EMULATION, REDUCTION, AND EMERGENCE IN DYNAMICAL SYSTEMS[†] [‡]

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Abstract

The received view about emergence and reduction is that they are incompatible categories. I argue in this paper that, contrary to the received view, emergence and reduction can hold together. To support this thesis, I focus attention on dynamical systems and, on the basis of a general representation theorem, I argue that, as far as these systems are concerned, the emulation relationship is sufficient for reduction (intuitively, a dynamical system DS_1 emulates a second dynamical system DS_2 when DS_1 exactly reproduces the whole dynamics of DS_2). This *representational* view of reduction, contrary to the standard *deductivist* one, is compatible with the existence of structural properties of the reduced system that are not also properties of the reducing one. Therefore, under this view, by no means are reduction and emergence incompatible categories but, rather, complementary ones.

1. Introduction

Emergence and reduction are traditionally viewed as incompatible categories (Beckermann 1992; Kim 1992). A property of a high level system is said to be emergent if it cannot be explained in terms of properties of the system's constitutive parts or, more precisely, if it is not one of the properties of more basic parts, which, together, make up the system. Thus, in order to speak of an emergent property P of system S_2 we need to verify, first, that S_2 is made up of another system S_1 (intuitively, S_1 is the system of the constitutive parts of S_2 taken in isolation, or in relations different from those typical of S_2; see Broad 1925, ch. 2) and, second, that P is not one of the properties of S_1. But then, the concept of emergence seems to yield a paradox: On the one hand, since S_2 *is made up of* S_1, S_2 is reduced to S_1; on the other one, since the property P of S_2 *is not one of the properties of* S_1, S_2 is not reduced to S_1. The traditional solution denies that the constitution relationship (S_2's being made up of S_1) is sufficient for reduction. By contrast, the second horn of the dilemma is not questioned, for it is taken for granted that S_2's reduction to S_1 entails that any property of S_2 is also a property of S_1.

This paper maintains that the traditional solution is irremediably flawed. In fact, there are pairs of systems, S_2 and S_1, for which both the constitution relationship (S_2 is made up of S_1) and the reduction one (S_2 is reduced to S_1) clearly hold together. Moreover, for these pairs of systems, it also turns out that some property of S_2 is not a property of S_1, so that any such property is emergent. It follows that, contrary to the received view, emergence and reduction by no means are incompatible categories but, rather, complementary ones.

To support this thesis, I will consider some simple examples of dynamical systems for which the emulation relationship holds. As intended here (Arnold 1977; Szlensk 1984; Giunti 1997), a *dynamical system* is a kind of mathematical model that expresses the idea of an arbitrary deterministic system, either reversible or irreversible, with discrete or continuous time or state space. Such models allow us to study in a precise way a whole series of typical phenomena in complex systems. Among them, in recent years, the phenomenon of emulation has gained growing attention (Wolfram 1983a, 1983b, 1984a, 1984b, 2002). Intuitively, a dynamical system DS_1 *emulates* a second dynamical system DS_2 when the first one exactly reproduces the whole dynamics of the second one.

The emulation relationship can be defined in a precise way for any two arbitrary dynamical systems, and it has also been shown (Giunti 1997, ch.1, th. 11) that, if DS_1 emulates DS_2, there is a third system DS_3 such that (i) DS_2 is isomorphic to DS_3; (ii) all states of DS_3 are states of DS_1; (iii) any state transition of DS_3 is constructed out of state transitions of DS_1. In this paper, I will prove a more general version of this theorem [*Virtual System Theorem VST*]; such a proof is based on a weaker and simpler definition of emulation. Because of this result, it makes perfect sense to claim that, if DS_1 emulates DS_2, then DS_2 is *made up* of DS_1, as well as DS_2 is *reduced* to DS_1. Therefore, to show that both reduction and emergence can hold together, it will suffice to exhibit two dynamical systems DS_1 and DS_2, and a property P, such that DS_1 emulates DS_2, DS_2 has P, but DS_1 does not have P. Finally, I will show that this situation already obtains for two pairs of simple finite discrete systems and that, in either case, the emergent property P is a strong form of irreversibility of system DS_2.

