

Particles before symmetry

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Abstract

The standard model of particle physics is usually cast in symmetry-first terms. Recently, a geometry-first picture has been proposed, in which the relevant symmetries do not appear explicitly at the ground level of ontology (Gomes, 2024). In this paper I extend this approach to two central mechanisms of the standard model: spontaneous symmetry breaking and the Yukawa coupling, both essential for particles to acquire mass. These reformulations offer alternative explanations cast in purely geometric terms. For example, a particle’s quantum numbers correspond to the internal space it inhabits and to the geometric type of object it is (e.g. an (n, m) -tensor). I argue that a symmetry-first account in terms of principal and associated bundles admits a genuine geometry-first counterpart only when the group’s representation coincides with the automorphism group of the fibre—a condition that cuts the slack tolerated by the symmetry-first view.

1 Introduction

Should we value mathematically equivalent formulations of a theory? Feynman (1994, p. 127) gives the gist of my preferred answer to this question:

Every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics. He knows that they are all equivalent, and that nobody is ever going to be able to decide which one is right at that level, but he keeps them in his head, hoping that they will give him different ideas for guessing.

And he further reflected on the value of alternative ways of thinking about a theory in his Nobel Prize Lecture (“The Development of the Space-Time View of QED”, 1965), in which he discussed his path integral formalism, which was mathematically equivalent to Schwinger’s earlier approach to quantum field theory:

Theories of the known, which are described by different physical ideas may be equivalent in all their predictions and are hence scientifically indistinguishable. However, they are not psychologically identical when trying to move from that base into the unknown. [...] If every individual student follows the same current fashion in expressing and thinking about electrodynamics [...], then the variety of hypotheses being generated ... is limited.

The subsequent career of Feynman’s path integrals is a testament to the validity of his arguments.

Another classic case further illustrates the point: Minkowski’s 1908 recasting of Einstein’s special relativity into the language of four-dimensional spacetime geometry. Again, the underlying physics was unchanged, but the shift in formulation was decisive for future developments. Einstein initially dismissed Minkowski’s treatment as “überflüssige Gelehrsamkeit” (superfluous erudition), yet by 1912 he had conceded that only the spacetime formulation revealed the true essence of the theory.¹ What Minkowski introduced was not new predictions but a new ontology: space and time no longer standing apart, but merged into a single structure. And it was precisely this geometrical vantage point that enabled the later generalisation to general relativity (cf. (Stachel, 2002, p. 226)).

In the current paper, I aim to provide such an alternative description, though here of classical gauge theory as it applies to the standard model of particle physics, not quantum electrodynamics or special relativity, and of a much more humble and less radical nature than either Feynman’s or Minkowski’s reformulations. Namely, I aim to provide an alternative formulation of the standard model that does not rely on symmetry. I will call this formulation *geometry-first*.

The knowledgeable reader will be quick to point out that gauge theory is already highly geometrical: principal fiber bundles and connections are, after all, the stock-in-trade of the geometer. However, the standard formulation also incorporates symmetry into its foundations, and with it, an ontology that extends beyond the spaces where matter fields actually reside; it extends the ontology to include the so-called principal fiber bundles. By *geometry-first* I mean a formulation that employs only those spaces where matter fields reside, and does not rely on symmetry to get off the ground. This definition is broad, but here we will focus on the case of particle physics.

For theories of particle physics, I will assume the spaces where matter fields reside are vector bundles and that the spaces that introduce symmetry at a ground level are principal fiber bundles. Principal bundles can still appear in a geometry-first formulation, but if they do they must be entirely supervenient on the structure of the vector bundles, which is the subvenience basis for the geometry-first formulation. I do *not* demand that this picture must

¹In 1912 Einstein wrote to Sommerfeld: “I have come to value greatly the four-dimensional formalism of Minkowski, which I had previously considered unnecessary erudition. In the meantime, I have also become convinced that only this formalism brings out the true essence of the theory.” (quoted in (Holton, 1974, p. 263))

be superior in every respect, or practically advantageous. I will start with a more modest aim: to offer an alternative perspective that clarifies some features of particle theory while omitting symmetry at the base of the explanatory chain. And, lest I sound too irenic, in Section 5 I will defend my preference for the geometry-first formulation more openly.

Needless to say, symmetry is the cornerstone of particle physics. Representations of Lie groups, Casimir invariants, spontaneous symmetry breaking, gauge-fixing: these are the daily bread of the standard model. (This much will be obvious to anyone familiar with the field, so I need not belabor the point.) That the associated principal bundles—and with them the explicit appeal to symmetry at the base of the explanatory chain—might be dispensed with is therefore anything but trivial.

However, a new geometry-first formulation has recently been proposed, in which symmetries are not postulated and principal fiber bundles are unnecessary (Gomes, 2024, 2025a). In the alternative formulation, the symmetry groups are only implicit: they arise as the automorphism groups of vector bundles. The geometry-first formulation is generally available as an alternative only for gauge groups that are linear, and for representations obtained from the fundamental representation (when it is unique) via tensor and direct products, symmetrisation, and other similar operations. Thus, from the bat, theories whose symmetry groups have no linear representation and groups that have no unique fundamental linear representation are outside the scope of an equivalent geometry-first formulation. But more importantly, the geometry-first formulation demands an alignment between symmetry groups and structure of the vector spaces where matter resides that is not always guaranteed from the symmetry-first perspective, with its largely independent principal fiber bundles and representation spaces (I will have more to say about these restrictions to equivalence in Section 5).

Even in the cases that admit the two formulations as mathematically equivalent, the geometry-first one comes with a significantly different ontology: for the standard model of particle physics, it consists of three fundamental vector bundles over spacetime where the various matter fields reside (as sections of tensor products). There is no need for a separate space to encode the principal connections.

Change the formulation, and the explanations change with it. Three examples show how familiar features of particle physics acquire alternative interpretations. First, in a non-Abelian vacuum Yang–Mills theory with Lie group G , the fundamental dynamical object is no longer a connection ω on a G -principal fibre bundle (or its spacetime representative A_μ^I), but the covariant derivative D_μ on a vector bundle whose automorphism group corresponds to G —and this remains true even if no vector fields are present to be differentiated: the affine structure of the vector bundle can be dynamical. In this setting, a particle’s quantum numbers become geometric labels: the internal space it inhabits, and the tensor type it is within that space. (These ideas were explained in Gomes (2024); so this paper will focus on the next two.)

Second, once symmetry groups drop out of the base level of the explanatory chain, the very notion of ‘symmetry-breaking’ requires reinterpretation. Vector bosons A_μ^I are replaced by covariant derivatives of the fundamental bundles, which are not on the same footing as

matter fields, and it is no longer clear how they could ‘acquire mass’ in the usual sense. Third, consider the Yukawa couplings. In the standard formulation, Yukawa terms are scalars formed from sections of different associated bundles, requiring explicit ‘bridges’ between them. In the geometry-first picture, by contrast, the fundamental objects are vector bundles themselves, with particles emerging from the corresponding tensor bundles. Scalars then arise naturally through inner products and contractions between vectors and their duals.

These examples show how a geometry-first perspective reshapes explanations. But the real strength of the approach lies in the methodological discipline it suggests. Where the principal-bundle picture allows considerable slack between symmetry and geometry, the vector-bundle point of view ties the two tightly together. Section 5 argues that this apparent restriction is in fact a virtue: it narrows the space of admissible theories in a way that clarifies the ontology of gauge theory and still encompasses our best physics.

Here is how I will proceed. Section 2 introduces both the familiar principal-bundle formulation and the alternative *vector-bundle point of view*. Section 3 gives the alternative account of the Higgs mechanism. Section 4 does not attempt a full reformulation of the Yukawa mechanism, but argues that its interpretation is more transparent in the geometry-first approach. Section 5 develops the methodological defense of the VB-POV. Finally, Section 6 draws the broader morals.

2 Symmetry-first and geometry-first formulations of gauge theory

Here I will give brief overviews of both the familiar, symmetry-first, and of the less familiar, geometry-first formulations of gauge theory. I will start with the more familiar and then introduce the novel.

2.1 Gauge theory and principal fiber bundles: the symmetry-first formulation

In short, the **symmetry-first** formulation of gauge theory is the familiar one: each fundamental interaction is associated with a symmetry group, which is taken as the structure group of a principal fiber bundle. Connections on this bundle then play the role of the vector bosons—the “force carriers.”

Classical configurations of matter particles charged under a force are described by sections of vector bundles associated to the principal bundle whose group encodes that force. One may endow these associated bundles with additional structure (for instance, a Hermitian inner product on \mathbb{C}^n); in such cases, the representations of the structure group are only required to preserve that structure.

The connection on the principal fiber bundle induces parallel transport on all associated bundles. Crucially, it is the *same* connection that governs transport in each case, ensuring that different matter fields charged under the same interaction remain coordinated: they all “march in step” under parallel transport, probing the same distributions of electroweak or strong

forces. Thus, while associated vector bundles are distinct entities, they are tied together by the principal bundle, which acts as their common coordinator (see (Weatherall, 2016) and Figure 1). The primacy of the postulated structure group, and the central role it plays, is what makes this a “symmetry-first” formulation. More specifically, since in the case of gauge theories the symmetries are introduced as part of the principal fiber bundles, I will call this version of the symmetry-first picture the *principal bundle point of view* (PFB-POV). Technical details are provided in Appendix A.

Formally, a principal fiber bundle (P, M, G) is a smooth manifold P equipped with a smooth, free action of a Lie group G , projecting onto a base manifold M —spacetime. Intuitively, such a bundle codifies the ways in which the symmetry group G can act on geometric objects defined over M . In this paper, I will focus on one especially important class of such objects: *vector bundles*. A vector bundle (E, M, V) assigns to each spacetime point $x \in M$ a copy of a fixed vector space V , called the *typical fiber*. Sections of vector bundles are smooth assignments of an element of V to each point of M , and matter fields are precisely such sections.

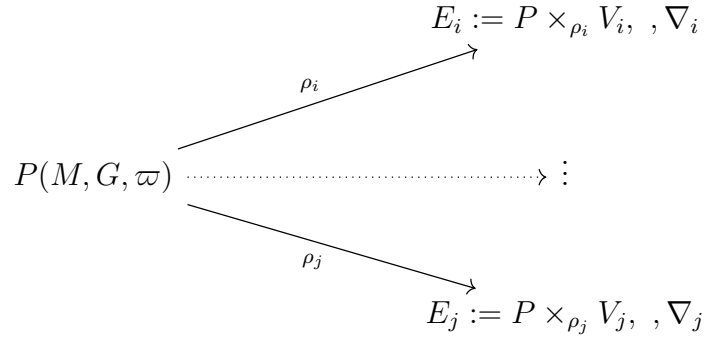


Figure 1: The principal G -bundle P , with structure group G , over the manifold M , with a principal connection ϖ (a \mathfrak{g} -valued one-form on P), abbreviated by $P(M, G, \omega)$, and its associated vector bundles $E_i := P \times_{\rho_i} V_i$, where $\rho_i : G \rightarrow V_i$ is a representation of the Lie group onto the vector space representing the typical fiber, V_i which is linearly isomorphic to $\pi_i^{-1}(x)$, for $x \in M$ and $\pi_i : E \rightarrow M$ the projection of the vector bundle onto its base space (spacetime). The covariant derivatives ∇_i are the ones induced by ϖ , as per Equation (A.6). See Appendix A for more details.

The principal connection, ϖ , is another main character in the principal bundle formalism. This object determines how orbits of the group residing over neighboring points of M are related; it can be used to determine parallel transport and covariant derivatives in the vector bundle (say, by determining which frame over a point in M gets mapped to which frame at an adjacent point).

Given a typical fiber V , say $V \simeq \mathbb{C}^n$, a principal bundle (P, M, G) , and a representation $\rho : G \rightarrow GL(V)$,² one can construct particular kinds of vector bundles, called *associated*

²Here ρ is understood as an embedding that is, in general, only a partial homomorphism, since it may fail to be faithful (injective) or surjective. This will be important in Section 5.

bundles. The idea is to think of the points of P as providing frames for V over each point of M , so that the group action corresponds to a change of frames on P , and a corresponding change of components of a vector in V (e.g. in \mathbb{C}^n). Since vector fields are frame-independent, one quotients pairs (p, v) , with $v \in V$ and $p \in P$, by the simultaneous action of the group on P and on V . The resulting object is the associated vector bundle, defined as

$$E := P \times_{\rho} V.$$

Particles that are sensitive to a given force are described by sections of vector bundles associated to the principal fiber bundle with the corresponding structure group³.

