \text{ is a proper ideal on } \mathcal{P} \text{ and } h(U') \subseteq Q\}$, are different, iff $U \neq U'$. □

The lemma leads, via Pospíšil's (1937) result, to the sufficient condition on $\mathcal{P}$ having a large number of proper maximal ideals:

Theorem 2. *Let $\mathcal{B}$ be a poset obtained from a complete atomic Boolean algebra with κ atoms by removing the top element, where κ is infinite, and assume that $\mathcal{B}$ is cofinally-embeddable in the poset $\mathcal{P}$. Then the set of free proper maximal ideals on $\mathcal{P}$ has cardinality of at least 2^{2^κ}.*

The theorem says that a poset $\mathcal{P}$ obtained by cofinal embedding, as described above, has a large number of free proper maximal ideals. Fact 3 below says, however, that such a poset is not a BST structure. Recall that PCP abbreviates the prior choice principle (see Definition 4).

Fact 3. *Let $\mathcal{B}$ be a poset obtained from a complete atomic Boolean algebra with κ atoms by removing the top element, where κ is infinite, and assume that $\mathcal{B}$ is cofinally-embeddable in the poset $\mathcal{P}$. Then $\mathcal{P}$ does not satisfy the PCP, so it is not a BST structure.*

Proof. By properties of cofinal embeddings, it is enough to prove the statement for $\mathcal{B}$. Free histories in $\mathcal{B}$ are maximal ideals extending the Fréchet ideal. These histories share a common segment which is just the Fréchet ideal. Now, two free histories h_1 and h_2 are different when one contains, and the other does not contain, some infinite subset of atoms, call this subset K. So, we have a prerequisite of PCP, $K \subset h_1 \setminus h_2$. For PCP to hold, there must be a maximal element of $h_1 \cap h_2$. This requires the Fréchet ideal to have a maximal element, which does not exist. □

As we mentioned before Definition 4, there is an alternative version of PCP, paying heed to alternative topological intuitions. In the present context, this second version requires the existence of the smallest cofinite subset in $B \setminus K$, which does not exist either.

To summarize the results of this section, some exceptions aside, a BST history is a free maximal ideal on a poset satisfying the BST axioms. This does not imply that being a BST history is an intangible property. Histories in some BST structures can be introduced without an appeal to AC and easy example of this sort is the structure obtained by pasting together an initial interval of two copies of the real line, so that each copy of the real line comes out as a BST history. More sophisticated examples of this sort, again constructed by the pasting technique, are Minkowskian Branching Structures (Belnap et al., 2022, ch. 9.1), whose basic building blocks are functions from $\mathbb{R}^4$ to a set of states. Such functions are used to first introduce an ordering and then, by taking appropriate equivalence classes – histories, which are isomorphic to Minkowski space-time. To obtain intangible free maximal ideals on a poset we next attempted to import them, via cofinal embeddings, from infinite complete Boolean algebras. This technique produces an abundance of intangible free maximal ideals on a resulting poset. However, this poset turns out to violate the prior choice principle, so they are not BST structures. Thus, this section brings a soothing message that perhaps one need not be worried by BST historiers being intangible. Our next section, however, conjures about the opposite result.

6 Histories and the axiom of choice

In this section we prove that the axiom of choice is equivalent to the statement that every BST structure has a history, and then an analogous theorem concerning histories in the theory of Branching Time (BT).

6.1 The existence of BST histories and the axiom of choice

It is a standard application of the ZK lemma (an equivalent of the axioms of choice) that every BST structure has a history. Take the ideal generated by any element of the poset, and extend it to a maximal one. Hence, the set of histories can never be empty. In some cases, a history of a BST structure is necessarily not proper: Take, for example, the real interval $[0, 1)$. This is a BST structure, but every maximal ideal must be the entire poset. The argument above that every BST structure has a history relied on the axiom of choice. It could be asked whether in ZF set theory (so, without the axiom of choice) there are BST structures with an empty set of histories. This is exactly what this section is about: we show (in ZF) that if every BST structure has a history, then the axiom of choice follows. This result contributes to the intangibility question mentioned in Section 1.

