Particles without symmetry

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Abstract

The standard model of particle physics relies heavily on the idea of *symmetry*. But recently, a new geometry-first picture has been proposed in which the relevant symmetries do not appear explicitly (Gomes, 2024). Here, I extend the initial correspondence to include alternative—and more transparent—accounts of spontaneous symmetry breaking and of the Yukawa coupling. These reformulations allow us to replace explanations that employ typical particle physics ideas and terminology with ones that are phrased geometrically, e.g. in terms of vector fields and their angles.

1 Introduction

Should we value mathematically equivalent formulations of a theory? Feynman (1994, p. 127) gives the gist of my preferred answer to this question:

> Every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics. He knows that they are all equivalent, and that nobody is ever going to be able to decide which one is right at that level, but he keeps them in his head, hoping that they will give him different ideas for guessing.

And he further reflected on the value of alternative ways of thinking about a theory in his Nobel Prize Lecture ("The Development of the Space-Time View of QED", 1965), in which he discussed his path integral formalism, which was mathematically equivalent to Schwinger's earlier approach to quantum field theory:

> Theories of the known, which are described by different physical ideas may be equivalent in all their predictions and are hence scientifically indistinguishable. However, they are not psychologically identical when trying to move from that base into the

> unknown. [...] If every individual student follows the same current fashion in expressing and thinking about electrodynamics [...], then the variety of hypotheses being generated ... is limited.

The subsequent career of Feynman's path integrals is a testament to the validity of his arguments.

Another classic case further illustrates the point: Minkowski's 1908 recasting of Einstein's special relativity into the language of four-dimensional spacetime geometry. Again, the underlying physics was unchanged, but the shift in formulation was decisive for future developments. Einstein initially dismissed Minkowski's treatment as "überflüssige Gelehrsamkeit" (superfluous erudition), yet by 1912 he had conceded that only the spacetime formulation revealed the true essence of the theory.[1] What Minkowski introduced was not new predictions but a new ontology: space and time no longer standing apart, but merged into a single structure. And it was precisely this geometrical vantage point that enabled the later generalisation to general relativity (cf. (Stachel, 2002, p. 226)).

In the current paper, I want to provide such an alternative description, though here of gauge theory, not quantum electrodynamics or special relativity, and of a much more humble and less radical nature than either Feynman's or Minkowski's reformulations. Namely, I want to provide an alternative formulation of particle theory without the use of symmetry. I will call this formulation *geometry-first*.

The knowledgeable reader will be quick to point out that gauge theory is already highly geometrical: principal fiber bundles and connections are, after all, the stock-in-trade of the geometer. But the standard formulation also builds symmetry into the foundations, and with it an ontology that extends beyond the spaces where matter fields actually live, the so-called principal fiber bundles. By *geometry-first* I mean a formulation that works directly with those spaces—the vector bundles of matter fields—and does not rely on principal fiber bundles and symmetry to get off the ground. I do *not* claim that this picture is superior in every respect, or practically advantageous. The aim is more modest: to offer an alternative perspective that clarifies some features of particle theory while omitting symmetry at the base of the explanatory chain.

Needless to say, symmetry is the cornerstone of particle physics. Representations of Lie groups, Casimir invariants, spontaneous symmetry breaking, gauge-fixing: these are the daily bread of the standard model. (This much will be obvious to anyone familiar with the field, so I need not belabor the point.) That the associated principal bundles—and with them the explicit appeal to symmetry at the base of the explanatory chain—might be dispensed with is therefore anything but trivial.

But recently, a new geometry-first formulation has been proposed in which the symmetries

[1] In 1912 Einstein wrote to Sommerfeld: "I have come to value greatly the four-dimensional formalism of Minkowski, which I had previously considered unnecessary erudition. In the meantime, I have also become convinced that only this formalism brings out the true essence of the theory." (quoted in (Holton, 1974, p. 263))

are not postulated and principal fiber bundles are unnecessary (Gomes, 2024, 2025a). In the alternative formulation, which I called 'geometry-first' above, the symmetry groups are only implicit: they arise as the automorphism groups of vector bundles. The geometry-first formulation is available as an alternative only for gauge groups that are linear, and for representations that are obtained from the fundamental representation (when it is unique) via tensor and direct products, symmetrisation, etc. Thus theories whose symmetry groups have no linear representation (such as infinite simple or torsion groups), or groups that have no unique fundamental linear representation, such as some exceptional groups, are outside the scope of an equivalent geometry-first formulation. But even in the cases that admit the two mathematically equivalent formulations, the geometry-first one comes with a significantly different ontology: for the standard model of particle physics, it consists of three fundamental vector bundles over spacetime where the various matter fields reside (as sections of tensor products). There is no need for a separate space to encode the principal connections.

Change the formulation, and the explanations change with it. Three examples illustrate how features of particle physics acquire alternative interpretations. First, in a non-Abelian vacuum Yang–Mills theory with Lie group G, the fundamental dynamical object is no longer a connection ω on a G-principal fiber bundle (or its spacetime representative A^I_μ), but rather the covariant derivative D_μ on a vector bundle whose automorphism group corresponds to G—and this remains true even if no vector fields on that bundle are present to be covariantly differentiated. Second, once symmetry groups drop out of the base level of the explanatory chain, the very notion of 'symmetry-breaking' must be reinterpreted. Third, in this formulation, vector bosons A^I_μ appear only as coordinate-dependent representations of the covariant derivatives of the fundamental bundles. They are not on the same footing as other particle fields, and it becomes unclear how they could 'acquire mass' through symmetry-breaking. The alternative explanation of the first feature was developed in Gomes (2024); here I will concentrate on the latter two.

A further case in point concerns the Yukawa couplings. In the standard formulation, Yukawa terms are scalars formed from sections of different associated bundles, which requires the introduction of explicit maps or 'bridges' between those bundles. By contrast, in the geometry-first formulation, the fundamental objects are vector bundles themselves, with different particles emerging from the corresponding tensor bundles. Scalars then arise naturally by combining inner products and contractions between vectors and their duals.

Here is how I will proceed: in Section 2 I will introduce both the familiar picture of principal and associated bundles and the alternative, *vector bundle point of view*, which I referred to as 'the geometry-first formulation' above. In section 3 I will provide the alternative explanations for the Higgs mechanism, and in Section 4, I will not attempt a full reformulation of the Yukawa mechanism, but will argue that its interpretation is more transparent in the geometry-first formulation. Finally, in Section 5 I will conclude with some methodological morals.

2 Symmetry-first and geometry-first formulations of gauge theory

Here I will give brief overviews of both the familiar, symmetry-first, and of the less familiar, geometry-first formulations of gauge theory. I will start with the more familiar and then introduce the novel.

2.1 Gauge theory and principal fiber bundles: the symmetry-first formulation

In short, the symmetry-first formulation of gauge theory is the familiar one, in which the symmetry group in question—one per fundamental interaction—is fixed as the structure group of a principal fiber bundle. Connections of this principal fiber bundle then play the role of vector bosons of the theory: they are the 'force-carriers'. Each classical configuration for each type of matter particle that interacts with each force is given as a section of an associated vector bundles, i.e. associated with the principal bundle whose group encodes that force.

The connection of the principal fiber bundle determines parallel transport in each of the associated vector bundles. The fact that it is the very same connection responsible for parallel transport on all the vector bundles (associated to the corresponding principal bundle) ensures that different matter fields that are charged under the same fundamental force march in step under parallel transport. In other words, this fact ensures such charged matter fields probe the same distributions of electroweak and strong forces. In this view, associated vector bundles are separate entities, but they are connected by the principal fiber bundle, which 'coordinates' them (see (Weatherall, 2016) and Figure 1). The fact that the structure group is postulated first, and that it plays such a pivotal role, is what makes this a 'symmetry-first' formulation. I call this the *principal bundle point of view* on gauge theory (PFB-POV). (To make this a relatively self-contained paper, I provide more mathematical details in Appendix A.)

It will prove useful to know that, given any vector bundle (E, M, V) (see Definition 4 in Appendix A), the bundle of frames for E, called $L(E)$, is itself a principal fiber bundle $(L(E), M, GL(V))$: here elements of $\pi^{-1}(x)$ are linear frames of E_x, and $G \simeq GL(F)$ acts via ρ on the typical fibers. By construction, $E \simeq L(E) \times_\rho V$. Now, for $G' \subset G \simeq GL(V)$ we can partition the points of each orbit in P, $\mathcal{O}_p := Gp$, into orbits of G'. Each such choice gives a principal bundle with group G' and it induces further structure on the associated vector bundle, e.g. an inner product, by selecting which frames are considered orthonormal. This is also a principal fiber bundle, $(L'(E), M, G')$, whose structure group is a proper subgroup of the general linear group, $G' \subset GL(V)$, taken to be the group that preserves the structure of V. This is called the *bundle of admissible frames*, e.g. of orthonormal frames. Conversely, if V has more than just the structure of a linear vector space, e.g. if it is endowed with an inner product, it will induce a subgroup $G' \subset GL(V)$ on P that respects that structure.