The advantage of defining vector bundles in this way is that it becomes transparent that fields inhabiting bundles associated to the same principal bundle covary under the action of a symmetry group. A further, related question is whether such vector bundles merely covary under the group action, or whether they in fact stand in a canonical relation to one another. Thus, suppose we are given:

$$E_1 = P \times_{\rho_1} V, \quad E_2 = P \times_{\rho_2} V \quad (2.1)$$

Given a local section of P , i.e. for $U \subset M$ a map $\sigma_U : U \rightarrow P$ such that $\pi(\sigma(x)) = x$, for all $x \in U$ (see Appendix A), we can write, for ξ_1 a local section of E_1 :

$$\xi_1(x) = [\sigma(x), v(x)]_1, \quad v : U \rightarrow V. \quad (2.2)$$

Then the obvious map to consider is:⁴

$$\begin{aligned} T : E_1 &\rightarrow E_2 \\ \xi_1 &:= [\sigma(x), v(x)]_1 \mapsto [\sigma(x), v(x)]_2 =: \xi_2. \end{aligned} \quad (2.3)$$

So the map acts as the identity on both entries, but nonetheless maps between sections in distinct vector bundles. However, on the right-hand side of (2.3), the representation under which we take equivalence classes is different: it is \sim_2 and not \sim_1 . So is this map well-defined for arbitrary representations ρ_1, ρ_2 ? The map should be invariant under gauge transformations (cf. Eq (A.5)) on both the domain and image. So consider a different representative of the equivalence class on the domain; according to (2.3) we must have:

$$[g(x) \cdot \sigma(x), \rho_1(g^{-1}(x))v(x)]_1 \mapsto [g(x) \cdot \sigma(x), \rho_1(g^{-1}(x))v(x)]_2 \quad (2.4)$$

for any $g : U \rightarrow G$. But on E_2 , we have the representation ρ_2 , and so we must have (omitting dependence on $x \in M$ for clarity):

$$(\sigma, v) \sim_1 (g \cdot \sigma, \rho_1^{-1}(g)v) \sim_2 (\sigma, \rho_2(g)\rho_1^{-1}(g)v) \not\sim_2 (\sigma, v). \quad (2.5)$$

Where the last inequivalence holds iff $\rho_1(g)\rho_2^{-1}(g) \neq \mathbb{1}$, $\forall g$, i.e. the equivalence holds iff $\rho_1 = \rho_2$. Thus we find that for the map (2.3) to be well-defined, we must have $\rho_1 = \rho_2$.

³For example, $SU(3)$ for the strong force.

⁴I thank Jim Weatherall for suggesting this.

Indeed, in physics, we are often faced with situations in which E_1 and E_2 have the same typical fiber, are associated to the same group, and yet have different representations. A simple example is when one of the representations is the trivial, or singleton, one, and the other is the fundamental (or any other).⁵ This occurs many times in the standard model: for fermions to acquire mass, one must relate sections of bundles that have different representations, since they represent different particles.

In contrast, in the vector-bundle point of view, all the vector bundles that, in the symmetry-first formulation, would be associated to the same principal bundle, are already endowed with a natural relation, as we will now see.

In Section 4 we will see how the issue of relating different vector bundles which, in the PFB-POV are associated to the same principal bundle, can arise in practice, and how both the symmetry-first and the geometry-first formulations deal with it.

2.2 Gauge theory and vector bundles: the geometry-first formulation

The geometric perspective I want to develop aims to dispense with the principal fiber bundle altogether. In this Section I set out a formulation of gauge theory that proceeds without gauge potentials, principal bundles, or explicit appeal to gauge symmetries.

The analogy with spacetime clarifies what is at stake. Consider (M, g, Ξ_i) , where (M, g) is a smooth Lorentzian manifold and the Ξ_i are various tensor fields on M , i.e. objects living in spaces constructed from the tangent bundle TM . The automorphism group of a typical fiber $T_x M$ is $O(3, 1)$ (or $SO(3, 1)$ if orientation is treated as background structure). This group becomes explicit once we introduce orthonormal frames. Yet much can be said about the Ξ_i in a purely geometric, frame-independent manner, without any reference to $SO(3, 1)$. If instead we were to posit a different group acting on TM —say $O(2)$ and not $SO(3, 1)$ —a geometrical rationale would be required to justify that action.

In gauge theory, by contrast, an analogous “frame-free” formulation for the behavior of matter remains largely undeveloped (cf. (Weatherall, 2016)), and the very idea of a geometric interpretation of the groups and their representations—for example, the adjoint action of $SU(2)$ on \mathbb{C}^3 endowed with an inner product, as opposed to the fundamental representation of $SU(3)$ —is seldom raised. We are after a formulation of gauge theories for which these interpretations are transparent.

I will introduce a realisation of the geometry-first formulation of gauge theory, which I will call *the vector bundle point of view* (VB-POV).⁶ To motivate the VB-POV from interpretational issues with the PFB-POV, let me recall that the main role of ω in (P, M, G) is to coordinate covariant derivatives between different associated vector bundles. But what is the physical status of ω ? Jacobs (2023, p. 41) convincingly argues they don’t have one; he concludes:

⁵A slightly more sophisticated example is as follows. Let $G = U(1)$, $V = \mathbb{C}^k$, and $\rho_i = n_i$, which acts as $e^{in_i} \mathbf{1}$ on \mathbb{C}^k . Then for $n_i \neq n_j$ for $i \neq j$ the map (2.3) is not well-defined, as can easily be verified.

⁶Other kinds of theories could also have a geometry-first formulation, e.g. those based on Cartan geometry, but here my focus is on particle physics, for which the relevant value spaces are vector bundles.

Neither the principal bundle nor the [principal] connection on its own represent anything physical. Rather, it is the induced connection on the associated bundle that represents the Yang-Mills field. [But] This approach has difficulties in accounting for distinct matter fields coupled to the same Yang-Mills field.

The issue, as he sees it, is that

there is no independent Yang-Mills field that the associated bundle connections supervene on. This makes it seem somewhat mysterious that these connections are equivalent. The coordination between associated bundles begs for a ‘common cause’ in the form of an independently existing Yang-Mills field.⁷

I agree with Jacobs that this is an issue and in (Gomes, 2024) I showed that it can be overcome. The introduction of (P, M, G) is unnecessary if particles that interact are all sections of the same vector bundles or of tensor products of the same vector bundles. Tensor products over a vector bundle inherit the same covariant derivatives by construction. In this case, parallel transport of the vector bundles in question automatically march in step. In this case we have at a hand a natural ‘common cause’ for the coordination of covariant derivatives, without the introduction of principal bundles. I will here call this *the vector bundle point of view* of gauge theory (VB-POV).

In more detail, given two vector bundles, E, E' , a covariant derivative on E will induce a covariant derivative on E' whenever E' is equal to a general tensor product involving E and its algebraic dual, E^* . In more detail, given E a vector bundle with covariant derivative D , and E^* its dual, we define, for sections $\kappa \in \Gamma(E)$ and $\xi \in \Gamma(E^*)$:

$$d(\langle \xi, \kappa \rangle)(X) = \langle \nabla_X^* \xi, \kappa \rangle + \langle \xi, \nabla_X \kappa \rangle, \quad (2.6)$$

where here angle brackets represent contraction. The generalisation to arbitrary tensor products is straightforward due to multilinearity.

On this view, no gauge groups need to be postulated at the ground level—the groups of automorphisms of vector bundles, $\mathbf{Aut}(E) \subset \mathbf{End}(E)$ are implied by the relevant structure. If principal fiber bundles are ever invoked, they are supervenient on the structure of the vector bundles, which forms the subvenience basis. The familiar distinction between Abelian and non-Abelian theories is then simply a distinction between different kinds of automorphism groups. In particular, one-dimensional vector bundles, whose typical fiber is isomorphic to \mathbb{C} , generate Abelian automorphism groups.

This vantage point also reframes the earlier question of whether there exist canonical maps between distinct vector bundles. In the PFB-POV, the natural candidate (equation (2.3)) is well-defined only within the same representation. Matters look different here. We assume that

⁷Jacobs instead defends the ‘inflationary approach’, which: “reifies not the principal bundle but the so-called ‘bundle of connections’. The inflationary approach is preferable because it can explain the way in which distinct matter fields couple to the same Yang-Mills field.” As I have argued in Gomes (2024), I don’t believe it is preferable in this sense, but I won’t rehash those arguments here.

all vector bundles charged under a given force descend from a single “fundamental” bundle. In the cases that I will explore each such fundamental bundle E^n , will have typical fiber \mathbb{C}^n , and will be equipped with inner product $\langle \cdot, \cdot \rangle$, and possibly a complex orientation (or volume-form) ε . Different associated bundles then appear not as unrelated objects in need of ad hoc identifications, but as systematic constructions from E^n . Their relations are fully accounted for by the usual functorial machinery: tensor products, (anti)symmetrization, dualization, projections into tensor factors, contractions, interior products, inner products, and so on. There is thus no mystery about how these bundles fit together—the geometry itself provides the correspondences. For instance, to contract an element of E^n with one of $E^{n*} \wedge E^{n*} \otimes E^n$, we can use the interior product, which generally is a map:

$$\begin{aligned} \iota : E^n \otimes \Lambda^m(E^{n*}) &\rightarrow \Lambda^{m-1}(E^{n*}) \\ (\xi, \Omega) &\mapsto \iota_\xi \Omega, \end{aligned} \tag{2.7}$$

where Λ is the anti-symmetric product, with $\Omega \in \Lambda^m(E^{n*})$, and, for any $m-1$ -tuple $(\xi_1, \dots, \xi_{m-1})$ gives

$$\iota_\xi \Omega(\xi_1, \dots, \xi_{m-1}) = \Omega(\xi, \xi_1, \dots, \xi_{m-1}), \tag{2.8}$$

etc. Similarly, we could use the inner product to map between E^n and E^{n*} , and so on.

One might object that a parallel, representation-theoretic argument for associated vector bundles could be mounted, mirroring the geometric one I have just given. Perhaps given arbitrary different representations of the same group, for arbitrary vector representation spaces, there are systematic ways to relate these representation spaces that mirror the ones I displayed above. That may well be true, but it is beside the point. Even if such arguments exist—and I have not found or worked one out, and am skeptical that one exists, for representations that are not faithful and full—the virtue of the geometric route is that it trades purely on geometrical language, and so it speaks directly to a community trained in geometry rather than in group and representation theory. The mere availability of a geometric formulation that sidesteps representation theory or other more technical algebraic machinery is already a win. My aim, after all, is to broaden the borders of the subject, making it accessible to different habits of thought.

Still, at first pass the VB-POV may seem too narrow to capture the full menagerie of gauge theories employed in physics. Some theories—those built from the exceptional Lie groups, for example—fall outside its reach. And even when a gauge group G is given, it is often a nontrivial matter to “reverse-engineer” a vector space structure for which $\text{Aut}(E_x) \simeq G$. How, for instance, does one coax $SO(4)$ out of a space whose typical fiber is \mathbb{C}^3 ; or $U(1)$ out of a space whose typical fiber is \mathbb{C}^n with $n \neq 1$? I will have much more to say about all this in Section 5.⁸

For all that, the standard model of particle physics fits neatly within the VB-POV. In the PFB-POV in which the standard model is usually described, every particle field is a section of

⁸The Peter–Weyl theorem guarantees that $U(n)$ admits nontrivial representations on \mathbb{C}^m , but extracting from this a natural structure on \mathbb{C}^m that renders the action geometrically meaningful is anything but straightforward.

an associated bundle for a principal fiber bundle whose structure group is $SU(3) \times SU(2) \times U(1)$, and the fundamental representation of each one of these component subgroups ($SU(n)$ or $U(n)$ for appropriate n) appears for some such section or other. The VB-POV alternative is available because under any representation of $U(n)$, the corresponding associated bundles can just as well be constructed by geometric means from the fundamental vector bundle—via tensor and exterior products, (anti)symmetrization, determinants, and the like. The VB-POV alternative is compelling (as I will expand on in Section 5), because the particular combination of representation, groups, and vector spaces is particularly suited for a geometrical interpretation. In such cases, a covariant derivative on a single vector bundle suffices to encode one fundamental interaction, while the various particle fields appear as sections of the appropriate derived bundles (e.g. tensor products).

Having surveyed both approaches to gauge theory—the symmetry-first PFB-POV and the geometry-first VB-POV—I now turn to the Higgs mechanism. My aim is to present it from within the VB-POV, while relegating to Appendix B a sketch of the more familiar PFB-POV treatment, which can be found in any standard textbook.

