Is spacetime locally flat?

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September 19, 2025

Abstract

I review three senses in which the slogans, 'spacetime is locally flat', and-or 'spacetime is locally approximately flat' can be justified. The background is a recent paper of Fletcher and Weatherall, that focuses on two of these senses: (i) that each tangent space is 'like Minkowski', and (ii) that around any geodesic we can construct Riemann-normal-coordinates in which Christoffel symbols vanish. They argue, against the orthodoxy, that these senses cannot be given a substantive content. I will here, if not entirely disagree, attempt to qualify their verdicts about (i) and (ii). I will also introduce a third sense, which I take to be the most cogent defense of the special role played by Minkowski spacetime in general relativity. This sense of local flatness concerns the extent to which tidal effects can be ignored, i.e. the extent to which deviation becomes linear (i.e. like Minkowski), when geodesics are very close to each other.

1 Introduction

1.1 Interpretations of 'local flatness'

Einstein's equivalence principle relates the dynamics on a small enough region (around a segment of a timelike geodesic) in a generic spacetime to the dynamics on a flat spacetime. But what mathematical sense can we give to this relation? Recently, Fletcher & Weatherall (2023a) have argued that flat spacetime has no special role to play in general relativity, and here I will engage with their arguments.[1]

I shall begin by issuing a note to the reader: this paper does not aim to review all the ways in which one could construe spacetime as being locally flat. I will focus on three such ways, two of which were discussed in Fletcher & Weatherall (2023a), and a third that was not.[2]

[1]During writing, a second paper tackling the issue of local flatness has appeared in the literature: (Teh et al., 2024). My arguments mostly agree with theirs, but are more focussed on a particular issue, as I discuss in Section 2.3.

[2]Neither is this paper meant as a review of Lorentzian or Riemannian geometry. I will assume basic acquaintance with some of the concepts, and will at most sketch the idea behind proofs of theorems. For the particular issues discussed here, I suggest the textbooks (Hawking & Ellis, 1975) and (Poisson, 2004).

The first two senses are:

(i) **Tangent Space Interpretation:** "The tangent space at a point of spacetime is, or is somehow equivalent to, Minkowski spacetime." (Fletcher & Weatherall, 2023a, p. 7).

(ii) **Coordinate Chart Interpretation:** "At any point of any relativistic spacetime (or along certain curves), local coordinates may be chosen so that, at that point (or along that curve), (a) the components of the metric agree with the Minkowski metric in standard coordinates and (b) all Christoffel symbols vanish."(Fletcher & Weatherall, 2023a, p. 10)

And the third, which is not included in Fletcher & Weatherall (2023a) is:

(iii) **Negligible geodesic deviation (tidal effects):** Given neighboring geodesics within any congruence of time-like geodesics on any relativistic spacetime, the acceleration of their deviation vector approximates, linearly, for small distances, the Minkowski—and only the Minkowski—behavior of *arbitrary* geodesics.

I will make each one of these three senses more precise, as I proceed.

Of the three, Fletcher & Weatherall (2023a) are dismissive of (i), agree with (ii) but argue that it needs clarification (which they seek to provide), and omit (iii). In contrast, I will: defend (i), agree that (ii) needs clarification and that they provide it, and champion (iii) as the most important sense of local flatness.[3]

Before we begin, I will briefly review the necessary mathematical and physical facts about Lorentzian manifolds and their interpretation in general relativity.

1.2 Mathematical background

General relativity models vacuum spacetime by Lorentzian manifolds, that is, a doublet $\langle M, g_{ab}\rangle$, where M is a smooth differentiable manifold and g_{ab} is a symmetric, bilinear, non-degenerate tensor, with signature $(1,3)$. I will use abstract index notation with Roman letters and coordinate components will be denoted with Greek letters.

Signature.

The first mathematical object that is important for us here is the *signature*: it is the number (counted with multiplicity) of positive and negative eigenvalues of the real symmetric matrix g_{ab} of the metric tensor *at each point*, with respect to a basis.[4]

Two comments are in order: first, that the concept of signature is applied pointwise, and thus one could consider manifolds that have a metric of varying signature.[5] Second, by 'Sylvester's law of inertia', the signature does not depend on the basis, and, moreover, at each point one can find a basis in which g_{ab} is diagonal, with all elements being -1 and 1.[6]

[3]In the second part of their paper, Fletcher & Weatherall (2023b) treat the dynamics of matter under the same topic of investigation. So one might have thought that geodesic deviation would be treated there. But the paper is instead focused on equations of motions of sections of vector bundles, and does not treat this case, which I think is indeed more germane to Fletcher & Weatherall (2023a).

[4]I am here focussing on non-degenerate metrics, and so ignore the possibility of zero eigenvalues.

[5]If spacetime is allowed to change signature, there are possible physical differences between signature (1,3) and (3,1); see Gibbons (1994).

[6]It is this fact that is behind the tetrad formalism for general relativity.

Curvature

Here I take parallel transport in M to be defined by a Levi-Civita connection, Γ, which at any point is a function of the metric and its first derivatives there. The curvature is a tensor that encodes the rotation due to parallel transport around an infinitesimal parallelogram, and it is at any point a function of the Christoffel symbols and its first derivatives. Here is a reminder of these relations (for a vector field X^a):

$$(\nabla_b\nabla_c - \nabla_c\nabla_b)X_d = R^a{}_{bcd}X_a; \tag{1.1}$$

$$R^a{}_{bcd} = \partial_c\Gamma^a{}_{db} - \partial_d\Gamma^a{}_{cb} + \Gamma^a{}_{cf}\Gamma^f{}_{db} - \Gamma^a{}_{df}\Gamma^f{}_{cb}, \tag{1.2}$$

where Γ's are the Christoffel symbols and ∇ are the Levi-Civita covariant derivative. The curvature $R^a{}_{bcd}$ is a tensor, and so transforms linearly under coordinate transformations: if it vanishes at a point in one coordinate system, it will vanish at that point for all coordinate systems. On the other hand, $\Gamma^a{}_{cb}$ is a pseudo-tensor, and thus, at any given point, it can vanish in one coordinate system without vanishing in all. Differences between Christoffel symbols will transform as tensors; this is what allows the Riemann curvature as defined in (1.1) to transform as a tensor; this is the reason we slightly abused notation and wrote Christoffel symbols in the abstract index notation in (1.2). More strictly, in a given coordinate system x^α, Chistoffel symbols are given by:

$$\Gamma^\alpha{}_{\beta\gamma} = g^{\alpha\sigma}(\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma}) \tag{1.3}$$