2. Dynamical systems and emulation

A dynamical system is a kind of mathematical model that expresses the idea of an arbitrary deterministic system, either reversible or irreversible, with discrete or continuous time or state space. Let Z be the integers, Z^+ the non-negative integers, R the reals and R^+ the non-negative reals; below is the exact definition of a dynamical system.

[1] *DS is a dynamical system* iff there is M, T, $(g^t)_{t\in T}$ such that $DS = (M, (g^t)_{t\in T})$ and

1. M is a non-empty set; M represents all the possible states of the system, and it is called the *state space*;
2. T is either Z, Z^+, R, or R^+; T represents the time of the system, and it is called the *time set*;
3. $(g^t)_{t\in T}$ is a family of functions from M to M; each function g^t is called a *state transition* or a *t-advance* of the system;
4. for any $t, v \in T$, for any $x \in M$, $g^0(x) = x$ and $g^{t+v}(x) = g^v(g^t(x))$.

[2] A *discrete dynamical system* is a dynamical system whose state space is finite or denumerable, and whose time set is either Z or Z^+; examples of discrete dynamical systems are Turing machines and cellular automata. [3] A *continuous dynamical system* is a dynamical system that is not discrete; examples of continuous dynamical systems are iterated mappings on R, and systems specified by ordinary differential equations.

[4] $DS = (M, (g^t)_{t\in T})$ is a *possible dynamical system* iff DS satisfies the first three conditions of definition [1]. We can now define the concept of an isomorphism between two possible dynamical systems as follows. [5] *u is an isomorphism of DS_1 in DS_2* iff $DS_1 = (M, (g^t)_{t\in T})$ and $DS_2 = (N, (h^v)_{v\in V})$ are possible dynamical systems, $T = V$, $u: M \to N$ is a bijection and, for any $t \in T$, for any $x \in M$, $u(g^t(x)) = h^t(u(x))$.

[6] DS_1 *is isomorphic* to DS_2 iff there is u such that u is an isomorphism of DS_1 in DS_2. It is easy to verify that the isomorphism relation is an equivalence relation on any given set of possible dynamical systems. (The concept of *set of all possible dynamical systems* is inconsistent, and we must then take as the basis of the theory of dynamical systems a specific, sufficiently large, set of possible dynamical systems.)

It is also not difficult to prove that the relation of isomorphism is a congruence with respect to the property of being a dynamical system, that is to say: if DS_1 is isomorphic to DS_2 and DS_1 is a dynamical system, then DS_2 is a dynamical system. This allows us to speak of abstract dynamical systems in exactly the same sense we talk of abstract groups, fields, lattices, order structures, etc. We can thus define: [7] an *abstract dynamical system* is any equivalence class of isomorphic dynamical systems.

Dynamical systems are appropriate models to study several interesting phenomena in complex systems. The one of emulation is typical of computational systems (Wolfram 2002), but it can in principle involve any two dynamical systems. The intuitive idea is that a dynamical system DS_1 emulates a second dynamical system DS_2 when the first one exactly reproduces the whole dynamics of the second one. Here are some examples. A universal Turing machine emulates any Turing machine; for any Turing machine *TM* there is a cellular automaton *CA* such that *CA* emulates *TM* (Smith 1971, th. 3), and vice versa; the simple cellular automaton specified by Wolfram's rule 18 emulates the one specified by rule 90 (both *CA* are monodimensional, with 2 possible values for cell, and neighborhood of radius 1; see Wolfram 1983b, 20).

Giunti 1997 (ch. 1, def. 4) gave a formal definition of the emulation relationship that applies to any two arbitrary dynamical systems. Here, I will employ a weaker and simpler definition, which nevertheless suffices for the present purposes.