3 The Higgs mechanism in the geometry-first formulation

The proof, they say, is in the eating of the pudding. So here, to prove that the geometry-first perspective embodied by the VB-POV is sufficiently different to the PFB-POV to merit attention, I will provide a stand-alone derivation of the (classical) Higgs mechanism.

In the standard presentation (cf. e.g. (Tong, 2025, Ch. 2), the Higgs mechanism is often described in terms of spontaneous symmetry breaking, and one must employ Goldstone’s theorem, gauge fixing (e.g. unitary gauge), etc. I give a brief overview of that presentation in Appendix B. Here I will outline an alternative approach, phrased purely in the geometric language of vector bundles, which makes the essential structure transparent without appeal to symmetry-breaking jargon.

3.1 The non-linearised Higgs field

Let $(E^n, M, \mathbb{C}^n, \langle \cdot, \cdot \rangle_n, \nabla_n)$ be a Hermitian vector bundle over a manifold M , with fibers $E_x^n \simeq \mathbb{C}^n$ and $\langle \cdot, \cdot \rangle_n$ an inner product on E^n , which is compatible with ∇_n , the covariant derivative on E^n . We will omit the subscript when it is understood from context, as it will be in this Section, so for now we take $\varphi \in \Gamma(E)$ (the generalisation to $\varphi \in \Gamma(E^i \otimes \cdots E^j)$ is straightforward, as we will see). So φ is a vector-valued spacetime scalar field, satisfying

$$\min_{x \in M} \|\varphi(x)\| = v', \tag{3.1}$$

for some constant $v' > 0$. We write $\|\varphi(x)\| = (\Delta + v')$, for $\Delta \in C_+^\infty(M)$ (the positive real-valued smooth scalar functions on M), and get

$$\varphi(x) = \|\varphi(x)\|e_0 = (\Delta(x) + v')e_0, \tag{3.2}$$

where $e_0 = \frac{\varphi}{\|\varphi\|}$ is a unit section, well-defined since $\|\varphi\| > v' > 0$, and $\langle e_0, e_0 \rangle = 1$.

The potential term in the Lagrangian—the Higgs potential, $V(\varphi)$ —is assumed to enforce such a nonzero minimum, but it need not coincide with v' : we call v the minimum of the potential. Our focus will be on the kinetic term. Note that:

$$\nabla \langle e_0, e_0 \rangle = 2\text{Re} \langle e_0, \nabla e_0 \rangle = 0, \quad \text{and} \quad \nabla v' = 0, \quad (3.3)$$

where Re takes the real component. Using (3.2) and (3.3) the kinetic term reads

$$\langle \nabla \varphi, \nabla \varphi \rangle = \|\nabla \varphi\|^2 = (\partial \Delta)^2 + (\Delta + v')^2 \langle \nabla e_0, \nabla e_0 \rangle, \quad (3.4)$$

where ∂ is the exterior derivative acting on scalars; i.e. it is the gradient.

When we introduce a connection, it will clearly appear quadratically in the term $v'^2 \langle \nabla e_0, \nabla e_0 \rangle$ (see Equation 3.6 below). But of course, ∇e_0 won't contain all the information in ∇ . The part of ∇ that doesn't appear in the kinetic term will thus remain 'massless'. This geometric presentation of the Higgs mechanism makes the key features clear: the Higgs field picks out a direction in the bundle, and vector bosons associated with directions orthogonal to it acquire mass. Since we have expressed everything in terms of abstract index notation, with vector and tensor fields, it is hard to see how one could 'break the symmetry'. (Indeed, the mass terms for the gauge potentials will arise out of a combination of $v e_0$ and the gauge potentials, and these are perfectly gauge-covariant.)

Moreover, it is important to note that this is a geometric characterization that can be stated outside of the linearised regime. In this remarkably simple derivation, we are already able to glimpse all the general features of the mechanism. Again, no mention of stabilisers, gauge orbits, gauge-fixing, etc, was made, as they would have in order to reach a similar point in the standard or familiar derivation (see (Hamilton, 2017, Ch. 8.1) or (Tong, 2025, Ch. 2.2) for a comparison). For instance, the fact that perturbations of the Higgs field are orthogonal to the orbits of the vacuum is replaced by the orthogonality relation, (3.3), and so on. This concludes the classical, non-linearised account of the 'mass acquisition' mechanism.⁹

⁹Had one been considering a whole configuration space of Higgs field, one would have had to restrict the analysis to a sector $\Gamma_0(E)$ for which one of the configurations had an absolute minimum (i.e. $\min_{\varphi \in \Gamma_0(E), x \in M} \|\varphi(x)\| = v'$). But this entire paper concerns the classical domain, and so one may reasonably argue that these symmetry concepts—such as gauge-fixing—may be required when we introduce quantum mechanics, or the entire sector of configurations. Here is how far my concession would go: in a sum over configurations, we use e_0 as the anchor, or 'representational scheme' across physical possibilities; cf. (Gomes, 2025b; Kabel et al., 2025). And indeed, representational schemes can be compared to gauge-fixings (cf. (Gomes, 2025b, Sec. 3.3)). A translation of this idea to the gauge terminology would go as follows: consider $\Gamma(E^2)$, and its sector $\Gamma_0(E^2)$. Let $\varphi, \varphi' \in \Gamma_0(E^2)$. The group $\text{Aut}(E^2)$ acts transitively on the unit normal sections: it can take any internal direction into any other. Therefore, we could, by a suitable gauge transformation on φ , make it collinear with φ' . Once they are collinear, it is a trivial matter to separate out the part that has a given norm from the rest.

3.2 Mass Generation in the Linearised Theory

Introduce a connection $\nabla = d + \omega$ such that $de_0 = 0$ and $\omega \in \Gamma(T^*M \otimes \text{End}(E))$, where $\text{End}(E)$ are the linear endomorphisms of E ; so for $\xi \in \Gamma(E)$, we have $\omega \cdot \xi \in \Gamma(T^*M \otimes E)$. Defining $v' - v =: c$, for v a spacetime-independent (i.e. ‘translation-invariant’) minimum of the Higgs potential, we rewrite (3.2) as

$$\varphi(x) = (H(x) + v)e_0, \quad (3.5)$$

where $H(x) = \Delta(x) + c$. If we assume that c and Δ are of the same order, since $c = (v' - v) < 0$ and $\Delta(x) > 0$, $H(x)$ can be both positive or negative, i.e. $H \in C^\infty(M)$.¹⁰ Then from (3.4)

$$\|\nabla\varphi\|^2 = (\partial H)^2 + (H^2 + 2Hv + v^2) \|\omega \cdot e_0\|^2, \quad (3.6)$$

where, as usual, the norm of a tensor product factorises linearly, i.e. for each basis element $\lambda \otimes \xi \in \Gamma(T^*M \otimes E)$, we have:

$$\|\lambda \otimes \xi\| := \|\lambda\|_M \|\xi\|_E. \quad (3.7)$$

But to unclutter notation I will omit the subscripts when understood from context.

Further assuming that $\mathcal{O}(H) = \mathcal{O}(\omega) = \varepsilon$,¹¹ yields

$$\|\nabla\varphi\|^2 = (\partial H)^2 + v^2 \|\omega \cdot e_0\|^2 + \mathcal{O}(\varepsilon^3). \quad (3.8)$$

Here we see clearly how the quadratic terms in the connection ω would correspond to vector bosons ‘acquiring masses’; again, without invoking unitary gauge or Goldstone’s theorem.

But as I said, not all components of ω contribute to $\|\omega \cdot e_0\|^2$ in (3.8). In a basis $\{e_I\}$ adapted to e_0 , we have

$$\nabla e_I = \omega^J{}_I e_J, \quad \text{and so} \quad \nabla e_0 = \omega^i{}_0 e_i, \quad \text{with} \quad i \neq 0, \quad (3.9)$$

from the anti-symmetry of the connection. Then

$$\|\nabla\varphi\|^2 = (\partial H)^2 + v^2 \sum_{i \neq 0} (\omega^i{}_0)^2 + \mathcal{O}(\varepsilon^3). \quad (3.10)$$

Hence, only those components of ω that move e_0 (onto the orthogonal directions) ‘acquire mass’. The components that preserve e_0 , e.g. $\omega^i{}_j, i \neq j$, remain massless. In the group-theoretic language, these would correspond precisely to the stabiliser subgroup of e_0 .

This concludes the geometric derivation of the Higgs mechanism. Let us now see how it reproduces standard results from the familiar or standard approach to gauge theory. The missing ingredient for the comparison is to write the connection ω in terms of preferred representations of the Lie algebras in question. I will start by providing an example (that is indeed isomorphic to $su(2)$) before showing how the usual endpoint of the Higgs mechanism for gauge bosons is recovered.

¹⁰We could of course have started directly from (3.5), by again assuming that: (i) the potential depended only on the norm of the Higgs field; (ii) that the minimum of the potential was non-zero and spacetime independent; and (iii) that the norm of the Higgs field did not deviate too much from this minimum, in particular, that it was also non-zero everywhere. I find the order of assumptions made in my presentation clearer, because they can be easily stated outside the linearised regime.

¹¹In the comparative sense: that $\frac{|H|}{v} \sim \varepsilon \ll 1$, and *mutatis mutandis* for the appropriate norm on ω .

3.2.a Example: (M^3, g)

Suppose we are dealing with three-dimensional Riemannian manifold. Here a general $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$ connection has the form

$$\omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}. \quad (3.11)$$

If the Higgs unit vector is $e_0 = (1, 0, 0)^T$ (where T here is the transpose, and it allows us to write column-vectors in-line!), then

$$\omega \cdot e_0 = \begin{pmatrix} 0 \\ \omega_z \\ -\omega_y \end{pmatrix}. \quad (3.12)$$

Thus, we would get:

$$\|\nabla\varphi\|^2 = v^2(\omega_y^2 + \omega_z^2). \quad (3.13)$$

So ω_y and ω_z would ‘acquire mass’, while ω_x would remain ‘massless’.

3.2.b Electroweak Example: $\mathbb{C}^2 \otimes \mathbb{C}^1$

The covariant derivative on an element $\mathbf{v} \otimes \mathbf{w} \in V \otimes W$ is given by

$$\nabla(\mathbf{v} \otimes \mathbf{w}) = (\nabla^V \mathbf{v}) \otimes \mathbf{w} + \mathbf{v} \otimes \nabla^W \mathbf{w}, \quad (3.14)$$

where ∇^V, ∇^W are covariant derivatives on, in what follows, $V \simeq \mathbb{C}^2, W \simeq \mathbb{C}^1$, respectively.

For the electroweak theory, let $e_0 = e_0^2 \otimes e_0^1 \in \Gamma(E^2 \otimes E^1)$ with $e_0^2 = (0, 1)$, $e_0^1 = 1$. And so we get:

$$\nabla e_0 = \omega \cdot e_0^2 + e_0^2 Z = (\omega + iZ\mathbb{1})e_0^2, \quad (3.15)$$

where ω is the connection for the covariant derivative on \mathbb{C}^2 and Z is the connection on \mathbb{C} . To complete the comparison with the standard formalism, we choose the weak-isospin eigenbasis, on which the third generator of the $\mathfrak{su}(2)$ algebra, \mathbb{T}_3 , is diagonal. Omitting the coupling constants for brevity, we can write ω as:¹²

$$\omega = \begin{pmatrix} iW_3 & iW_1 - W_2 \\ iW_1 + W_2 & -iW_3 \end{pmatrix}, \text{ and } iZ\mathbb{1} = \begin{pmatrix} iZ & 0 \\ 0 & iZ \end{pmatrix}. \quad (3.16)$$

¹²Note that this is not the ω written in terms of the spin coefficients, i.e. in terms of an orthonormal frame that includes e_0 . That could also be done, and indeed it was done in the previous example $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$, with an orthonormal frame $(0, 1), (0, i), (1, 0), (i, 0)$, for the inner product $\text{Re}\langle \cdot, \cdot \rangle$, which is effectively what appears in Lagrangians, due to the use of the complex conjugate terms, cf. (Hamilton, 2017, Ch. 8). Here we are attempting to make contact with the standard notation and formalism and so are using its conventions.