Riemann-normal coordinates

In order to describe Riemann-normal coordinates, and their physical interpretation, we first go back to the context of flat spacetime. Let a particle trajectory in Minkowski spacetime be given, in inertial coordinates, by $x^\alpha(\lambda)$.[7] In these coordinates, the equations of motion for a unit-parametrized geodesic are:

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\lambda^2} = 0. \tag{1.4}$$

More generally, in these coordinates, a test-particle under forced motion will follow Newton's second law, according to

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\lambda^2} = F^\alpha, \tag{1.5}$$

for some source F.

But even in the absence of external forces, under more general coordinate systems, the unit-parametrized geodesic (1.4) of flat spacetime becomes:

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\lambda} = 0. \tag{1.6}$$

The difference between (1.6) and (1.4) is the presence of the Christoffel symbols. Their presence is necessary because, while equation (1.4) is invariant under Poincaré transformations, it is not invariant under general coordinate transformations. But (1.6) *is* invariant under general

[7] Here, since we are discussing particular coordinate components, we have dropped the abstract index notation.

coordinate transformations; the presence of Γ and their transformation properties is what guarantees general coordinate-invariance, even in flat spacetime.

Comparing (1.5) with (1.6), we get to the idea of 'fictional forces' in a flat spacetime: (1.6) gives differences in linear motion that could, naively, be interpreted as a force—as in (1.5)), albeit a sourceless one—but which actually vanish in an appropriate choice of coordinates (namely, inertial). Again, this is possible because Γ is not a tensor, and so it can vanish in one coordinate system but not in the other.

The question is: does the same argument apply to curved spacetime, thereby supporting a construal of geodesic motion as 'inertial'? That is, if the Riemann curvature is non-vanishing, can we similarly find coordinates around a geodesic curve in which the terms encoding deviation from linear behavior (for the coordinates of a test-particle travelling a geodesic) vanish?

The answer is a qualified 'yes', and it is a corollary of the existence of Riemann-normal coordinates.But unlike the flat case, Riemann-normal coordinates only guarantee that the Christoffel symbols vanish on the geodesic itself; they do not vanish in a neighborhood of the geodesic (otherwise their derivatives, and thus the Riemann curvature, would vanish there as well); they only 'approximately vanish' in small neighborhoods of the geodesic. So, I will take the Riemann-normal coordinates associated to the time-like curve γ, to be a choice of coordinates for which

$$g_{\mu\nu}|_\gamma = \eta_{\mu\nu}|_\gamma, \text{ and } \Gamma^\alpha{}_{\beta\gamma}|_\gamma = 0. \quad \text{(Riemann-normal coordinates)} \tag{1.7}$$

In such coordinates x^μ, we can expand the metric around γ on a sufficiently small neighborhood as:

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3}R_{\alpha\beta\mu\nu}x^\mu x^\nu + \mathcal{O}(|x|^3), \tag{1.8}$$

where $|x|$ has dimensions of length, and $|R_{\alpha\beta\mu\nu}|$ has dimensions of length^{-2}, so the approximation is good while the curvature scale is very small compared to the distance from the geodesic. We will come back to these points in Section 2.4.

Geodesic deviation

Suppose we are given a two-parameter family of non-intersecting unit-parametrized geodesics, spanning a two-dimensional surface, $\gamma(s,t)$, where t is the parameter along each geodesic, and the geodesics are labelled by (ie distinguished from each other by) the parameter s, i.e. at fixed s the curve $\gamma(s,t)$ is a geodesic. Using bold-face to denote vector fields, we define two families of vector fields on this surface:

$$\frac{\mathrm{D}\gamma}{\mathrm{d}t}|_{(t',s')} := \mathbf{v}(t',s'), \text{ which satisfies } v^a\nabla_a v^b = 0,\ v^a v_a = -1; \tag{1.9}$$

and

$$\frac{\mathrm{D}\gamma}{\mathrm{d}s}|_{(t',s')} := \mathbf{r}(t',s') \quad \text{which is called the } \textit{geodesic deviation vector.} \tag{1.10}$$

Since these are tangent vectors to a surface, $[\mathbf{r},\mathbf{v}] = 0$, and since we have zero torsion,

$$v^a\nabla_a r^b = r^a\nabla_a v^b. \tag{1.11}$$

Using (1.11) and (1.9), we can set $r_a v^a = 0$.[8]

[8]We obtain that it is constant along the geodesics, and so we we set it to zero. That is:

$$v^a\nabla_a(r_b v^b) = v^b v^a\nabla_a r_b = v^b r^a\nabla_a v_b = r^a\nabla_a(v^b v_b) = 0.$$

Then, using (1.9) and (1.11), it is easy to show, by commuting derivatives, that the *acceleration of the geodesic deviation* is given by:

$$\frac{\mathrm{D}^2 r^a}{\mathrm{d}t^2} := v^c \nabla_c (v^d \nabla_d r^a) = R^a{}_{bcd} v^b v^d r^c \tag{1.12}$$

2 In what sense is a generic spacetime locally flat?

At several places, Fletcher & Weatherall (2023a) criticize notions of local flatness around a point for "telling us nothing about the curvature tensor" at that point. In discussing interpretation (i), that 'the tangent space at a point of spacetime is, or is somehow equivalent to, Minkowski spacetime', they say:

> An advocate for this interpretation might reply that "local flatness" means that infinitesimal neighborhoods of each point—that is, the tangent space—should be thought of as equivalent to Minkowski spacetimes with a distinguished point, since after all we are representing a neighborhood of a particular point. Fine. But even if we set aside the structural differences between Minkowski spacetime and the tangent space at a point of a relativistic spacetime, if the tangent space interpretation is all that is meant by "local flatness", it is strikingly weak. This is because Riemann curvature is a tensor field, and so it determines a tensor acting on the tangent space at each point. Thus, there is a sense in which, even from the perspective of the tangent space at a point, one can "see" the curvature of spacetime near that point, by considering the curvature tensor there.[9]

Of course, all hands agree that the curvature tensor is, indeed, a tensor, and, in general, it is non-zero on most points. Obviously, there is no sense in which this tensor could be (generically) "approximately zero" at any given point. But being locally flat can be given different senses, in particular the senses (i-iii) I listed in Section 1.1; and so saying that flat spacetime has global properties that every spacetime possesses locally—what is meant when we say spacetimes are 'locally flat'—need not mean spacetimes have everywhere 'approximately zero Riemann curvature'. Each of (i), (ii), and (iii) provide one way to construe *an analogy or relationship* between curved spacetime and Minkowski spacetime, and none of these ways should be contingent on the values of the Riemann curvature. For instance, it could be that such analogies hold only for certain types of fields. No matter: if the analogy between such local and global properties holds uniquely between generic spacetimes and Minkowski spacetime, I maintain that this would still count as spacetime being, in a certain sense—i.e. relative to certain particles—locally flat.

In Sections 2.1 and 2.2 I will report and assess Fletcher & Weatherall (2023a)'s critiques of the two interpretations of local flatness listed as (i) and (ii) in Section 1.1. In Section 2.3 I will briefly summarise some of the relevant arguments of a different critique of (Fletcher & Weatherall, 2023a), given by Teh et al. (2024). Then, in Section 2.4 I will assess the interpretation (iii), of flatness via geodesic deviation arguing that it is the most cogent among

[9] And in (p. 10, Ibid), they apply a similar criterion to the coordinate chart intepretation, (ii): "What does it tell us about local curvature?"

the three; and in 2.5 I will show that the effects of curvature could, in principle be witnessed even over a single geodesic, but, in fact, no classical matter could realize these effects for arbitrarily short distances.

2.1 Local flatness according to the tangent space intepretation

In particular, the 'tangent space interpretation' provides a non-trivial sense in which a general spacetime is like Minkowski spacetime. In the quote above, Fletcher & Weatherall (2023a) describe Minkowski space as an affine metric space. But affine metric spaces are modeled, *uniquely*, on vector spaces with an inner product; and each such inner product has a signature. The noteworthy sense in which Lorentzian manifolds are 'locally like Minkowski', according to (i), is the following: *for any* p, T_pM has an inner product, $g_{ab}(p)$, and Minkowski space is the *unique* affine space modeled on (any of these) T_pM, with an inner product that has the same signature as $g_{ab}(p)$. The fact that each point of spacetime has a causal cone, separating space, time, and null directions is a consequence of this fact. And to reiterate: this is true for any choice of $p \in M$. In this sense—of being linear, inner product spaces of the same signature—tangent spaces of M are, mathematically, a 'microcosm' of Minkowski space. Thus sense (i) serves as a robust interpretation of spacetime being 'locally like Minkowski', at the mathematical level.

I believe Fletcher & Weatherall (2023a) undersell this property of Minkowski as unique among all of the other Lorentzian metrics: they say it pertains to a vague notion of 'simplicity'. But I believe in this case the price is not quite right: after all affine spaces are, in a strong mathematical sense, *like* vector spaces, and in this case mathematical uniqueness is a very precise attribute.

A different argument for the special status of Minkowski as associated to the tangent space is provided by Teh et al. (2024). As they point out, the tetrad formulation of GR relies on this association, and the formulation plays an ineliminable role in several theoretical applications of GR.

2.2 According to the Coordinate Chart Interpretation

In this Section, I will first, in Section 2.2.a, criticize the standard interpretation of Riemann-normal coordinates, agreeing with Fletcher & Weatherall (2023a); then, in Section 2.2.b I will summarize what I take to be the main result of Fletcher & Weatherall (2023a), assessing the geometric significance of Riemann-normal coordinates.

2.2.a In agreement with Fletcher & Weatherall (2023a)

As indicated above, I agree with Fletcher & Weatherall (2023a), that there are many ways to interpret the phrase 'spacetime is locally flat'.

As described in Section 1.2, the existence of Riemann-normal coordinates provides another analogy between 'inertial motion' in generally curved spacetimes and in flat spacetimes. This is a somewhat limited sense in which a general spacetime is, around a 'small neighborhood' of a geodesic, approximately 'like' flat spacetime.[10]

[10]Though this existence proof seems to realize Einstein's "happiest thought"—the equivalence principle—

The sense is limited because all it says is that there are *natural* coordinates in the neighborhood of a spacetime geodesic such that the equations of motion of a test-particle are approximately of a specific form, of vanishing *coordinate* acceleration (as in (1.4)).

Of course it does *not* say that the curvature vanishes along the geodesic: once more, the curvature in (1.1) is a tensor, and it depends not only on the Christoffel symbols, but also on their derivatives. Derivatives of Christoffel symbols, as per (1.3), are second order in derivatives of the metric, and thus the existence of Riemann-normal coordinates gives a sense in which metrics on the neighborhood of any curve are 'close to flat' up to its first derivatives.

Fleshing out this rough statement is the focus of Fletcher & Weatherall (2023a). As they say (ibid, p.10):

> it is true that on this interpretation one invokes special coordinates—viz., [Riemann-normal] ones—but the significance of those coordinates requires further commentary. What does the "form" of the metric in some coordinate system tell us about the metric or its derivatives, all of which are coordinate independent structures?