[8] DS_1 *emulates* DS_2 iff $DS_1 = (M, (g^t)_{t\in T})$ and $DS_2 = (N, (h^v)_{v\in V})$ are dynamical systems, and there is an injective function u: $N \rightarrow M$ such that, for any $c \in N$, for any $v \in V$, there is $t \in T$ such that $u(h^v(c)) = g^t(u(c))$. Any function u that satisfies the previous condition is called an *emulation of* DS_2 *in* DS_1.

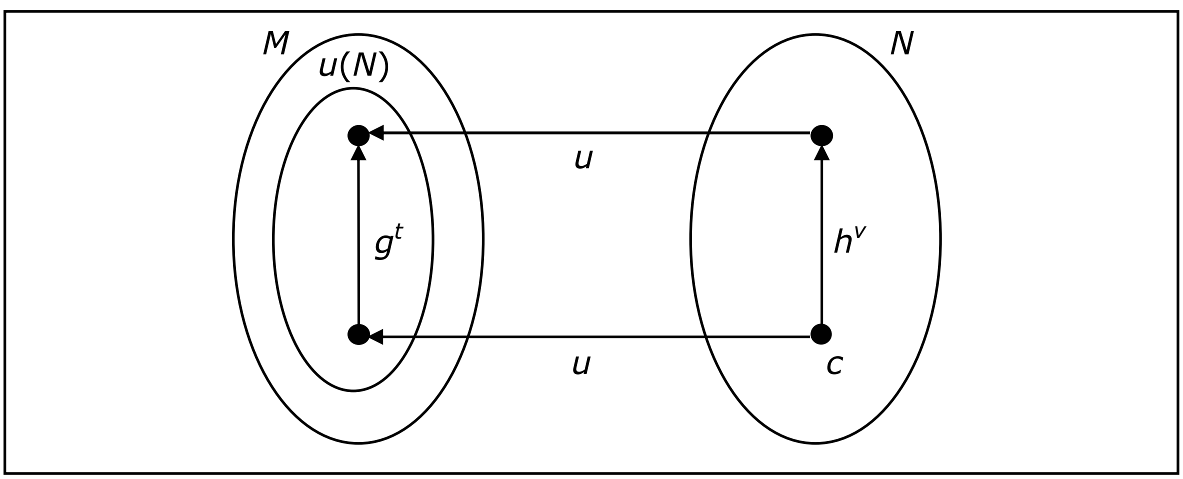

FIGURE 1 Emulation

3. Emulation, constitution, and reduction

Giunti 1997 (ch.1, th. 11) proved that, if u is an emulation of DS_2 in DS_1, there is a third system DS_3 such that (i) u is an isomorphism of DS_2 in DS_3; (ii) all states of DS_3 are states of DS_1; (iii) any state transition of DS_3 is constructed out of state transitions of DS_1. This result still holds for the weaker definition of emulation [8], as the following theorem shows.

Virtual System Theorem [*VST*]

- Let $DS_1 = (M, (g^t)_{t\in T})$ and $DS_2 = (N, (h^v)_{v\in V})$ be dynamical systems, and u be an emulation of DS_2 *in* DS_1;
- let $DS_3 = (\underline{N}, (\underline{h}^v)_{v\in V})$, where $\underline{N} = u(N)$ and, for any $a \in \underline{N}$, for any $v \in V$, $\underline{h}^v(a) = u(h^v(u^{-1}(a))$; the system DS_3 is called *the virtual u-system* DS_2 *in* DS_1 (see figure 2);

then:

(i) u is an isomorphism of DS_2 in DS_3;
(ii) all states of DS_3 are states of DS_1;
(iii) for any state transition $\underline{h}^v$ of DS_3, for any $a \in \underline{N}$, there is a state transition g^t of DS_1 such that $\underline{h}^v(a) = g^t(a)$.

Proof of (i)
By the definition of DS_3, for any $c \in N$, $u(h^v(c)) = u(h^v(u^{-1}(u(c))) = \underline{h}^v(u(c))$. Therefore, by the definition of isomorphism [5], u is an isomorphism of DS_2 in DS_3.

Proof of (ii)
Obvious, by the definition of DS_3.

