**RESCHER'S APORETICS AND A ROAD TO THE VOIGT TRANSFORMATION**

**Abstract**

With the classical distinction between context of discovery and context of justification considered by many to have been overcome, heuristics (understood in a broad sense) has increasingly rekindled the interest of philosophers of science. Building on this trend, a heuristic approach to the Voigt transformation (based on Rescher's Aporetics) is first presented - an issue on which there seem to be no precedents in the literature. Second, the value of this approach is defended from a philosophical (and, indirectly, pedagogical) viewpoint. By using this approach, several conceptual links in the theory of space-time can be highlighted (links which go unnoticed in classical hypothetical-deductive methods leading to the Voigt transformation). In particular an interesting connection with the Lorentz transformation becomes apparent.

**Keywords**

Rescher's Aporetics; Heuristics; Galileo Transformation; Voigt Transformation; Lorentz transformation.

The Voigt transformation is far more interesting than the literature published to date suggests. A heuristic argument can be made (the aporetic route, based on what can be termed the heuristics of distinction) which shows that the Voigt transformation is the natural development (natural destiny) of the Galilean transformation when the hypothesis of light speed invariance is added (without however taking into consideration the requirements for the principle of relativity). To this end, a heuristic resource will be deployed which philosophers have also made abundant use of throughout the course of history (such as to discern verbal disputes, Hospers (1967), among other things). Generically expressed, a heuristic procedure does not begin with basic postulates so as to proceed deductively, but with what is known so as to proceed through intuition and analogy. In this sense, it has played an important role both in the construction of science (which will be briefly recalled below, specifically focusing on the case of physics) and in the pedagogy and interpretation of science. This second aspect is of interest here, particularly in relation to a development that, in some respects, preluded special theory of relativity: the Voigt transformation.

**1. The Voigt transformation to the present day**

In 1887 Voigt published a paper (which went largely unnoticed at the time and for much of the 20th century) that showed for the first time how light speed invariance (strictly speaking, the covariance of the homogeneous wave equation) - an invariance highlighted the same year in the Michelson-Morley experiment - could be explained by altering the equations in the Galilean transformation between inertial systems. The success of the Lorentz transformation that came soon after (which is incompatible with it) likely eclipsed his work, despite Lorentz's own acknowledgement in 1909. Disagreements exist as to the true historical significance of Voigt's contribution on this point (Wesley 1986, Rott 1987,1988, Doyle 1988, Ernst & Hsu 2001, Chashchina et al. 2019). Kittel (1974) considers Voigt, Larmor and Lorentz historically and argues that Larmor is the premature discoverer of the Lorentz transformation. Keswani (1965) recalls that Dugas (1957), in his monumental "A History of Mechanics", suggests we speak of the Voigt-Lorentz transformation rather than the Lorentz transformation (however, Browne 2018 disputes that the Voigt transformation is relativistic). Acknowledging that, in addition to light speed invariance, Voigt was the first to propound that time is not universal, Browne (2020) however stresses that he did not clearly explain whether the transformation of time was more than a purely mathematical matter. Such disputes will not be discussed here. It does however seem clear that interest in the Voigt transformation has grown since the last quarter of the past century (in this respect Gluckman (1976), for example, proved that even Maxwell's own equations are covariant under the Voigt transformation). However, there exists an important deficit in the literature. Voigt did not clearly discuss the postulates leading to his transformation, and subsequent authors have tended to ignore this point, either by simply mentioning his work (Pauli 1921, Brown 2005) or by simply proving that it "works"; that the transformation he found leads to light speed invariance (Gluckman 1968, Hsu & Hsu 2006). To my knowledge, only Heras (2017) analyses Voigt's work in detail, clarifying it and proposing an alternative derivation. Yet the proof remains complex. This paper proposes a different - not strictly deductive but heuristic - road that leads to the Voigt transformation. As the reader will hopefully see, the road will be credited for its remarkable simplicity, and for the clarity with which light speed invariance is embedded in the underlying kinematics.