In the first step, for a non-empty family of non-empty sets A_i for $i \in I$ we construct (in ZF) the poset $\mathfrak{B}(A_i : i \in I)$. For the sake of keeping the notation simple, let us assume that the sets A_i are pairwise disjoint, and disjoint from I. A partial choice function is a mapping $f : X \to \cup_{i\in I} A_i$ for some finite non-empty $X \subseteq I$ such that $f(i) \in A_i$ for every $i \in I$. We start with an informal description of $\mathfrak{B}(A_i : i \in I)$, illustrated in Fig. 1. The poset has a smallest element $\perp$. Above $\perp$ in the first level we have the elements of I. The second level consists of all elements of the sets A_i. Each element from A_i is above i. On top of this we have all the partial choice functions. A partial choice function f is above all the $f(i)$'s for $i \in dom(f)$. The partial choice function g is above f if and only if g is an extension of f, that is, $dom(f) \subseteq dom(g)$ and on $dom(f)$ the two functions agree. In the figure h and f are incomparable and incompatible, but g extends f. In the Hasse diagram in Figure 1, the lines representing the ordering relation are copies of the $(0, 1)$ interval.

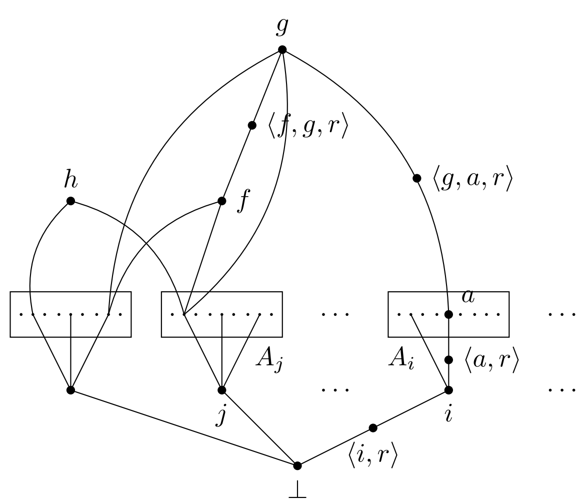

Figure 1: The poset $\mathfrak{B}(A_i : i \in I)$ corresponding to the family of sets A_i $(i \in I)$.

Formally, the underlying set of the poset $\mathfrak{B}(A_i : i \in I)$ is defined as follows. Here, f and g denote partial choice functions.

$$P = \Big\{\perp,\ \langle i, r\rangle,\ i,\ \langle a, r\rangle,\ a:\ i \in I,\ a \in \cup_{j\in I} A_j,\ r \in (0, 1)\Big\} \cup \Big\{\langle f, a, r\rangle,\ \langle f, g, r\rangle,\ f,\ g:\ \text{for } f, g \text{ s.t. } a \in ran(f),\ f \subseteq g,\ r \in (0, 1)\Big\}.$$

The ordering $<$ on P is the smallest transitive ordering such that

$$\bot < \langle i, r\rangle < i, \qquad i < \langle a, r\rangle < a, \qquad a < \langle f, a, r\rangle < f$$

for all $i \in I$, $r \in (0,1)$, $a \in A_i$, and f partial choice function such that $a \in ran(f)$; and

$$f < \langle f, g, r\rangle < g$$

for all $r \in (0,1)$ and partial choice functions f and g such that g extends f.

Note that $\mathfrak{B}(A_i : i \in I)$ has a single minimal element, but has no maximal elements. This is because every partial choice function has a finite domain, and thus can be extended.

Claim 1. *Every maximal ideal of the poset $\mathfrak{B}(A_i : i \in I)$ corresponds to a choice function of the family A_i $(i \in I)$.*

Proof. As every element of $\mathfrak{B}(A_i : i \in I)$ has a proper extension, a fixed ideal cannot be maximal. Thus, a maximal ideal is free. It is also proper, because two partial choice functions with overlapping domains, and disagreeing on the overlapping part cannot both belong to the ideal, as ideals are upward directed, and two such functions cannot have a common extension. For a maximal ideal J let F consists of the partial choice functions belonging to J. Then F contains compatible functions, and thus $c = \bigcup F$ is a function. By maximality of J, the domain of c is I, and thus c is the desired choice function.