An important question is whether different vector bundles with the same typical fiber and associated to the same principal bundle P are canonically related. Thus suppose we are given:

$$E_1 = P \times_{\rho_1} V, \quad E_2 = P \times_{\rho_2} V \tag{2.1}$$

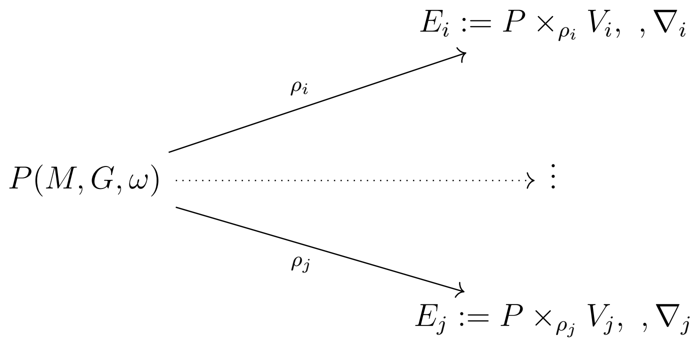

Figure 1: The principal G-bundle , with structure group G, over the manifold M, with a principal connection ω (a $\mathfrak{g}$-valued one-form on P), abbreviated by $P(M, G, \omega)$, and its associated vector bundles $E_i := P \times_{\rho_i} V_i$, where $\rho_i : G \to V_i$ is a representation of the Lie group onto the vector space representing the typical fiber, V_i which is linearly isomorphic to $\pi_i^{-1}(x)$, for $x \in M$ and $\pi_i : E \to M$ the projection of the vector bundle onto its base space (spacetime). The covariant derivatives ∇_i are the ones induced by ω, as per Equation (A.6). See Appendix A for more details.

Given a local section of P, i.e. for $U \subset M$ a map $\sigma_U : U \to P$ such that $\pi(\sigma(x)) = x$, for all $x \in U$ (see Appendix A), we can write, for ξ_1 a local section of E_1:

$$\xi_1(x) = [\sigma(x), v(x)]_1, \quad v : U \to V. \tag{2.2}$$

Then the obvious map to consider is:[2]

$$\begin{aligned} T : E_1 &\to E_2 \\ [\sigma(x), v(x)]_1 &\mapsto [\sigma(x), v(x)]_2. \end{aligned} \tag{2.3}$$

So the map acts as the identity on both entries.

But on the right-hand side the representation under which we take equivalence classes, $\sim_2$ and not $\sim_1$, is different. So is this map well-defined for arbitrary representations ρ_1, ρ_2? The map should be invariant under gauge transformations (cf. Eq (A.5)) on both the domain and image. So consider a different representative of the equivalence class on the domain; according to (2.3) we must have:

$$[g(x) \cdot \sigma(x), \rho_1(g^{-1}(x))v(x)]_1 \mapsto [g(x) \cdot \sigma(x), \rho_1(g^{-1}(x))v(x)]_2 \tag{2.4}$$

for any $g : U \to G$. But on E_2, we have the representation ρ_2, and so we must have (omitting dependence on $x \in M$ for clarity):

$$(\sigma, v) \sim_1 (g \cdot \sigma, \rho_1^{-1}(g)v) \sim_2 (\sigma, \rho_2(g)\rho_1^{-1}(g)v) \not\sim_2 (\sigma, v). \tag{2.5}$$

Where the last inequivalence holds iff $\rho_1 \neq \rho_2$. Thus we find that for the map (2.3) to be well-defined, we must have $\rho_1 = \rho_2$.

[2] I thank Jim Weatherall for suggesting this.

Indeed, in physics, we are often faced with situations in which E_1 and E_2 have the same typical fiber, are associated to the same group, and yet have different representations. A simple example is when one of the representations is the trivial, or singleton, one, and the other is the fundamental (or any other).[3] This occurs many times in the standard model: for fermions to acquire mass, one must relate sections of bundles that have different representations, since they represent different particles (see Section 4).

We will see how this issue can arise in practice, and how both the symmetry-first and the geometry-first formulations deal with it in Section 4.

2.2 Gauge theory and vector bundles: the geometry-first formulation

The geometric perspective I want to develop aims to dispense with the principal fiber bundle altogether. In this Section I set out a formulation of gauge theory that proceeds without gauge potentials, principal bundles, or explicit appeal to gauge symmetries. On this approach, the gauge potential is nothing but a coordinate expression of an affine covariant derivative on a vector bundle—just as Christoffel symbols are the coordinate expressions of the Levi-Civita connection in differential geometry.

The analogy with spacetime clarifies what is at stake. Consider (M, g, O_i), where (M, g) is a smooth Lorentzian manifold and the O_i are various tensor fields on M, i.e. objects living in spaces constructed from the tangent bundle TM. The automorphism group of a typical fiber T_xM is $O(3,1)$ (or $SO(3,1)$ if orientation is treated as background structure). This group becomes explicit once we introduce orthonormal frames. Yet much can be said about the O_i in a purely geometric, frame-independent manner, without any reference to $SO(3,1)$. If instead we were to posit a different group acting on TM—say $O(2)$—a geometrical rationale would be required to justify that choice. In gauge theory, by contrast, an analogous "frame-free" formulation for the behavior of matter remains largely undeveloped (cf. (Weatherall, 2016)), and the very idea of a geometric interpretation of the groups and their representations—for example, the adjoint action of $SU(2)$ on $\mathbb{C}^3$—is seldom raised.[4]

Now, to connect the vector bundle point of view (which is a geometry-first formulation) to the symmetry-first formulation provided by the principal bundle, let me recall that (P, M, G, ω) are used for coordinating covariant derivatives. But what is the physical status of these objects?

[3] A slightly more sophisticated example is as follows. Let $G = U(1), V = \mathbb{C}^k$, and $\rho_i = n_i$, which acts as $e^{in_i}\mathbb{1}$ on $\mathbb{C}^k$. Then for $n_i \neq n_j$ for $i \neq j$ the map (2.3) is not well-defined, as can easily be verified.

[4] Here is a more complete example: given $\mathbb{C}^n$ and $U(m)$, we can examine two cases. If $m \geq n$, there exists a *faithful* representation, given by the block-diagonal inclusion. If $m < n$, there still exist non-trivial (though not faithful) representations. For $m = 1$, the determinant map $\det : U(n) \to U(1)$ yields a non-trivial one-dimensional representation. For general m, one may take symmetric or exterior powers of the defining representation of $U(n)$ on $\mathbb{C}^n$, or tensor products of these, and then project onto an m-dimensional invariant subspace. The point here is that these representations would require, from the geometry-first perspective, a geometric justification: e.g. how do we geometrically interpret the subspace corresponding to the block-diagonal inclusion, or even the determinant map; how do we choose the m-dimensional invariant subspace onto which to project; etc.

Jacobs (2023, p. 41) convincingly argues they don't have one; he concludes:

> Neither the principal bundle nor the [principal] connection on its own represent anything physical. Rather, it is the induced connection on the associated bundle that represents the Yang-Mills field. [But] This approach has difficulties in accounting for distinct matter fields coupled to the same Yang-Mills field.

The issue, as he sees it, is that

> there is no independent Yang-Mills field that the associated bundle connections supervene on. This makes it seem somewhat mysterious that these connections are equivalent. The coordination between associated bundles begs for a 'common cause' in the form of an independently existing Yang-Mills field.

I agree with Jacobs that this is an issue and in (Gomes, 2024) I showed that it can be overcome. The introduction of PFBs is unnecessary if particles that interact are all sections of the same vector bundles or of tensor products of the same vector bundles. Tensor products over a vector bundle inherit the same covariant derivatives by construction. In this case, parallel transport of the vector bundles in question automatically march in step. In this case we have at a hand a natural 'common cause' for the coordination of covariant derivatives, without the introduction of principal bundles. I will here call this *the vector bundle point of view* of gauge theory (VB-POV).

In more detail, given two vector bundles, E, E', a covariant derivative on E will induce a covariant derivative on E' whenever E' is equal to a general tensor product involving E and its algebraic dual, E^*. In more detail, given E a vector bundle with covariant derivative D, and E^* its dual, we define, for sections $\kappa \in \Gamma(E)$ and $\xi \in \Gamma(E^*)$:

$$d(\langle \xi, \kappa \rangle)(X) = \langle \nabla^*_X \xi, \kappa \rangle + \langle \xi, \nabla_X \kappa \rangle, \tag{2.6}$$

where here angle brackets represent contraction. The generalisation to arbitrary tensor products is straightforward due to multilinearity.

On this view, there are no "gauge groups" at all—only groups of automorphisms of vector bundles, $\mathsf{Aut}(E) \subset \mathsf{End}(E)$. The familiar distinction between Abelian and non-Abelian theories is then simply a distinction between different kinds of automorphism groups. In particular, one-dimensional vector bundles, whose typical fiber is isomorphic to $\mathbb{C}$, generate Abelian automorphism groups.

This vantage point also reframes the earlier question of whether there exist canonical maps between distinct vector bundles. In the PFB-POV, the natural candidate (equation (2.3)) is well-defined only within the same representation. Matters look different here. We assume that all vector bundles charged under a given force descend from a single "fundamental" bundle, E^n, whose typical fiber is $\mathbb{C}^n$ equipped with inner product and orientation. Different associated bundles then appear not as unrelated objects in need of ad hoc identifications, but as systematic constructions from E^n. Their relations are fully accounted for by the usual functorial machinery: tensor products, (anti)symmetrization, dualization, projections into tensor factors, contractions, interior products, inner products, and so on. There is thus no mystery about how these bundles fit together—the geometry itself provides the correspondences. For instance, to contract an element of E^n with one of $E^{n*} \wedge E^{n*} \otimes E^n$, we can use the interior product, which generally is a map:

$$\begin{aligned} \iota : E^n \otimes \Lambda^m(E^{n*}) &\to \Lambda^{m-1}(E^{n*}) \\ (\xi, \Omega) &\mapsto \iota_\xi \Omega, \end{aligned} \tag{2.7}$$

where Λ is the anti-symmetric product, with $\Omega \in \Lambda^m(E^{n*})$, and, for any $m-1$-tuple $(\xi_1, \cdots, \xi_{m-1})$ gives

$$\iota_\xi \Omega(\xi_1, \cdots, \xi_{m-1}) = \Omega(\xi, \xi_1, \cdots, \xi_m), \tag{2.8}$$

etc. Similarly, we could use the inner product to map between E^n and E^{n*}, and so on.

One might object that a parallel, representation-theoretic argument for associated vector bundles could be mounted, mirroring the geometric one I have just given. That may well be true—but it is beside the point. Even if such arguments exist (and I have not found or worked one out), the virtue of the geometric route is that it speaks directly to a community trained in geometry rather than in group and representation theory. The mere availability of a geometric formulation that sidesteps representation theory is already a win. My aim, after all, is to broaden the borders of the subject, making it accessible to different habits of thought.