Proof of (iii)
By the definition of DS_3, for any $v \in V$, for any $a \in \underline{N}$, $\underline{h}^v(a) = u(h^v(u^{-1}(a))$. Let $c = u^{-1}(a)$. Since u is an emulation of DS_2 in DS_1, by definition [8], there is $t \in T$ such that $u(h^v(c)) = g^t(u(c))$. Therefore, $\underline{h}^v(a) = g^t(u(c)) = g^t(a)$. Q.E.D.

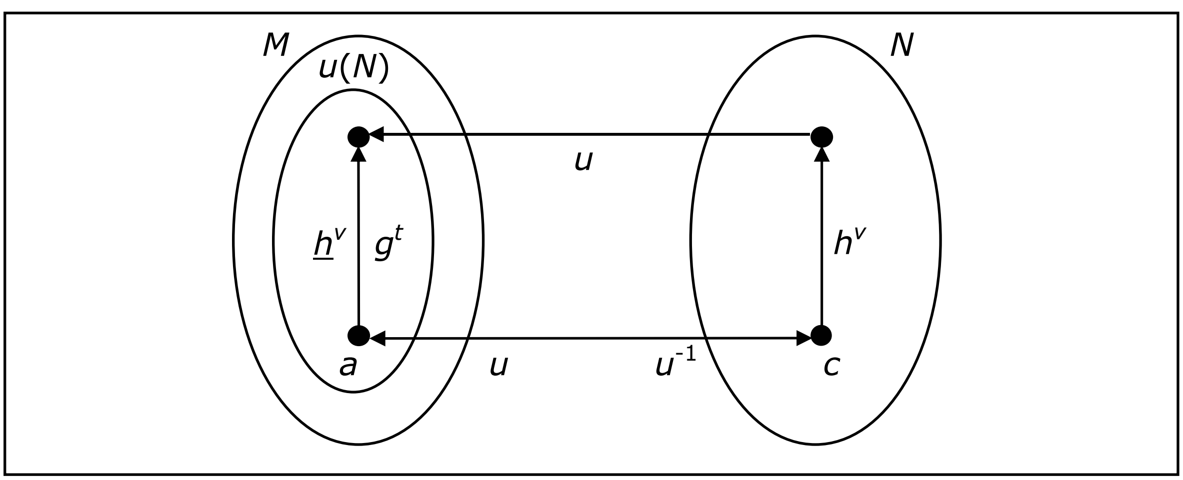

FIGURE 2 The virtual u-system DS_2 in DS_1

Because of $[VST]$, if a dynamical system DS_1 emulates a second system DS_2, it makes perfect sense to claim that DS_2 *is made up of* DS_1, as well as that DS_2 *is reduced to* DS_1. In other words, I maintain that emulation[1] is sufficient for both *constitution* and *reduction*. In fact, in virtue of $[VST]$, the system DS_2 turns out to be isomorphic to DS_3 (i.e. the virtual *u*-system DS_2 in DS_1), for whose constitutive parts (namely, its states and state-transitions) conditions (ii) and (iii), respectively, hold. This, I maintain, is sufficient for claiming that both the constitution and the reduction relationship hold between DS_2 and DS_1.

4. Emergence and reduction

A property P of a high level system S_2 is said to be *emergent with respect to a lower level system* S_1 just in case (a) S_2 is made up of S_1 (intuitively, S_1 is the system of the constitutive parts of S_2 taken in isolation, or in relations different from those typical of S_2; see Broad 1925, ch. 2) and (b) P is not one of the properties of S_1.[2]

Therefore, since emulation is sufficient for both constitution and reduction, in order to show that emergence and reduction can hold together, it is sufficient to exhibit a pair of dynamical systems DS_1 and DS_2, as well as a property P, such that DS_1 emulates DS_2, DS_2 has P, but DS_1 does not have P. In the next section, I will give two examples of such pairs of systems. For each pair, both DS_1 and DS_2 are small finite discrete systems (with just three states), while the emergent property P is the strong irreversibility[3] of system DS_2.