Conversely, if $c : I \to \cup_{i \in I} A_i$ is a global choice function, then the restrictions of c to finite domains are partial choice functions belonging to the poset. The downward closure of the set of such partial choice functions is a maximal proper ideal. □

Claim 2. $\mathfrak{B}(A_i : i \in I)$ *is a BST structure satisfying the prior choice principle.*

Proof. We need to check if $\mathfrak{B}(A_i : i \in I)$ satisfies the BST axioms. Clearly, $\mathfrak{B}(A_i : i \in I)$ is non-empty and by construction – a strict partial order. Given the associated real intervals $(0,1)$, the ordering is dense. Notice, that if there are no histories in the structure, then the rest of the BST axioms, including PCP, are automatically satisfied. We thus discuss below only $\mathfrak{B}(A_i : i \in I)$ with histories. Before turning to the remaining axioms, we make two observations.

Choice points: Recall (see Definition 4) that for histories h_1 and h_2 a point that is maximal in $h_1 \cap h_2$ is called a choice point. Suppose h_1 and h_2 are histories (i.e., maximal ideals). Maximal ideals correspond to choice functions of the family A_i $(i \in I)$. Two different choice functions must disagree on some $k \in I$. Hence, in the corresponding maximal ideals h_1 and h_2 there are partial choice functions f and g, resp., such that $a = f(k) \neq g(k) = a'$. Accordingly, $a \in h_1 \setminus h_2$ and $a' \in h_2 \setminus h_1$. A maximal element of the intersection $h_1 \cap h_2$ is thus k. Although we already found a maximal element, let us note that the rest of the choice points of h_1 and h_2 can be described similarly: take any two partial choice functions $f \in h_1 \setminus h_2$ and $g \in h_2 \setminus h_1$ that are incompatible (that is, there is k such that $f(k) \neq g(k)$). Let p be the function $p = f \cap g$, that is, the largest function which is below both f and g. If p is not the empty set, then p is a partial choice function, and thus it is a choice point. Therefore we have two types of choice points, the first type consists of the elements k for $k \in I$ for which $f(k) \neq g(k)$; and the second type contains elements of the form p, as above, for $p \neq \emptyset$.

Intersection of two histories: Consider two distinct histories (maximal ideals) h_1 and h_2 (suppose they exist). They correspond to choice functions f_1 and f_2, respectively. The set $h_1 \cap h_2$ can be described as the downward closure of

$$\Big\{h \subseteq f_1 \cap f_2 : \; h \text{ is a partial choice function}\Big\} \cup I$$

It follows that any two histories have a non-empty intersection: the elements $\{i, \langle i, r\rangle, \bot : \; r \in (0,1), i \in I\}$ belong to all histories.

Upper bounded chains: Take an upper bounded chain C. For each point of the chain, consider the entire line segment in which the point is contained. E.g. if $\langle a, r\rangle \in C$ for some $i \in I$ such that $a \in A_i$ then consider

$$[i, a] := \{i, \langle a, r\rangle, a : \; r \in (0,1)\}.$$

Similarly, if $\langle f, a, r\rangle \in C$, then consider the line segment

$$[a, f] := \{a, \langle f, a, r\rangle, f : \; r \in (0,1)\},$$

etc. For the cases $i, a, f \in C$ take any line segment which contains these elements and make the union of the line segments a linear ordering. Each such line segment is homeomorphic to $[0, 1]$. As C is upper bounded, there are finitely many line segments considered, therefore the union of all such line segments is also homeomorphic to $[0, 1]$. The rest follows from that every bounded chain in $[0, 1]$ has a supremum. Note that the same argument carries out when it comes to history-relative supremum: the supremum does not depend on the particular history h for which $C \subseteq h$, because if C belongs to two histories, then so too are all the considered line segments.

Infima: The argument is essentially the same as in the case of upper bounded chains.

Weiner's postulate: This follows from the fact that the supremum of an upper bounded chain is history-independent.

Prior choice principle: Recall from above that the downward closure of the points I belong to any history. If $C \subseteq (h_1 \setminus h_2)$ is a chain fully belonging to history h_1 and not intersecting history h_2, then there must be an $i \in I$ such that $i < C$. This i is the desired choice point. □

Theorem 3. *In ZF set theory the following statements are equivalent.*

(A) Every BST structure has a history.