Applying this to e_0^2 in (3.15) gives

$$\nabla e_0 = \begin{pmatrix} iW_1 - W_2 \\ -iW_3 + iZ \end{pmatrix}. \quad (3.17)$$

Hence the corresponding quadratic term appearing in (3.8) is

$$\|\nabla e_0\|^2 = W_1^2 + W_2^2 + (Z - W_3)^2. \quad (3.18)$$

Thus W_1, W_2 and the combination $Z - W_3$ acquire mass, while $Z + W_3$ remains massless. The latter is identified with the photon. Of course, had we chosen a different form for e_0^2 , we would have obtained different combination of massive and massless bosons. For instance, for $e_0^2 = (1, 0)$ it is easy to see that it would have been $Z + W_3$ that would acquire mass, while $Z - W_3$ would remain massless.

4 The Yukawa mechanism

Whereas the Higgs mechanism is used to ‘endow mass’ to the gauge potentials, the Yukawa form is used to endow mass to the matter fields—here we needn’t use scare-quotes!

In the Standard Model fermion masses cannot be introduced as they can for real or complex-valued scalar fields. First of all, a Dirac mass term must couple left- and right-handed chiral fermions; moreover, the two chiralities are mapped into internal spaces that transform differently under the gauge group $G = SU(3) \times SU(2) \times U(1)$, so coupling them would violate gauge invariance: this is related to the issue we saw in Section 2.1 about canonical isomorphisms between associated vector bundles with different representations. The solution is to introduce the Higgs field ϕ , in such a way that gauge invariance is preserved, while the fermions acquire effective masses. This is the *Yukawa mechanism*.

Here I will essentially follow the treatment given in (Hamilton, 2017, Ch. 8), whose notation and general approach is already much closer to the geometric approach that I’m pursuing here (as compared to the treatment of more familiar textbooks, for instance, the one given in (Weinberg, 2005, Ch. 21), which uses representation theory more heavily). So I will call the treatment to be followed here ‘the standard’ treatment of the Yukawa mechanism. In Section 4.1 I will describe the obstruction to the formulation of mass terms for fermions, and its resolution in this, geometric-friendly but still ‘standard’, exposition. Then in Section 4.2 I will discuss what I think is explanatorily unsatisfactory about this resolution, and say why I take the VB-POV to provide a more transparent explanation.

4.1 The ‘standard’ presentation of the Yukawa mechanism

In more detail, here is the obstruction to the formulation of mass terms for fermions. Fermions are spinors, but for Weyl spinors, the inner product is anti-diagonal in the left and right basis: $\bar{\psi}_R \psi_R = 0$, and so, in order to extract mass terms we must couple left to right-handed spinors:

$\bar{\psi}_R \psi_L$. Thus, if both ψ_L and ψ_R are valued in the same internal space, i.e. in the same vector bundle, and are in the same representation, one may add mass terms of the form:

$$\mathcal{L}_{\text{mass}} = -m \bar{\psi} \psi = -m \text{Re}(\bar{\psi}_L \psi_R) \quad (4.1)$$

and this will be gauge invariant since ψ_L and ψ_R transform in the same representation of the gauge group. I.e. locally, $\psi_L \in \Gamma(S_L \otimes E)$, where (E, M, V) is the vector bundle with the representation space V of the gauge group in question, and S_L is the bundle of left-handed spinors over spacetime, whose typical fiber space is called Δ_L (*mutatis mutandis* for right-handed spinors).

In the Standard Model, however, fermions are both twisted and chiral: left- and right-handed components transform in inequivalent representations of the gauge group. For instance,

$$e_L \in (\mathbf{1}, \mathbf{2}, -1), \quad e_R \in (\mathbf{1}, \mathbf{1}, -2).$$

These internal vector bundles are representationally inequivalent; e.g. $\psi_L \in \Gamma(S_L \otimes E_L)$ and $\psi_R \in \Gamma(S_R \otimes E_R)$, with different representation spaces, $V_L \not\cong V_R$. Thus a bilinear such as $\bar{e}_L e_R$ is not gauge-invariant, and a bare mass term as in (4.1) is forbidden. (Table 1, reproduced from (Hamilton, 2017, Table 8.2), shows the representations of $SU(2)_L \times U(1)_Y$ for the fermions and the Higgs in the standard model.)

Moreover, for V_R, V_L irreducible, unitary, non-isomorphic representations of G , mass pairings, defined as G -invariant maps, $\kappa : V_L \times V_R \rightarrow \mathbb{C}$, complex antilinear in the first variable and complex linear in the second (so that they form mass terms), are necessarily trivial (see (Hamilton, 2017, Theorem 7.6.11)).

The remedy is a *Yukawa form*, defined as follows. Let V_L, V_R, W be representation spaces for $G = SU(3) \times SU(2) \times U(1)_Y$. A Yukawa form is a G -invariant trilinear map

$$\tau : V_L \otimes W \otimes V_R \longrightarrow \mathbb{C},$$

antilinear in V_L , real linear in W , linear in V_R . What do these maps look like, more precisely? Let us look at an example. Consider the $SU(2) \times U(1)$ representations for the leptons (taken from Table 1):

$$V_L = \mathbb{C}^2 \stackrel{\rho_L}{=} \mathbf{2}_{-1}, \quad (4.2)$$

$$V_R = \mathbb{C} \stackrel{\rho_R}{=} \mathbf{1}_{-2}, \quad (4.3)$$

$$W = \mathbb{C}^2 \stackrel{\rho_W}{=} \mathbf{2}_1. \quad (4.4)$$

Then, for $l_L : U \rightarrow V_L, \phi : U \rightarrow W, l_R : U \rightarrow V_R$, it is standard to define the Yukawa form as:

$$\tau : V_L \times W \times V_R \longrightarrow \mathbb{C}, \quad (4.5)$$

$$(l_L, \phi, l_R) \longmapsto l_L^\dagger \phi l_R, \quad (4.6)$$

which is $SU(2) \times U(1)$ invariant by construction.

Sector	$SU(2)_L \times U(1)_Y$ rep.	Basis vectors	Particle	T_3	Y	Q
Q_L	$\mathbb{C}^2 \otimes \mathbb{C}_{1/3}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	\mathbf{u}_L	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{2}{3}$
		$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	\mathbf{d}_L	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{3}$
Q_R	$\mathbb{C} \otimes \mathbb{C}_{4/3}$	1	\mathbf{u}_R	0	$+\frac{4}{3}$	$+\frac{2}{3}$
	$\mathbb{C} \otimes \mathbb{C}_{-2/3}$	1	\mathbf{d}_R	0	$-\frac{2}{3}$	$-\frac{1}{3}$
L_L	$\mathbb{C}^2 \otimes \mathbb{C}_{-1}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	ν_{eL}	$+\frac{1}{2}$	-1	0
		$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	e_L	$-\frac{1}{2}$	-1	-1
L_R	$\mathbb{C} \otimes \mathbb{C}_{-2}$	1	e_R	0	-2	-1
Higgs φ	$\mathbb{C}^2 \otimes \mathbb{C}_1$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	φ^+	$+\frac{1}{2}$	+1	+1
		$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	φ^0	$-\frac{1}{2}$	+1	0
Higgs $_{\perp}$ φ_c	$\mathbb{C}^2 \otimes \mathbb{C}_{-1}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\overline{\varphi}^0$	$+\frac{1}{2}$	-1	0
		$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$-\overline{\varphi}^+$	$-\frac{1}{2}$	-1	-1

Table 1: First-generation fermion representations under $SU(2)_L \times U(1)_Y$, together with the Higgs doublet and its conjugate. Here boldface on the quarks means each such term is a vector in \mathbb{C}^3 . (φ^0, φ^+) as well as the left-handed particles are doublets: they can be rotated into each other by an $SU(2)$ transformation. Y is the hypercharge, and T_3 is weak isospin. Here $Q = T_3 + \frac{1}{2}Y$.

τ is still a map on vector spaces, and so it relies on trivialisations of the vector bundles (i.e. of the local decompositions of the form $E|_U \simeq U \times V$). Therefore, we must invariantly extend this map to sections of associated vector bundles, but that is easy to do. Given a section $\sigma(x)$ of an $SU(2) \times U(1)$ principal bundle (cf. (2.2)), we can use the local maps $l_L : U \rightarrow V_L$, $\varphi : U \rightarrow W$, $l_R : U \rightarrow V_R$ above to form sections of the corresponding vector bundles, $e_L \in \Gamma(S_L \otimes E_L)$, $\varphi \in \Gamma(F)$, $e_R \in \Gamma(S_R \otimes E_R)$ that are independent of the trivialisation of the vector bundle. For instance, a left-handed electron (I will have more to say about the ‘up’ and ‘down’ components of this particle in a second) would be given by:

$$e_L = \psi_L \otimes [\sigma, l_L], \quad (4.7)$$

where ψ_L is a left-handed Weyl spinor, $\psi_L \in \Gamma(S_L)$, and $\lambda_L := [\sigma, l_L] \in \Gamma(E_L)$, where E_L is the vector bundle with typical fiber V_L in (4.2), *mutatis mutandis* for the right-handed electron, and $\varphi := [\sigma, \phi]$.

In any case, since τ is invariant under $SU(2) \times U(1)$, we can define:

$$T(e_L, \varphi, e_R) := \tau(l_L, \phi, l_R) = l_L^\dagger \phi l_R, \quad (4.8)$$

which is gauge-invariant. But why this map? What kind of map would have been analogous

to this for completely different vector spaces, groups and representations? To answer this question, I will first translate it into the VB-POV.

4.2 The VB-POV presentation of the Yukawa mechanism

In Section 2.1 I argued that there was no canonical map between associated vector bundles corresponding to different representations of the principal bundle, and yet I have just presented a map from different vector bundles into a gauge-invariant scalar. But there is no real mystery here: we don't need a canonical map between associated vector bundles. All we need is that T , given in (4.8), is a map between associated vector bundles, with τ a map between the representation spaces; and presenting one such map is sufficient for comparison with experiments. Nonetheless, I find this answer unsatisfactory, because opaque: why this particular map? Couldn't we have found others? What are the possible maps, would they equally apply for different groups and representation spaces; and how should we interpret them?

I take the geometric, VB-POV, to provide a more transparent interpretation of what the map T represents, and what other choices would represent. Again, in the geometry-first formulation, all we have are structures in the fundamental vector bundle spaces. The fundamental vector spaces are given by $(E^n, M, \mathbb{C}^n, \langle \cdot, \cdot \rangle_n)$, for $n = 1, 2, 3$ (we will include orientation as further structure below, when we look at the Yukawa form for quarks). Different particles are merely different sections of different tensor products for these fundamental vector spaces. We replace 'quantum numbers' by a geometric characterisation of a given particle. Thus, for instance, a down-right-handed quark (of any of the three generations, but here we assume the first) is given by:

$$\mathbf{d}_R \in \Gamma(E^3 \otimes (E^{1*} \otimes E^{1*})), \quad (4.9)$$

whereas vector bosons are replaced by the corresponding affine covariant derivatives, e.g. $\nabla^1, \nabla^2, \nabla^3$ (see (Gomes, 2024, 2025a) for more details).