**2. The role of heuristics in physics**

As a method of demonstration and/or discovery, heuristics has a long (and rather covert) tradition in physics. In "Symmetry. A Mathematical Exploration" Tapp (2021) states that beautiful new ideas are often discovered when we strive to put visual proofs and vague heuristic arguments on a more solid and rigorous footing. On this point, Einstein's "On a Heuristic Point of View Concerning the Production and Transformation of Light" may come to mind, which was the first step towards a quantum theory of radiation. Also, the heuristic quantization rule ("promote Poisson brackets to commutators", Ivancevic, V. & Ivancevic, T. 2008), which ensures that Heisenberg's formulation of quantum mechanics has the correct classical limit. Similarly, Schrödinger's formulation of the same theory was constructed as a generalisation of classical mechanics following, as a guiding heuristic principle, the way in which wave optics generalises geometrical optics (Teta, 2018). Further interesting examples include the Correspondence Principle as a heuristic guiding principle (Radder 1991) in the formation of new theoretical concepts in Bohr's quantum theory, the Born-Oppenheimer approach as a "heuristic axiom" (Vancik, 2021), which enables chemical structure to be referred to as molecular geometry (by considering atomic nuclei immovable) or Planck's heuristic derivation of the expression for the energy density per frequency, giving birth to quantum mechanics. Even Bohr's own atomic model has been put forward as an example of a hypothesis reached through a heuristic (non-deductive) method of metaphor-based problem-solving (Cellucci 2018).

For obvious reasons, the pedagogical uses of heuristics are not as widely known as its scientific uses, but they do indeed exist - and some of them are of great interest. For example, Hassani 2008 proceeded along this path by introducing an ingenious heuristic means of reaching the Minkowski metric. Visser 2005 arrives at Schwarzschild geometry no less, based on intuitive Newtonian foundations. Finally, the classical Laplacian "black hole" model (a widely reported model, e.g. D'Inverno 1992) must be recalled here, which leads to the correct estimation of the Schwarzschild radius by surprisingly simple means. All this needs to be stated because, based on the abundance of historical examples mentioned above, the wrong impression could be given that heuristics is only justifiable when no theory is available - when a theory is under construction. Unlike in physics, these pedagogical uses of heuristics have been theorised in mathematics since the middle of the past century, at least. The most compelling example of the importance of methodological heuristics as a tool for both learning and understanding is perhaps, in mathematics, Polya's work (1945, 1968), (not to mention Lakatos' masterpiece, 1976). And such importance is accredited by their undeniable influence on current studies on creativity, analogical reasoning and non-deductive methods in mathematics. Something similar is missing in physics, and it is not a task that can be tackled in a short paper of this kind. The following section will be limited to proposing a heuristic tool (to be subsequently used) that has proven its worth in domains other than physics.

**3. Aporetics as a heuristic resource**

Aporetics, the theory of rational deliberation in the face of inconsistencies, is an inexhaustible source of heuristic resources in philosophy. In the face of an inconsistency, at least one of the propositions involved must be dropped. Yet, in doing so, it is almost always possible to introduce a distinction that allows at least some of what must be dropped to be retained. At this point it is pertinent to recall the old scholastic adage that W. James revives in his exposition of pragmatism: whenever we encounter a contradiction, we must make a distinction (James 1975). The key importance of making a distinction when faced with contradictions, aporias or inconsistencies was analysed in particular detail by Rescher [[1]](#footnote-1). Rescher (2009) states: "Distinctions enable the philosopher to remove inconsistencies not just by brute negativism of thesis *rejection*, but by the more subtle and constructive device of thesis *qualification*. The crux of a distinction is not mere negation or denial, but the amendment of an untenable thesis into something positive that does the job better" (pp. 121- 122). Also, "Once apory breaks out, we can thus salvage our philosophical commitments by *complicating* them, through revisions in light of appropriate distinctions, rather than abandoning them altogether." (p. 123). Rescher connects the role of distinctions with what he calls 'a heuristic maxim', the so-called Ramsey’s Maxim (Ramsey, 2001). However, it seems fairer and more accurate to refer specifically to Rescher's Maxim as the heuristic maxim based on the role of distinctions in the face of inconsistency [[2]](#footnote-2). Rescher's Maxim reflects what can be called a heuristic of distinction, of qualification. This heuristic of distinction will be the core of the aporetic route to the Voigt transformation.