5. Examples of dynamical systems DS_1 and DS_2 such that (i) DS_2 is reduced to DS_1 and (ii) DS_2 has emergent properties with respect to DS_1

To state the examples, we first need a few more general concepts of dynamical systems theory. [9] A *cascade* is a dynamical system with discrete time, i.e., whose time set is either Z or Z^+. [10] A dynamical system is *reversible* iff its time set is either Z or R; [11] it is *irreversible* iff its time set is either Z^+ or R^+. Note that any t-advance g^t ($t > 0$) of an irreversible cascade $(M, (g^t)_{t \in Z^+})$ can always be thought as being generated by iterating t

[1] I recall that emulation, as defined here, is an exact relationship between two mathematical models; this sense of the term "emulation" is standard in both dynamical systems theory and computation theory, and it should not be confused with a common use of the same term, which refers to the relationship involved in the simulation of a physical system (e.g. a water flow) by a second one (e.g. a digital computer that, by means of appropriate software, implements a mathematical model of the water flow).

[2] In order to avoid trivial cases, it is also intended that P be a *structural property* of the mathematical kind that both S_1 and S_2 share. This means the following. (i) The two systems S_1 and S_2 are systems of the same mathematical kind K (for example, they are both dynamical systems, or groups, rings, etc.); (ii) the appropriate isomorphism relationship $\equiv$ is defined for the kind of system K; (iii) the property P is specific to the kind K, that is to say, for any system S, if $S \notin K$, then S has not P; (iv) the property P is preserved by the isomorphism $\equiv$, that is to say, for any two systems S_1 and $S_2 \in K$, if S_1 has P and $S_1 \equiv S_2$, then S_2 has P.

Also note that the characterization of an *emergent property* given in the text is not a formal definition; it is rather an explicit formulation of one of the senses of the term "emergence", which is quite standard in either the philosophical or systems science literature.

[3] Strong irreversibility is defined in the next section. It is easy to verify that strong irreversibility is a structural property (see note 2) of dynamical systems.

times a given function g: $M \rightarrow M$ (where $g^1 = g$).[4] Therefore, as far as an irreversible cascade is concerned, the whole dynamics of the system reduces to the behavior of its first t-advance g^1.

[12] A dynamical system is *logically reversible* iff it is irreversible, but all its state-transitions are injective; [13] it is *logically irreversible* iff it is irreversible and at least one of its state-transitions is not injective; [14] it is *strongly irreversible* iff there are two different states a and b and a state-transition g^v such that $g^v(a) = g^v(b)$ and, for any state-transition g^t, $g^t(a) \neq b$ and $g^t(b) \neq a$. Obviously, by definitions [12], [13] and [14], if a dynamical system is logically reversible, it is not strongly irreversible.[5]

Figure 3 shows a pair of cascades $DS_1 = (M, (g^t)_{t \in Z^+})$ and $DS_2 = (N, (h^v)_{v \in Z^+})$. The state space of DS_1 is $M = \{x, y, z\}$, and that of DS_2 is $N = \{a, b, c\}$. Each state-transition g^t of DS_1 is obtained by applying t-times the state-transition g^1, defined by: $g^1(x) = \mathrm{y}$, $g^1(y) = \mathrm{z}$, $g^1(z) = z$; analogously, an arbitrary state-transition h^v of DS_2 is obtained from the first state-transition h^1, defined by: $h^1(a) = c$, $h^1(b) = c$, $h^1(c) = c$. The function u: $N \rightarrow M$ is defined as follows: $u(a) = x$, $u(b) = y$, $u(c) = z$. Figure 3 then shows that (a) u is an emulation of DS_2 is DS_1, (b) DS_1 is logically irreversible but not strongly irreversible, (c) DS_2 is strongly irreversible. From this, since emulation is sufficient for both constitution and reduction, it follows that (i) DS_2 is reduced to DS_1 and (ii) the property P of strong irreversibility is an emergent property of DS_2 with respect to DS_1.