In this formulation, weak isospin T_3 —defined only with respect to a chosen basis of the Lie algebra—has no independent geometrical meaning (see 3.2.b and footnote 12). Accordingly, left-handed fermions, together with φ^+ and φ^0 (also $SU(2)$ -doublets), are best understood simply as components of the vector fields Q_L and L_L . The familiar distinction between, say, the electron and the electron-neutrino, or between the up- and down-left quarks, does not arise at this level: it appears only through their couplings with the Higgs. The Higgs field φ provides a frame within \mathbb{C}^2 that endows T_3 —and hence these component fields—with physical significance. The charges listed in Table 1 are already adapted to this frame, since they presuppose the choice $\varphi = \varphi^0 = (0, 1)^T$ (i.e. $\varphi^+ = 0$); for example, only in that frame do the left-handed up-quark components appear as $(u_L^I, 0)^T$.¹³

But geometrically it makes more sense to define the left-handed components of both leptons

¹³This explains why Table 1, reproduced from (Hamilton, 2017, Table 8.2), can be misleading: if both components of the Higgs are retained, the up and down components of the left-handed quarks and leptons do not yet have any physical meaning.

and quarks as parallel and orthogonal to the Higgs according to the inner product on E^2 , i.e.:

$$\mathbf{e}_L := \langle L_L, e_0 \rangle_2 e_0, \quad \text{with} \quad e_L = \langle L_L, e_0 \rangle_2; \quad \boldsymbol{\nu}_{eL} := L_L - e_L, \quad (4.10)$$

$$\mathbf{u}_L^I := \langle Q_L^I, e_0 \rangle_2 e_0 \quad \text{with} \quad u_L^I = \langle Q_L^I, e_0 \rangle_2; \quad \mathbf{d}_L := Q_L - \mathbf{u}_L, \quad (4.11)$$

where capital I indicates color components (i.e. red, green and blue) in an orthonormal frame of \mathbb{C}^3 and I used the notation e_0 for the unit-direction of the Higgs, introduced in Section 3.1 (not to be confused with the left-handed electron, \mathbf{e}_L).¹⁴

Before we give the geometric interpretation of (4.8), and of the corresponding form for quarks, note that, given an orthonormal basis for E^2 , we can form duals: for $\xi = \xi^\perp e_\perp + \xi^\parallel e_0 = (\xi^\perp, \xi^\parallel)^T$ (e.g. $\mathbf{e}_L = L_L^\parallel$, $\boldsymbol{\nu}_{eL} = L_L^\perp$) the dual takes the conjugate of the transpose, so:

$$((\xi^\perp, \xi^\parallel)^T)^* = (\bar{\xi}^\perp, \bar{\xi}^\parallel). \quad (4.12)$$

Using (4.12) and an orthonormal frame aligned with the Higgs (3.5), the Yukawa term for the leptons in Equation (4.8) now can be stated directly using (trivialisation-independent) sections of the vector bundles, without the need to involve the sections σ of the principal bundles, and reads (including a coupling constant, g_e):

$$T(L_L, \varphi, e_R) = g_e \langle \langle L_L, \varphi \rangle_2, e_R \rangle_1 = g_e (v + H) \bar{e}_L e_R, \quad (4.13)$$

where the first equality gives the ‘standard’ definition; e_L is a Weyl left-handed spinor and internal scalar (i.e. the magnitude of the vector field along the Higgs); $\langle \cdot, \cdot \rangle_2$ is complex anti-linear in the first entry and maps elements of $E^2 \otimes E^1 \times E^2$ into E^1 in the obvious way (by taking inner products among the E^2 components); and $\langle \cdot, \cdot \rangle_1$ is just the scalar inner product in \mathbb{C} .¹⁵ From (4.13) we can see how mass terms, proportional to $g_e v$ (as well as interactions with the Higgs field) emerge for the electron.

Geometrically, the inner products in (4.13) are a very natural way to obtain scalars: we are measuring ‘internal angles’ between the different particles seen as vector fields on the same spaces. I take this form of (4.13), namely $\langle \langle L_L, \varphi \rangle_2, e_R \rangle_1$, to be a more transparent interpretation of the Yukawa term for leptons.

Chirality here shows up in the fact that right-handed particles don’t couple directly to the Higgs: e_R couples to the \mathbb{C}^1 component of L_L . This is geometrically explained by the fact that only left-handed particles have components in E^2 . Note, moreover, that in this convention the neutrinos don’t acquire mass. First, because the left-handed electron-neutrinos are orthogonal to the Higgs, but more fundamentally, because we have not included right-handed neutrinos in our particle content. (Because of this feature, the Yukawa terms for leptons are diagonal in generations: these mass terms don’t mix, say electrons with muons and taus.)

In the case of quarks (or also for the leptons if we include right-handed neutrinos), things are different: we add another field, which is orthogonal to, but not independent from, the

¹⁴Not many textbooks that I have encountered emphasise this point—(Tong, 2025) is an exception. And none describe it geometrically as I did here.

¹⁵This is slightly misleading: what we have here is that $\varphi \in \Gamma(E^2 \otimes^3 E^1)$, i.e. the third tensor product of E^1 , which is still one-dimensional, $e_L^* \in \Gamma(E^{2*} \otimes^3 E^{1*})$, and $e_R \in \Gamma(\otimes^6 E^1)$. This is why they match to a scalar.

Higgs, and generations mix. This new field, called φ_c on Table 1, is obtained by recruiting another geometric structure that we can equip \mathbb{C}^2 with (besides the Hermitean inner product): an orientation. This implies we can use the totally anti-symmetric form, or the volume form, ϵ_{ab} , as part of the geometrical structure. In other words, whereas the Higgs mechanism, described in Section 3, used the structure $(E^2, M, \mathbb{C}^2, \langle \cdot, \cdot \rangle_2)$, here we extend that to the structure $(E^2, M, \mathbb{C}^2, \langle \cdot, \cdot \rangle_2, \epsilon)$.¹⁶

Now, besides the metric, we can use ϵ_{ab} and its inverse ϵ^{ab} to raise or lower indices.¹⁷ Thus if we call the isomorphism $J : E^2 \rightarrow E^{2*}$ which acts as $\xi \mapsto \langle \xi, \cdot \rangle$ we have:

$$C := \epsilon^\sharp \circ J : E^2 \mapsto E^2 \quad (4.14)$$

$$\xi^a \mapsto \epsilon^{ac} h_{cb} \xi^b \quad (4.15)$$

where we used, in abstract index notation, h_{ab} as the inner product on E^2 . Thus we call

$$\varphi_c := C(\varphi); \quad (4.16)$$

it can be seen as a measure on the ‘areas’ orthogonal to φ . (see Appendix C for more details on how this definition relates to the standard one).

Denoting the generation by an index $i = 1, 2, 3$, we then have, for the total Yukawa coupling term for quarks:¹⁸

$$T(Q_L, \varphi, d_R) := Y_{ij}^d \overline{Q}_L^i \varphi \mathbf{d}_R^j + Y_{ij}^u \overline{Q}_L^i \varphi_c \mathbf{u}_R^j = Y_{ij}^d \langle \langle Q_L^i, \mathbf{d}_R^j \rangle_3, \varphi \rangle_2 \rangle_1 + Y_{ij}^u \langle \langle Q_L^i, \mathbf{u}_R^j \rangle_3, \varphi_c \rangle_2 \rangle_1. \quad (4.17)$$

The boldface on lowercase letters is used to indicate that these are vector fields (and so are the capital Q and L , and so is the Higgs φ , but omitting boldface here doesn’t conflict with our notation in what follows). Again, the first equality in (4.17) gives the ‘standard’ definition (cf. (Hamilton, 2017, Lemma 8.8.4)); the second gives the geometric form of that definition: it is, in the VB-POV, what really counts.¹⁹

Nonetheless, as often is the case in physics, we can glean more by introducing a frame: here, once more it is convenient, in order to compare with standard presentations, to choose

¹⁶Under $A \in U(2)$, ϵ_{ab} is taken to transform as $\epsilon_{ab} \mapsto \det(A) \epsilon_{ab}$. So $SU(2)$ preserves it. Moreover, since $AA^\dagger = \mathbb{1}$ for any $A \in U(n)$, we know that $\det(A) \det(A^\dagger) = |\det(A)| = 1$, so $\det(A) = e^{i\theta}$ denotes an orientation change the \mathbb{C}^n . Using ϵ_{ab} as a geometric datum then implies we have a fixed orientation, as well as an inner product, on \mathbb{C}^2 .

¹⁷Indeed, in standard differential geometry, we can find a similar sort of operator acting on two dimensions: the Hodge star: which would take a basis $e_0, e_1 \mapsto -e_1, e_0$, respectively, so *its action on vectors* can be written in this frame as a matrix operator: $*$ = $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which is of the same form as ϵ_{ab} .

¹⁸Note that here, unlike for the leptons and the left-handed quarks, the up and down right-handed quarks are genuinely different particles, since they have different components in E^1 .

¹⁹It is a little disappointing that, unlike their left-handed counterparts, up and down right-handed quarks can’t be straightforwardly understood as components of a single vector field, due to their different components in \mathbb{C}^1 . If they could be so understood, in place of (4.17), we would have ther simpler:

$$T(Q_L, \varphi, Q_R) = Y_{ij}^d \langle \langle Q_L^i, Q_R^j \rangle_3, \varphi \rangle_2 \rangle_1 + Y_{ij}^u \langle \langle Q_L^i, Q_R^j \rangle_3, \varphi_c \rangle_2 \rangle_1, \quad (4.18)$$

which only takes the components of the same inner product along and orthogonal to the Higgs.

the orthonormal frame (3.5) for the Higgs, which gives the components for the quarks along and orthogonal to the Higgs (given in Equation (4.10)) as in Table 1, as well as $\varphi^+ = 0$. Then:

$$Y_{ij}^d \langle \langle Q_L^i, \mathbf{d}_R^j \rangle_3, \varphi \rangle_2 \rangle_1 + Y_{ij}^u \langle \langle Q_L^i, \mathbf{u}_R^j \rangle_3, \varphi_c \rangle_2 \rangle_1 = (H + v) \left(Y_{ij}^d d_L^{Ii} d_R^{Ij} + Y_{ij}^u u_L^{Ii} u_R^{Ij} \right), \quad (4.19)$$

where now all variables are scalar (and we are summing over the color indices, I , as well as over the generations i, j).

Lastly, the Yukawa matrices Y are generically non-diagonal, i.e. they mix generations of quarks. One can always find linear combinations of quarks such that, say, Y^u is diagonal; this defines what is called the *mass basis*. But Y^u and Y^d cannot be diagonalised simultaneously, and the residual mixing is encoded in the *Cabibbo–Kobayashi–Maskawa* (CKM) matrix. Most textbooks (cf. (Hamilton, 2017, p. 515)) then explain that the CKM matrix describes the physical effects of left-handed quark mixing across generations, from the ‘mass eigenstate basis’ to the ‘weak eigenstate basis’ (the latter being the one we have used here). It then “follows that the interactions with the W-bosons can connect quarks from different generations if the CKM matrix is not diagonal” (ibid).

In the geometric perspective the situation is more transparent. If the up and down left-handed quarks were truly independent particles—i.e. distinct fields rather than components of the same field (usually called a doublet) in E^2 —we *could* diagonalise Y_u and Y_d separately. But because they are components of the same E^2 -field, and because these components are coupled to different fields (e.g. φ, φ_c), we can’t. Correspondingly, the W bosons represent ∇^2 , the covariant derivative on E^2 , and so they, too, naturally mix generations when they couple to the relevant currents.

5 A defense of the geometry-first formulation

To motivate the methodological defense of the VB-POV, let me recall an illustrative example from the previous Section. The Yukawa coupling for quarks depends essentially on the orientation of \mathbb{C}^2 : it requires the introduction of φ_c , which encodes the ‘oriented area’ orthogonal to the Higgs. This shows how, in the VB-POV, group-theoretic distinctions (here, the reduction from $U(2)$ to $SU(2)$) arise directly from geometric structures. But no parallel mechanism seems to exist for \mathbb{C}^3 : why is the orientation of color space geometrically important, so as to enforce $SU(3)$ rather than $U(3)$? I will return to this question below, but the broader moral is already clear: the VB-POV ties the existence of symmetry groups much more tightly to the underlying geometry than the symmetry-first formulation actually requires. And while this may look like a liability of the VB-POV, it in fact strengthens its case over the PFB-POV. In what follows, I will make that case explicit: first, by examining how the PFB-POV allows symmetry and geometry to come apart; second, by showing that this “slack” is not available within the VB-POV; and finally, by arguing that the tighter fit demanded by the VB-POV is not a weakness but a methodological advantage.

The first step is to recall how the two points of view differ in their basic ingredients. In

the VB-POV, one can only define (sub)groups G as isomorphic to $\mathbf{Aut}(V)$. By contrast, in the PFB-POV there are in principle three separate ingredients: not only G (or P) and V , but also the representation ρ .

Here again the spacetime analogy is instructive. The automorphism group of the tangent bundle equipped with a Lorentzian metric is $SO(3, 1)$ (or $O(3, 1)$), a fact that becomes explicit once orthonormal frames are introduced. Yet most of spacetime geometry can be developed without ever invoking $SO(3, 1)$; this is precisely why it serves as an analogue for the geometry-first VB-POV of gauge theory. And in the spacetime case, if one were to posit some other group acting on TM —say $O(2)$ or $SU(n)$ —a clear geometric rationale would be required: which feature of TM could elicit such an action?

The situation is different in the symmetry-first approach to gauge theory. In general, one posits a vector space V , a group G , and an action ρ , without requiring that G transparently reflect the structure of V . In the equivalence relation that defines associated bundles, (A.5), g is an element of the structure group of the PFB, but $\rho(G)$ need not coincide with the automorphism group of the typical fibre V . The only requirement is invariance:

$$\rho(G) \subseteq \mathbf{Aut}(V). \quad (5.1)$$

By contrast, in the spacetime case we do have an identification,

$$SO(3, 1) = G \simeq \rho(G) \simeq \mathbf{Aut}(T_x M), \quad (5.2)$$

whereas in general gauge theories

$$G \simeq \rho(G) \simeq \mathbf{Aut}(V) \quad (5.3)$$

fails unless the symmetry group is reconstructed from the geometry, as the VB-POV requires. But when (5.3) holds—e.g. for $G \simeq O(4)$ acting on $V \simeq \mathbb{R}^4$ with the Euclidean inner product via the fundamental representation—one can reconstruct (P, G, M) from $L_\rho(E)$ (the bundle of admissible frames for E , see Section 2.1).