Let us first briefly look at an example of its application to the case of Grandi's series S = 1 - 1 + 1 - 1 - 1 + ... , which led to considerable confusion in 18th century mathematics. Its sum can be "justified" as being S = (1 -1) + (1 - 1) + (1 - 1) + (1 - 1) + ... = 0, also S = 1 + (-1 + 1) + (-1 + 1) + ... = 1, or even S = 1/2, as Euler argued. However, the contradiction vanishes when two different meanings of the term "sum" are distinguished (Waismann 1965), depending on whether it refers to a finite or infinite number of summands. A clear specification of the second meaning (in terms of the idea of convergence) was made in the 19th century - and the distinction made formed the basis of today's standard solution to the paradox.

**4. The aporetic route to the Voigt transformation**

I concur with Ippoliti (2018) that the heuristic view draws on the idea that theory-building is problem-solving. This is in keeping with the idea that heuristics should also be of value as a pedagogical instrument. Hence, the Voigt transformation problem is approached from the perspective of a "Newtonian beginner" facing the resolution of a problem: the inconsistency of Newtonianism with light speed invariance.

My heuristic route leading to the Voigt transformation will be the result of a series of distinctions to be made which are analogous to the distinction that led to convergence theory, as we have just seen.

We begin with the Galilean transformation written for the standard two-observer case O (X,Y, Z axis system) and O' (X',Y', Z' axis system) whose origins coincide (x = x' = 0) at instant t = t' = 0, with O' moving at velocity v with respect to O along the X-axis of O (which coincides with the X' axis of O'). The coordinates that O and O' assign to any event (coordinates x, y, z, t in the case of O and coordinates x', y', z', t' in the case of O') are related by

x' = x - vt (1-1), y' = y (1-2), z' = z (1-3), t' = t (1-4)

The law of velocity composition w' = w - v follows from here, where w and w' are the velocities of an object moving along the X and X' axes measured respectively by O and O'. Consequently, for a photon’s (photon I) motion in the same direction as O', c' = c - v, with c being its velocity with respect to O.

The road leading from here to the Voigt transformation involves just three simple steps suggested by the new requirement that the speed of light must be the same for all inertial observers. Consider the natural way in which a "Newtonian" student might incorporate this new condition into a typical problem-solving exercise.

Step 1

Suppose that photon I above arises at t = 0 at the common origin of both reference frames. The law of motion follows from the Galilean description: x = ct and x' = c't' = c't = (c - v)t. Since the speed of light is then imposed as being the same for observer O and observer O', c = c', inconsistency c = c - v (with v ≠ 0) is reached. The role of Rescher's Maxim in the face of inconsistency (the heuristics of distinction) should now be recalled. When following this maxim, the Galilean transformation should not simply be dismissed, but rather a pertinent distinction that is not acknowledged by the latter should be made explicit: the distinction between t and t'. The fact that the distinction to be made is precisely this one is not prescribed by Rescher's Maxim, as befits its role as a heuristic maxim, a heuristic of distinction. Rescher indeed acknowledges that antinomy resolution by means of new distinctions is a matter of creative innovation, whose outcome cannot be foreseen. Nevertheless, introducing a new time t' ≠ t is the most plausible distinction. We now have x' = ct' together with x = ct (note that (1-1) is incompatible with the conjunction of x' = ct and x = ct). Then, by replacing x = ct and x' = ct' in (1-1), it can be deduced that ct' = ct - (vx/c), in other words, t' = t - (vx/c2). This is the new precise relationship between times t and t', which is certainly more complicated than the original t = t' identity. The new transformation law replacing the Galilean transformation must therefore be

x' = x - vt (2-1), y' = y (2-2), z' = z (2-3), t' = t - (vx/c2) (2-4)

Finally, it should be noted that (2-4), t' = t - (vx/c2), follows directly from (2-1), x' = x - vt, by simply using x' = ct' and x = ct. There is thus complete symmetry between the expressions for x' ((2-1)) and t' ((2-4)). This is especially clear if units are taken in which the speed of light is equal to the unit, c = 1. So (2-1) and (2-4) read as follows: x' = x - vt and t' = t - vx. One expression transforms into the other by simply switching the roles of x and t on the one hand, and x' and t' on the other.