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4 When time is discrete (either Z or Z^+, but let us just consider the simpler case of Z^+), dynamical systems reduce to iterated mappings. In fact, on the one hand, if g: $M \rightarrow M$ is an arbitrary function, we can define the n-th ($n > 0$) iteration of g as $g \circ g \circ ... \circ g$ (n times), where $\circ$ is function composition; futhermore, by definition, the 0-th iteration is the identity function. Now, if we take the family $(g^n)_{n \in Z^+}$ of all the n-th iterations of g, it is immediate to verify that $(M, (g^n)_{n \in Z^+})$ is a dynamical system in the sense of definition 1 (also note that $g^1 = g$). Conversely, if $(M, (g^n)_{n \in Z^+})$ is a dynamical system whose time set is Z^+, then, by condition 4 of def. 1, and by the definition of family of the n-th iterations, $(g^n)_{n \in Z^+}$ is the family of the n-th iterations of g^1.

5 Irreversibility is a complex concept, and dynamical systems theory allows us to make fine distinctions. The use of just a non-negative time-set (either Z^+ or R^+) give us the weakest and most general concept of an irreversible system.

To understand this, we must take into account the algebraic structure $(\{g^t: t \in Z^+\}, \circ)$, i.e., the set of all state transitions together with the composition operation $\circ$. By condition 4 of def. 1, it is immediate to verify that: (1) if $T = Z$ or R, $(\{g^t: t \in Z^+\}, \circ)$ is a commutative group, whose unity is g^0. Furthermore, for any t, the algebraic inverse of g^t (which exists and is unique because $(\{g^t: t \in Z^+\}, \circ)$ is a group) is g^{-t}; this also entails that all state transitions are bijections, and that g^{-t} is the inverse function of g^t (i.e. the two concepts of *algebraic inverse* and *inverse function* coincide); (2) if $T = Z^+$ or R^+, $(\{g^t: t \in Z^+\}, \circ)$ is a commutative monoid, i.e., a commutative semigroup with unity; in this case too the unity is g^0 but, since negative times are lacking, no state transition has an inverse.

This is the situation from the formal point of view. Intuitively, this means that, if we consider just a non-negative time set, the system does not have the internal resources (i.e. the negative state transitions) to retrieve its past from the current state, even though it might be logically possible (it is possible if all state transitions are injective, i.e. if the system is logically reversible). To put it in a different way. The difference between a logically reversible system and a (fully) reversible one is that the second can *itself* retrieve its past. For a logically reversible system, instead, retrieving its past is just possible, but it cannot be made by the system itself (we need to employ external resources to do it).