As I will now discuss, Fletcher & Weatherall (2023a) provide a geometric understanding of Riemann-normal coordinates—that does not mention coordinates—and construe this understanding as 'approximate flatness' in a very literal sense.[11]

2.2.b 'Geometrizing' Riemann-normal coordinates

Fletcher & Weatherall (2023a)'s main result is that a spacetime is locally flat in the sense that an arbitrary metric locally—i.e. for some small neighborhood of a geodesic—approximates a flat metric, where 'approximate' is understood as arbitrarily close according to a distance function between metrics that only takes into account contributions up to first-order derivatives of the metrics. I will give the technical statement below, in Theorem 1.

Crucially, the result relies on the standard proof of existence for Riemann-normal coordinates of Equation (1.8). Namely, as they discuss at the end of (p. 14, ibid), that same proof can be used to find a flat metric on γ whose derivative operator $\overline{\nabla}$ agrees with ∇ on γ.[12] The idea here (see (Iliev, 2006, Theorem II.3.2)) is to first find coordinates in which the Christoffel symbols vanish along the curve, then use those coordinates to pull-back the metric to an en-

the relationship between Riemann-normal coordinates and the weak equivalence principle is not direct: see (Fletcher & Weatherall, 2023a, Footnotes 2 and 6).

[11] I believe Fletcher & Weatherall (2023a) at one point slightly under-sell the significance of Riemann-normal coordinates; namely, when they say "If it is not clear what features of these coordinates are supposed to be salient, it is hopeless to try to establish the general existence or uniqueness conditions for such coordinates." This is under-selling because it is clear that existence is supposed to be the relevant fact, not uniqueness, and *that* is well established; moreover, the salient features of these coordinates are also clear. But I think this is a momentary lapse, for they get to the heart of the matter when they go on to say that: "it is unclear whether spatial flatness is meant to be a claim about the existence of certain coordinate systems, *which in turn expresses something about the local geometry of relativistic spacetimes, or if it is supposed to include a further interpretive claim about idealized measurement apparatuses of natural motion, which, of course, would go far beyond any facts about curvature, local or otherwise*" (my emphasis).

[12] Indeed, a crucial ingredient of their Theorem 1, which plays an importart role in all subsequent results, is a corollary of the usual proof of existence of Riemann-normal coordinates: see (Iliev, 2006, Theo. II.3.2).

tire neighborhood of the geodesic.[13] That pull-back will produce a metric whose Levi-Civita derivative $\overline{\nabla}$ is flat everywhere but nonetheless agrees with the original Levi-Civita derivative of g_{ab}, ∇, on the curve itself. Thus there is a sense in which the standard proof of existence of Riemann-normal coordinates works 'under the hood' of Fletcher & Weatherall (2023a)'s geometric construction.

Now, to discuss approximations of metrics one to another, one first needs to introduce the idea of an auxiliiary positive definite metric, h_{ab}. This auxiliary metric is then used to measure distances between metrics, order by order in derivatives, and it is applied to measure the distance between the original metric and a flat metric on a neighborhood of the geodesic.

In more detail:— given some compact set U, they define a k-distance between two tensors, T, T', as

$$d_{U,h,k}(T,T') := \max_{j\in\{0,\cdots,k\}} \sup_{x\in U} |(\nabla)^j_h(T-T')|_h, \tag{2.1}$$

where here the subscript h denotes that both the covariant derivative and the norm are taken with respect to h_{ab}, and j denotes the number of covariant derivatives taken of the difference between tensors. Then, given metrics g_{ab} in U and g'_{ab} in U', a diffeomorphism $f : U \to U'$ is defined to be an (h, k, δ)-isometry if $d_{U,h,k}(g_{ab}, f^*g'_{ab}) < \delta$.

Now they show:

Theorem 1 *Given any spacetime (M, g_{ab}), embedded curve $\gamma : I \to M$, point $p \in \gamma[I]$, compact neighborhood U of p, Riemannian metric h_{ab} on U, real $\delta > 0$, and point p' in Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$, there exist neighborhoods $O \ni p$ and $O' \ni p'$, an embedded curve $\gamma' : I \to \mathbb{R}^4$ with $p' \in \gamma'[I]$, and an $(h, 1, \delta)$-isometry $f : O \to O'$ between (O, g_{ab}) and (O', η_{ab}) satisfying $f \circ \gamma = \gamma'$ on I and $f^*(\eta_{ab}|_{\gamma'}) = g_{ab}|_\gamma$.*

Moreover, they show that the corollary extends to $k > 1$ (i.e. to derivative higher than the first) if and only if the curvature of g_{ab} vanishes everywhere.

Their upshot is that:

> while one can isolate a precise and accurate statement to the effect that spacetime is locally approximately Minkowskian, this statement is misleadingly specific given that local approximation is pervasive. Perhaps a better way of characterizing the situation is that, to first order, all spacetimes with the same metric signature have a universal character, in the sense that they all locally approximate one another. It is only at second order and higher that differences in structure between different spacetimes can be seen in arbitrarily small neighborhoods of a point or curve. That spacetimes cannot approximate one another arbitrarily well to second order is, of course, closely related to the fact that curvature is a tensor. (Fletcher & Weatherall, 2023a, p. 27,28)

Which is in line with the proof of existence of Riemann-normal coordinates.