Still, at first pass the VB-POV may seem too narrow to capture the full menagerie of gauge theories employed in physics. Some theories—those built from the exceptional Lie groups, for example—fall outside its reach. And even when a gauge group G is given, it is often a nontrivial matter to "reverse-engineer" a vector space structure for which $\mathsf{Aut}(E_x) \simeq G$. How, for instance, does one coax $U(1)$ out of a space whose typical fiber is $\mathbb{C}^n$ with $n \neq 1$?[5]

For all that, the standard model of particle physics fits neatly within this framework. Every particle field is a section of an associated bundle for some principal fiber bundle whose structure group is $SU(n)$ or $U(n)$ for appropriate n. Moreover, under any representation of $U(n)$, the corresponding associated bundles can just as well be constructed by geometric means from the fundamental vector bundle—via tensor and exterior products, (anti)symmetrization, determinants, and the like. In such cases, a covariant derivative on a single vector bundle suffices to encode one fundamental interaction, while the various particle fields appear as sections of the appropriate derived bundles (e.g. tensor products).

Having surveyed both approaches to gauge theory—the symmetry-first PFB-POV and the geometry-first VB-POV—I now turn to the Higgs mechanism. My aim is to present it from within the VB-POV, while relegating to Appendix B a sketch of the more familiar PFB-POV treatment, which can be found in any standard textbook.

[5] The Peter–Weyl theorem guarantees that $U(n)$ admits nontrivial representations on $\mathbb{C}^m$, but extracting from this a natural structure on $\mathbb{C}^m$ that renders the action geometrically meaningful is anything but straightforward.

3 The Higgs mechanism in the geometry-first formulation

The proof, they say, is in the eating of the pudding. So here, to prove that the geometry-first perspective embodied by the VB-POV is sufficiently different to the PFB-POV to merit attention, I will provide a stand-alone derivation of the Higgs mechanism.

In the standard presentation, the Higgs mechanism is often described in terms of spontaneous symmetry breaking, and one must employ Goldstone's theorem, gauge fixing (e.g. unitary gauge), etc. I give a brief overview of that presentation in Appendix B. Here I will outline an alternative approach, phrased purely in the geometric language of vector bundles, which makes the essential structure transparent without appeal to symmetry-breaking jargon.

3.1 The non-linearised Higgs field

Let $(E^n, M, \mathbb{C}^n, \langle\cdot,\cdot\rangle_n, \nabla_n)$ be a Hermitian vector bundle over a manifold M, with fibers $E^n_x \simeq \mathbb{C}^n$ and $\langle\cdot,\cdot\rangle_n$ an inner product on E^n, which is compatible with ∇_n, the covariant derivative on E^n. We will omit the subscript when it is understood from context, as it will be in this Section, so for now we take $\varphi \in \Gamma(E)$ (the generalisation to $\varphi \in \Gamma(E^i \otimes \cdots E^j)$ is straightforward, as we will see). So φ is a vector-valued spacetime scalar field, satisfying

$$\min_{x\in M} \|\varphi(x)\| = v', \tag{3.1}$$

for some constant $v' > 0$. We write $\|\varphi(x)\| = (\Delta + v')$, for $\Delta \in C^\infty_+(M)$ (the positive real-valued smooth scalar functions on M), and get

$$\varphi(x) = \|\varphi(x)\| e_0 = (\Delta(x) + v')e_0, \tag{3.2}$$

where $e_0 = \frac{\varphi}{\|\varphi\|}$ is a unit section, well-defined since $\|\varphi\| > v' > 0$, and $\langle e_0, e_0\rangle = 1$.

The potential term in the Lagrangian—the Higgs potential, $V(\varphi)$—is assumed to enforce such a nonzero minimum, but it need not coincide with v': we call v the minimum of the potential. Our focus will be on the kinetic term. Note that:

$$\nabla\langle e_0, e_0\rangle = 2\mathsf{Re}\,\langle e_0, \nabla e_0\rangle = 0, \quad \text{and} \quad \nabla v' = 0, \tag{3.3}$$

where Re takes the real component. Using (3.2) and (3.3) the kinetic term reads

$$\langle\nabla\varphi, \nabla\varphi\rangle = \|\nabla\varphi\|^2 = (\partial\Delta)^2 + (\Delta + v')^2\langle\nabla e_0, \nabla e_0\rangle, \tag{3.4}$$

where ∂ is the exterior derivative acting on scalars; i.e. it is the gradient.

When we introduce a connection, it will clearly appear quadratically in the term $v'^2\langle\nabla e_0, \nabla e_0\rangle$ (see Equation 3.6 below). But of course, ∇e_0 won't contain all the information in ∇. The part of ∇ that doesn't appear in the kinetic term will thus remain 'massless'. This geometric presentation of the Higgs mechanism makes the key features clear: the scalar vev picks out a direction in the bundle, and vector bosons associated with directions orthogonal to it acquire mass. Since we have expressed everything in terms of abstract index notation, with vector and tensor fields, it is hard to see how one could 'break the symmetry'. Indeed, the mass terms for the gauge potentials will arise out of a combination of ve_0 and the gauge potentials, and these are perfectly gauge-covariant.

Moreover, it is important to note that this is a geometric characterization that can be stated outside of the linearised regime. In this remarkably simple derivation, we are already able to glimpse all the general features of the mechanism. Again, no mention of stabilisers, gauge orbits, gauge-fixing, etc, was made, as they would have in order to reach a similar point in the standard or familiar derivation (see (Hamilton, 2017, Ch. 8.1) for a comparison). For instance, the fact that perturbations of the Higgs field are orthogonal to the orbits of the vacuum is replaced by the orthogonality relation, (3.3), and so on. This concludes the non-perturbative account of the 'mass acquisition' mechanism.[6]

3.2 Mass Generation in the Linearised Theory

Introduce a connection $\nabla = d+\omega$ such that $de_0 = 0$ and $\omega \in \Gamma(T^*M \otimes \mathsf{End}(E))$, where $\mathsf{End}(E)$ are the linear endomorphisms of E; so for $\xi \in \Gamma(E)$, we have $\omega \cdot \xi \in \Gamma(T^*M \otimes E)$. Defining $v' - v =: c$, for v a spacetime-independent (i.e. 'translation-invariant') minimum of the Higgs potential, we rewrite (3.2) as

$$\varphi(x) = (H(x) + v)e_0, \tag{3.5}$$

where $H(x) = \Delta(x)+c$. If we assume that c and Δ are of the same order, since $c = (v'-v) < 0$ and $\Delta(x) > 0$, $H(x)$ can be both positive or negative, i.e. $H \in C^\infty(M)$.[7] Then from (3.4)

$$\|\nabla\varphi\|^2 = (\partial H)^2 + (H^2 + 2Hv + v^2)\,\|\omega \cdot e_0\|^2, \tag{3.6}$$

where, as usual, the norm of a tensor product factorises linearly, i.e. for each basis element $\lambda \otimes \xi \in \Gamma(T^*M \otimes E)$, we have:

$$\|\lambda \otimes \xi\| := \|\lambda\|_M \|\xi\|_E. \tag{3.7}$$

But to unclutter notation I will omit the subscripts when understood from context.

[6] This entire paper concerns the classical domain, and so one may reasonably argue that these symmetry concepts—such as gauge-fixing—may be required when we introduce quantum mechanics. Here is how far my concession would go: in a sum over configurations, we use e_0 as the anchor, or 'representational scheme' across physical possibilities; cf; (Gomes, 2025b; Kabel et al., 2025). And indeed, representational schemes can be compared to gauge-fixings (cf. (Gomes, 2025b, Sec. 3.3)). A translation of this idea to the guage terminology would go as follows: consider $\Gamma(E^2)$, and its sector of nowhere vanishing elements, $\Gamma_0(E^2)$. Let $\varphi, \varphi' \in \Gamma_0(E^2)$. The group $\mathsf{Aut}(E^2)$ acts transitively on the unit normal sections: it can take any internal direction into any other. Therefore, we could, by a suitable gauge transformation on φ, make it collinear with φ'. Once they are collinear, it is a trivial matter to separate out the part that has a given norm from the rest.

[7] We could of course have started directly from (3.5), by again assuming that: (i) the potential depended only on the norm of the Higgs field; (ii) that the minimum of the potential was non-zero and spacetime independent; and (iii) that the norm of the Higgs field did not deviate too much from this minimum, in particular, that it was also non-zero everywhere. I find the order of assumptions made in my presentation clearer, because they can be easily stated outside the linearised regime.

Further assuming that $\mathcal{O}(H) = \mathcal{O}(\omega) = \varepsilon$,[8] yields

$$\|\nabla\varphi\|^2 = (\partial H)^2 + v^2\|\omega \cdot e_0\|^2 + \mathcal{O}(\varepsilon^3). \tag{3.8}$$

Here we see clearly how the quadratic terms in the connection ω would correspond to vector bosons 'acquiring masses'; again, without invoking unitary gauge or Goldstone's theorem.

But as I said, not all components of ω contribute to $\|\omega \cdot e_0\|^2$ in (3.8). In a basis $\{e_I\}$ adapted to e_0, we have

$$\nabla e_I = \omega^J{}_I\, e_J, \quad \text{and so} \quad \nabla e_0 = \omega_0^i e_i, \quad \text{with} \quad i \neq 0, \tag{3.9}$$

from the anti-symmetry of the connection. Then

$$\|\nabla\varphi\|^2 = (\partial H)^2 + v^2 \sum_{i\neq 0} (\omega^i{}_0)^2 + \mathcal{O}(\varepsilon^3). \tag{3.10}$$

Hence, only those components of ω that move e_0 (onto the orthogonal directions) 'acquire mass'. The components that preserve e_0, e.g. $\omega^i{}_j, i \neq j$, remain massless. In the group-theoretic language, these would correspond precisely to the stabiliser subgroup of e_0.