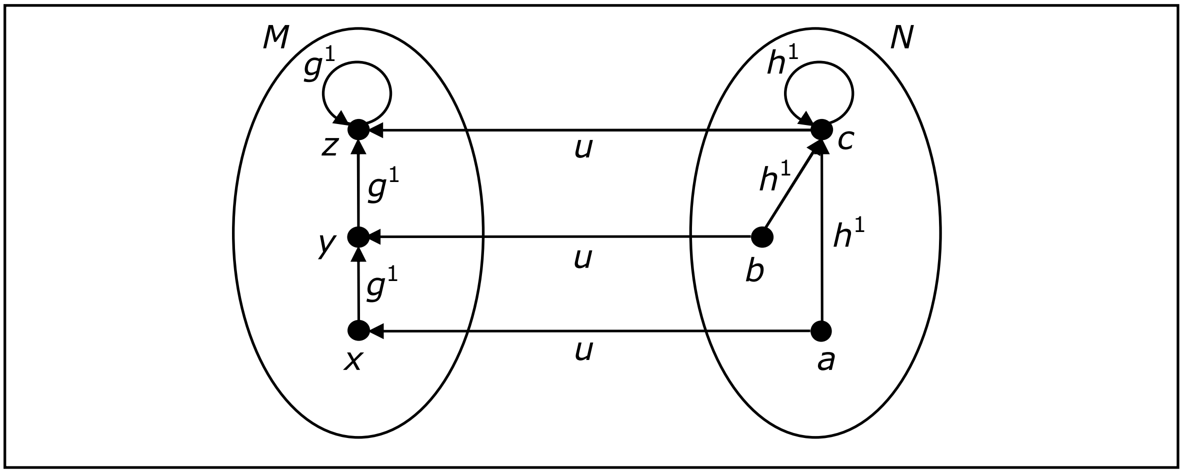

FIGURE 3 DS_1 emulates DS_2, DS_1 is logically irreversible but not strongly irreversible, and DS_2 is strongly irreversible

Figure 4 shows a second pair of cascades $DS_1 = (M, (g^t)_{t \in Z^+})$ and $DS_2 = (N, (h^v)_{v \in Z^+})$ such that (i) DS_2 is reduced to DS_1 and (ii) the property *P* of strong irreversibility is an emergent property of DS_2 with respect to DS_1. Note that DS_2 is identical to the corresponding system in figure 3. As for DS_1, the one in figure 4 is a logically reversible system.

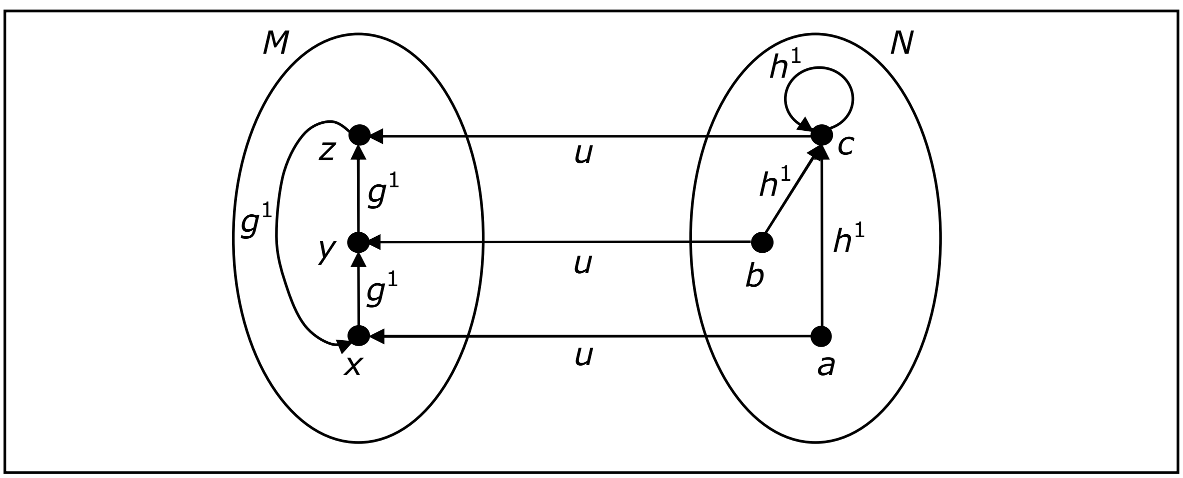

FIGURE 4 DS_1 emulates DS_2, DS_1 is logically reversible (thus, not strongly irreversible), and DS_2 is strongly irreversible

6. Concluding remarks: Toward a general representational theory of reduction and emergence

Traditionally, reduction has been analyzed in terms of a *deductive* relationship between two empirically interpreted formal *theories*, via correspondence rules between the terms of the two theories (Nagel 1961; Churchland 1979, 1985; Hooker 1981). By shifting attention from formal theories to mathematical *models*, it is natural to think of reduction in terms of some kind of *representation* relationship between two models. This paper has argued that, if the two models are dynamical systems, the relationship of emulation is sufficient for reduction (in virtue of [*VST*]).