Step 2

Now consider another photon (photon II) which follows a rectilinear trajectory but not along superposed axes OX and OX', rather in an arbitrary direction on plane X-Y (however, like photon I, it arises at t = 0 at the common origin of both reference frames). Since its velocity c is the same for O and O', after time t' in O' and t in O (related by (2-4): t' = t - (vx/c2)) photon II must be at coordinate point (x', y') for O' and coordinate point (x, y) for O. Therefore,

x'2+ y'2= c2t'2 (3) y'2 = c2t'2 - x'2 (3')

and

x2+ y2= c2t2 (4) y2 = c2t2 - x2 (4')

From y' = y (which is (2-2)), (3') and (4'), it follows that x2- c2t2 = x'2- c2t'2. However, this leads to a contradiction because from (2-1) and (2-4), x'2- c2t'2 = (x - vt )2 - c2(t - vx/c2)2 = x2 + v2t2 - 2xvt - c2(t2 + v2x2/c4 - 2vxt/c2) = (x2 - c2t2)(1 - v2/c2) ≠ (x2 - c2t2) (with v ≠ 0). The inconsistency found can be addressed heuristically in a way that is analogous to that followed in Step 1 by introducing a new distinction with regard to (2-2): y' ≠ y. So, from (3') and (4'): y'2 = c2t'2 - x'2 = (c2t2 - x2)(1 - v2/c2) = y2(1 - v2/c2), in other words, y' = y√(1 - v2/c2). This is the new, precise relationship between the coordinates that are transversal to movement y and y', which is certainly more complicated than the original y = y' identity. The new transformation law replacing (2-1) to (2-4) must therefore be

x' = x - vt (5-1), y' = y√(1 - v2/c2) (5-2), z' = z (5-3), t' = t - (vx/c2) (5-4)

Step 3

By finally considering a photon (photon III) that moves in a similar way to photon II, the only difference being that it moves on plane X-Z (instead of plane X-Y), the aporetic route now requires (as expected) that z' ≠ z in the precise form z' = z√(1 - v2/c2). In so doing, the new transformation law replacing (5-1) to (5-4) must therefore be

x' = x - vt (6-1), y' = y√(1 - v2/c2) (6-2), z' = z√(1 - v2/c2) (6-3), t' = t - (vx/c2) (6-4)

This is precisely the Voigt transformation.

**5. Advantages of the aporetic route**

Since the aporetic route is a heuristic route, the latter gives it a clear pedagogical advantage over strictly deductive procedures in general. Namely, instead of beginning with general principles (which are usually abstract and thus more difficult to grasp), the starting point is from some principles that are already known (such as the Galilean transformation in the present case), making the path ahead easier and more intuitive. Note that, in contrast, Voigt's purely deductive procedure begins by assuming a rather special kind of transformation equations between inertial systems, from which he develops a complex mathematical derivation toward the transformation equations bearing his name, which significantly hinders any attempt to explain the genesis of these equations from a more conceptual point of view. In this respect, the alternative derivation of the Voigt transformation presented by Heras in 2017 is of little help. While mathematically more tractable, it begins, from the outset, by assuming a very particular form for the conformal character of the D'Alembert operator. As a result, its explanatory value, in the aforementioned sense, suffers in the same way.

The fact that the Voigt transformation has been derived from the Galilean transformation and light speed invariance is also interesting because it qualifies the role of the Lorentz transformation in relation to the postulate of this invariance. It allows the specific contribution made by the invariance of c to be seen in a very simple way when separated from the principle of relativity (which is not done when studying the Lorentz transformation), since only the former plays a role in the heuristic argument.

It is interesting to note that (2-1) to (2-4) constitute the linear approximation of special relativity (Günther 2020), that is, the linear approximation to the Lorentz transformation (and to the Voigt transformation, which at first order in v/c coincides with it). There are in fact infinite limit cases of the Lorentz transformation (series development can be halted in the n-th order in v/c, with non-negative integer n), and it is notable that the first of these that is not trivial (case n = 1), can be reached so simply (naturally and directly) from the Galilean transformation along the heuristic road proposed. Indeed, this was done in Section 4 (Step 1) solely on the basis of the heuristic of distinction (Rescher's Maxim) in Rescher's aporetics. This result connects (2-1) to (2-4) very directly with Galileo, unlike conventional (non-heuristic) treatments, which tend to see it rather as a limit case of Lorentz.