This concludes the geometric derivation of the Higgs mechanism. Let us now see how it reproduces standard results from the familiar or standard approach to gauge theory. The missing ingredient for the comparison is to write the connection ω in terms of preferred representations of the Lie algebras in question. I will start by providing an example (that is indeed isomorphic to $su(2)$) before showing how the usual endpoint of the Higgs mechanism for gauge bosons is entirely recovered.

3.2.a Example: $so(3) \simeq su(2)$

A general $so(3)$ connection has the form

$$\omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}. \tag{3.11}$$

If the Higgs unit vector is $e_0 = (1,0,0)^T$ (where T here is the transpose, and it allows us to write column-vectors in-line!), then

$$\omega \cdot e_0 = \begin{pmatrix} 0 \\ \omega_z \\ -\omega_y \end{pmatrix}. \tag{3.12}$$

Thus, we would get:

$$\|\nabla\varphi\|^2 = v^2(\omega_y^2 + \omega_z^2). \tag{3.13}$$

So ω_y and ω_z would 'acquire mass', while ω_x would remain 'massless'.

[8] In the comparative sense: that $\frac{|H|}{v} \sim \varepsilon << 1$, and *mutatis mutandis* for the appropriate norm on ω.

3.2.b Electroweak Example: $SU(2) \times U(1)$

The covariant derivative on an element $\mathbf{v} \otimes \mathbf{w} \in V \otimes W$ is given by

$$\nabla(\mathbf{v} \otimes \mathbf{w}) = (\nabla^V \mathbf{v}) \otimes \mathbf{w} + \mathbf{v} \otimes \nabla^W \mathbf{w}, \tag{3.14}$$

where ∇^V, ∇^W are covariant derivatives on V, W, respectively.

For the electroweak theory, let $e_0 = e_0^2 \otimes e_0^1 \in \Gamma(E^2 \otimes E^1)$ with $e_0^2 = (0,1)$, $e_0^1 = 1$. And so we get:

$$\nabla e_0 = \omega \cdot e_0^2 + e_0^2 Z = (\omega + iZ\mathbb{1})e_0^2, \tag{3.15}$$

where ω is the connection for the covariant derivative on $\mathbb{C}^2$ and Z is the connection on $\mathbb{C}$. To complete the comparison with the standard formalism, we choose the weak-isospin eigenbasis, on which the third generator of the $\mathfrak{su}(2)$ algebra, $\mathbb{T}_3$, is diagonal. Omitting the coupling constants for brevity, we can write ω as:[9]

$$\omega = \begin{pmatrix} iW_3 & iW_1 - W_2 \\ iW_1 + W_2 & -iW_3 \end{pmatrix}, \text{and} \quad iZ\mathbb{1} = \begin{pmatrix} iZ & 0 \\ 0 & iZ \end{pmatrix}. \tag{3.16}$$

Applying this to e_0^2 in (3.15) gives

$$\nabla e_0 = \begin{pmatrix} iW_1 - W_2 \\ -iW_3 + iZ \end{pmatrix}. \tag{3.17}$$

Hence the corresponding quadratic term appearing in (3.8) is

$$\|\nabla e_0\|^2 = W_1^2 + W_2^2 + (Z - W_3)^2. \tag{3.18}$$

Thus W_1, W_2 and the combination $Z - W_3$ acquire mass, while $Z + W_3$ remains massless. The latter is identified with the photon. Of course, had we chosen a different form for e_0^2, we would have obtained different combination of massive and massless bosons. For instance, for $e_0^2 = (1,0)$ it is easy to see that it would have been $Z + W_3$ that would acquire mass, while $Z - W_3$ would remain massless.

4 The Yukawa mechanism

Whereas the Higgs mechanism is used to 'endow mass' to the gauge potentials, the Yukawa form is used to endow mass to the matter fields—here we needn't use scare-quotes!

In the Standard Model fermion masses cannot be introduced as they can for real or complex-valued scalar fields. First of all, a Dirac mass term must couple left- and right-handed chiral

[9] Note that this is not the ω written in terms of the spin coefficients, i.e. in terms of an orthonormal frame that includes e_0. That could also be done, and indeed it was done in the previous example $so(3) \simeq su(2)$, with an orthonormal frame $(0,1), (0,i), (1,0), (i,0)$, for the inner product $\mathrm{Re}\langle \cdot, \cdot \rangle$, which is effectively what appears in Lagrangians, due to the use of the complex conjugate terms, cf. (Hamilton, 2017, Ch. 8). Here we are attempting to make contact with the standard notation and formalism and so are using its conventions.

fermions; moreover, the two chiralities are mapped into internal spaces that transform differently under the gauge group $G = SU(3) \times SU(2) \times U(1)$, so coupling them would violate gauge invariance: this is related to the issue we saw in Section 2 about canonical isomorphisms between associated vector bundles with different representations. The solution is to introduce the Higgs field ϕ, in such a way that gauge invariance is preserved, while the fermions acquire effective masses. This is the *Yukawa mechanism*.

Here I will essentially follow the treatment given in (Hamilton, 2017, Ch. 8), whose notation and general approach is already much closer to the geometric approach that I'm pursuing here (as compared to the treatment of more familiar textbooks, for instance, the one given in (Weinberg, 2005, Ch. 21), which uses representation theory more heavily). So I will call the treatment to be followed here 'the standard' treatment of the Yukawa mechanism. In Section 4.1 I will describe the obstruction to the formulation of mass terms for fermions, and its resolution in this, geometric-friendly but still 'standard', exposition. Then in Section 4.2 I will discuss what I think is explanatorily unsatisfactory about this resolution, and say why I take the VB-POV to provide a more transparent explanation.

4.1 The 'standard' presentation of the Yukawa mechanism

In more detail, here is the obstruction to the formulation of mass terms for fermions. Fermions are spinors, but for Weyl spinors, the inner product is anti-diagonal in the left and right basis: $\overline{\psi}_R\psi_R = 0$, and so, in order to extract mass terms we must couple left to right-handed spinors: $\overline{\psi}_R\psi_L$. Thus, if both ψ_L and ψ_R are valued in the same internal space, i.e. in the same vector bundle, and are in the same representation, one may add mass terms of the form:

$$\mathcal{L}_{\text{mass}} = -m\,\overline{\psi}\psi = -m\text{Re}(\overline{\psi}_L\psi_R) \tag{4.1}$$

and this will be gauge invariant since ψ_L and ψ_R transform in the same representation of the gauge group. I.e. locally, $\psi_L \in \Gamma(S_L \otimes E)$, where (E, M, V) is the vector bundle with the representation space V of the gauge group in question, and S_L is the bundle of left-handed spinors over spacetime, whose typical fiber space is called Δ_L (*mutatis mutandis* for right-handed spinors).

In the Standard Model, however, fermions are both twisted and chiral: left- and right-handed components transform in inequivalent representations of the gauge group. For instance,

$$e_L \in (\mathbf{1}, \mathbf{2}, -1), \qquad e_R \in (\mathbf{1}, \mathbf{1}, -2).$$

These internal vector bundles are representationally inequivalent; e.g. $\psi_L \in \Gamma(S_L \otimes E_L)$ and $\psi_R \in \Gamma(S_R \otimes E_R)$, with different representation spaces, $V_L \not\simeq V_R$. Thus a bilinear such as $\overline{e}_L e_R$ is not gauge-invariant, and a bare mass term as in (4.1) is forbidden. (Table 1, reproduced from (Hamilton, 2017, Table 8.2), shows the representations of $SU(2)_L \times U(1)_Y$ for the fermions and the Higgs in the standard model.)

Moreover, for V_R, V_L irreducible, unitary, non-isomorphic representations of G, mass pairings, defined as G-invariant maps, $\kappa : V_L \times V_R \to \mathbb{C}$, complex antilinear in the first variable

Sector	$SU(2)_L \times U(1)_Y$ rep.	Basis vectors	Particle	T_3	Y	Q
Q_L	$\mathbb{C}^2 \otimes \mathbb{C}_{1/3}$	$\binom{1}{0}$	$\mathbf{u}_L$	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{2}{3}$
		$\binom{0}{1}$	$\mathbf{d}_L$	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{3}$
Q_R	$\mathbb{C} \otimes \mathbb{C}_{4/3}$	1	$\mathbf{u}_R$	0	$+\frac{4}{3}$	$+\frac{2}{3}$
	$\mathbb{C} \otimes \mathbb{C}_{-2/3}$	1	$\mathbf{d}_R$	0	$-\frac{2}{3}$	$-\frac{1}{3}$
L_L	$\mathbb{C}^2 \otimes \mathbb{C}_{-1}$	$\binom{1}{0}$	ν_{eL}	$+\frac{1}{2}$	-1	0
		$\binom{0}{1}$	e_L	$-\frac{1}{2}$	-1	-1
L_R	$\mathbb{C} \otimes \mathbb{C}_{-2}$	1	e_R	0	-2	-1
Higgs φ	$\mathbb{C}^2 \otimes \mathbb{C}_1$	$\binom{1}{0}$	φ^+	$+\frac{1}{2}$	$+1$	$+1$
		$\binom{0}{1}$	φ^0	$-\frac{1}{2}$	$+1$	0
Higgs$_\perp$ φ_c	$\mathbb{C}^2 \otimes \mathbb{C}_{-1}$	$\binom{1}{0}$	$\overline{\varphi}^0$	$+\frac{1}{2}$	-1	0
		$\binom{0}{1}$	$-\overline{\varphi}^+$	$-\frac{1}{2}$	-1	-1

Table 1: First-generation fermion representations under $SU(2)_L \times U(1)_Y$, together with the Higgs doublet and its conjugate. Here boldface on the quarks means each such term is a vector in $\mathbb{C}^3$. (φ^0, φ^+) as well as the left-handed particles are doublets: they can be rotated into each other by an $SU(2)$ transformation. Y is the hypercharge, and T_3 is weak isospin. Here $Q = T_3 + \frac{1}{2}Y$.

and complex linear in the second (so that they form mass terms), are necessarily trivial (see (Hamilton, 2017, Theorem 7.6.11)).