(B) Axiom of choice.

Proof. The implication (B)⇒(A) is a standard application of the Zorn lemma. (A)⇒(B) follows from putting together Claim 1 and Claim 2. □

We end this section with some remarks.

Remark 1. Let us assume that the axiom of choice fails, witnessed by the family A_i $(i \in I)$. Then $\mathfrak{B}(A_i : i \in I)$ is an example of a BST structure in which there are no histories at all. This is because, by Claim 1, histories of $\mathfrak{B}(A_i : i \in I)$ are in a one-to-one correspondence with the choice functions of the family A_i $(i \in I)$, and thus if this family is a witness for the failure of the axiom of choice, then the family has no choice functions at all, and thus the corresponding BST structure has no histories either.

Remark 2 (No Modal Funny Business in $\mathfrak{B}(A_i : i \in I)$). Belnap et al. (2022)) To recall (cf. Belnap et al. (2022)), a BST structure exhibits MFB iff it has two SLR point events e and e' and possibilities $[h]_e$ and $h_{e'}$ open at e and e', resp., that do not intersect. If AC fails, no two events in $\mathfrak{B}(A_i : i \in I)$ are SLR, and hence the structure does not exhibit MFB. If AC is assumed, in the poset $\mathfrak{B}(A_i : i \in I)$ we have a large degree of freedom in constructing histories, identified with global choice functions: for any given SLR events e_1 and e_2 and histories $h_1 \ni e_1$ and $h_2 \ni e_2$, which define open possibilities $[h_1]_{e_1}$ and $[h_2]_{e_2}$ one can define a history h that contains partial choice functions c and d such that

$$e_1 < c \leq h_1, \quad \text{and} \quad e_2 < d \leq h_2 \,.$$

hold. Since $e_1 < c$ and $c \in h_1 \cap h$ and $e_2 < d$ and $d \in h_2 \cap h$, it follows that $h \in [h_1]_{e_1}$ and $h \in [h_2]_{e_2}$. This shows that $\mathfrak{B}(A_i : i \in I)$ does not constitute a case of modal funny business. This argument can be extended to the infinite cases as well.

6.2 The existence of BT histories and the axiom of choice

BT structures provide a framework for modeling the indeterministic evolution of time while ignoring its spatial aspects, where multiple possible futures can stem from a given moment. These structures employ a tree-like ordering to distinguish between an open future of possibilities and a fixed past. This ordering is backwards-linear, ensuring that branching occurs only toward the future, not the past. In a BT structure, a history is defined as a maximal chain (a maximal linearly ordered subset), and BT structures can be seen as BST structures of a particularly simple kind, namely, BST structures without SLR elements.

Formally, a BT structure is a partially ordered set $(W, <)$, where for all $w \in W$ the set $\{v \in W : \ v < w\}$ is linearly ordered. A *history* of a BT structure is a maximal linearly ordered subset.

Let us recall the statement DC_κ for an infinite cardinal κ (working in ZF, κ is an "aleph"), see Jech (2008).

> (DC_κ) Let S be a nonempty set, and R a binary relation such that for any $\alpha < \kappa$ and every α-sequence $s : \alpha \to S$ there is $y \in S$ such that $s \; R \; y$. Then there is a κ-sequence $f : \kappa \to S$ such that $(f \restriction \alpha) \; R \; f(\alpha)$ for every $\alpha < \kappa$.

In ZF, the statement $\forall\kappa \mathrm{DC}_\kappa$ is equivalent to AC, see (Jech, 2008, Theorem 8.1(c)).

Theorem 4. *In ZF the following are equivalent.*

(A) Axiom of choice.

(B) Every BT structure has a history.

Proof. (A)⇒(B) is a standard application of the ZK lemma. As for (B)⇒(A) it is enough to show that for every aleph κ, DC_κ holds. Take any nonempty set S and a relation R satisfying the conditions of DC_κ. For $\alpha < \kappa$ an α-sequence $s : \alpha \to S$ is R-increasing if $(s \restriction \beta)\ R\ s(\beta)$ for every $\beta < \alpha$. Let W be the set of R-increasing sequences of length α for all $\alpha < \kappa$, and let the ordering $<$ on W be defined as sequence extension. Then $(W, <)$ is a BT-structure (a rooted tree).