An important point needs to be stressed. If we think of S_2's reduction to S_1 as a form of *deduction* of theory S_2 from theory S_1 (more precisely, the deduction of a *relevantly isomorphic image* of S_2 from S_1; see Churchland 1985, sec. 1; Beckermann 1992, 108), then it is obvious that all the properties of S_2 (more precisely, the properties referred to by statements of the relevantly isomorphic image of S_2) are a fortiori properties of S_1. Therefore, if we take *this kind* of approach to reduction, there cannot be two theories S_2 and S_1 such that S_2 is reduced to S_1 and S_2 has emergent properties with respect to S_1.

But this need not be the case if we think of reduction as a form of *representation* between two models S_1 and S_2, which grants the construction, within the representing model S_1, of an isomorphic (or, perhaps, just homomorphic) image of S_2. In fact, as I have just shown for the special case of dynamical systems, this view of reduction is compatible with the existence of structural properties of the reduced system that are not also properties of the reducing one. Therefore, under this view, reduction and emergence no longer are incompatible relationships but, rather, complementary ones.

At present, the *representational theory* of reduction and emergence has a precise formulation only if the models involved are dynamical systems. Even though many interesting models in real science are of this kind, by no means is this special formulation sufficient to account for all relevant cases of reduction or emergence. What we need is a *general* representational theory, as precise as the one restricted to dynamical systems, which apply to *arbitrary models*. The formulation of such a general theory, however, is not an easy matter, for it involves a preliminary investigation of fairly hard questions like: What is, *in general*, a mathematical structure? What is, *in general*, a mathematical model? What is an isomorphism between two *arbitrary* models? What is the relationship between two *arbitrary* models that generalizes the one of emulation between dynamical systems?

References

Arnold, Vladimir I. (1977), *Ordinary Differential Equations*. Cambridge: The MIT Press.

Beckermann, Ansgar (1992), "Supervenience, Emergence and Reduction", in *Emergence or Reduction? Essays on the Prospects of Nonreductive Physicalism*, 94-118. See Beckermann, Toffoli, and Kim (eds.) 1992.

Beckermann, Ansgar, Tommaso Toffoli, and Jaegwon Kim (eds.) (1992), *Emergence or Reduction? Essays on the Prospects of Nonreductive Physicalism*. Berlin: Walter de Gruyter.

Broad, Charlie Dunbar (1925), *The Mind and its Place in Nature*. London, Routledge and Kegan Paul.

Churchland, Paul M. (1979), *Scientific Realism and the Plasticity of Mind*. Cambridge: Cambridge University Press.

——— (1985), "Reduction, Qualia, and the Direct Introspection of Brain States", *Journal of Philosophy*, 82, 1:8-28.

Giunti, Marco (1997), *Computation, Dynamics, and Cognition*. New York: Oxford University Press.

Hooker, Clifford Alan (1981), "Towards a General Theory of Reduction", *Dialogue* 20:38-60, 201-236, 496-529.

Kim, Jaegwon (1992), "Downward Causation in Emergentism and Non-reductive Physicalism", in *Emergence or Reduction? Essays on the Prospects of Nonreductive Physicalism*, 119-138. See Beckermann, Toffoli, and Kim (eds.) 1992.

Nagel, Ernest (1961), *The Structure of Science*. New York: Harcourt, Brace & World.

Smith, Alvy Ray III (1971), "Simple Computation-universal Cellular Spaces", Journal of the Association for Computing Machinery 18, 3:339-353.

Szlensk, Wieslaw (1984), *An Introduction to the Theory of Smooth Dynamical Systems*. Chichister, England: John Wiley and Sons.

Wolfram, Stephen (1983a), "Statistical Mechanics of Cellular Automata", *Reviews of Modern Physics* 55, 3:601-644.

——— (1983b), "Cellular Automata", *Los Alamos Science* 9:2-21.

——— (1984a), "Computer Software in Science and Mathematics", *Scientific American* 56:188-203.

——— (1984b), "Universality and Complexity in Cellular Automata", in Doyne Farmer, Tommaso Toffoli, and Stephen Wolfram (eds.), *Cellular Automata*. Amsterdam: North Holland Publishing Company, 1-35.

——— (2002), *A New Kind of Science*. Champaign, IL: Wolfram Media, Inc.