The remedy is a *Yukawa form*, defined as follows. Let V_L, V_R, W be representation spaces for $G = SU(3) \times SU(2) \times U(1)_Y$. A Yukawa form is a G-invariant trilinear map

$$\tau : V_L \otimes W \otimes V_R \longrightarrow \mathbb{C},$$

antilinear in V_L, real linear in W, linear in V_R. What to these maps look like, more precisely? Let us look at an example. Consider the $SU(2) \times U(1)$ representations for the leptons (taken from Table 1):

$$V_L = \mathbb{C}^2 \overset{\rho_L}{=} \mathbf{2}_{-1}, \tag{4.2}$$
$$V_R = \mathbb{C} \overset{\rho_R}{=} \mathbf{1}_{-2}, \tag{4.3}$$
$$W = \mathbb{C}^2 \overset{\rho_W}{=} \mathbf{2}_1. \tag{4.4}$$

Then, for $l_L : U \to V_L, \phi : U \to W, l_R : U \to V_R$, it is standard to define the Yukawa form as:

$$\tau : V_L \times W \times V_R \longrightarrow \mathbb{C}, \tag{4.5}$$
$$(l_L, \phi, l_R) \longmapsto l_L^\dagger \phi \, l_R, \tag{4.6}$$

which is $SU(2) \times U(1)$ invariant by construction.

This is still a map on vector spaces and we must invariantly extend it to sections of associated vector bundles, but this is easy to do. Given a section $\sigma(x)$ of an $SU(2) \times U(1)$ principal bundle, we can use the local maps $l_L : U \to V_L$, $\varphi : U \to W$, $l_R : U \to V_R$ above to form sections of the corresponding vector bundles, $e_L \in \Gamma(S_L \otimes E_L)$, $\varphi \in \Gamma(F)$, $e_R \in \Gamma(S_R \otimes E_R)$. For instance, a left-handed quark (I will have more to say about the 'up' and 'down' components of this particle in a second) would be given by:

$$e_L = \psi_L \otimes [\sigma, l_L], \tag{4.7}$$

where ψ_L is a left-handed Weyl spinor, $\psi_L \in \Gamma(S_L)$, and $\lambda_L := [\sigma, l_L] \in \Gamma(E_L)$, where E_L is the vector bundle with typical fiber V_L in (4.2), *mutatis mutandis* for the right-handed electron, and $\varphi := [\sigma, \phi]$. So we can define:

$$T(e_L, \varphi, e_R) := l_L^\dagger \phi\, l_R, \tag{4.8}$$

which is gauge-invariant. But why this map? Now I will provide more details, and include a Yukawa term for quarks. But first, to answer this question, I will first translate it into the VB-POV.

4.2 The VB-POV presentation of the Yukawa mechanism

In Section 2 I argued that there was no canonical map between associated vector bundles corresponding to different representations of the principal bundle, and yet I have just presented a map from different vector bundles into a gauge-invariant scalar. But there is no real mystery here: we don't need a canonical map between associated vector bundles. All we need is that T, given in (4.8), is *a* map between associated vector bundles, with τ a map between the representation spaces; and presenting one such map is sufficient for comparison with experiments. Nonetheless, I find this answer unsatisfactory, because opaque: why this particular map? Couldn't we have found others? What are the possible maps, and how should we interpret them?

I take the geometric, VB-POV, to provide a more transparent interpretation of what the map T represents, and what other choices would represent. Again, in the geometry-first formulation, all we have are structures in the fundamental vector bundle spaces. The fundamental vector spaces are given by $(E^n, M, \mathbb{C}^n, \langle\cdot,\cdot\rangle_n)$, for $n = 1, 2, 3$ (we will include orientation as further structure below, when we look at the Yukawa form for quarks). Different particles are merely different sections of different tensor products for these fundamental vector spaces. Thus, for instance, a down-right-handed quark (of any of the three generations, but here we assume the first) is given by:

$$\mathbf{d}_R \in \Gamma(E^3 \otimes (E^{1*} \otimes E^{1*})), \tag{4.9}$$

whereas vector bosons are replaced by the corresponding affine covariant derivatives, e.g. $\nabla^1, \nabla^2, \nabla^3$ (see (Gomes, 2024, 2025a) for more details).

In this formulation, it is clear that weak isospin T_3 has no independent geometrical meaning (see 3.2.b and footnote 9), and so left-handed fermions (as well as φ^+, φ^0, which are also $SU(2)$-doublets), are better understood merely as components of the vector fields Q_L, L_L. Thus, though left-handed leptons and up and down quarks are usually seen as different particles—with different masses and electric charges—the (intra-lepton and intra-quark) distinction occurs only via their couplings with the Higgs. According to (4.10) and (4.11), we should instead see both left-handed leptons and quarks as single particles, L_L, Q_L^I, respectively, whose decomposition relative to the Higgs field becomes physically important. The point is that φ gives a frame within $\mathbb{C}^2$ which imparts meaning to T_3 and so to left-handed up and down quarks and electrons and electron-neutrinos. As to the charges exhibited in a frame for $\mathbb{C}^2$ on Table 1, they are already adapted to the Higgs in the form $\varphi = \varphi^0 = (0,1)^T$ (i.e. when $\varphi^+ = 0$); e.g. only then do the up-left handed quark components become $(u_L^I, 0)^T$.[10]

Indeed, geometrically, it makes more sense to define the left-handed components of both leptons and quarks as parallel and orthogonal to the Higgs according to the inner product on E^2, i.e.:

$$\mathbf{e}_L := \langle L_L, e_0\rangle_2\, e_0, \quad \text{with} \quad e_L = \langle L_L, e_0\rangle_2\,; \quad \boldsymbol{\nu}_{eL} := L_L - e_L, \tag{4.10}$$

$$\mathbf{u}_L^I := \langle Q_L^I, e_0\rangle_2\, e_0 \quad \text{with} \quad u_L^I = \langle Q_L^I, e_0\rangle_2\,; \quad \mathbf{d}_L := Q_L - \mathbf{u}_L, \tag{4.11}$$

where capital I indicates color components (i.e. red, green and blue) in an orthonormal frame of $\mathbb{C}^3$ and I used the notation e_0 for the unit-direction of the Higgs, introduced in Section 3.1 (not to be confused with the left-handed electron, $\mathbf{e}_L$).

Before we give the geometric interpretation of (4.8), and of the corresponding form for quarks, note that, given an orthonormal basis for E^2, we can form duals: for $\xi = \xi^\perp e_\perp + \xi^\parallel e_0 = (\xi^\perp, \xi^\parallel)^T$ (e.g. $\mathbf{e}_L = L_L^\parallel$, $\boldsymbol{\nu}_{eL} = L_L^\perp$) the dual takes the conjugate of the transpose, so:

$$((\xi^\perp, \xi^\parallel)^T)^* = (\bar{\xi}^\perp, \bar{\xi}^\parallel). \tag{4.12}$$

Using (4.12) and an orthonormal frame aligned with the Higgs (3.5), the Yukawa term for the leptons in Equation (4.8) reads (including a coupling constant, g_e):

$$T(L_L, \varphi, e_R) := g_e L_L^* \phi e_R = g_e \langle\langle L_L, \varphi\rangle_2, e_R\rangle_1 = g_e(v + H)\bar{e}_L e_R, \tag{4.13}$$

where the first equality gives the 'standard' definition; e_L is a Weyl left-handed spinor and internal scalar (i.e. the magnitude of the vector field along the Higgs); $\langle .\,, .\rangle_2$ is complex anti-linear in the first entry and maps elements of $E^2 \otimes E^1 \times E^2$ into E^1 in the obvious way (by taking inner products among the E^2 components); and $\langle .\,, .\rangle_1$ is just the scalar inner product in $\mathbb{C}$.[11] From (4.13) we can see how mass terms, proportional to $g_e v$ (as well as interactions with the Higgs field) emerge for the electron.

[10]Therefore the table, reproduced from (Hamilton, 2017, Table 8.2), is slightly misleading: if one includes both components for the Higgs, the up and down components for the left-handed quarks and leptons would not have any physical meaning.

[11]This is slightly misleading: what we have here is that $\varphi \in \Gamma(E^2 \otimes^3 E^1)$, i.e. the third tensor product of E^1, which is still one-dimensional, $e_L^* \in \Gamma(E^{2*} \otimes^3 E^{1*})$, and $e_R \in \Gamma(\otimes^6 E^1)$. This is why they match to a scalar.

Geometrically, the inner products in (4.13) are a very natural way to obtain scalars: we are measuring 'internal angles' between the different particles seen as vector fields on the same spaces. I take this form of (4.13), namely $\langle\langle L_L, \varphi\rangle_2, e_R\rangle_1$, to be a more transparent interpretation of the Yukawa term for leptons.

Note that, here, chirality—the fact that right-handed particles don't couple to the Higgs—is explained by the fact that only left-handed particles have components in E^2. Note, moreover, that in this convention the neutrinos don't acquire mass. First, because they are orthogonal to the Higgs, but more fundamentally, because we have not included right-handed neutrinos in our particle content. Because of this feature, the Yukawa terms for leptons are diagonal in generations: these mass terms don't mix, say electrons with muons and taus.