By assumption, $(W, <)$ has a history, that is a branch of the tree: $f : \kappa \to S$, such that every restriction $f \restriction \alpha$ to some $\alpha < \kappa$ is R-increasing. But this is exactly what is needed to verify that DC_κ holds for S and R. □

This theorem gives us the following example. Assume that AC fails. Then there is a BT-structure without having a history. Add the ordering of the natural numbers to the side of this BT-structure, i.e. glue it as a new branch of the tree. In the resulting BT-structure the chain extension principle fails (that is, not every chain can be extended to a maximal chain), yet the structure has a history (the glued copy of the natural numbers). More is true: there is a model of ZF in which the Ideal Extension Theorem fails, but every infinite set has a nontrivial maximal ideal (cf. (Jech, 2008, p.132)).

Let us modify the construction in the proof of Theorem 4 above as follows. Take any nonempty set S and a relation R satisfying the conditions of DC_κ. In the poset $(W, <)$ (from the proof of Theorem 4) we stipulated $s < t$ for two R-increasing $<\kappa$-sequences, if $dom(s) \subsetneq dom(t)$, and $s = t \restriction dom(s)$. By the assumption of DC_κ, every R-increasing $<\kappa$-sequence s has at least one immediate successor, and the set of immediate successors of s are the sequences $s^\frown y$ for $y \in S$ such that $s\ R\ y$.

Let us modify the construction of W by adding a copy of the unit interval $(0, 1)$ between every element s and $s^\frown y$ for $y \in S$ such that $s\ R\ y$. See Figure 2 for an illustration. The resulting ordering remains a tree, but the ordering becomes dense. Maximal ideals of this poset are maximal branches of the tree, and one can verify that the poset satisfies the BST axioms.

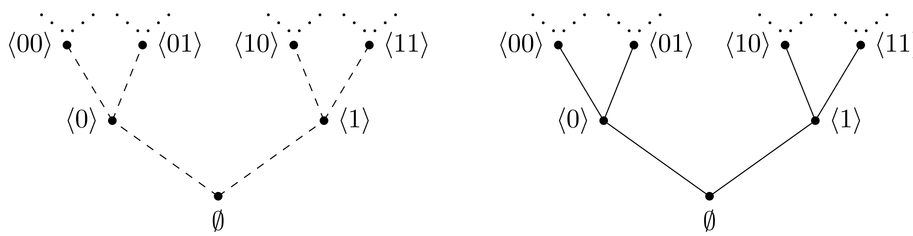

Figure 2: The poset $(W, <)$ with $S = \{0, 1\}$ and R full on the left, and its modified version on the right, where the line segment $(0, 1)$ is added between all consecutive points. The dashed lines represent the ordering relation.

This construction yields the following corollary.

Corollary 1. *Assume that we change the definition of histories in BST structures such that in the modified version, a history of a BST structure is a maximal chain. In this case, in ZF, the axiom of choice is equivalent to that every BST structure has a history.*

7 Conclusions

The observation underlying this paper is that BST histories are maximal ideals on a partially ordered set satisfying the BST axioms. We began with the task of finding necessary and sufficient conditions for a maximal ideal on a poset to be free. Our Theorem 1 identifies these conditions with a poset having a maximal chain with no upper bound. The consequence is that, barring some exceptions, BST histories are free maximal ideals.

A natural question then is whether BST histories are intangible, that is, to assure their existence, the axiom of choice (AC) is required. An analogy with maximal ideals over infinite sets would suggest a positive answer here; on the other hand, some BST structures (notably Minkowskian Branching Structures) are defined without appeal to AC. We investigated one way of producing posets with intangible maximal ideals by "importing" them, via cofinal embedding, from posets isomorphic to complete Boolean algebras, with the unit element removed. The procedure is successful, as witnessed by Theorem 2. However, the resulting posets turn out to not satisfy the prior choice principle, so this technique does not produce intangible BST histories. To put it more loosely, there are no BST histories built on the Fréchet ideal.