A particularly bad (and ubiquitous) subcase. In many gauge theories—and routinely in the Standard Model—the fibres of the associated bundles are built from direct sums or tensor products of distinct factors, e.g. $V = V_1 \otimes V_2 \oplus V_3$, with the gauge group acting differently on different factors and often *trivially* on some. In such cases the geometry of a single associated bundle cannot, by itself, fix the structure group: kernels appear factorwise, so typically $G \not\simeq \rho(G)$ while still $\rho(G) \subset \mathbf{Aut}(V)$. Electrons, for instance, carry no colour and are unaffected by $SU(3)$; right-handed quarks are unaffected by $SU(2)$; and so on. This is precisely the kind of degeneration that makes the PFB-POV ill-suited to tying groups to geometry in the presence of factorwise (and trivial) actions.

Both the general situation and this subcase imply a loose link between symmetry and geometry: neither determines the other. Let us now make this looseness concrete with three perfectly ordinary PFB-POV examples: $U(1)$ acting on \mathbb{C}^3 by scalar multiples of the identity;

$SO(4)$ acting on \mathbb{C}^2 via a spinor map; and the trivial action of $SU(n)$ on \mathbb{C}^m . Each case packs a lesson.

Example 1 (faithful but not surjective). The condition $G \simeq \rho(G)$ holds iff ρ is *faithful* (injective), so that only $\mathbf{1}$ acts trivially. This is satisfied when the gauge group is $U(1)$, the fibres are \mathbb{C}^3 with Hermitian inner product, and the representation is $\rho_y(\theta) = e^{iy\theta}\mathbf{1}$ with $y \in \mathbb{N}$. But $U(1)$ is clearly not isomorphic to the full automorphism group of \mathbb{C}^3 , which is $U(3)$: ρ is not surjective onto $\mathbf{Aut}(V)$. From the PFB-POV this is perfectly admissible: one may posit a group that preserves the relevant geometric structures without exhausting them. Still, there is a geometric interpretation: the action rotates the complex volume form of \mathbb{C}^3 .²⁰ Thus in this case we have

$$G \simeq \rho(G) \subset \mathbf{Aut}(V), \quad \dim(G) < \dim(\mathbf{Aut}(V)). \quad (5.4)$$

Lesson: even with a faithful representation, the full geometry of the fibre cannot be recovered from $\rho(G)$ alone.²¹

Example 2 (trivial representation). Consider the trivial action of $SU(n)$ on $V = \mathbb{C}^m$. In this case

$$\rho(G) = \mathbf{1} \subset \mathbf{Aut}(V), \quad \dim(G) \leq \dim(\mathbf{Aut}(V)) \text{ if } m \leq n, \quad \dim(G) \geq \dim(\mathbf{Aut}(V)) \text{ if } m \geq n. \quad (5.5)$$

Even if $n = m$, one cannot reconstruct G from its representation on V , since the action is trivial. Thus we have

$$G \not\simeq \rho(G), \quad \rho(G) \subset \mathbf{Aut}(V), \quad (5.6)$$

so both conditions in (5.3) fail. Intrinsically, G may be either larger or smaller than $\mathbf{Aut}(V)$, and the group can neither be recovered from, nor recover, the geometry of the fibre.

Lesson: with a trivial representation, the fibre carries no information about G , and G imposes no structure on the fibre.

Example 3 (non-faithful but geometrically admissible). The reconstruction of G may be elusive even when the group acts non-trivially. Consider $G = SO(4)$ with fibres $V = \mathbb{C}^2$. $SO(4)$ admits two inequivalent irreducible representations on \mathbb{C}^2 , corresponding to left- and right-handed spinors under $SU(2)$. If we pick one of these factors to act, $SO(4)$ does preserve the structure of \mathbb{C}^2 , and so the situation is admissible from the PFB-POV. In this case we have

$$G \not\simeq \rho(G) \simeq \mathbf{Aut}(V), \quad \dim(G) > \dim(\mathbf{Aut}(V)). \quad (5.7)$$

The image of the representation matches $\mathbf{Aut}(V)$, yet the full group G is strictly larger. Lesson: even when $\rho(G) \simeq \mathbf{Aut}(V)$, the embedding of G may lack a clear geometric rationale.

²⁰That is, it rotates $\Lambda^3 \mathbb{C}^3$ (equivalently, the determinant line): scalar multiplication $e^{i\theta}\mathbf{1}$ acts by $e^{i3\theta}$ on $\Lambda^3 \mathbb{C}^3$. Fixing a unit complex volume form reduces $U(3)$ to $SU(3)$; the residual $U(1)$ is the phase on the determinant line. See footnote 16. In cases like this, the principal connection would, via (A.6), only determine the parallel transport of a phase.

²¹On the broader question of recovering the geometry of a vector space from the action of subgroups of $GL(V)$, see Gomes et al. (2024).

In the $U(1)$ case on \mathbb{C}^3 (Eq. (5.4)), the representation was faithful, so $G \simeq \rho(G)$. Though the full geometric data of the fibre could not be recovered, it was at least plausible that the group could be grounded in appropriate geometric structures on the fibre, as we did.²² By contrast, in the cases of $SO(4)$ acting on \mathbb{C}^2 and $SU(n)$ acting trivially on \mathbb{C}^m , the representation is not faithful: $G \not\simeq \rho(G)$, so no unique reconstruction of the group from the fibre is possible, even in principle. These cases illustrate the PFB-POV's tolerance for symmetry groups that exceed what can be informed by geometry.

Summing up: there are two necessary conditions for the group to reflect the geometry of the fibre, as expressed in Equation (5.3), and they can fail independently. We may have $G \not\simeq \rho(G)$, or $\rho(G) \subset \mathbf{Aut}(V)$ without surjectivity. And even when $G \simeq \mathbf{Aut}(V)$, the link between V and G —namely $\rho(G)$ —may still fail.

A particularly bad—and in the Standard Model, ubiquitous—subcase occurs when whole factors of the group act trivially on components of V . In such cases, $G \not\simeq \rho(G)$ for every individual multiplet, and at best we have only $\rho(G) \subset \mathbf{Aut}(V)$. The Standard Model abounds with these situations. For example, electrons are unaffected by $SU(3)$, right-handed quarks by $SU(2)$, and so on. In other words, the V 's appearing in associated bundles are often direct sums or tensor products such as $V = V_1 \oplus V_2$, with entire factors of the gauge group acting trivially on some component, under the representations defined by the particles' quantum numbers. These are precisely of the kind illustrated in Eq. (5.6). There is therefore no prospect that any single associated vector bundle, with ρ given by its quantum numbers (i.e. its representation labels), could, on its own, tie the total PFB group to the geometry of its fibre.

That the PFB-POV accommodates such cases is, from its perspective, a feature rather than a bug. It is necessary in the standard approach, where each particle type is represented as a section of a particular associated vector bundle, but without any attempt to reconstruct the gauge group from the geometrical structure of those bundles.

But if one nonetheless tries to recover the group from the automorphisms of the associated bundles themselves, not only would one fail to get the right verdict; in general, the recovery would not even be consistent.

Suppose we are given a collection of associated vector bundles that are claimed to come from the same principal bundle, but we are not told what the gauge group is; we are asked to recover it from the automorphism groups of the associated bundles.

For concreteness, take $V \simeq \mathbb{C}^3 \otimes \mathbb{C}^2$, and consider two representations: $\rho_1 = \mathbf{3} \otimes \mathbf{1}$ (a colour triplet, singlet under $SU(2)$), and $\rho_2 = \mathbf{1} \otimes \mathbf{2}$ (a weak doublet, singlet under $SU(3)$). Assume,

²²Faithfulness is not, of course, sufficient. The VB-POV strategy is to fix a fibre V together with invariant geometric or algebraic data (e.g. inner product, volume/symplectic forms, higher-degree tensors), and then define the gauge group as $G = \mathbf{Aut}(V, \text{data}) \subset GL(V)$. This works canonically for the classical families, but for exceptional groups it typically fails to be unique: (i) distinct, equally natural choices of data can select the same abstract G ; (ii) small changes of the data may alter the real form or enlarge the stabiliser; and (iii) the data alone need not determine isogeny or centre. Hence a unique grounding of an exceptional gauge group via $G = \mathbf{Aut}(V, \text{data})$ is generally not available.

per impossibile, that the collection

$$(P, M, G, \{\rho_i\}_i, \{V_i\}_i), \quad i = 1, 2 \quad (5.8)$$

is consistent, and that we can reconstruct (P, G, M) from each $L_{\rho_i}(E_i)$ (the bundle of admissible frames for E_i , see Section 2.1). The problem is that we would in general recover *different* groups for different i : constructing the bundle of admissible frames forces a subgroup $G' \subset G \simeq GL(V)$ to act trivially on some subspaces of E_i , so the resulting principal bundle reflects only the subspaces where G' acts non-trivially—and these differ across the associated bundles. Thus in this case we would recover $G'_1 = SU(3)$ from one bundle, and $G'_2 = SU(2)$ from the other. What we would like, of course, is $SU(3) \times SU(2)$; but the product structure is nowhere to be found at the level of any single associated bundle.²³

The VB-POV avoids this difficulty by shifting the level at which geometry fixes symmetry. It requires each gauge-group *factor* to arise as the automorphism group of a *fundamental* (or atomic) vector bundle, not of the composite associated bundles corresponding to particle multiplets. These fundamental bundles are posited as the basic building blocks: from them one can reconstruct each factor ($SU(3)$, $SU(2)$, etc.), and then deduce their actions on tensor products and direct sums. Once the gauge group is in place, each particle type may indeed live in an associated vector bundle, but at this composite level the group need not be recoverable from geometry. By contrast, in the PFB-POV one begins with associated bundles for each particle, and these typically combine several of the VB-POV's building blocks at once.²⁴

The upshot is that, in general, the PFB-POV allows a certain “slack” between P 's symmetry group G and the geometry of the associated vector bundles V . This slack is no problem within the PFB-POV itself, since one is not attempting to *extract* symmetries from geometry, but merely postulating both, requiring only that the former preserve the latter. On the other hand—thankfully for the geometry-first view—in our world, the symmetry group realised in Nature happens to fit the VB-POV account of symmetry quite snugly.

²³One might try to rescue the PFB-POV by restricting admissible theories to those in which no group element is superfluous—i.e. such that every non-identity element acts non-trivially on some associated vector bundle. Formally, for $E_i := P \times_{\rho_i} V_i$ one would require

$$\forall g \in G, \exists v \in \bigcup_i E_i \text{ such that } \rho_i(g)v \neq v. \quad (5.9)$$

This would at least guarantee a faithful action of the *entire* group on the *total* field content. But relating this reconstructed group back to the automorphisms of the individual fibres V_i is obscure. And in any case the restriction fails in the Standard Model: the action of $SU(3) \times SU(2) \times U(1)$ on all multiplets has a common central kernel $\Gamma \cong \mathbb{Z}_6$, generated (with the convention $Q = T_3 + \frac{1}{2}Y$) by $(\omega_3 \mathbf{1}_3, -\mathbf{1}_2, e^{i\pi})$, where $\omega_3 = e^{2\pi i/3}$. Thus the only faithful group available on the total field content is

$$(SU(3) \times SU(2) \times U(1))/\Gamma,$$

which cannot be read off from the geometry of any single associated bundle.

²⁴Once one has the fundamental representation of a given linear group, one can construct any other representation. But this already presupposes the fundamental representation itself, which is exactly what the VB-POV singles out via $G \simeq \rho(G) \simeq \text{Aut}(V)$.

To close, let me underscore the point with a simple case. In discussing item (i) above, I gave several examples where the VB-POV would fail. It is easy to imagine such a world—admissible under the PFB-POV—in which there is a single particle valued in V , together with a principal bundle (P, M, G) where $\dim(G) < \dim(\text{Aut}(V))$. Suppose it were physically possible in this world to obtain independent evidence for a non-trivial connection ϖ —say, through an Aharonov–Bohm-type experiment. Then one would have independent evidence for a symmetry group inferred from ϖ , alongside a geometrical structure for the fermions invariant under a larger automorphism group.²⁵

But that’s not our world.