In the case of quarks (or also for the leptons if we include right-handed neutrinos), things are different: we add another field, which is orthogonal to, but not independent from, the Higgs, and generations mix. This new field, called φ_c on Table 1, is obtained by recruiting another geometric structure that we can equip $\mathbb{C}^2$ with (besides the Hermitean inner product): an orientation. This implies we can use the totally anti-symmetric form, or the volume form, ϵ_{ab}, as part of the geometrical structure. In other words, whereas the Higgs mechanism, described in Section 3, used the structure $(E^2, M, \mathbb{C}^2, \langle\cdot,\cdot\rangle_2)$, here we extend that to the structure $(E^2, M, \mathbb{C}^2, \langle\cdot,\cdot\rangle_2, \epsilon)$.[12]

Now, besides the metric, we can use ϵ_{ab} and its inverse ϵ^{ab} to raise or lower indices.[13] Thus if we call the isomorphism $J : E^2 \to E^{2*}$ which acts as $\xi \mapsto \langle\xi, \cdot\rangle$ we have:

$$C := \epsilon^\sharp \circ J : E^2 \mapsto E^2 \tag{4.14}$$

$$\xi^a \mapsto \epsilon^{ac} h_{cb} \xi^b \tag{4.15}$$

where we used, in abstract index notation, h_{ab} as the inner product on E^2. Thus we call

$$\varphi_c := C(\varphi); \tag{4.16}$$

it can be seen as a measure on the 'areas' orthogonal to φ. (see Appendix C for more details on how this definition relates to the standard one).

Denoting the generation by an index $i = 1, 2, 3$, we then have, for the total Yukawa coupling term for quarks:[14]

$$T(Q_L, \varphi, d_R) := Y^d_{ij} \overline{Q}^i_L \varphi \mathbf{d}^j_R + Y^u_{ij} \overline{Q}^i_L \varphi_c \mathbf{u}^j_R = Y^d_{ij} \langle\langle\langle Q^i_L, \mathbf{d}^j_R\rangle_3, \varphi\rangle_2\rangle_1 + Y^u_{ij} \langle\langle\langle Q^i_L, \mathbf{u}^j_R\rangle_3, \varphi_c\rangle_2\rangle_1. \tag{4.17}$$

[12]Under $A \in U(2)$, ϵ_{ab} is taken to transform as $\epsilon_{ab} \mapsto \det(A)\epsilon_{ab}$. So $SU(2)$ preserves it. Moreover, since $AA^\dagger = \mathbb{1}$ for any $A \in U(n)$, we know that $\det(A)\det(A^\dagger) = |\det(A)| = 1$, so $\det(A) = e^{i\theta}$ denotes an orientation change the $\mathbb{C}^n$. Using ϵ_{ab} as a geometric datum then implies we have a fixed orientation, as well as an inner product, on $\mathbb{C}^2$.

[13]Indeed, in standard differential geometry, we can find a similar sort of operator acting on two dimensions: the Hodge star: which would take a basis $e_0, e_1 \mapsto -e_1, e_0$, respectively, so *its action on vectors* can be written in this frame as a matrix operator: $* = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, which is of the same form as ϵ_{ab}.

[14]Note that here, unlike for the leptons and the left-handed quarks, the up and down right-handed quarks are genuinely different particles, since they have different components in E^1.

The boldface on lowercase letters is used to indicate that these are vector fields (and so are the capital Q and L, and so is the Higgs φ, but omitting boldface here doesn't conflict with our notation in what follows). Again, the first equality in (4.17) gives the 'standard' definition (cf. (Hamilton, 2017, Lemma 8.8.4)); the second gives the geometric form of that definition: it is, in the VB-POV, what really counts.[15]

Nonetheless, as often is the case in physics, we can glean more by introducing a frame: here, once more it is convenient, in order to compare with standard presentations, to choose the orthonormal frame (3.5) for the Higgs, which gives the components for the quarks along and orthogonal to the Higgs (given in Equation (4.10)) as in Table 1, as well as $\varphi^+ = 0$. Then:

$$Y^d_{ij}\langle\langle\langle Q^i_L, \mathbf{d}^j_R\rangle_3, \varphi\rangle_2\rangle_1 + Y^u_{ij}\langle\langle\langle Q^i_L, \mathbf{u}^j_R\rangle_3, \varphi_c\rangle_2\rangle_1 = (H+v)\left(Y^d_{ij} d^{Ii}_L d^{Ij}_R + Y^u_{ij} u^{Ii}_L u^{Ij}_R\right), \tag{4.19}$$

where now all variables are scalar (and we are summing over the color indices, I, as well as over the generations i, j).

Lastly, the Yukawa matrices Y are generically non-diagonal, i.e. they mix generations of quarks. One can always find linear combinations of quarks such that, say, Y^u is diagonal; this defines what is called the *mass basis*. But Y^u and Y^d cannot be diagonalised simultaneously, and the residual mixing is encoded in the *Cabibbo–Kobayashi–Maskawa* (CKM) matrix. Most textbooks (cf. (Hamilton, 2017, p. 515)) then explain that the CKM matrix describes the physical effects of left-handed quark mixing across generations, from the 'mass eigenstate basis' to the 'weak eigenstate basis' (the latter being the one we have used here). It then "follows that the interactions with the W-bosons can connect quarks from different generations if the CKM matrix is not diagonal" (ibid).

From the geometric perspective, however, the situation is more transparent. If the up and down left-handed quarks were truly independent particles—i.e. distinct fields rather than components of the same field (usually called a doublet) in E^2—we *could* diagonalise Y_u and Y_d separately. But because they are components of the same E^2-field, we cannot. Correspondingly, the W bosons represent ∇^2, the covariant derivative on E^2, and so they, too, naturally mix generations when they couple to the relevant currents.

5 Conclusions

Feynman's Nobel prize lecture, with which I began, reflected on his alternative formulation of quantum electrodynamics via path integrals. That formulation, like Minkowski's introduction of spacetime—and indeed many other mathematically equivalent yet conceptually transformative innovations scattered through the history of physics—proved invaluable. I make no claim

[15] It is a little disappointing that, unlike their left-handed counterparts, up and down right-handed quarks can't be straightforwardly understood as components of a single vector field, due to their different components in $\mathbb{C}^1$. If they could be so understood, in place of (4.17), we would have ther simpler:

$$T(Q_L, \varphi, Q_R) = Y^d_{ij}\langle\langle\langle Q^i_L, Q^j_R\rangle_3, \varphi\rangle_2\rangle_1 + Y^u_{ij}\langle\langle\langle Q^i_L, Q^j_R\rangle_3, \varphi_c\rangle_2\rangle_1, \tag{4.18}$$

which only takes the components of the same inner product along and orthogonal to the Higgs.

that the geometry-first formulation of gauge theory developed here will ascend to comparable heights, nor do I expect it to become orthodoxy, as Feynman's and Minkowski's did.

But I don't want to understate what has been gained: alternative, 'symmetry-free' explanations of familiar mechanisms and features of gauge theory. In brief: in this new formulation, as in the familiar one, the Higgs field is a nowhere-vanishing section of a vector bundle with approximately constant norm. The component of the Higgs field carrying this constant nonzero norm plays the role of the *Higgs vacuum*. And although symmetries and vector bosons no longer appear at the fundamental level, the existence of such a section is enough: the geometry alone performs the explanatory work that symmetry was thought indispensable for.

Goldstone modes never appear here, and so never require elimination. The reason is simple: the constant magnitude of the Higgs vacuum section ensures that it is orthogonal to its covariant derivative. What in the symmetry-first formulation is described as the 'acquisition of mass' by vector bosons is, in this geometry-first account, nothing more than the non-vanishing of the (covariant) kinetic energy of the Higgs vacuum. In other words, the kinetic term of the Higgs depends on the affine structure of the vector bundle.

Moreover, the covariant derivative along a single section of a vector bundle does not depend on all the affine degrees of freedom of the bundle (for $\dim(E_x) \geq 2$). The absent degrees of freedom correspond, in the symmetry-first idiom, to the unbroken gauge group, giving rise at the perturbative level to the massless photons. In this formulation, then, talk of 'mass acquisition' may strike a geometry-first militant—say, a relativist—as misplaced.[16]

Turning to the Yukawa mechanism: I argued that standard presentations are explanatorily 'opaque,' and offered instead a more transparent geometric version of the Yukawa form itself. I readily admit that my sense of opacity may stem from a general preference for geometric explanations, *simpliciter*. But the point remains: as emphasised in Section 2.2, the mere availability of a geometric argument that bypasses representation theory is grist to my mill. The aim, after all, is to open the subject to a different community, with different, more geometric ways of thinking.

In this spirit, the geometric formalism already reveals distinctions between the Higgs and Yukawa mechanisms that, to my knowledge, have not been emphasised in the literature. (That does not mean they are controversial; perhaps they are simply too minor to warrant mention in standard presentations.)

First, the acquisition of mass by fermions through the Yukawa mechanism does not require any linearised expansion of the fields; nor does it require any choice of local frames or bundle trivialisation. By contrast, the 'acquisition of mass' by vector bosons, to the extent that vector bosons are expressed tensorially, relies on expanding the covariant derivative into a flat background plus a gauge potential; it requires the choice of a frame or trivialisation of the bundle.

[16] To be sure, some would hesitate to say that gravitons acquire mass merely because a spacetime, or a collection thereof, admits a kinetic term for a vector field of constant norm; yet that is precisely the consensus for such theories (Jacobson, 2008).

Second, from the geometric perspective, the left-handed up and down quarks, and the electron and electron-neutrino, are not separate particles at all, but components of the (first-generation) left-handed quark fields and leptons, respectively, resolved parallel and orthogonal to the Higgs.

Third, the Higgs mechanism is blind to the orientation of $\mathbb{C}^2$: nowhere does one need to invoke the volume form ϵ_{ab}. In group-theoretic terms, under $U(1)$ rotations, the Higgs carries precisely the hypercharge required to combine with the $SU(2)$ action into the standard two-component complex representation of $U(2)$. Similarly, the Yukawa term for leptons does not invoke ϵ_{ab} explicitly. However, the Yukawa term for (up-)quarks, does. The reason is that, unlike leptons, the quark Yukawa coupling is sensitive to the components orthogonal to the Higgs—namely, the up-type quarks. This sensitivity requires the introduction of the field φ_c, which encodes the 'oriented area' orthogonal to the Higgs. Put differently: up-quarks interact with, or 'measure,' the areas orthogonal to the Higgs.