Moreover, even as early as Step 1, and with transformation equations (2-1) to (2-4) in mind, some of the most important results in the theory of relativity can be advanced. In particular, the law of velocity composition in one-directional motion (conventionally along the x-axis) is a direct consequence of these. Indeed, written in terms of increments, (2-1) to (2-4) can be expressed as

Δx' = Δx - vΔt (7-1), Δy' = Δy (7-2), Δz' = Δz (7-3), Δt' = Δt - (vΔx/c2) (7-4)

It then follows that

(Δx'/Δt') = [Δx - vΔt]/[Δt - (vΔx/c2)] = [(Δx/Δt) - v]/[1 - (v/c2)(Δx/Δt)] (8)

However, Δx/Δt is velocity w measured in reference frame O of an object moving one-directionally in the prescribed manner. And Δx'/Δt' is the velocity of the same object when measured from system O'. Therefore,

w' = (w - v)/[1 - (vw/c2)] (9),

which is exactly the known relativistic law of velocity composition (from which light speed c invariance is trivially retrieved).

I would like to conclude this section by stating one last virtue of the aporetic route presented in this paper. Proving that the Voigt transformation fulfils its task of making light speed c invariant is not the same as explaining why it does so. The heuristic argument provides this kind of explanation (in three Steps, each showing how the invariance of c is reflected in the transformation equations) in the simplest way: an algebraically elementary explanation (without any recourse, unlike Voigt or Heras, to the partial derivative properties). In other words, rather than proving the conformal character of Voigt's space-time metric, my heuristic method explains in the simplest way (in three Steps) why this metric is conformal.

Despite the advantages mentioned, the interest in using heuristic principles in order to arrive at the Voigt transformation (starting from the Galilean transformation and making use of light speed invariance) might be called into question. The reason being that a theory already exists that deals with this invariance in detail: the special theory of relativity (which, unlike the Voigt and Galilean transformations, is correct). This position does not seem convincing, and there are many examples that argue against it. For example, although a detailed and correct theory exists for the deflection of light rays by a gravitational field (general theory of relativity), there is a Newtonian approach to the problem that continues to arouse interest despite producing the wrong result (Mahajan 2021, Perez & Lamine 2018). The ultimate reason for this wrong result lies in (Will 1988) the fact that the heuristic model of light being bent by gravity does not incorporate the curvature of space (which gravity does entail in the framework of general theory of relativity). So, in this respect, the Newtonian approach helps to clarify the importance of said curvature. In parallel, it can be stated that the ultimate reason why my aporetic route, based on light speed invariance, yields the wrong result (the Voigt transformation) is because it does not incorporate the principle of relativity (as does the special theory of relativity). So, in this respect, it helps to clarify the importance of said principle. In the section that follows, however, an interesting relationship between the principle of relativity, Voigt and Lorentz is highlighted.

**6. Where is Lorentz in all this?**

It is common knowledge that the Lorentz transformation is obtained directly from the Voigt transformation ((6-1) to (6-4)) by multiplying each of the space-time coordinates x, y, z, t which appear therein (but not x', y', z', t') by factor 1/√(1 - v2/c2). This is a purely formal connection, with no apparent physical content. A different connection will be proposed here which does emphasise physical principles, and which may help to arouse more interest in the Voigt transformation (and, consequently, in the heuristic route to it). By clearing the space-time coordinates x', y', z', t' measured by O' in (6-1) to (6-4) as a function of those measured by O, the following is obtained

x = (x' + vt')/(1 - v2/c2) (10-1), y = y'/√(1 - v2/c2) (10-2), z = z'/√(1 - v2/c2) (10-3),

t = (t' + (vx'/c2))/(1 - v2/c2) (10-4)

Furthermore, if w = 0 is made in (9), w' = - v is obtained. w' is therefore the observer (reference frame) O's velocity with respect to O'. Since it was assumed from the outset that O' moves at velocity v in relation to O, it can be seen that the reciprocity condition is fulfilled. It is now time to introduce the principle of relativity (equivalence of all inertial frames as far as the laws of physics are concerned) for the first time in this paper. From this and the reciprocity condition, it follows that the relationship between the x, y, z, t coordinates of O and x', y', z', t' coordinates of O' should follow from (6-1) to (6-4) by simply switching the roles of the primed and unprimed coordinates therein whilst replacing v for - v (making sure that the reciprocity condition is satisfied is essential because its fulfilment does not automatically follow from the principle of relativity, as Moylan 2021 clearly emphasised). Therefore, such a relationship should be

x = x' + vt' (11-1), y = y'√(1 - v2/c2) (11-2), z = z'√(1 - v2/c2) (11-3),

t = t' + (vx'/c2) (11-4)