This all seems like good news for local flatness. But Fletcher & Weatherall (2023a) quickly dampen the initial enthusiasm by showing that η_{ab} plays no special role here: the same theorem

[13] Here they say "parallel transport" the metric along those coordinates: but I believe that is just typo (the only metric at one's disposal thus far in the argument is the original g_{ab}, and it is already parallel transported along those curves, since its covariant derivative vanishes).

can be proven using any other metric of the same signature in place of η_{ab}. The idea is merely to use the triangle inequality with respect to (2.1), with the Minkowski metric as the intermediate vertex of the triangle: if g_{ab} and g'_{ab} are each a certain distance from η_{ab}, $d(g, \eta)$ and $d(g', \eta)$, then the distance between them, $d(g, g')$, is bounded by the sum $d(g, \eta) + d(g', \eta)$, which, in the context of the proof, is arbitrarily small.

In sum, the original existence proof of Riemann-normal coordinates says, between the lines, that all metrics are similar to Minkowski along a curve, up to zero derivatives of the Christoffel symbols (i.e. up to first order in derivatives of the metric). And since all metrics are similar to Minkowski, from a straightforward application of the triangle inequality, they are all similar to each other in this way.

2.2.c Being explicit about other metrics

While Fletcher & Weatherall (2023a)'s main result, described in the previous Section, is technically correct, there is no direct proof of Theorem 1 for g'_{ab} in place of η_{ab}. That is, there is no proof that does not explicitly invoke η_{ab} at any step. The fact that the construction of Riemann-normal coordinates is operating under the hood of the results—it is necessary to construct the flat metric in the first place—should give one pause about how such a direct proof would proceed for different metrics.

Here is a tentative sketch: as I mentioned above, (see (Iliev, 2006, Theorem II.3.2)) to prove Theorem 1, we first must find coordinates in which the Christoffel symbols vanish along the curve, then use those coordinates to pull-back the metric to an entire neighborhood of the geodesic, resulting in a flat metric in that neighborhood. So supposedly one could choose some other set of Chistoffel symbols Γ', and find the coordinates that pick it out along γ. But now, following the analogous step in the construction of the local flat metric on a neighborhood of γ, we need to pull back the metric along the coordinate curves to an entire neighborhood of the curve. But how could the pull-back of the metric on the curve along these coordinates give rise to a general, *non-homogeneous* metric, e.g. one that varies along the 'radial' directions? It is not clear to me. In a similar vein, Teh et al. (2024) point out that only Riemann normal coordinates provide an expansion in terms of physical parameters, i.e. relative curvature scale, as described in the next Section.

2.3 Teh et al 2024

Teh et al. (2024) give several arguments for the special status of Minkowski spacetime in the formulation of GR. Apart from the one mentioned in Section 2.1, about the tetrad formulation of GR, they also discuss the origin of the special status of Riemann normal coordinates and give an argument focused on the local symmetries of spacetime. My focus will be on their argument for Riemann normal coordinates, since it is the one that most overlaps with my own.[14]

They argue that the main property of Riemann normal coordinates, given in (1.8), is not related to coordinates at all: it is purely geometric. To show this, they employ Synge's world function, $\sigma(x, x') \in \mathbb{R}$ defined for $x \in M$ and x' in U a geodesically convex neighborhood of

[14] They argue that the approximate Killing symmetries are directly related to geodesic deviation equations, which is also correct, but takes a different angle on the issue.

x, and there giving the geodesic distance between x and x'. It can be shown that $\nabla^{a'}\sigma(x,x')$ gives (minus) the vector at x' that is tangent to the geodesic that connects to x.

Now, using Synge's world function, and defining $\sigma_x^{a'}(x') := \nabla^{a'}\sigma(x,x')$one can expand the metric around x without employing any coordinates; instead of (1.8), we obtain:

$$g^x_{ab}(x') = \eta_{\mu\nu} - \frac{1}{3}R_{a'b'c'd'}(x')\sigma_x^{c'}(x')\sigma_x^{d'}(x') + \mathcal{O}(\sigma(x,x')^3), \tag{2.2}$$

And indeed, this shows that there is a very strong sense in which Riemann normal coordinates reflect the local geodesic structure of spacetime. Consequently, due to the appearance of the Minkowski flat metric in (2.2), it also shows an important sense in which spacetimes are locally flat, and *not* locally isometric to other metrics.[15]

Of course, as I described above, it still might be possible to expand the metric in terms of other geometrical quantities. But I take the onus of the construction of such an expansion for an arbitrary metric to be on the side of those who defend the thesis that the Minkowski metric plays no special role in the general relativistic theory.

2.4 Negligible geodesic deviation

In this Section, I want to address a point that Fletcher & Weatherall (2023a) do not consider: local flatness as understood in terms of the geodesic deviation equation, (1.12). The equation provides a physical interpretation of curvature. It is used to model, among other things, how a spherical dust of particles is deformed under free-fall; the effects we usually call 'tidal' due to their historical roots.

What does equation (1.12) have to do with spacetime being locally flat? When the curvature vanishes, i.e. in flat spacetime, the rate of change of the deviation between any two geodesics stays constant, as expected. In other words, the spatial deviation between time-like geodesics (according to any flat spatial hypersurface) grows linearly along the geodesics. The analogy to more generally curved backgrounds is simply that the smaller the deviation is at some initial time, the closer the evolution of the deviation is to being linear, for any (fixed) Riemann tensor.

Although the main message of this Section can be glimpsed from (2.2), here I will give a mathematical result that more directly reflects the physical intuition behind local flatness. The physical interpretation of what I shall prove is that sufficiently close test-particles accelerate towards each other as if they were in a flat spacetime.