Even in the Aharonov–Bohm case, as I have argued elsewhere [Gomes \(2025a\)](#), in the VB-POV the evidential link for a connection runs through the covariant derivative ∇ on a *fundamental vector bundle*. It is not *directly* tied to the principal connection, since that does not figure in the ontology. In this view, what experiments probe is the $\text{End}(E)$ -valued representative of ∇ , not ϖ . Therefore, from the geometry-first perspective, no such slack is possible.

The slack-world scenario is implausible; but its very implausibility points to an implicit assumption already built into our use of principal-associated formalisms, which in practice never exploit the full flexibility of the PFB-POV. The assumption is precisely that such formalisms ultimately rest on underlying fundamental vector bundles for which

$$G \simeq \rho(G) \simeq \text{Aut}(V). \quad (5.10)$$

In other words, the standard use of principal and associated fibre bundles tacitly presupposes the commitments of the geometry-first formulation of gauge theory.

Summing up: upon reconstructing symmetry groups, the VB-POV insists that each gauge subgroup factor be the automorphism group of its corresponding fundamental vector bundle. I do not see this tight identification as a limitation of the VB-POV; on the contrary, it is its chief virtue. By grounding group actions in geometry, the VB-POV rules out many possible theories. That restriction is methodologically valuable — and it still encompasses our best physics: the Standard Model itself falls within its bounds.

6 Conclusions

Feynman’s Nobel prize lecture, with which I began, reflected on his alternative formulation of quantum electrodynamics via path integrals. That formulation, like Minkowski’s introduction of spacetime—and indeed many other mathematically equivalent yet conceptually transformative innovations scattered through the history of physics—proved invaluable. They provided

²⁵I need not get bogged down in operational questions about what kind of experiment could in principle do this. Electromagnetic fields, after all, are only discernible via their action on charges; does that make their physical reality supervenient on the motion of particles? The point is simply that the gauge-invariant content of the principal connection is part of the physical apparatus of a gauge theory formulated in the PFB-POV.

new explanations and opened new directions for development.²⁶ I do not expect the geometry-first formulation of gauge theory developed here will ascend to comparable heights, nor do I expect it will become orthodoxy, as Feynman’s and Minkowski’s did.

But I do not want to understate what has been gained. The geometry-first formalism postulates a different ontology, offers independent ‘symmetry-free’ explanations of familiar mechanisms in gauge theory, and, above all, *cuts the slack* between symmetry and geometry. That tighter fit might be taken to explain why certain group–geometry correspondences are realised in our world and others are not.

Let me summarise the virtues of the independent explanations. In brief, in this new formulation, as in the familiar one, the Higgs field is a nowhere-vanishing section of a vector bundle with approximately constant norm. The component of the Higgs field carrying this constant nonzero norm plays the role of the *Higgs vacuum*. And although symmetries and vector bosons no longer appear at the fundamental level, the existence of such a section is enough: the geometry alone performs the explanatory work that symmetry was thought indispensable for.

Goldstone modes never appear here, and so never require elimination. The reason is simple: the constant magnitude of the Higgs vacuum section ensures that it is orthogonal to its covariant derivative. What in the symmetry-first formulation is described as the ‘acquisition of mass’ by vector bosons is, in this geometry-first account, nothing more than the non-vanishing of the (covariant) kinetic energy of the Higgs vacuum. In other words, it is nothing more than the run-of-the-mill idea that the kinetic term of the Higgs depends on the affine structure of the vector bundle.

Moreover, the covariant derivative along a single section of a vector bundle does not depend on all the affine degrees of freedom of the bundle (for $\dim(E_x) \geq 2$). The absent degrees of freedom correspond, in the symmetry-first idiom, to the unbroken gauge group, giving rise at the perturbative level to the massless photons. In this formulation, then, talk of ‘mass acquisition’ may strike a geometry-first militant—say, a relativist—as misplaced.²⁷

Turning to the Yukawa mechanism: I argued that standard presentations are explanatorily ‘opaque,’ and offered instead a more transparent geometric version of the Yukawa form itself. I readily admit that my sense of opacity may stem from a general preference for geometric explanations, *simpliciter*. But the point remains: as emphasised in Section 2.2, the mere availability of a geometric argument that bypasses representation theory is grist to my mill. The aim, after all, is to open the subject to a different community, with different, more geometric ways of thinking.

In this spirit, the geometric formalism already brings out three points about the Higgs and Yukawa mechanisms that, to my knowledge, have not been emphasised in the literature. (That

²⁶This kind of independent explanation is not always available for mathematically equivalent theories. If you disagree, I suggest you try to prove general relativity’s focussing theorem in the alternative language of Einstein algebras (Geroch, 1972).

²⁷To be sure, some would hesitate to say that gravitons acquire mass merely because a spacetime, or a collection thereof, admits a kinetic term for a vector field of constant norm; yet that is precisely the consensus for such theories (Jacobson, 2008).

does not mean they are controversial; perhaps they are simply too minor to warrant mention in standard presentations.)

First, fermion masses arise through the Yukawa mechanism without any need for a linearised expansion of the fields, or for the choice of local frames or bundle trivialisations. By contrast, the ‘mass acquisition’ of vector bosons, insofar as they are expressed tensorially, does require such choices: one expands the covariant derivative into a flat background plus a gauge potential. From the VB-POV, what would it even mean for the affine structure, or the covariant derivative ∇ , to ‘acquire mass’?

Second, in the geometric picture, the left-handed up and down quarks, and likewise the electron and electron-neutrino, are not distinct particles at all. They are simply the parallel and orthogonal components, with respect to the Higgs direction, of the first-generation left-handed quark fields and leptons.²⁸

Third, the quark Yukawa term depends essentially on the orientation of \mathbb{C}^2 . Up-type quarks couple to φ_c , which encodes the oriented area orthogonal to the Higgs. This explains why the geometry-first picture singles out $SU(2)$ rather than $U(2)$. By contrast, for \mathbb{C}^3 I know of no analogous mechanism: why does the Standard Model employ $SU(3)$ rather than $U(3)$?²⁹ This open question is emblematic of the broader methodological point: when symmetry is reconstructed from geometry, its explanatory role becomes sharper—and more constrained—than in the symmetry-first account.

Finally, Section 5 argued that the geometry-first formulation earns its keep by cutting the slack between symmetry and geometry. Where the PFB-POV tolerates a loose fit between the structure group and the geometry of its associated bundles, the VB-POV requires each gauge-group factor to coincide with the automorphism group of a fundamental vector bundle, whose sections must figure in the description of particles. The demand is restrictive, but the restriction is clarifying: it ties symmetries tightly to the spaces where matter fields live, and narrows the range of admissible theories. Fortunately, that narrowing is a virtue rather than a liability, since our best physics—the Standard Model—falls squarely within its bounds.

The methodological lesson of cutting the slack between symmetry and geometry points to two broader morals. First, that future developments of gauge theory might do well to begin with structured vector bundles and the tensors they carry. Second, that Occam’s razor, if it has an edge here, cuts with the VB-POV.

²⁸That they cannot represent physically distinct particles before symmetry breaking is noted in some textbooks—(Tong, 2025, p. 185) is an exception more than the rule—but I have not seen this parallel/orthogonal decomposition relative to the Higgs section made explicit.

²⁹Indeed, there is a canonical isomorphism $U(3) \simeq (SU(3) \times U(1))/\mathbb{Z}^3$, but the representations of $U(1)$ in the Standard Model do not appear to realise this isomorphism. Benjamin Muntz has suggested that the place to look may be in the triality constraints on baryon coupling: colourless states built from three quarks would not be invariant under the full $U(3)$.

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APPENDIX

A Principal and associated fiber bundles

I will start with the definition of a principal bundle:

Definition 1 (Principal Fiber Bundle) (P, M, G) consists of a smooth manifold P that admits a smooth free action of a (path-connected, semi-simple) Lie group, G : i.e. there is a map $G \times P \rightarrow P$ with $(g, p) \mapsto g \cdot p$ for some right (or left, with appropriate changes throughout) action \cdot and such that for each $p \in P$, the isotropy group is the identity (i.e. $G_p := \{g \in G \mid g \cdot p = p\} = \{e\}$). P has a canonical, differentiable, surjective map, called a projection, under the equivalence relation $p \sim g \cdot p$, such that $\pi : P \rightarrow P/G \simeq M$, where here \simeq stands for a diffeomorphism.

It follows from the definition that $\pi^{-1}(x) = \{G \cdot p\}$ for $\pi(p) = x$. And so there is a diffeomorphism between G and $\pi^{-1}(x)$, fixed by a choice of $p \in \pi^{-1}(x)$. It also follows (more subtly) from the definition, that local sections of P exist. A local section of P over $U \subset M$ is a map, $\sigma : U \rightarrow P$ such that $\pi \circ \sigma = \text{Id}_U$.

Given an element ξ of the Lie-algebra \mathfrak{g} , and the action of G on P , we use the exponential to find an action of \mathfrak{g} on P . This defines an embedding of the Lie algebra into the tangent space at each point, given by the *hash* operator: $\sharp_p : \mathfrak{g} \rightarrow T_p P$. The image of this embedding we call *the vertical space* V_p at a point $p \in P$: it is tangent to the orbits of the group, and is linearly spanned by vectors of the form

$$\text{for } \xi \in \mathfrak{g}: \quad \xi^\sharp(p) := \frac{d}{dt}\bigg|_{t=0}(\exp(t\xi) \cdot p) \in V_p \subset T_p P. \quad (\text{A.1})$$

Vector fields of the form ξ^\sharp for $\xi \in \mathfrak{g}$ are called *fundamental vector fields*.³⁰

The vertical spaces are defined canonically from the group action, as in (A.1). But we can define an ‘orthogonal’ projection operator, \widehat{V} such that:

$$\widehat{V}|_V = \text{Id}|_V, \quad \widehat{V} \circ \widehat{V} = \widehat{V}, \quad (\text{A.2})$$

and defining $H \subset TP$ as $H := \ker(\widehat{V})$. It follows that $\widehat{H} = \text{Id} - \widehat{V}$ and so $\widehat{V} \circ \widehat{H} = \widehat{H} \circ \widehat{V} = 0$. Moreover, since $\pi_* \circ \widehat{V} = 0$ it follows that:

$$\pi_* \circ \widehat{H} = \pi_*. \quad (\text{A.3})$$

³⁰It is important to note that there are vector fields that are vertical and yet are not fundamental, since they may depend on $x \in M$ (or on the orbit).

The connection-form should be visualized essentially as the projection onto the vertical spaces: given some infinitesimal direction, or change of frames, the vertical projection picks out the part of that change that was due solely to a translation across the group orbit. The only difference between \widehat{V} and ϖ is that the latter is \mathfrak{g} -valued, Thus we get it via the isomorphism between V_p and \mathfrak{g} (ϖ 's inverse is $\sharp : \mathfrak{g} \mapsto V \subset TP$). We can define it directly as:

Definition 2 (An principal connection-form) ϖ is defined as a Lie-algebra valued one form on P , satisfying the following properties:

$$\varpi(\xi^\sharp) = \xi \quad \text{and} \quad L_g^* \varpi = \text{Ad}_g \varpi, \quad (\text{A.4})$$

where the adjoint representation of G on \mathfrak{g} is defined as $\text{Ad}_g \xi = g \xi g^{-1}$, for $\xi \in \mathfrak{g}$; L_g^* is the pull-back of TP induced by the diffeomorphism $g : P \rightarrow P$.

Now, in possession of an principal connection, we can induce a notion of covariant derivative on associated vector bundles:

Definition 3 (Associated Vector Bundle) A vector bundle over M with typical fiber V , is associated to P with structure group G , is defined as:

$$E = P \times_\rho V := P \times V / \sim \quad \text{where} \quad (p, v) \sim (g \cdot p, \rho(g^{-1})v), \quad (\text{A.5})$$

where $\rho : G \rightarrow GL(V)$ is a representation of G on V .

One can get a covariant derivative on an associated vector bundle E from ϖ as follows: let $\gamma : I \rightarrow M$ be a curve tangent to $\mathbf{v} \in T_x M$, and consider its horizontal lift, γ_h . Suppose $\kappa(x) = [p, v]$. Then

$$\nabla_{\mathbf{v}} \kappa = \frac{d}{dt} [\gamma_h, v]. \quad (\text{A.6})$$

Conversely, we can define a horizontal subspace from the covariant derivatives as follows. For $p = e_1, \dots, e_n \in L(E)$, and for all curves $\gamma \in M$ such that $\mathbf{v} = \dot{\gamma}(0) \in T_x M$, with $\pi(p) = x$, let $\{e_1(t), \dots, e_n(t)\}$ be curves in E such that $\nabla_{\mathbf{v}}(e_i(t)) = 0$. Doing this for each v defines a horizontal subspace.