There is a blatant contradiction between equations (10) and (11) (except in the trivial case where v = 0). This contradiction is a measure of the incompatibility between the Voigt transformation and the principle of relativity, and explains why my heuristic road does not make use of the latter. However, it may be avoided by adopting a compromise solution, some kind of mean between the expressions of (10) and (11). So, the geometric mean of the corresponding expressions in both sets of equations is in fact taken. The final appendix provides justification for why the geometric mean (and no other average) should be used here, and also explains how its use is suggested by a new application of Rescher's Aporetics and the heuristics of distinction associated with it. The result from taking geometric means is evidently

x = √[(10-1)⋅(11-1)] = √[(x' + vt')(x' + vt')/(1 - v2/c2)] = (x' + vt')/√(1 - v2/c2) (12-1)

and analogously

y = y' (12-2), z = z' (12-3), t = (t' + (vx'/c2))/√(1 - v2/c2) (12-4)

Surprisingly, this is the Lorentz transformation. Inverted, it takes on the conventional form

x' = (x - vt)/√(1 - v2/c2) (13-1), y' = y (13-2), z' = z (13-3),

t' = (t - (vx/c2))/√(1 - v2/c2) (13-4)

The Lorentz transformation then appears as the result of a very precise compromise (in the form of a geometric mean) between the Voigt transformation and the principle of relativity requirements. At least the need to resort to some mysterious multiplicative factor 1/√(1 - v2/c2) no longer exists.

**APPENDIX: Why a geometric mean?**

The contradiction observed between equations (10) and equations (11) suggests a further application of Rescher's Aporetics which introduces a distinction between the space-time coordinates measured by O according to (10-1) to (10-4) and those measured by O according to (11-1) to (11-4). The space-time coordinates measured by O (which appear above in equations (10-1) to (10-4) as a function of the space-time coordinates measured by O') are rewritten below and distinguished by subscript (1) (and thus referred to as coordinates-1):

x(1) = (x' + vt')/(1 - v2/c2) (10-1), y(1) = y'/√(1 - v2/c2) (10-2), z(1) = z'/√(1 - v2/c2) (10-3),

t(1) = (t' + (vx'/c2))/(1 - v2/c2) (10-4)

Similarly, the space-time coordinates measured by O (shown above in equations (11-1) to (11-4) as a function of the space-time coordinates measured by O'), are rewritten below and distinguished by subscript (2) (and thus referred to as coordinates-2):

x(2) = x' + vt' (11-1), y(2) = y'√(1 - v2/c2) (11-2), z(2) = z'√(1 - v2/c2) (11-3),

t(2) = t' + (vx'/c2) (11-4)

How can a compromise between both pairs of equations be justified?

By using coordinates-1, velocity w(1) measured by O is w(1) = x(1) / t(1). Similarly, by using coordinates-2, velocity w(2) measured by O is w(2) = x(2) / t(2).

The actual space-time coordinates x, y, z, t measured by O must be some kind of Average (which will be called Average\*, Average\*\*, Average\*\*\* and Average\*\*\*\* respectively) between coordinates-1 and coordinates-2:

x = Average\*(x(1), x(2)) (14-1), y = Average\*\*(y(1), y(2)) (14-2),

z = Average\*\*\*(z(1), z(2)) (14-3), t = Average\*\*\*\*(t(1), t(2)) (14-4)

Similarly, the actual velocity w = x/t measured by O must be some kind of Average (which will be called Average\*\*\*\*\*) between velocities w(1) = x(1) / t(1) and w(2) = x(2) / t(2). Namely

w = Average\*\*\*\*\*( x(1) / t(1), x(2) / t(2)) (15)

Since w = x/t, from (14-1), (14-4) and (15), it follows that

Average\*\*\*\*\*(x(1) / t(1), x(2) / t(2)) = [Average\*(x(1), x(2))] / [Average\*\*\*\*(t(1), t(2))] (16)

The simplest solution to this functional equation is one in which all Average functions coincide, i.e., Average\*( , ) ≡ Average\*\*( , ) ≡ ... ≡ Average\*\*\*\*\*( , ). The arguments of these functions are also assumed to be positive numbers (which is always feasible when dealing with space-time coordinates).