I am not aware of any similar mechanism involving the orientation of $\mathbb{C}^3$, and so here is an interesting question, that crops up from the VB-POV: where is the orientation of $\mathbb{C}^3$, that forces the standard model to employ $SU(3)$ as opposed to $U(3)$, geometrically important?[17]

This puzzle points to a broader methodological issue, best framed by returning to the spacetime analogy. The automorphism group of a tangent space is $SO(3,1)$ (or $O(3,1)$), and this becomes explicit once we introduce orthonormal frames. Yet a vast amount of spacetime geometry can be developed without ever invoking $SO(3,1)$ directly. By the same token, if one were to posit a different group acting on TM—say $O(2)$—a clear geometrical rationale would be required, perhaps the presence of a plane of symmetry. In the symmetry-first formulation of gauge theory, by contrast, it is common to posit group actions that do not transparently reflect the geometry of the underlying vector bundles—for example, the action of $U(1)$ on $\mathbb{C}^2$ equipped with an inner product. In such cases we rarely ask what additional structure, if any, constrains or justifies these actions. A geometry-first formulation makes that question unavoidable: as in the spacetime analogy, any admissible group action must be grounded in geometry. Of course, much can be achieved without mentioning symmetry groups at all. But when it is convenient to invoke a representation of a linear group, this too can be accommodated—so long as the representation is first anchored in geometric structure.[18]

Lastly, as a possible new use of the VB-POV, I want to point out that the geometry-first mechanisms I described here make sense already at a non-linearised—though here strictly

[17] Indeed, there is a canonical isomorphism between $U(3)$ and $SU(3) \times U(1)/\mathbb{Z}^3$, but the representations of $U(1)$ don't seem to realise this isomorphism in the standard model. Benjamin Muntz has suggested to me that the place to look for the involvement of the volume-form of E^3, and therefore the reduction of the structure group from $U(3)$ to $SU(3)$, may be the triality constraints on baryon coupling: colourless states built out of three quarks which would not be invariant under the full $U(3)$.

[18] In many respects, the contrast between this geometry-first alternative and the familiar, symmetry-first formulation is analogous to the contrast between what is often called the Riemannian (geometry-first) and the Kleinian (symmetry-first) formulations of geometrical objects (cf. e.g. Barrett & Manchak (2024) for a recent discussion).

classical—level, and so may help clarify which properties or mechanisms rely on the validity of quantum field-theoretic perturbation theory. This broader use remains for further exploration.

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APPENDIX

A Principal and associated fiber bundles

I will start with the definition of a principal bundle:

Definition 1 (Principal Fiber Bundle) (P, M, G) *consists of a smooth manifold* P *that admits a smooth free action of a (path-connected, semi-simple) Lie group,* G*: i.e. there is a map* $G \times P \to P$ *with* $(g, p) \mapsto g \cdot p$ *for some right (or left, with appropriate changes throughout) action* $\cdot$ *and such that for each* $p \in P$*, the isotropy group is the identity (i.e.* $G_p := \{g \in G \mid g \cdot p = p\} = \{e\}$*).* P *has a canonical, differentiable, surjective map, called a projection, under the equivalence relation* $p \sim g \cdot p$*, such that* $\pi : P \to P/G \simeq M$*, where here* $\simeq$ *stands for a diffeomorphism.*

It follows from the definition that $\pi^{-1}(x) = \{G \cdot p\}$ for $\pi(p) = x$. And so there is a diffeomorphism between G and $\pi^{-1}(x)$, fixed by a choice of $p \in \pi^{-1}(x)$. It also follows (more subtly) from the definition, that local sections of P exist. A local section of P over $U \subset M$ is a map, $\sigma : U \to P$ such that $\pi \circ \sigma = \mathrm{Id}_U$.

Given an element ξ of the Lie-algebra $\mathfrak{g}$, and the action of G on P, we use the exponential to find an action of $\mathfrak{g}$ on P. This defines an embedding of the Lie algebra into the tangent space at each point, given by the *hash* operator: $\sharp_p : \mathfrak{g} \to T_pP$. The image of this embedding we call *the vertical space* V_p at a point $p \in P$: it is tangent to the orbits of the group, and is linearly spanned by vectors of the form

$$\text{for} \quad \xi \in \mathfrak{g} : \quad \xi^\sharp(p) := \frac{d}{dt}_{|t=0}(\exp(t\xi) \cdot p) \in V_p \subset T_pP. \tag{A.1}$$

Vector fields of the form $\xi^\sharp$ for $\xi \in \mathfrak{g}$ are called *fundamental vector fields.*[19]

The vertical spaces are defined canonically from the group action, as in (A.1). But we can define an 'orthogonal' projection operator, $\widehat{V}$ such that:

$$\widehat{V}_{|V} = \mathsf{Id}_{|V}, \quad \widehat{V} \circ \widehat{V} = \widehat{V}, \tag{A.2}$$

[19] It is important to note that there are vector fields that are vertical and yet are not fundamental, since they may depend on $x \in M$ (or on the orbit).

and defining $H \subset TP$ as $H := \mathsf{ker}(\widehat{V})$. It follows that $\widehat{H} = \mathsf{Id} - \widehat{V}$ and so $\widehat{V} \circ \widehat{H} = \widehat{H} \circ \widehat{V} = 0$. Moreover, since $\pi_* \circ \widehat{V} = 0$ it follows that:

$$\pi_* \circ \widehat{H} = \pi_*. \tag{A.3}$$

The connection-form should be visualized essentially as the projection onto the vertical spaces: given some infinitesimal direction, or change of frames, the vertical projection picks out the part of that change that was due solely to a translation across the group orbit. The only difference between $\widehat{V}$ and ϖ is that the latter is $\mathfrak{g}$-valued, Thus we get it via the isomorphism between V_p and $\mathfrak{g}$ (ϖ's inverse is $\sharp : \mathfrak{g} \mapsto V \subset TP$). We can define it directly as:

Definition 2 (An principal connection-form) *ϖ is defined as a Lie-algebra valued one form on P, satisfying the following properties:*

$$\varpi(\xi^\sharp) = \xi \qquad \textit{and} \qquad {L_g}^* \varpi = \mathrm{Ad}_g \varpi, \tag{A.4}$$

where the adjoint representation of G on $\mathfrak{g}$ is defined as $\mathrm{Ad}_g \xi = g \xi g^{-1}$, for $\xi \in \mathfrak{g}$; ${L_g}^*$ is the pull-back of TP induced by the diffeomorphism $g : P \to P$.

Now, in possession of an principal connection, we can induce a notion of covariant derivative on *associated vector bundles*:

Definition 3 (Associated Vector Bundle) *A vector bundle over M with typical fiber V, is associated to P with structure group G, is defined as:*

$$E = P \times_\rho V := P \times V / \sim \quad \textit{where} \quad (p, v) \sim (g \cdot p, \rho(g^{-1})v), \tag{A.5}$$

where $\rho : G \to GL(V)$ is a representation of G on V.

One can get a covariant derivative on an associated vector bundle E from ϖ as follows: let $\gamma : I \to M$ be a curve tangent to $\mathbf{v} \in T_x M$, and consider its horizontal lift, γ_h. Suppose $\kappa(x) = [p, v]$. Then

$$\nabla_{\mathbf{v}} \kappa = \frac{d}{dt}[\gamma_h, v]. \tag{A.6}$$

Conversely, we can define a horizontal subspace from the covariant derivatives as follows. For $p = e_1, ...e_n \in L(E)$, and for all curves $\gamma \in M$ such that $\mathbf{v} = \dot{\gamma}(0) \in T_x M$, with $\pi(p) = x$, let $\{e_1(t), ..., e_n(t)\}$ be curves in E such that $\nabla_{\mathbf{v}}(e_i(t)) = 0$. Doing this for each v defines a horizontal subspace.

But we can also obtain the vector bundles more directly as follows:

Definition 4 (Vector Bundle) *A vector bundle (E, M, V) consists of: E a smooth manifold that admits the action of a surjective projection $\pi_E : E \to M$ so that any point of the base space M has a neighborhood, $U \subset M$, such that, for all proper subsets of U, E is locally of the form $\pi^{-1}(U) \simeq U \times V$, where V is a vector space (e.g. $\mathbb{R}^k$, or $\mathbb{C}^k$) which is linearly isomorphic to $\pi^{-1}(x)$, for any $x \in M$.*

Note that the isomorphism between $\pi^{-1}(U)$ and $U \times V$ is not unique, which is why there is no canonical identification of elements of fibers over different points of spacetime. Each choice of isomorphism is called 'a trivialization' of the bundle.

Definition 5 (A section of E) *A section of E is a map $\kappa : M \to E$ such that $\pi_E \circ \kappa = \mathrm{Id}_M$. We denote the space of smooth sections by $\kappa \in \Gamma(E)$ (see Figure 2 for a formulation of such a section).*

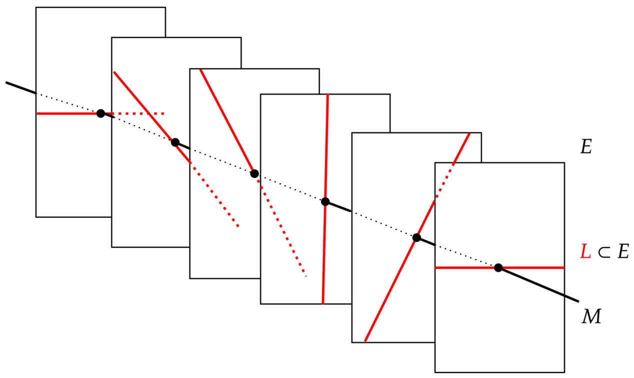

Figure 2: A vector bundle with a two-dimensional fiber over a one-dimensional base space, with a section here called L. (Figure taken from Wikipedia)

Given a vector bundle (E, M, V) a covariant derivative D is an operator:

$$\mathrm{D} : \Gamma(E) \to \Gamma(T^*M \otimes E) \tag{A.7}$$

such that the product rule

$$\mathrm{D}(f\kappa) = \mathrm{d}f \otimes \kappa + f\mathrm{D}\kappa \tag{A.8}$$

is satisfied for all smooth, real (or complex)-valued functions $f \in \Gamma(M)$.