But we can also obtain the vector bundles more directly as follows:

Definition 4 (Vector Bundle) A vector bundle (E, M, V) consists of: E a smooth manifold that admits the action of a surjective projection $\pi_E : E \rightarrow M$ so that any point of the base space M has a neighborhood, $U \subset M$, such that, for all proper subsets of U , E is locally of the form $\pi^{-1}(U) \simeq U \times V$, where V is a vector space (e.g. \mathbb{R}^k , or \mathbb{C}^k) which is linearly isomorphic to $\pi^{-1}(x)$, for any $x \in M$.

Note that the isomorphism between $\pi^{-1}(U)$ and $U \times V$ is not unique, which is why there is no canonical identification of elements of fibers over different points of spacetime. Each choice of isomorphism is called ‘a trivialization’ of the bundle.

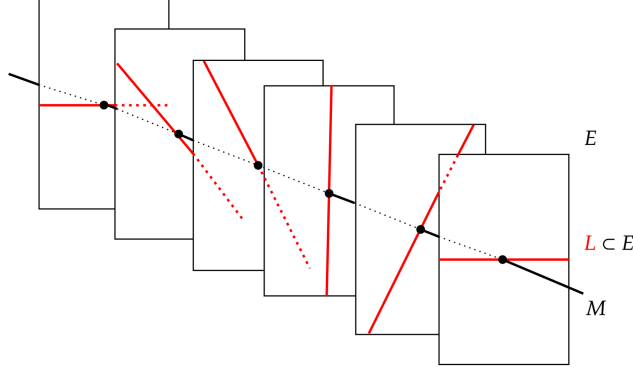


Figure 2: A vector bundle with a two-dimensional fiber over a one-dimensional base space, with a section here called L . (Figure taken from Wikipedia)

Definition 5 (A section of E) A section of E is a map $\kappa : M \rightarrow E$ such that $\pi_E \circ \kappa = \text{Id}_M$. We denote the space of smooth sections by $\kappa \in \Gamma(E)$ (see Figure 2 for a formulation of such a section).

Given a vector bundle (E, M, V) a covariant derivative D is an operator:

$$D : \Gamma(E) \rightarrow \Gamma(T^*M \otimes E) \quad (\text{A.7})$$

such that the product rule

$$D(f\kappa) = df \otimes \kappa + fD\kappa \quad (\text{A.8})$$

is satisfied for all smooth, real (or complex)-valued functions $f \in \Gamma(M)$.

Thus we can define parallel transport as follows:

Definition 6 (Parallel transport in a vector bundle) Let D be a covariant derivative on (E, M, V) , $\mathbf{v} \in E_x$ and $\gamma(t)$ a curve in M such that $\gamma(0) = x$. Then we define the parallel transport along γ as the unique section $\mathbf{v}_h(t)$ of $E|_\gamma$ such that:

$$D_{\gamma'} \mathbf{v}_h = 0. \quad (\text{A.9})$$

The existence and uniqueness of this map is guaranteed for $\gamma \subset U$ some open subset of M , and it follows from properties of solutions of ordinary differential equations (cf. (Kobayashi & Nomizu, 1963, Ch. II.2)).

Here D is an operator, not a tensor. But by introducing a coordinate frame or basis, we can represent it as such. This is the same as for spacetime covariant derivatives, ∇ : it is only upon the introduction of a frame or basis that we find an explicit representation.

It will prove useful to know that, given any vector bundle (E, M, V) the bundle of frames for E , called $L(E)$, is itself a principal fiber bundle $(L(E), M, GL(V))$: here elements of $\pi^{-1}(x)$ are linear frames of E_x , and $G \simeq GL(V)$ acts via ρ on the typical fibers. By construction, $E \simeq L(E) \times_\rho V$. Now, for $G' \subset G \simeq GL(V)$ we can partition the points of each orbit in P , $\mathcal{O}_p := Gp$, into orbits of G' . Each such choice gives a principal bundle with group G' and it induces further structure on the associated vector bundle, e.g. an inner product, by selecting

which frames are considered orthonormal. This is also a principal fiber bundle, $(L'(E), M, G')$, whose structure group is a proper subgroup of the general linear group, $G' \subset GL(V)$, taken to be the group that preserves the structure of V . This is called the *bundle of admissible frames*, e.g. of orthonormal frames. Conversely, if V has more than just the structure of a linear vector space, e.g. if it is endowed with an inner product, it will induce a subgroup $G' \subset GL(V)$ on P that respects that structure.

B The Standard Group-Theoretic Exposition of the Higgs Mechanism

Before turning to our geometric reformulation, we briefly review the conventional mathematical account of the Higgs mechanism, as in Hamilton (Hamilton, 2017, Ch. 8). This will allow us to highlight the points at which symmetry groups, stabilisers, and coset spaces enter essentially.

We begin with a compact Lie group G acting unitarily on a complex vector space W (the Higgs vector space). A Higgs potential of the form

$$V(w) = -\mu\|w\|^2 + \lambda\|w\|^4, \quad \mu, \lambda > 0,$$

is G -invariant, and has minima along a sphere

$$\mathcal{M}_{\text{vac}} = \{w \in W : \|w\| = v\}, \quad v = \sqrt{\mu/2\lambda}.$$

Thus the set of vacua is itself a homogeneous G -space:

$$\mathcal{M}_{\text{vac}} \cong G/H,$$

where $H = G_{w_0}$ is the stabiliser (isotropy subgroup) of a chosen vacuum vector $w_0 \in W$. Already here the reasoning is group-theoretic: the possible vacua are classified by subgroup data (G, H) .

A vacuum configuration is given by a constant section Φ_0 of the Higgs bundle, with $\Phi_0(x) = w_0$ for all $x \in M$. The unbroken subgroup H is compact (as a closed subgroup of G). If $H \subsetneq G$, the gauge theory is said to be *spontaneously broken* (Hamilton, 2017, Def. 8.1.6). The Higgs condensate Φ_0 is the non-zero background field in which other particles propagate, and is invariant only under $H \subset G$. Again, the classification of broken versus unbroken symmetries is a stabiliser argument.

Perturbations of the Higgs field $\Phi = \Phi_0 + \tilde{\phi}$ decompose relative to the tangent space at w_0 :

$$T_{w_0}W \cong T_{w_0}(G \cdot w_0) \oplus (T_{w_0}(G \cdot w_0))^\perp.$$

Group theory guarantees this orthogonal splitting (Hamilton, 2017, Lem. 8.1.12). One then expands $\tilde{\phi}$ in an eigenbasis of the Hessian:

$$\tilde{\phi} = \frac{1}{\sqrt{2}} \sum_{i=1}^d \pi_i e_i + \frac{1}{\sqrt{2}} \sum_{j=1}^{2n-d} \sigma_j f_j,$$

with $\{e_i\}$ tangent to the orbit $G \cdot w_0$ and $\{f_j\}$ orthogonal. The π_i are massless scalar fields: the Nambu–Goldstone bosons. The σ_j are massive scalars: the Higgs bosons (Hamilton, 2017, Def. 8.1.14). This is precisely Goldstone’s theorem: $\dim(G/H)$ massless scalars, deduced from the group structure of the vacuum manifold.

Physically the Goldstone bosons are unobservable, since they can be gauged away. Mathematically this is formalised by the *unitary gauge* (Hamilton, 2017, Def. 8.1.18, Thm. 8.1.20). One uses a physical gauge transformation $\gamma : M \rightarrow G$ to rotate the Higgs field entirely into the fixed direction w_0 :

$$\Phi(x) \mapsto \gamma(x) \cdot \Phi(x) = (0, \dots, 0, v + h(x)).$$

By definition, in unitary gauge the shifted Higgs field is orthogonal to the orbit $G \cdot w_0$, and the Nambu–Goldstone bosons vanish. This step is again an essentially group-theoretic argument, exploiting the transitivity of the G -action on the vacuum manifold.

Let $\mathfrak{g} = \mathfrak{g}$ be the Lie algebra of G , with $\mathfrak{h} = \text{Lie}(H)$ the subalgebra of unbroken generators. With respect to an invariant scalar product, decompose

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp.$$

The \mathfrak{h}^\perp directions correspond to broken generators. It is exactly these components of the gauge field A_μ that acquire mass terms from the kinetic energy of the Higgs:

$$\|D\Phi\|^2 \supset v^2 \sum_{X \in \mathfrak{h}^\perp} \|A_\mu^X\|^2.$$

Conversely, the \mathfrak{h} -components remain massless. This is the algebraic re-expression of the stabiliser picture.

In the electroweak theory $G = SU(2)_L \times U(1)_Y$ acts on $W = \mathbb{C}^2$. Choosing a vacuum vector $w_0 = (0, v)$, the stabiliser is a diagonal $U(1)$ subgroup, which is identified with electromagnetism. The Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ decomposes accordingly, and a change of basis (the Weinberg angle) diagonalises the mass form, producing massive W^\pm , Z^0 and a massless photon. Each of these identifications rests on the subgroup structure of G and the group-theoretic decomposition of its representation on W (Hamilton, 2017, Ch. 8.3).

Summary

The group-theoretic presentation of the Higgs mechanism thus depends essentially on:

1. Identifying the vacuum manifold as G/H , a homogeneous space.
2. Invoking Goldstone’s theorem: $\dim(G/H)$ massless modes.
3. Using unitary gauge to remove Goldstone bosons by G -action.
4. Decomposing \mathfrak{g} into $\mathfrak{h} \oplus \mathfrak{h}^\perp$ to classify massive and massless gauge bosons.

5. In the electroweak case, applying these steps to $SU(2)_L \times U(1)_Y$, producing W^\pm, Z^0 , and the photon.

These symmetry-based arguments provide the conventional foundation. In the next section we shall see how the same results can be obtained directly from the geometry of vector bundles, without recourse to stabilisers, cosets, or gauge fixing.

C Connecting the geometric and the standard interpretations of φ_c

Let $V \cong \mathbb{C}^2$ be the fundamental $SU(2)$ doublet space. We use only:

- the $SU(2)$ -invariant Hermitian form $h : V \times V \rightarrow \mathbb{C}$,
- the $SU(2)$ -invariant complex symplectic form $\varepsilon : V \times V \rightarrow \mathbb{C}$ (bilinear, antisymmetric),
- and the natural dual $V^* = \text{Hom}(V, \mathbb{C})$.

Write the “Hermitian dual” (transpose+conjugate) map

$$J : V \longrightarrow V^*, \quad J(v) := h(v, \cdot).$$

This map is *antilinear* and $SU(2)$ -equivariant into the contragredient representation:

$$J(Uv) = J(v) \circ U^{-1} \quad (U \in SU(2)).$$

Next, use ε to identify V with its dual *linearly*:

$$\varepsilon^\flat : V \rightarrow V^*, \quad \varepsilon^\flat(w) := \varepsilon(w, \cdot), \quad \text{with inverse } \varepsilon^\sharp := (\varepsilon^\flat)^{-1} : V^* \rightarrow V.$$

Equivariance of ε is the identity $U^T \varepsilon U = \varepsilon$, which is equivalent to

$$\varepsilon^\sharp \circ \alpha^* = \alpha \circ \varepsilon^\sharp \quad \text{for all } \alpha \in \text{End}(V).$$

Definition. Define the antilinear, $SU(2)$ -equivariant map

$$C := \varepsilon^\sharp \circ J : V \longrightarrow V, \quad \tilde{v} := C(v).$$

Equivariance follows immediately:

$$C(Uv) = \varepsilon^\sharp(J(v) \circ U^{-1}) = (\varepsilon^\sharp \circ J(v)) U^{-1} = U(\varepsilon^\sharp \circ J(v)) = U C(v).$$

If v has hypercharge Y , then J (being the Hermitian dual) implements the phase $e^{iY\theta} \mapsto e^{-iY\theta}$, so C flips $Y \mapsto -Y$. Hence, for the Higgs doublet $\phi \in (\mathbf{2}, +1)$ we set

$$\tilde{\phi} := C(\phi) = \varepsilon^\sharp(h(\phi, \cdot)) \in (\mathbf{2}, -1).$$

Component check. Choose an orthonormal basis so that h is the identity and $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $J(\phi)$ is the Hermitian row vector ϕ^\dagger , and

$$\tilde{\phi} = \varepsilon^\#(J(\phi)) = \varepsilon^{-1} \phi^\dagger = \varepsilon \phi^* = i\sigma_2 \phi^*,$$

i.e. the usual $\tilde{\phi}$.

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