In any event, (16) becomes

Average\*( x(1) / t(1), x(2) / t(2)) = [Average\*(x(1), x(2))] / [Average\*(t(1), t(2))] (17)

Generally speaking, for any positive numbers a, b, c, d:

Average\*(a / b, c / d) = [Average\*(a, c)] / [Average\*(b, d)]

which can also be written as:

Average\*((a/b)·b, (c/d)·d) = Average\*(a, c) = [Average\*( a / b, c / d)]·[Average\*(b, d)]

That is, renaming terms (a/b ≡ A; c/d ≡ C):

Average\*(A·b, C·d) = [Average\*(A, C)]·[Average\*(b, d)] (18)

Two additional properties of any Average (and also, therefore, of Average\*) are evidently (for any numbers a and b):

Average\* (a, a) = a (19)

Average\*(a, b) = Average\*(b, a) (20)

Now, from (18), (19) and (20) it easily follows that (see, for example, the slightly more general argument in Fleming and Wallace 1986, p. 221) Average\*(a, b) = √(a·b). That is, function Average\*( , ) is the geometric mean. This is the result used in Section 6 to find (not deductively but heuristically) the Lorentz transformation on the basis of the Voigt transformation.

**REFERENCES**

Brown, H. R. Physical Relativity. Space-time structure from a dynamical perspective. Clarendon Press. Oxford. 2005.the

Browne, K. M. Classical and special relativity in four steps. *European Journal of Physics.* **39** (2018) 025601.

Browne, K. M. Galilei proposed the principle of relativity, but not the "Galilean transformation". *American Journal of Physics.* **88** (2020), pp. 207-213.

Cellucci, Carlo. Theory Building as Problem Solving. In David Danks & Ippoliti, Emiliano (Eds.) *Building Theories. Heuristics and Hypotheses in Sciences*. Springer. Switzerland. 2018. Pp. 63-79.

Chashchina, Olga; Dudisheva, Natalya and Silagadze, Zurab K. Voigt transformation in retrospect. missed opportunities?. One more essay on the Einatein-Poincaré dispute. *Annales de la Fondation louis de Broglie*. **44** (2019), pp. 39-109.

D'Inverno, Ray. Introducing Einstein's Relativity. Clarendon Press. Oxford. 1992.

Doyle W. T. Recognition for Woldemar Voigt. *Physics Today.* **41** (1988),p. 102.

Dugas, René. A History of Mechanics. Routledge & Kegan Paul. London. 1957.

Ernst A. and Hsu J-P. First Proposal of the Universal Speed of Light by Voigt in 1887. *Chinese Journal of Physics.* **39** (2001), pp. 211–230.

Fleming, Philip J. and Wallace, John J. How not to lie with statistics: the correct way to summarize benchmark results. *Communications of the ACM*. **29**, 3(1986), pp. 218-221.

Gluckman, Albert G. Coordinate Transformations of W. Voigt and the Principle of Special Relativity. *American Journal of Physics.* **36** (1968), pp. 226-231.

Gluckman, Albert G. Voigt Kinematics and Electrodynamic Consequences. *Foundations of Physics* **6** (1976), pp. 305-316.

Günther, Helmut. Elementary Approach to special Relativity. Springer. 2020.

Hassani, Sadri. A heuristic derivation of Minkowski distance and Lorentz transformation. *European Journal of Physics*. **29** (2008), pp. 103-111.

Heras, R. A review of Voigt's transformations in the framework of special relativity. [arXiv:1411.2559v4](https://arxiv.org/abs/1411.2559v4)**[physics.hist-ph]**.2017.

Hospers, John. An Introduction to Philosophical Analysis. Prentice-Hall, Inc. Englewood Cliffs, N. J. 1967.

Hsu, J. P. & Hsu, L. A Broader View of Relativity. General Implications of Lorentz and Poincaré Invariance. World Scientific. New Jersey. 2006.