More rigorously, we could say:

Definition 1 (δ-flatness) *A d-dimensional Lorentzian spacetime $\langle M, g_{ab}\rangle$ is δ-flat for $\delta > 0$, if for each and every $x \in M$, there is a neighborhood $\nu(x)$ of x, such that for any timelike geodesic γ going through x:*
(i) There is a segment of the timelike geodesic $\gamma : [-1,1] \to \nu$, such that the normal exponential map along γ, $\exp : T^\perp_\gamma M \to M$, i.e. the exponential map restricted to vectors orthogonal to γ, gives a diffeomorphism between B_ρ and the open neighborhood $\nu_\rho \subset \nu$, where $B_\rho := \{\mathbf{r} \in T^\perp_\gamma M, \|\mathbf{r}\| < \rho\}$ is the open 'cylinder' in $T^\perp_\gamma M \simeq [0,1]\times\mathbb{R}^{d-1}$ with radius $\rho > 0$ (ν_ρ is called a

[15]It is also worth pointing out that Teh et al (ibid) explicitly link their use of Synge world function techniques to geodesic deviation – thus there is a connection between their discussion and the present paper's discussion in the upcoming Section 2.4.

tubular neighborhood *of the segment* γ*); and*
(ii) given any other (connected segment of) timelike geodesic $\gamma' : [-\epsilon, \epsilon] \to \nu_\rho$*, (where* ϵ *can be defined by the parameter times in which* γ' *enters and exits* ν_ρ*) there is a* ρ *such that the displacement vector* $\mathbf{r}(t) \in T^\perp_\gamma M$*, for which* $\gamma'(t) = \exp \mathbf{r}(t)$*, has its acceleration at* $t = 0$ *bounded by* δ*:*

$$\|\frac{\mathrm{D}^2\mathbf{r}(t)}{\mathrm{d}t^2}\| \leq \delta. \tag{2.3}$$

Now we have

Proposition 1 *Every spacetime is* δ*-flat for all* $\delta > 0$*.*

The proof follows from the existence of tubular neighborhoods (cf. e.g. (Hirsch, 1976, Ch. 4)), the geodesic deviation equation (1.12) and its local expansions, and Minkowski's inequality, as we will now see.

Given any $x \in M$ and finite time-like geodesic segment γ through x, take $\exp(B_{\rho'})$ the tubular neighborhood of γ with radius ρ', which exists for some ρ'. Now consider the vectors in $T^\perp_\gamma M$ with unit norm, i.e. $N_\gamma := \{\mathbf{e} \in T^\perp_\gamma M, \|\mathbf{e}\| = 1\}$. Identify the set $\{\mathbf{e}_o\}$ as:

$$\{\mathbf{e}_o\} = \sup_{\mathbf{e}\in N_\gamma} \|R^a{}_{bcd} v^b v^d e^c\|. \tag{2.4}$$

Of course, in this equation, the supremum also quantifies on the point along the curve γ, and so takes into account the differences in the tangent vectors v^a and the tensor $R^a{}_{bcd}$ along γ. It is also useful to designate a constant for this number, which sets a kind of 'curvature scale' along γ:

$$\kappa := \|R^a{}_{bcd} v^b v^d e^c_o\|. \tag{2.5}$$

Assuming there are no curvature singularities along γ, we can affirm that $\kappa < \infty$.

Take a vector $\mathbf{r}_o = s(0)\mathbf{e}_o \in B_{\rho'}$ for one such $\mathbf{e}_r$ and a geodesic $\gamma'(t) \in \exp(B_{\rho'})$ through $\exp(\mathbf{r}_o)$. We can write $\gamma'(t) = \exp(\mathbf{r}_t)$ for $\mathbf{r}_t = s(t)\mathbf{e}_t$ a curve in $B_{\rho'}$ (and $\mathbf{e}_t$ a curve in N_γ). By construction, $0 < s(t) < \rho'$. Assume without loss of generality, by affinely reparametrizing the geodesic, that $\gamma'(0) = \exp(\mathbf{r}_o)$, so $\mathbf{e}_0 = \mathbf{e}_o$.

Let us first give the heuristic argument, and then the more complete one.

The geodesic equation is infinitesimal: it is valid for a congruence of geodesics and has as one of its inputs the infinitesimal deviation between two neighboring geodesics. Nonetheless, assuming acceleration is continuous, with some further assumptions, we can use the geodesic deviation equation as the first term in an expansion. If we expand the geodesic deviation for terms of order higher than linear in the deviation, we also obtain terms as follows:

$$\frac{\mathrm{D}^2 r^a}{\mathrm{d}t^2} = R^a{}_{bcd} v^b v^d r^c + C_1 r^{i_1} \nabla_{i_1} R^a{}_{bcd} v^b v^d r^c + \;\dots\; + C_j r^{i_1} ... r^{i_j} \nabla_{i_1} ... \nabla_{i_j} R^a{}_{bcd} v^b v^d r^c + ... \tag{2.6}$$

all the way to $j \to \infty$, with certain coefficients C_j (see (Muller et al., 1999, Eq. 27)). These coefficients are not important to what follows. [16]

[16] These curvature terms and coefficients are sufficient if we assume the curvature is an analytic function. If we loosen that condition to smoothness, we can't avoid computing integrals of these terms along $\mathbf{r}$. But these are compact intervals, and so the corresponding curvature scalars will still admit maxima, which can be replaced in the integrals to obtain numbers, which is all that is required for my argument.

But let us first assume only the first term in this expansion is important. We thus assume that the deviation is much smaller than the curvature scale: $\rho' << 1/\kappa$. Physically, this makes sense: we are taking the tubular neighborhood as much smaller than the curvature scale around (that segment of) γ. The larger the curvature, the smaller the radius of the tubular neighborhood. As we will see, our result will be merely a modulation of this condition. Now, assuming that the acceleration vector of the spacelike $\mathbf{r}_o$ 'very small' in the sense above and non-null,[17] and since $s(0) < \rho' << 1/\kappa$, we can expand:

$$\begin{aligned}\|\frac{\mathrm{D}^2 r_t^a}{\mathrm{d}t^2}\|_{|t=0} &= s(0)\|R^a{}_{bcd}v^b v^d e_o^c\| + \mathcal{O}(s(0)^2)\\ &= s(0)\kappa + \mathcal{O}(\|s(0)\|^2) < \rho'\kappa + \mathcal{O}(\rho'^2).\end{aligned} \tag{2.7}$$

Of course, the right-hand side of (2.7) could still be greater than a given δ. But the acceleration of the deviation can be made arbitrarily small by taking geodesics in strictly smaller tubular neighborhoods. That is, by taking $\rho' \to \rho$ sufficiently small, the right-hand side of (2.7) can be made arbitrarily small. More explicitly, ignoring terms of higher order in ρ, it suffices that $0 < \rho << \delta/\kappa$, which just requires us to take the radius of the tubular neighborhood sufficiently smaller than the curvature scale.