Thus we can define parallel transport as follows:

Definition 6 (Parallel transport in a vector bundle) *Let* D *be a covariant derivative on (E, M, V), $\mathbf{v} \in E_x$ and $\gamma(t)$ a curve in M such that $\gamma(0) = x$. Then we define the parallel transport along γ as the unique section $\mathbf{v}_h(t)$ of $E|_\gamma$ such that:*

$$\mathrm{D}_{\gamma'}\mathbf{v}_h = 0. \tag{A.9}$$

The existence and uniqueness of this map is guaranteed for $\gamma \subset U$ some open subset of M, and it follows from properties of solutions of ordinary differential equations (cf. (Kobayashi & Nomizu, 1963, Ch. II.2)).

Here D is an operator, not a tensor. But by introducing a coordinate frame or basis, we can represent it as such. This is the same as for spacetime covariant derivatives, ∇: it is only upon the introduction of a frame or basis that we find an explicit representation.

B The Standard Group-Theoretic Exposition of the Higgs Mechanism

Before turning to our geometric reformulation, we briefly review the conventional mathematical account of the Higgs mechanism, as in Hamilton (Hamilton, 2017, Ch. 8). This will allow us to highlight the points at which symmetry groups, stabilisers, and coset spaces enter essentially.

We begin with a compact Lie group G acting unitarily on a complex vector space W (the Higgs vector space). A Higgs potential of the form

$$V(w) = -\mu\|w\|^2 + \lambda\|w\|^4, \quad \mu, \lambda > 0,$$

is G-invariant, and has minima along a sphere

$$\mathcal{M}_{\text{vac}} = \{w \in W : \|w\| = v\}, \quad v = \sqrt{\mu/2\lambda}.$$

Thus the set of vacua is itself a homogeneous G-space:

$$\mathcal{M}_{\text{vac}} \cong G/H,$$

where $H = G_{w_0}$ is the stabiliser (isotropy subgroup) of a chosen vacuum vector $w_0 \in W$. Already here the reasoning is group-theoretic: the possible vacua are classified by subgroup data (G, H).

A vacuum configuration is given by a constant section Φ_0 of the Higgs bundle, with $\Phi_0(x) = w_0$ for all $x \in M$. The unbroken subgroup H is compact (as a closed subgroup of G). If $H \subsetneq G$, the gauge theory is said to be *spontaneously broken* (Hamilton, 2017, Def. 8.1.6). The Higgs condensate Φ_0 is the non-zero background field in which other particles propagate, and is invariant only under $H \subset G$. Again, the classification of broken versus unbroken symmetries is a stabiliser argument.

Perturbations of the Higgs field $\Phi = \Phi_0 + \tilde{\phi}$ decompose relative to the tangent space at w_0:

$$T_{w_0}W \cong T_{w_0}(G \cdot w_0) \oplus (T_{w_0}(G \cdot w_0))^\perp.$$

Group theory guarantees this orthogonal splitting (Hamilton, 2017, Lem. 8.1.12). One then expands $\tilde{\phi}$ in an eigenbasis of the Hessian:

$$\tilde{\phi} = \frac{1}{\sqrt{2}}\sum_{i=1}^{d} \pi_i e_i \; + \; \frac{1}{\sqrt{2}}\sum_{j=1}^{2n-d} \sigma_j f_j,$$

with $\{e_i\}$ tangent to the orbit $G \cdot w_0$ and $\{f_j\}$ orthogonal. The π_i are massless scalar fields: the Nambu–Goldstone bosons. The σ_j are massive scalars: the Higgs bosons (Hamilton, 2017, Def. 8.1.14). This is precisely Goldstone's theorem: $\dim(G/H)$ massless scalars, deduced from the group structure of the vacuum manifold.

Physically the Goldstone bosons are unobservable, since they can be gauged away. Mathematically this is formalised by the *unitary gauge* (Hamilton, 2017, Def. 8.1.18, Thm. 8.1.20). One uses a physical gauge transformation $\gamma : M \to G$ to rotate the Higgs field entirely into the fixed direction w_0:

$$\Phi(x) \mapsto \gamma(x) \cdot \Phi(x) = (0, \ldots, 0, v + h(x)).$$

By definition, in unitary gauge the shifted Higgs field is orthogonal to the orbit $G \cdot w_0$, and the Nambu–Goldstone bosons vanish. This step is again an essentially group-theoretic argument, exploiting the transitivity of the G-action on the vacuum manifold.

Let $g = \mathfrak{g}$ be the Lie algebra of G, with $h = \mathrm{Lie}(H)$ the subalgebra of unbroken generators. With respect to an invariant scalar product, decompose

$$g = h \oplus h^{\perp}.$$

The $h^{\perp}$ directions correspond to broken generators. It is exactly these components of the gauge field A_μ that acquire mass terms from the kinetic energy of the Higgs:

$$\|D\Phi\|^2 \supset v^2 \sum_{X \in h^{\perp}} \|A_\mu^X\|^2.$$

Conversely, the h-components remain massless. This is the algebraic re-expression of the stabiliser picture.

In the electroweak theory $G = SU(2)_L \times U(1)_Y$ acts on $W = \mathbb{C}^2$. Choosing a vacuum vector $w_0 = (0, v)$, the stabiliser is a diagonal $U(1)$ subgroup, which is identified with electromagnetism. The Lie algebra $su(2) \oplus u(1)$ decomposes accordingly, and a change of basis (the Weinberg angle) diagonalises the mass form, producing massive $W^{\pm}$, Z^0 and a massless photon. Each of these identifications rests on the subgroup structure of G and the group-theoretic decomposition of its representation on W (Hamilton, 2017, Ch. 8.3).

Summary

The group-theoretic presentation of the Higgs mechanism thus depends essentially on:

1. Identifying the vacuum manifold as G/H, a homogeneous space.
2. Invoking Goldstone's theorem: $\dim(G/H)$ massless modes.
3. Using unitary gauge to remove Goldstone bosons by G-action.
4. Decomposing $\mathfrak{g}$ into $h \oplus h^{\perp}$ to classify massive and massless gauge bosons.
5. In the electroweak case, applying these steps to $SU(2)_L \times U(1)_Y$, producing $W^{\pm}$, Z^0, and the photon.

These symmetry-based arguments provide the conventional foundation. In the next section we shall see how the same results can be obtained directly from the geometry of vector bundles, without recourse to stabilisers, cosets, or gauge fixing.

C Connecting the geometric and the standard interpretations of φ_c

Let $V \cong \mathbb{C}^2$ be the fundamental $SU(2)$ doublet space. We use only:

- the $SU(2)$-invariant Hermitian form $h : V \times V \to \mathbb{C}$,
- the $SU(2)$-invariant complex symplectic form $\varepsilon : V \times V \to \mathbb{C}$ (bilinear, antisymmetric),
- and the natural dual $V^* = \mathrm{Hom}(V, \mathbb{C})$.

Write the "Hermitian dual" (transpose+conjugate) map

$$J : V \longrightarrow V^*, \qquad J(v) := h(v, \cdot).$$

This map is *antilinear* and $SU(2)$–equivariant into the contragredient representation:

$$J(Uv) = J(v) \circ U^{-1} \qquad (U \in SU(2)).$$

Next, use ε to identify V with its dual *linearly*:

$$\varepsilon^\flat : V \to V^*, \qquad \varepsilon^\flat(w) := \varepsilon(w, \cdot), \qquad \text{with inverse } \varepsilon^\sharp := (\varepsilon^\flat)^{-1} : V^* \to V.$$

Equivariance of ε is the identity $U^T \varepsilon U = \varepsilon$, which is equivalent to

$$\varepsilon^\sharp \circ \alpha^* = \alpha \circ \varepsilon^\sharp \qquad \text{for all } \alpha \in \mathrm{End}(V).$$

Definition. Define the antilinear, $SU(2)$–equivariant map

$$C \;:=\; \varepsilon^\sharp \circ J \;:\; V \longrightarrow V, \qquad \tilde{v} \;:=\; C(v).$$

Equivariance follows immediately:

$$C(Uv) = \varepsilon^\sharp\big(J(v) \circ U^{-1}\big) = \big(\varepsilon^\sharp \circ J(v)\big)\, U^{-1} \;=\; U\, \big(\varepsilon^\sharp \circ J(v)\big) = U\, C(v).$$

If v has hypercharge Y, then J (being the Hermitian dual) implements the phase $e^{iY\theta} \mapsto e^{-iY\theta}$, so C flips $Y \mapsto -Y$. Hence, for the Higgs doublet $\phi \in (\mathbf{2}, +1)$ we set

$$\tilde{\phi} \;:=\; C(\phi) \;=\; \varepsilon^\sharp\big(h(\phi, \cdot)\big) \;\in\; (\mathbf{2}, -1).$$

Component check. Choose an orthonormal basis so that h is the identity and $\varepsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then $J(\phi)$ is the Hermitian row vector $\phi^\dagger$, and

$$\tilde{\phi} \;=\; \varepsilon^\sharp(J(\phi)) \;=\; \varepsilon^{-1}\, \phi^\dagger \;=\; \varepsilon\, \phi^* \;=\; i\sigma_2\, \phi^*,$$

i.e. the usual $\tilde{\phi}$.

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