Ippoliti, Emiliano. Building Theories: The Heuristic Way. In Danks, David & Ippoliti, Emiliano (Eds.) *Building Theories. Heuristics and Hypotheses in Sciences*. Springer. Switzerland. 2018. Pp. 3-20.

Ivancevic, Vladimir G. & Ivancevic, Tijana T. Quantum Leap. From Dirac and Feynman, Across the Universe, to Human Body and Mind. World Scientific. New Jersey. 2008.

James, W. Pragmatism: A New Name for some Old Ways of Thinking. Harvard University Press. Cambridge, MA. 1975.

Keswani, G. H. Origin and Concept of Relativity (I). *The British Journal for the Philosophy of Science.* **16 (**1965), pp. 286-306.

Kittel, Ch. Larmor and the Prehistory of the Lorentz Transformations. *American Journal of Physics.* **42** (1974), pp. 726-729.

Lakatos, Imre. Proofs and Refutations: The Logic of Mathematical Discovery. J. Worrall and E. Zahar (eds.). Cambridge University Press. Cambridge. 1976.

Lorentz, H. A. The Theory of Electrons. Teubner. Leipzig. 1909, p. 198.

Luhmann, Niklas. The Third Question: The Creative Use of Paradoxes in Law and Legal History. *Journal of Law and Society.* **15** (2) 1988, pp. 153-165.

Mahajan, Sanjoy. Bending of starlight by gravity. *American Journal of Physics.* **89** (2021), pp. 749-750.

Moylan, Patrick. Velocity reciprocity and the relativity principle. *American Journal of Physics.* **90** (2022), pp. 126-134.

Pauli, W. Theory of Relativity. Dover. New York. 1921.

Pérez, José-Philippe & Lamine, Brahim. Phase velocity and a light bending in a gravitational potential. European Journal of Physics. **39** (2018) 055602.

Polya, George. How to Solve It. Princeton University Press. Princeton. 1945.

Polya, George. Patterns of Plausible Inference. 2 vols. Princeton University Press. Princeton. 1968.

Radder, Hans. Heuristics and the Generalized Correspondence Principle. *The British Journal for the Philosophy of Science*. **42** (1991), 195-226.

Ramsey, Frank P. The foundations of Mathematics and Other Logical Essays. Ed. R. B. Braithwaite. Routledge. Taylor & Francis group. London and New York. 2001.

Rescher, Nicholas. Aporetics. Rational Deliberation in the Face of Inconsistency. University of Pittsburgh Press. Pittsburgh. 2009.

Rott, N. Relativity's forgotten figure. *Physics Today.* **40** (1987), p. 11.

Rott, N. Rott replies. *Physics Today.* **41** (1988)pp. 102-103.

Tapp, Kristopher. Symmetry. A Mathematical Exploration. Springer. Switzerland. 2021.

Teta, Alessandro. A Mathematical Primer on Quantum Mechanics. Springer. Switzerland. 2018.

Vancik, Hrvoj. Philosophy of Chemistry. Springer. Switzerland. 2021.

Voigt, W. Ueber das Doppler'sche Princip. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg-Augustus-Universität zu Göttingen*. **2** (1887), pp. 41-51.

Visser, M. 2005. Heuristic approach to the Schwarzschild geometry. *International Journal of Modern Physics D*. **14** (2005), pp. 2051-2067

Waismann, F. The Principles of Linguistic Philosophy. MacMillan. London. 1965.

Wesley J. P. Michelson-Morley result, a Voigt-Doppler effect in absolute space-time. *Foundations of Physics.* **16** (1986), pp. 817–824.

Will, Clifford M. Henry Cavendish, Johann von Soldner, and the deflection of light. *American Journal of Physics.* **56** (1988), pp. 413-415.

1. The technique of making a distinction to resolve a paradox has also been defended in detail in the field of normative disciplines. See, for example, Luhmann (1988), who states that:

""Saving distinction" - this is the recipe for solving the paradox, and "saving" should be taken in the double sense of saving the system in spite of the paradox by using a distinction and saving the distinction itself by the operation that makes use of them." (p. 162). [↑](#footnote-ref-1)
2. Rescher notes that the history of philosophy is replete with distinctions introduced to avert aporetic difficulties. [↑](#footnote-ref-2)