That is, for any such $\mathbf{r}_o \in B_\rho$ and $\delta > 0$, there is a $\rho > 0$ such that the deviation vector for a geodesic γ'' that goes through a point in the direction of $\mathbf{e}_o$ but within a shorter distance of γ, for which $\mathbf{r}'_o = s'(0)\mathbf{e}_o \in B_\rho$ and $0 < s'(0) < \rho$, necessarily accelerates towards/away from γ at a slower rate than δ.

Now, take any other geodesic in this neighborhood, $\tilde\gamma(t) = \exp(\tilde{\mathbf{r}}_t) \in \exp(B_\rho)$, with $\tilde{\mathbf{r}}_t := \tilde s(t)\tilde{\mathbf{e}}_t$ and $\tilde{\mathbf{e}}_t$ a curve in N_γ; and suppose it goes through the point $\exp(\tilde s(0)\tilde{\mathbf{e}}_o)$. Again, $0 < \tilde s(t) < \rho$, and we have:

$$\|\frac{\mathrm{D}^2 \tilde r^a}{\mathrm{d}t^2}\| = \tilde s(0)\|\tilde R^a{}_{bcd}\tilde v^b \tilde v^d \tilde e_o^c\| + \mathcal{O}(\rho^2) \leq \rho\kappa + \mathcal{O}(\rho^2) \leq \delta, \tag{2.8}$$

since $\|\tilde R^a{}_{bcd}\tilde v^b \tilde v^d \tilde e_o^c\| \leq \|R^a{}_{bcd}v^b v^d e_o^c\| = \kappa$, $\tilde s(0) < \rho$, and we have chosen ρ such that $\rho\kappa + \mathcal{O}(\rho^2) \leq \delta$. So in the new tubular neighborhood, with radius ρ sufficiently smaller than the curvature scale, to first order in curvature scale, the acceleration of any deviation vector to another segment of a timelike geodesic, at any point in this neighborhood, is bounded by δ.[18]

We can now extend our argument to higher-order contributions to the geodesic deviation equation. Analogously to (2.4) and (2.5), we define:

$$\{\mathbf{e}_j\} = \sup_{\mathbf{e}\in N_\gamma} \|e^{i_1}...e^{i_j}e^c\nabla_{i_1}...\nabla_{i_j}R^a{}_{bcd}v^b v^d\|, \tag{2.9}$$

and

$$\kappa_j := \|e_j^{i_1}...e_j^{i_j}e_j^c\nabla_{i_1}...\nabla_{i_j}R^a{}_{bcd}v^b v^d\|, \tag{2.10}$$

where, since we assume there are no curvature singularities at γ,

$$\kappa_j < \infty, \quad \forall j. \tag{2.11}$$

[17] This is the generic case. A similar proof can be done for null acceleration using affine parametrisations: I leave it as an exercise to the reader.

[18] I keep repeating that these are segments of timelike geodesics. Of course, most geodesics would quickly exit the tubular neighborhood given by $\exp(B_\rho)$, and so no longer have the corresponding acceleration of the deviation bounded by δ.

Finally, analogously to (2.8), it follows from Minkowski's inequality, Equations (2.6), (2.9) and (2.10), that:

$$\|\frac{\mathrm{D}^2 r^a}{\mathrm{d}t^2}\| < \sum_j (\rho)^j |C^j| \kappa_j, \tag{2.12}$$

(where the fist superscript is an exponent). Clearly, the righ-hand side of this inequality can be made arbitrarily small with a suitable radius

$$\rho << \min\{(\kappa_j)^{-1/j}, j = 1, ..., \infty\} \tag{2.13}$$

i.e. smaller than the (generalised) curvature scale.[19] □

Thus every spacetime is δ-flat for $\delta > 0$. But Minkowski spacetime is the only spacetime that has this property for the limiting value of $\delta = 0$ (note that we keep $\rho > 0$). That is, in order to satisfy the inequality for $\delta = 0$ and $\rho > 0$, for every direction e^a, and at every point of every geodesic, we must have $R^a{}_{bcd} = 0$ everywhere. Moreover, the tubular neighborhood for which this limit is respected becomes infinite. This is of course compatible with the only constraint on ρ as $\kappa_j \to 0$ being $\rho < \infty$.

Thus my conclusion is that the geodesic deviation equation *does* provide a geometric sense, more interesting than (ii), in which spacetime is locally flat. Indeed, the tubular neighborhood $\exp(B_\rho)$ of γ can be pictured along the lines of 'Einstein's freely-falling elevator'.

To see that this behavior under the shortening of the deviation says something interesting, contrast this case with that of electromagnetic charges, in which we bring together the trajectories of two particles with the same charge (or similarly, if we bring together two massive particles in Newtonian gravity). In that case, the acceleration of the deviation is inversely proportional to the separation between the particles; we could not obtain an analogue of (2.7): the more we shorten the tube, the greater the acceleration.

Now one might say that δ-flatness is not that special a property. That we could find, for any spacetime g_{ab}, a similar Property A, which only g_{ab} has in the limit, but which any metric arbitrarily approximates. First off, it seems to me that such a Property A would have to be substantially different than δ-flatness and so it would be beside the point of the question we are addressing here.[20] Moreover, in order for g_{ab} to have any interesting global properties, g_{ab} would have to be homogeneous, which already severely undercuts the main thesis of Fletcher & Weatherall (2023a).