Yet, there are intangible BST histories as well as intangible BT histories, as Section 6 shows. We construct there, in ZF, posets that satisfy the BST axioms, and then ask whether they have BST histories. It turns out that they have histories iff the axiom of choice holds. Theorem 3 states that in ZF set theory, the axiom of choice is equivalent to the claim that every BST structure has a history and thus, BST histories are sometimes intangible. An analogous result, concerning BT histories, is stated as Theorem 4.

Should the metaphysician be worried by BST histories being intangible? Needless to say, they would be worried if they had misgivings about using AC in mathematics. More interesting (and perhaps more typical) is a position that accepts the use of AC in mathematics, but objects to its applications in reasonings concerning the non-mathematical realm. A glimpse at the construction of space-times of General Relativity might be a case in point. An interesting class of such objects, known as maximal global hyperbolic developments, are proved to exist by appealing to the axiom of choice.[15] There is research that aims to prove these space-times' existence without invoking the axiom of choice (see e.g., Wong (2013); Sbierski (2016)). We may speculate what the physicists' reaction would be if the opposite fact was proved, that the existence of general-relativistic space-times entailed the axiom of choice. An underlying feeling is that such intangible space-times are not scrutinizable in the sense that no finitely accessible data can determine which space-time we live in. A different example suggesting caution in using the axiom of choice in physical reasoning concerns an interplay between AC and probabilistic non-signaling in quantum mechanics (Baumeler et al., 2025). Roughly, if experimenters implement a global choice function, it violates a certain constraint derivable from probabilistic non-signaling strategies.

Thus, the apprehension of intangible physical objects is understandable to some extent, even though the line between mathematics and theoretical physics is often blurred. Now, are possible worlds/histories, which are posited in analytic metaphysics, more like mathematical structures or like posited structures of physics? Are they more akin to extensions of the Fréchet ideal, or like physically reasonable space-times of general relativity? Perhaps there is no clear-cut answer to this query, yet, we finish this paper with offering the reader some premonitions (rather than solid arguments).

[15]See e.g., Ringström (2009, ch. 16).

Possible worlds/histories are posited to encode and systematize our justified beliefs about possibilities and impossibilities. In some cases, the justification is supported by common sense alone, but in others it is based on our best scientific theories, which provide, for instance, the information about which measurement results are possible and which are not. Clearly, since possible worlds/histories are global, they cannot be fully grasped. They contain too many details, or, since they represent temporal dimension, there are parts in a possible world/history that cannot be now ascertained, as they refer to things in our future. These kinds of limitations can be mitigated, however. The world might have an orderly set of laws of nature that permit one derive the future facts from the knowledge about some data in the past or the present. Or, the multiple details that we cannot scrutinize might have no impact on what is possible and what is not. Yet, intangibility pushes the inaccessibility to the utmost limit. Imagine that your modal data are encoded in the BST structure $\mathfrak{B}(A_i : i \in I)$ of Section 6 and you try to figure what is possible and what is not for you. To address this question, you need to get a grasp on possible histories in this structure. This amounts to discovering what global choice function is implemented. Clearly, no amount of finite data will decide it. The possibilities and impossibilities will be hidden to you, and they will be so for a different reason than the disorderly flow of time.

Finally, it may be asked if it is just the branching-style histories that are prone to being intangible, whereas possible worlds/histories of other possible-worlds theories escape this problem. To investigate this question, the compossibility criteria of these theories have to be formulated in a mathematically manageable manner, and typically they are not. Furthermore, Fact 3 that free maximal ideals on a poset fail to satisfy a BST axiom of PCP, recommends caution: it shows that some further constraints on possible worlds might enforce their tangibility. But too little is known about possible worlds studied in philosophy to have such formally specified constraints. This predicament calls for mathematically rigorous studies of possible worlds. In these circumstances we can only share our hunch. Short of a decisive argument, but inspired by the proofs above, we conjecture that in any possible-worlds theory positing infinitely many entities, there will be histories that are intangible in the sense of Theorems 3 and 4. Metaphysicians wary of the axiom of choice should thus use possible worlds with the utmost caution.

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