2.5 Witnessing curvature on a spacetime geodesic

Of course, the argument of Section 2.4 is not contingent on the values of the curvature. But of course, the relevant scales of separation at which we can ignore curvature effects *are* contingent on the curvature.Physically, it says that one can always find a small enough freely-falling 'elevator' in which dust particles will behave as if there were no curvature, i.e. as in Minkowski

[19] I here preferred to prove this using an expansion in derivative os the curvature, but one could have similarly used instead integrals of contracted Riemann curvature over the radius: we can similarly there take maxima over this radius and bound those integrals.

[20] For instance, could we pick out, to first order, another, non-flat, constant Riemann tensor in a geodesic deviation equation, perhaps by employing a different limiting procedure? In other words, can we provide a limiting procedure that, for any spacetime, approximates (linearly? quadratically?) a constant, non-zero Riemann curvature? I am skeptical.

spacetime: this is, in my opinion, a more faithful mathematical rendition of Einstein's happiest thought than the existence of Riemann-normal coordinates.

But it is a fact that is physically pertinent to point-like scalar particles. It leaves open the possibility that certain fields or bodies, even at very relative small scales, do not behave as if in a flat spacetime.

From a mathematical perspective, it is useful to ask why curvature appears in the geodesic deviation equation and not in the geodesic equation. First, note that the geodesic equation, as applied to test-particles, is an equation that is second-order in covariant derivatives of a scalar quantity (or first-order for vector quantities). Since covariant derivatives applied to a scalar function commute, i.e. $\nabla_{[a}\nabla_{b]}\varphi = 0$, we cannot extract from the evolution of these functions any information about the curvature, which depends on non-commuting covariant derivatives (see (1.1)). To extract information about the curvature from scalar test particles, we thus require neighboring geodesics and their deviation vector.

Things are different for test particles with intrinsic spin. The equation describing the acceleration of such particles can be found using the parallel transport along geodesic curves, and it will include curvature terms. If the quantity is vectorial, there will be no parameters that we can take to zero to reveal an approximately flat evolution equation, even if the quantity is had by a body occupying only a 1-dimensional curve. Indeed, geometric effects of curvature along a single geodesic could be witnessed if (classical) particles with intrinsic spin were admitted in our models.

In more detail, by supposing we have a time-like geodesic with tangent v^a, possessing an intrinsic vectorial quantity, s^a, with zero initial velocity,[21] with $s^a s_a = 1$,:

$$v^a\nabla_a v^b = 0, \;\; v^a v_a = -1, \;\; v^a\nabla_a s^b = s^a\nabla_a v^b, \tag{2.14}$$

we can recycle the derivation of (1.12), obtaining the same result for the parallel transport acceleration of s^a along the time-like direction determined by the geodesic:

$$\frac{\mathrm{D}^2 s^a}{\mathrm{d}t^2} := v^c\nabla_c(v^d\nabla_d s^a) = R^a{}_{bcd}v^b v^d s^c \tag{2.15}$$

The important difference between this and the geodesic deviation equation (1.12) is that here, by limiting the analysis to smaller and smaller neighborhoods of the geodesic, we don't simultaneously take the 'intrinsic spin' s^a to become smaller and smaller. That is, there is no correlation between s^a and the tubular radius ρ. Therefore, for such quantities, even if we were in a freely-falling 'arbitrarily small elevator', we should be able to distinguish the evolution of s^a in a curved background from that in a flat background. Indeed, this is essentially the source of the famous frame-dragging effect, recently experimentally verified by the Gravity Probe-B experiment (Everitt et al., 2011).

But of course, the instrumentation of Gravity Probe-B *has* physical extension. So the question arises: is it really possible to give physical meaning to a limiting procedure that leads to a point-like particle with intrinsic spin, *in classical general relativity*? And here the answer is *no*: as discussed in (Weatherall, 2018, Footnote 10), such an approximation would flout all the reasonable energy conditions expected of a classical material source:

[21] This term is here ambiguous: I mean such that the Lie derivative of $\mathbf{s}$ initially vanishes in the direction of $\mathbf{v}$: $\mathcal{L}_{\mathbf{v}}\mathbf{s} = 0$. This assumption is only in place so that we can recycle the proof of (1.12): different assumptions would generically (if no other constraints are imposed) also lead to equations of motion involving the curvature.

Although it is a side issue for present purposes, observe that this result points to a problem with certain approaches to treating the motion of rotating particles that represent "spin" by higher order distributions supported on a curve (Papapetrou, 1951; Souriau, 1974): such particles are incompatible with the energy condition. There is good physical reason for this. For ever smaller bodies to have large angular momentum (per unit mass), their rotational velocity must increase without bound—leading to superluminal velocities, which are incompatible with the energy condition.

3 Conclusion

I analysed three senses in which spacetime could be interpreted as locally like Minkowski, the third of which was not mentioned by Fletcher & Weatherall (2023a). I conclude that sense (i)—the tangent space interpretation—is robust, but not very informative; sense (ii)—the local coordinate chart interpretation—is indeed weak, and the more strictly geometric understanding of Riemann-normal coordinates, mentioned at Section 2.2.b as the 'great merit' of Fletcher & Weatherall (2023a), lays bare that weakness; and finally, (iii)—the geodesic deviation interpretation—is robust, and it is more informative than the tangent space interpretation, as it describes local scales under which the effects of curvature can be ignored. In more detail, apart from assessing Fletcher & Weatherall (2023a)'s claims, my positive argument in this paper is that a generic spacetime is locally flat in the sense that, for any geodesic and any point along that geodesic, we can always find a sufficiently small tubular neighborhood of this geodesic so that all putative effects of the Riemann curvature on the relations between freely-falling point-like particles remain arbitrarily small.

Acknowledgements

I would like to thank Jeremy Butterfield for reading and commenting on an early draft of this paper. I would also like to thank Sam Fletcher and Jim Weatherall for taking the time to patiently discuss their results with me.

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